Symmetric sign pattern matrices that require unique inertia

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Received 7 March 2000; accepted 16 May 2001

Abstract

A sign pattern matrix is a matrix whose entries are from the set \{+, −, 0\}. The purpose of this paper is to characterize symmetric sign patterns that require unique inertia, that is, all the real symmetric matrices with the given sign pattern must have the same inertia. Further, some constructions to obtain sign patterns that require unique inertia are provided. Sign patterns corresponding to some special graphs are also considered. Finally, extensions to complex sign patterns are mentioned. © 2001 Elsevier Science Inc. All rights reserved.

AMS classification: 15A18; 15A48

Keywords: Inertia; Sign pattern matrix; Inertia set; Unique inertia

1. Introduction

In qualitative and combinatorial matrix theory, we study the properties of a matrix based on combinatorial information such as the signs of entries in the matrix. A matrix whose entries are from the set \{+, −, 0\} is called a sign pattern matrix (or sign pattern, or pattern). We denote the set of all \(n \times n\) sign pattern matrices by \(Q_n\). For a real matrix \(B\), by \(\text{sgn}(B)\) we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry of \(B\) is replaced by + (respectively, −, 0). If \(A \in Q_n\), then the sign pattern class of \(A\) is defined by

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\[ Q(A) = \{ B : \text{sgn}(B) = A \}. \]

The inertia of a real symmetric matrix \( B \), written as \( i(B) \), is the triple of integers
\[ i(B) = (i_+(B), i_-(B), i_0(B)), \]
where \( i_+(B) \) (respectively, \( i_-(B), i_0(B) \)) denotes the number of positive (respectively, negative, zero) eigenvalues of matrix \( B \) counted with their algebraic multiplicities. The inertia of matrix \( B \) is said to be balanced if \( i_+(B) = i_-(B) \). Notice that the rank of a real symmetric matrix \( B \) is equal to \( i_+(B) + i_-(B) \). For a symmetric sign pattern \( A \), we define the inertia set of \( A \) to be
\[ i(A) = \{ i(B) : B = B^T \in Q(A) \}. \]
As a special case, if \( i(B_1) = i(B_2) \) for all real symmetric matrices \( B_1, B_2 \in Q(A) \), we say the sign pattern \( A \) requires unique inertia. There is an extensive literature on inertias of matrices, for instance see [6] and the recent survey paper [2]. However, little was known about the inertia of a matrix solely based on the knowledge of the signs of the entries of the matrix. In this paper, we characterize the symmetric sign patterns that require unique inertias.

If \( A = (a_{ij}) \) is an \( n \times n \) sign pattern matrix, then a formal product of the form
\[ \gamma = a_{i_1j_1}a_{i_2j_2} \ldots a_{i_kj_k}, \]
where each of the elements is nonzero and the index set \( \{i_1, i_2, \ldots, i_k \} \) consists of distinct indices, is called a simple cycle of length \( k \), or a \( k \)-cycle, in \( A \). A composite cycle \( \gamma \) in \( A \) is a product of simple cycles, say \( \gamma = \gamma_1\gamma_2 \ldots \gamma_m \), where the index sets of the \( \gamma_i \)'s are mutually disjoint. If the length of \( \gamma_i \) is \( l_i \), then the length of \( \gamma \) is \( \sum_{i=1}^{m} l_i \). If we say a cycle \( \gamma \) is an odd (respectively even) cycle, we mean that the length of the simple or composite cycle \( \gamma \) is odd (even). In this paper, the term cycle always refers to a composite cycle (which as a special case could be a simple cycle).

A matching of size \( k \) in a digraph \( D = (V, E) \) is a set of \( k \) arcs
\[ M = \{ (i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k) \} \subseteq E \]
such that the vertices \( i_1, i_2, \ldots, i_k \) are distinct and the vertices \( j_1, j_2, \ldots, j_k \) are distinct. \( M = \{ (i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k) \} \) is called a principal matching if \( \{i_1, i_2, \ldots, i_k \} = \{ j_1, j_2, \ldots, j_k \} \). Matchings and principal matchings in a matrix \( A = (a_{ij}) \) are defined similarly, as formal products of the form
\[ a_{i_1j_1}a_{i_2j_2} \ldots a_{i_kj_k}, \]
where each of the elements is nonzero. It is easy to see that a principal matching of \( A \) corresponds to a (composite) cycle of \( A \). Clearly, any subset of a matching is also a matching.

If \( A \) is a symmetric sign pattern, we define \( \text{smr}(A) \), the symmetric minimal rank of \( A \) by
\[ \text{smr}(A) = \min \{ \text{rank} B : B = B^T, B \in Q(A) \}. \]
Similarly, the symmetric maximal rank of \( A \), \( \text{SMR}(A) \), is
\[ \text{SMR}(A) = \max \{ \text{rank} B : B = B^T, B \in Q(A) \}. \]
For \( A \in Q_n \), the minimal rank of \( A \), denoted as \( \text{mr}(A) \), is defined by
\[ \text{mr}(A) = \min \{ \text{rank} B : B \in Q(A) \}. \]
The maximal rank of \( A \), \( \text{MR}(A) \), is given by
\[ \text{MR}(A) = \max \{ \text{rank} B : B \in Q(A) \}. \]
If \( S \in Q_n \) is a diagonal sign pattern matrix, each of whose diagonal entries is + or −, then \( S \) is called a signature pattern. A signature similarity on a pattern \( A \in Q_n \) is defined as a product of the form \( SAS \), where \( S \) is a signature pattern. A sign pattern matrix \( S \) is called a permutation pattern if exactly one entry in each row and column is equal to +, and all other entries are 0. A product of the form \( S^TAS \), where \( S \) is a permutation pattern, is called a permutation similarity. Note that if \( S \) is a signature pattern, then \( SS = I \) (the diagonal sign pattern with all diagonal entries equal to +); if \( S \) is a permutation pattern, then \( S^T S = SS^T = I \).

Suppose \( P \) is a property referring to a real matrix. Then a sign pattern \( A \) is said to require \( P \) if every real matrix in \( Q(A) \) has property \( P \), or to allow \( P \) if some real matrix in \( Q(A) \) has property \( P \).

A sign pattern \( A \in Q_n \) is said to be sign nonsingular if \( Q(A) \) requires nonsingularity. It is well known that \( A \) is sign nonsingular if and only if \( \det(A) = + \) or \( \det(A) = − \), that is, in the standard expansion of \( \det(A) \) into \( n! \) terms, there is at least one nonzero term, and all the nonzero terms have the same sign. \( A \) is said to be sign singular if \( Q(A) \) requires singularity, or equivalently, if \( \det(A) = 0 \).

For a symmetric \( n \times n \) sign pattern \( A \), by \( \bar{G}(A) \) we mean the undirected graph of \( A \), with vertex set \( \{1, \ldots, n\} \) and \( (i, j) \) is an edge if and only if \( a_{i,j} \neq 0 \).

In this paper, we mostly restrict our attention to real symmetric matrices. However, by generalizing the definition of \( i(B) = (i_+(B), i_−(B), i_0(B)) \) to arbitrary real square matrices \( B \), such that \( i_+(B) \) (respectively, \( i_−(B) \), \( i_0(B) \)) is the number of eigenvalues of \( B \) in the right half plane (respectively, left half plane, imaginary axis), some results about \( i(B) \) for the symmetric case can be generalized to the nonsymmetric case.

### 2. Some general results and observations

Let \( A \) be a symmetric \( n \times n \) sign pattern. It can be seen that there are exactly \( \frac{1}{2}(n+1)(n+2) \) triples that can be inertias of all \( n \times n \) matrices. Indeed, if \( k = i_+(B) \), then \( 0 \leq k \leq n \), \( 0 \leq i_−(B) \leq n - k \), and \( i_0(B) = n - (i_+(B) + i_−(B)) \) is uniquely determined by \( i_+(B) \) and \( i_−(B) \). It is clear that not every subset of the set of these \( \frac{1}{2}(n+1)(n+2) \) triples can be achieved by some \( i(A) \). For example, there is no \( n \times n \) symmetric sign pattern \( A \) with \( i(A) = \{(n,0,0), (0,n,0)\} \), since \( (n,0,0) \in i(A) \) means \( A \) allows positive definiteness and hence all diagonal entries of \( A \) are +, while \( (0,n,0) \in i(A) \) means \( A \) allows negative definiteness and hence all diagonal entries of \( A \) are −, a contradiction. In this paper, we focus our attention on symmetric sign patterns that require unique inertia.

Recall that \( i_+(B) + i_−(B) = \text{rank}(B) \) for a real symmetric matrix \( B \). The next theorem gives the maximum value of \( i_+(B) + i_−(B) \) for \( B = B^T \in Q(A) \). First, we need to prove a lemma.
Lemma 2.1. If $A = A^T$, then for any matching $\gamma = a_{i_1j_1} \ldots a_{i_kj_k}$ in $A$, there exists a principal matching $\beta$ in $A$ of size $\geq k$, whose index set is contained in $\{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_k\}$.

Proof. Recall that each nonzero entry $a_{ij}$ of $A$ may be identified with an arc of $D(A)$. Thus the entries in $\gamma = a_{i_1j_1} \ldots a_{i_kj_k}$ induce a subgraph $H$ of $D(A)$. Since $\gamma$ is a matching, each vertex in $H$ has in-degree and out-degree at most one. It follows that $H$ is a disjoint union of directed paths and cycles. Clearly, a cycle in $H$ corresponds to a principal matching in $A$. Without loss of generality, let $P = a_{12}a_{23} \ldots a_{s-1,s}$ be a path in $A$ corresponding to a maximal directed path in $H$. Then $P$ is also a matching of size $s - 1$ in $A$. Let $t = \lfloor \frac{s}{2} \rfloor$. Then the principal matching $P' = a_{12}a_{24}a_{34}a_{34} \ldots a_{2t-1,2t}a_{2t,2t-1}$ has size $s$ or $s - 1$ depending on whether $s$ is even or odd. Note that the index set of $P'$ is contained in the index set of $P$. Repeating this construction for the disjoint directed paths in $H$, and keeping the cycles in $H$, we arrive at a desired principal matching. □

Theorem 2.2. Let $A = A^T \in Q_n$. Then

$$\max \{i_+(B) + i_-(B) : B = B^T \in Q(A)\} = \text{SMR}(A) = \text{MR}(A).$$

Proof. The first part is obvious. To prove $\text{SMR}(A) = \text{MR}(A)$, note that $\text{SMR}(A) \leq \text{MR}(A)$, by definition. Suppose $\text{MR}(A) = r$, equivalently, the maximal size of matchings in $A$ is $r$. Then by Lemma 2.1, there exists a principal matching of size $k \geq r$. Without loss of generality, we may just consider the principal matching $\gamma = a_{12}a_{23} \ldots a_{k-1,k}a_{k1}$, where the elements are all $+$. Notice that if $k$ is odd, then

$$\left(\begin{array}{cccccc}
0 & + & 0 & \ldots & 0 & + \\
+ & 0 & + & 0 & 0 \\
0 & + & 0 & \ldots & \vdots & \\
\vdots & 0 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
+ & 0 & \ldots & 0 & + & 0
\end{array}\right)_{k \times k}$$

is sign nonsingular, and hence there exists a real symmetric matrix $B \in Q(A)$ with $\text{rank}(B) \geq k$.

If $k$ is even, by symmetrically emphasizing the entries $a_{i,i+1}$ and $a_{i+1,i}$ for odd $i$, we get a real symmetric matrix $B$ with $\text{rank}(B) \geq k$. Thus, $\text{SMR}(A) \geq \text{MR}(A)$. It follows that $\text{SMR}(A) = \text{MR}(A)$. □

In contrast to Theorem 2.2, we now show by example that $\text{mr}(A) = \text{smr}(A)$ is not true in general.
Example 2.3. Let
\[
A = \begin{pmatrix}
+ & 0 & + & + \\
0 & + & + & + \\
+ & + & - & 0 \\
+ & + & 0 & -
\end{pmatrix}.
\]
Then \(\text{smr}(A) = \text{SMR}(A) = 4\), since \(A\) requires unique inertia \(i(A) = (2, 2, 0)\), see Theorem 3.1. Note that \(A\) is not sign nonsingular, since in the usual expansion of \(\det(A)\), there are two oppositely signed terms \(a_{11}a_{22}a_{33}a_{44} = +\) and \(-a_{14}a_{23}a_{31}a_{42} = -\). Since the upper left \(3 \times 3\) submatrix of \(A\) can be seen to be sign nonsingular, it then follows that \(\text{mr}(A) = 3\). Thus we have \(3 = \text{mr}(A) < \text{smr}(A) = 4\).

Clearly, if \(A\) requires unique inertia, then \(\text{smr}(A) = \text{SMR}(A)\). In Section 3, we prove that the converse of this statement is also true.

A graph (or digraph) is said to be bipartite if its vertex set can be partitioned as \(V_1, V_2\) such that every edge of the graph is between a vertex in \(V_1\) and a vertex in \(V_2\). It is well known that a graph is bipartite if and only if it has no odd cycle. We say a matrix is bipartite if its graph is bipartite. Note that a symmetric sign pattern matrix \(A\) is bipartite if and only if it is permutation similar to a matrix of the form
\[
\begin{pmatrix}
0 & A_1 \\
A_1^T & 0
\end{pmatrix}.
\]
It is known that if there is no odd cycle in a (not necessarily symmetric) sign pattern \(A \in \mathbb{Q}_n\), then all the matrices \(B \in \mathbb{Q}(A)\) have balanced inertia.

In view of this, we have the following result, which will be used in Section 4.

Proposition 2.4. For a symmetric bipartite sign pattern
\[
A = \begin{pmatrix}
0 & A_1 \\
A_1^T & 0
\end{pmatrix},
\]
we have
\[
i(A) = \{ (k, k, n - 2k) \mid k = \text{rank}(B_1) \text{ for some } B_1 \in \mathbb{Q}(A_1) \}.
\]

We make the interesting observation that for a symmetric sign pattern \(A\) with \(\text{smr}(A) < \text{SMR}(A)\), it may happen that for some \(k\), \(\text{smr}(A) < k < \text{SMR}(A)\), there is no matrix \(B = B^T \in \mathbb{Q}(A)\) with \(\text{rank}(B) = k\).

Example 2.5. Let
\[
A = \begin{pmatrix}
0 & 0 & + & + \\
0 & 0 & + & + \\
+ & + & 0 & 0 \\
+ & + & 0 & 0
\end{pmatrix}.
\]
We have \( \text{smr}(A) = 2 \) and \( \text{SMR}(A) = 4 \). For \( k = 3 \), there is no matrix \( B = B^T \in Q(A) \) with \( \text{rank}(B) = 3 \). Indeed, by Proposition 2.4, every \( B = B^T \in Q(A) \) has balanced inertia and hence \( B \) has even rank.

A natural question is: Is there a symmetric sign pattern \( A \) for which \( \text{mr}(A) < \text{smr}(A) < \text{SMR}(A) \)? By taking \( A \) to be the direct sum of the patterns in Examples 2.3 and 2.5, we see that this can be the case.

3. Sign patterns that require unique inertia

In this section we explore sign patterns that require unique inertia.

**Theorem 3.1.** Let \( A \in Q_n \) be a symmetric sign pattern with all diagonal entries nonzero. Then \( A \) requires a unique inertia if and only if \( A \) is permutationally similar to a pattern of the form

\[
\begin{pmatrix}
I_k & W \\
W^T & -I_{n-k}
\end{pmatrix},
\]

where \( k \) is the number of positive diagonal entries of the matrix \( A \),

\[
I_k = \begin{pmatrix}
+ & & \\
& + & \\
& & +
\end{pmatrix}_{k \times k},
\]

and \( W \) stands for some sign pattern of size \( k \times (n - k) \). Further, \( i(A) = (k, n - k, 0) \).

**Proof.** Suppose that \( A \in Q_n \) requires unique inertia. Performing a permutation similarity if necessary, we may assume that the first \( k \) diagonal entries of \( A \) are \( + \), and the remaining diagonal entries are \( - \).

By emphasizing the diagonal entries, it can be seen that \( i(A) = (k, n - k, 0) \). Assume that the sign pattern \( A \) is not permutationally similar to a sign pattern of the form

\[
\begin{pmatrix}
I_k & W \\
W^T & -I_{n-k}
\end{pmatrix}.
\]

Without loss of generality, we can assume that there is a symmetric matrix \( B \in Q(A) \) such that the leading \( 2 \times 2 \) principal submatrix of \( B \) is

\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix},
\]

all the diagonal entries of \( B \) are 1 or \(-1\), and all the remaining entries of \( B \) have sufficiently small absolute values. Since the upper-left submatrix of order 2 of \( B \)
has eigenvalues 3 and $-1$, we have $i(B) = (k - 1, n - k + 1, 0)$, contradicting the assumption that $A$ requires unique inertia.

Conversely, if a sign pattern $A$ is permutationally similar to a pattern

$$E = \begin{pmatrix} I_k & W \\ W^T & -I_{n-k} \end{pmatrix},$$

then every real symmetric matrix in $Q(E)$ is diagonally congruent to a matrix of the form

$$B = \begin{pmatrix} I_k & W \\ W^T & -I_{n-k} \end{pmatrix}.$$  

Let

$$C = \begin{pmatrix} I_k & -W \\ 0 & I_{n-k} \end{pmatrix}.$$  

Then

$$C^TBC = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} - W^TW \end{pmatrix}.$$  

Note that $I_k$ is positive definite and $-I_{n-k} - W^TW$ is negative definite. By Sylvester’s law of inertia, we have $i(B) = (k, n - k, 0)$. Since $B = B^T \in Q(E)$ is arbitrary, we conclude that the sign pattern matrix $A$ requires unique inertia. \qed

As a consequence of Theorem 3.1, it can be seen that for $n \geq 3$, if a symmetric sign pattern $A$ of order $n$ with all diagonal entries nonzero requires unique inertia, then $A$ has at least $\lceil \frac{1}{2}n^2 - n \rceil$ zero entries. This shows that just as sign nonsingularity implies a certain degree of sparsity (see [8]), requiring unique inertia also implies a certain degree of sparsity.

For general symmetric patterns, we have the following characterization.

**Theorem 3.2.** A symmetric sign pattern $A$ requires unique inertia if and only if $\text{smr}(A) = \text{SMR}(A)$.

**Proof.** Necessity is clear. We now prove sufficiency.

Assume that $\text{smr}(A) = \text{SMR}(A)$. It is well known that the eigenvalues are continuous functions of the entries of a matrix (see [5, Appendix D]). Let $B_1$, $B_2$ be any two symmetric matrices in $Q(A)$. Define $B(t) = (1 - t)B_1 + tB_2$, $0 \leq t \leq 1$, $B(t) = B_1$ if $t < 0$ and $B(t) = B_2$ if $t > 1$. Then $B(t) = B(t)^T \in Q(A)$ for all $t$. The eigenvalues of $B(t)$ are $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$, where each $\lambda_k(t)$ is a continuous function of $t$ on $(-\infty, \infty)$. For each fixed $t \in [0, 1]$, if $i(B(t)) = (p, q, n - p - q)$, then $\lambda_{i_1}(t) > 0, \ldots, \lambda_{i_p}(t) > 0, \lambda_{j_1}(t) < 0, \ldots, \lambda_{j_q}(t) < 0$, and $\lambda_j(t) = 0$ if $j \notin \{i_1, \ldots, i_p, j_1, \ldots, j_q\}$. Since $\text{smr}(A) = \text{SMR}(A)$, we have $p + q = \text{smr}(A)$. Because of the continuity of the functions $\lambda_k(t)$, there is an open interval $(t - \delta(t), t + \delta(t))$ over which $\lambda_{i_1}(t) > 0, \ldots, \lambda_{i_p}(t) > 0, \lambda_{j_1}(t) < 0, \lambda_{j_2}(t) < 0, \ldots, \lambda_{j_q}(t) < 0$.  

Since \( \text{rank}(B(t)) = \text{smr}(A) \) is fixed throughout, we see that \( \lambda_j(t) = 0 \) over this open interval for all \( j \notin \{i_1, \ldots, i_p, j_1, \ldots, j_q\} \). This shows that the zero eigenvalues remain at zero. Thus \( i(B(t)) \) is constant over \((t - \delta(t), t + \delta(t))\). The set \( \{(t - \delta(t), t + \delta(t)) \mid t \in [0, 1]\} \) forms an open cover of the interval \([0, 1]\). Since \([0, 1]\) is a compact set, there exists a finite open cover \( \{(t_i - \delta(t_i), t_i + \delta(t_i)) \mid 1 \leq i \leq m\} \). Since \( i(B(t)) \) is constant on all the open intervals in this finite open cover, we see that \( i(B(t)) \) is constant on \([0, 1]\). Thus \( i(B_1) = i(B(0)) = i(B(1)) = i(B_2) \). Therefore, \( A \) requires unique inertia. \[\square\]

**Corollary 3.3.** If a symmetric sign pattern \( A \) requires fixed rank, then \( A \) requires unique inertia. In particular, if \( A \) is a sign nonsingular symmetric pattern, then \( A \) requires unique inertia.

Note that \( A \in Q_n \) requires \( n \) distinct eigenvalues does not imply \( A \) requires a unique inertia, as the following example shows.

**Example 3.4.** Let
\[
A = \begin{pmatrix}
+ & + & 0 \\
+ & + & + \\
0 & + & +
\end{pmatrix}.
\]

Then \( A \) requires three distinct eigenvalues (see [3, Lemma 2.1]). By emphasizing the 1-cycles, we see that \((3, 0, 0) \in i(A)\). By symmetrically emphasizing the cycle \( a_{12}a_{21}a_{33} \), we see that \((2, 1, 0) \in i(A)\). Thus \( A \) does not require unique inertia.

We can now give a further characterization of symmetric sign patterns that require unique inertia. This theorem, as well as Theorem 3.2, will be very useful in the following section on graphs and unique inertia.

**Theorem 3.5.** Let \( A \in Q_n \) be a symmetric sign pattern, with the maximum length of the (composite) cycles in \( A \) equal to \( m \geq 1 \). Then \( A \) requires unique inertia if and only if \( E_m(B) \) has the same sign for all \( B = B^T \in Q(A) \). In particular, if all the terms in \( E_m(B) \) have the same sign for any \( B \in Q(A) \), then \( A \) requires unique inertia.

**Proof.** Since the maximum length of cycles in \( A \) is \( m \), for any \( B \in Q(A) \), the characteristic polynomial of \( B \) is given by
\[
P_B(\lambda) = \lambda^n - E_1(B)\lambda^{n-1} + E_2(B)\lambda^{n-2} - \cdots + (-1)^m E_m(B)\lambda^{n-m},
\]
where \( E_k(B) \) is the sum of all cycles (simple or composite) of length \( k \) in \( B \) properly signed. By Theorem 2.2, it can be seen that \( \text{SMR}(A) = \text{MR}(A) = m \). Observe also that \( \text{smr}(A) = m \) if and only if \( E_m(B) \neq 0 \) for every \( B = B^T \in Q(A) \), since the rank of \( B \) is equal to the number of nonzero eigenvalues of \( B \). Suppose that \( E_m(B_1) < 0 \) and \( E_m(B_2) > 0 \) for some symmetric matrices \( B_1 \) and \( B_2 \) in \( Q(A) \). Since \( E_m(B) \) is
a continuous function of the entries in the matrix $B$, by considering convex combinations $(1 - t)B_1 + tB_2 \in Q(A), 0 \leq t \leq 1$, we get a matrix $\hat{B} = \hat{B}^T \in Q(A)$ with $E_m(\hat{B}) = 0$. It follows that $E_m(B) \neq 0$ for every $B = B^T \in Q(A)$ if and only if $E_m(B)$ has the same sign for all $B = B^T \in Q(A)$. The result now follows from Theorem 3.2. $\Box$

We note that the condition that all the terms in $E_m(B)$ have the same sign for any $B \in Q(A)$ is not a necessary condition for $A$ to require unique inertia. For example, by Theorem 3.1, the sign pattern

$$A = \begin{pmatrix}
+ & 0 & + & + \\
0 & + & + & + \\
+ & + & 0 & - \\
+ & + & 0 & - \\
\end{pmatrix}$$

requires unique inertia. However, for any $B \in Q(A)$, $E_4(B)$ has oppositely signed terms $b_{11}b_{22}b_{33}b_{44} > 0$, and $-b_{14}b_{24}b_{23}b_{31} < 0$. This is an example where $\det(B) \neq 0$ for all $B = B^T \in Q(A)$, yet $A$ is not sign nonsingular.

Furthermore, when $A \in Q_n$ is nonsymmetric, the condition that all the terms in $E_m(B)$ have the same sign for any $B \in Q(A)$ does not guarantee that $A$ requires unique inertia in the more general sense. For example, for the sign pattern

$$A = \begin{pmatrix}
+ & + & 0 \\
0 & + & + \\
+ & 0 & + \\
\end{pmatrix}$$

both terms in $E_3(B)$ are positive. However, by emphasizing the diagonal entries, we can obtain a matrix $B_1 \in Q(A)$ with $i(B_1) = (3, 0, 0)$; by emphasizing the $(1, 2)$, $(2, 3)$ and $(3, 1)$ entries, we can obtain a matrix $B_2 \in Q(A)$ with $i(B_2) = (1, 2, 0)$.

We shall now give conditions for a symmetric sign pattern $A$ to require unique inertia, which are more recognizable, especially for large and complicated symmetric sign patterns. For a cycle $\gamma$ in $A$, $l(\gamma)$ will denote the length of $\gamma$, and $\text{sign}(\gamma)$ will denote the actual product (+ or −) of the entries on $\gamma$. Further, we let $p(\gamma)$ denote the number of simple odd cycles $\beta$ in $\gamma$ such that $\text{sign}(\beta) = (-)^{[l(\beta) - 1]/2}$, and we let $q(\gamma)$ denote the number of simple odd cycles $\beta$ in $\gamma$ such that $\text{sign}(\beta) = (-)^{[l(\beta) + 1]/2}$.

By $X_A$ we mean the symmetric $n \times n$ matrix which is obtained from $A$ by replacing each nonzero entry $a_{ij}$ by a real variable $x_{ij}$, where we restrict $x_{ij}$ to take on values whose sign is $a_{ij}$. We emphasize that $x_{ij} = x_{ji}$. For a cycle $\gamma$ in $A$, we use $X_\gamma$ to denote the symmetric $n \times n$ matrix obtained from $X_A$ by setting the entries in $X_A$ off the $\gamma$ positions to zero.

**Lemma 3.6.** Let $\gamma$ be a cycle of length $m$ in a symmetric sign pattern $A$. Then exactly one of the following four conditions holds for the permissible values of the variables involved:

(i) $E_m(X_\gamma) > 0$;
(ii) \( E_m(X_{\gamma}) \geq 0 \), and equality is achieved by some permissible values of the variables;

(iii) \( E_m(X_{\gamma}) < 0 \);

(iv) \( E_m(X_{\gamma}) \leq 0 \), and equality is achieved by some permissible values of the variables.

**Proof.** Note that if \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_q \), where each \( \gamma_i \) is simple with length \( l_i \), then

\[
E_m(X_{\gamma}) = \prod_{i=1}^{q} E_{l_i}(X_{\gamma_i}).
\]

Hence, we may assume that \( \gamma \) is a simple cycle. If \( m = 1 \), then \( \gamma = a_{ii} \) for some \( i \) and \( E_m(X_{\gamma}) = x_{ii}^2 \). If \( m = 2 \), then \( \gamma = a_{ij}a_{jj} \) for some \( i \neq j \), and \( E_m(X_{\gamma}) = -x_{ij}^2 \). If \( m \geq 3 \) is odd, then \( \gamma \) has the form \( \gamma = a_{i_1i_2}a_{i_2i_3} \ldots a_{i_mi_1} \) and \( E_m(X_{\gamma}) = 2x_{i_1i_2}x_{i_2i_3} \ldots x_{i_{m-1}i_1} \). Finally, if \( m \geq 4 \) is even, then with \( \gamma \) of the form \( \gamma = a_{i_1i_2}a_{i_2i_3} \ldots a_{i_{m-1}i_1} \), it can be seen that

\[
E_m(X_{\gamma}) = (-1)^{m/2}x_{i_1i_2}^2x_{i_3i_4}^2 \cdots x_{i_{m-1}i_m}^2 + (-1)^{m/2}x_{i_2i_3}^2x_{i_4i_5}^2 \cdots x_{i_{m-1}i_1}^2 - 2x_{i_1i_2}x_{i_2i_3} \cdots x_{i_{m-1}i_1}.
\]

The results in the statement of Lemma 3.6 now follow easily. \( \square \)

For two \( m \)-cycles \( \gamma_1 \) and \( \gamma_2 \), we say that \( E_m(X_{\gamma_1}) \) and \( E_m(X_{\gamma_2}) \) are weakly of the same sign if \( E_m(X_{\gamma_1})E_m(X_{\gamma_2}) \geq 0 \), for the permissible values of the variables involved.

**Theorem 3.7.** Let \( A \in \mathbb{Q}_n \) be a symmetric sign pattern, with the maximum length of the (composite) cycles in \( A \) equal to \( m \geq 1 \). Then the conditions

(i) \( A \) requires unique inertia,

(ii) for any two \( m \)-cycles \( \gamma_1 \) and \( \gamma_2 \) in \( A \), \( p(\gamma_1) - q(\gamma_1) = p(\gamma_2) - q(\gamma_2) \),

(iii) for any two \( m \)-cycles \( \gamma_1 \) and \( \gamma_2 \) in \( A \), \( E_m(X_{\gamma_1}) \) and \( E_m(X_{\gamma_2}) \) are weakly of the same sign

satisfy (i) implies (ii) and (ii) implies (iii).

**Proof.** Let \( \beta \) be a simple cycle in \( A \). Without loss of generality, we may assume that \( \beta = a_{12}a_{23} \ldots a_{k-1,k}a_{1} \). If \( k = l(\beta) \) is even, by symmetrically emphasizing the entries \( a_{12}, a_{34}, \ldots, a_{k-1,k} \), we can obtain \( \frac{1}{2}k \) eigenvalues close to 1 and \( \frac{1}{2}k \) eigenvalues close to \(-1\).

If \( k = l(\beta) \) is odd, then the symmetric pattern of order \( k \) induced by \( \beta \) requires unique inertia, since it is sign nonsingular. Further, by symmetrically emphasizing the entries \( a_{12}, a_{34}, \ldots, a_{k-2,k-1} \), we can see that there are at least \( \frac{1}{2}(k-1) \) positive (respectively, negative) eigenvalues. The product of all the \( k \) eigenvalues has the
same sign as \( \text{sign}(\beta) = a_{12}a_{23} \ldots a_{k-1,k}a_{k1} \). Therefore, there is one more positive (respectively, negative) eigenvalue than negative (respectively, positive) eigenvalues if \( \text{sign}(\beta) = (-)^{\lfloor l(\beta)/2 \rfloor} \) (respectively, \( \text{sign}(\beta) = (-)^{\lceil l(\beta)/2 \rceil} \)).

Let \( \gamma_1 \) and \( \gamma_2 \) be two \( m \)-cycles in \( A \). By using the above constructions on the simple cycles in \( \gamma_1 \), we can obtain a symmetric matrix \( B_1 \in Q(A) \) of rank \( m \) whose signature (i.e., the number of positive eigenvalues minus the number of negative eigenvalues) is \( p(\gamma_1) - q(\gamma_1) \). Similarly, for \( \gamma_2 \), we can obtain a symmetric matrix \( B_2 \in Q(A) \) of rank \( m \) whose signature is \( p(\gamma_2) - q(\gamma_2) \). Now, if \( A \) requires unique inertia, then clearly (ii) holds. Thus we have shown that (i) implies (ii).

We now show that (ii) implies (iii). Let \( \gamma_1 \) and \( \gamma_2 \) be two \( m \)-cycles in \( A \). Note that \( E_m(X_{\gamma_1}) \) weakly has the same sign as the product of the \( m \) nonzero eigenvalues of a symmetric matrix \( B_1 \in Q(A) \) obtained by suitably symmetrically emphasizing the entries on \( \gamma_1 \), as in the previous paragraph. It is clear that the sign of the product of the \( m \) nonzero eigenvalues of \( B_1 \) is determined by \( m = \text{rank}(B_1) \) and the signature of \( B_1 \), which equals \( p(\gamma_1) - q(\gamma_1) \). Similarly, \( E_m(X_{\gamma_2}) \) weakly has the same sign as the product of the \( m \) nonzero eigenvalues of a symmetric matrix \( B_2 \in Q(A) \) obtained by suitably symmetrically emphasizing the entries on \( \gamma_2 \); and the sign of the product of the \( m \) nonzero eigenvalues of \( B_2 \) is determined by the signature of \( B_2 \), which equals \( p(\gamma_2) - q(\gamma_2) \). Thus if (ii) holds, then \( B_1 \) and \( B_2 \) have the same signature (and the same rank, \( m \)), and hence, \( B_1 \) and \( B_2 \) have the same number of positive (respectively, negative) eigenvalues. It then follows that \( E_m(X_{\gamma_1}) \) and \( E_m(X_{\gamma_2}) \) weakly have the same sign, namely, (iii) holds. \( \square \)

We point out that (ii) (and hence, (iii)) does not imply (i). For instance, for the sign pattern

\[
A = \begin{pmatrix}
0 & + & 0 & + \\
+ & 0 & + & 0 \\
0 & + & 0 & + \\
+ & 0 & + & 0
\end{pmatrix},
\]

it is clear that (ii) holds, since \( A \) does not have any simple odd cycles. Also, from the above theorem or from the proof of Lemma 3.6, we see that (iii) holds. However, it can be seen that \( 2 = \text{smr}(A) < \text{SMR}(A) = 4 \). Hence, \( A \) does not require unique inertia, namely, (i) fails to hold.

In view of Theorem 3.5, it can be seen that a symmetric sign pattern \( A \) with maximal cycle length \( m \) requires unique inertia if and only if \( E_m(X_A) \) is either always positive or always negative for all permissible values of the variables involved. Although polynomial functions that are positive on certain closed or compact semi-algebraic sets have been studied in real algebraic geometry and functional analysis (see for example [1,7]), there is yet no characterization of polynomial functions that are positive on open semi-algebraic sets. However, if \( E_m(X_A) \) can be written as a sum of the form \( \pm(P_1 + P_2) \), where \( P_1 \) is a sum of positive monomials (taking into consideration the signs of the variables) so that \( P_1 > 0 \) for all permissible values.
of the variables, and \( P_2 \) is a sum in which each term is the product of a positive monomial with the square of a polynomial, so that \( P_2 \geq 0 \) for all permissible values of the variables, then clearly \( E_m(X_A) \) has fixed sign, and hence, \( A \) requires unique inertia. We conjecture that the existence of such a decomposition of \( E_m(X_A) \) into \( \pm (P_1 + P_2) \) is also a necessary condition for \( A \) to require unique inertia.

**Example 3.8.** Let

\[
A = \begin{pmatrix}
  + & 0 & + & + \\
  0 & + & + & + \\
  + & + & - & 0 \\
  + & + & 0 & 0
\end{pmatrix}.
\]

Then \( E_4(X_A) = \det X_A = P_1 + P_2 \), where \( P_1 = -x_{11}x_{22}^2x_{33} - x_{22}x_{14}^2x_{33} > 0 \) since \( x_{33} < 0 \) while all the other variables are \( > 0 \), and \( P_2 = x_{13}x_{24}^2 - 2x_{13}x_{24}x_{14}x_{23} + x_{14}x_{23}^2 = (x_{13}x_{24} - x_{14}x_{23})^2 \geq 0 \). Therefore, \( \det X_A > 0 \) for all permissible values of the variables. It follows that \( A \) requires unique inertia.

4. **Graphs and unique inertia**

A sign pattern \( A \) is a symmetric tree sign pattern if \( A \) is symmetric and \( G(A) \) is a tree, possibly with 1-loops. We investigate which symmetric tree sign patterns require unique inertia. It is easy to see that a symmetric tree sign pattern with all diagonal entries equal to 0 is a symmetric bipartite sign pattern. The following theorem, which follows directly from Proposition 2.4, describes when a symmetric bipartite sign pattern requires unique inertia.

**Proposition 4.1.** An \( n \times n \) symmetric bipartite sign pattern

\[
A = \begin{pmatrix}
  0 & A_1^T \\
  A_1 & 0
\end{pmatrix}
\]

requires the unique inertia \( i(A) = (k, k, n - 2k) \) if and only if \( A_1 \) requires rank \( k \).

In particular, the sign pattern

\[
A = \begin{pmatrix}
  0 & A_1^T \\
  A_1 & 0
\end{pmatrix}
\]

has the unique inertia \( i(A) = (k, k, 0) \) if and only if \( A_1 \) is a sign nonsingular pattern of order \( k \). Note also that in Proposition 4.1, \( A_1 \) requires rank \( k \) is equivalent to \( A \) requires rank \( 2k \), that is, \( \text{mr}(A) = \text{MR}(A) = 2k \).

**Proposition 4.2.** Let \( A \in Q_n \) be a symmetric tree sign pattern with no loops. Then \( A \) requires unique inertia, and \( i(A) = (k, k, n - 2k) \) for some positive integer \( k \).
Proof. Since \( A \) is bipartite, there is no odd cycle in \( A \). The only simple cycles in the symmetric tree sign pattern \( A \) are positive 2-cycles. Let \( 2k \) be the maximum length of the cycles in \( A \). Then all the terms in \( E_{2k}(B) \) have the same sign as \((-1)^k\). Hence, by Theorem 3.5, \( A \) requires unique inertia. By Proposition 4.1, \( i(A) = (k, k, n - 2k) \).

We now consider symmetric tree sign pattern matrices \( A \) that possibly have some nonzero diagonal entries. The next result handles the case when \( G(A) \) is a “star”. As usual, a * entry in a sign pattern can be +, −, or 0.

**Theorem 4.3.** Up to permutation similarity, signature similarity, and negation, a symmetric tree sign pattern

\[
A = \begin{pmatrix}
* & + & + & \ldots & + \\
+ & * & 0 & \ldots & 0 \\
+ & 0 & * & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
+ & 0 & 0 & \ldots & * 
\end{pmatrix}
\]

requires unique inertia if and only if the diagonal of \( A \) has one of the following forms:

\((*, \ldots, *, 0), (0, +, \ldots, +), (+, -, \ldots, -)\).

Proof. Let \( m \) be the maximum length of the cycles in \( A \). If the diagonal of \( A \) has one of the three forms indicated in the theorem, then it can be seen that all terms in \( E_m(B) \) have the same sign for any \( B \in Q(A) \). (In the first two cases, a longest cycle in \( A \) consists of one 2-cycle and a number of 1-cycles.) Hence, by Theorem 3.5, \( A \) requires unique inertia.

Up to the operations mentioned in the theorem, the only other possible diagonals of \( A \) are of the forms:

\((+, +, *, \ldots, *)\) and \((0, +, -, *, \ldots, *)\),

where all the * entries are nonzero. In the first case, \( A \) does not require unique inertia by Theorem 3.1. In the second case, by symmetrically emphasizing \( b_{12}, b_{21}, b_{33}, b_{44}, \ldots, b_{nn} \) and \( b_{13}, b_{31}, b_{22}, b_{44}, \ldots, b_{nn} \), respectively, we can obtain two different inertias. Thus, \( A \) does not require unique inertia.

**Example 4.4.** A sign pattern of the form

\[
A = \begin{pmatrix}
* & + & 0 & \ldots & \ldots & 0 \\
+ & * & + & 0 & \ldots & 0 \\
0 & + & * & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & + & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & + \\
0 & 0 & \ldots & 0 & + & * 
\end{pmatrix}
\]
is a symmetric tree sign pattern. For each $B = B^T \in Q(A)$ and for each $\lambda \in \sigma(B)$, $\text{rank}(\lambda I - B) = n - 1$ so that (algebraic multiplicity of $\lambda$) = (geometric multiplicity of $\lambda$) = 1. Hence, $B$ has $n$ distinct eigenvalues, and $\text{smr}(A) \geq n - 1$. There are thus three cases:

**Case 1.** $\text{smr}(A) = \text{SMR}(A) = n$. By Theorem 3.2, $A$ requires unique inertia, so that by Theorem 3.5, $\det(B)$ has the same sign for all $B = B^T \in Q(A)$. In this case, since the only simple cycles in $A$ are 1- and 2-cycles, for any composite cycle $\gamma$ of length $n$ in $A$, by symmetrically emphasizing the entries on $\gamma$, we can obtain a $B = B^T \in Q(A)$ so that $\det(B)$ has the same sign as $(\text{sgn } \gamma)\gamma$. Hence all the terms in $E_n(B) = \det(B)$ have the same sign. Thus $A$ is sign nonsingular.

**Case 2.** $\text{smr}(A) = \text{SMR}(A) = n - 1$. By Theorem 3.2, $A$ requires unique inertia. Since $\text{MR}(A) = \text{SMR}(A) < n$, $A$ is sign singular. By modifying the argument in the above case, it can be seen that all the terms in $E_{n-1}(B)$ have the same sign. It follows that each matrix $B \in Q(A)$ has $n - 1$ nonzero eigenvalues, and hence, $A$ requires rank $n - 1$.

**Case 3.** $n - 1 = \text{smr}(A) < \text{SMR}(A) = n$. By Theorem 3.2, $A$ does not require unique inertia.

To illustrate Case 3 in Example 4.4, consider the sign pattern

$$A = \begin{pmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{pmatrix}.$$  

By emphasizing the diagonal entries, we can obtain a symmetric matrix $B_1 \in Q(A)$ with $i(B_1) = (1, 3, 0)$. By emphasizing the $(1, 2), (2, 1), (3, 4)$ and $(4, 3)$ entries, we can obtain a symmetric matrix $B_2 \in Q(A)$ with $i(B_2) = (2, 2, 0)$.

For a general symmetric tree sign pattern, we have the following result.

**Theorem 4.5.** Let $A$ be a symmetric tree sign pattern, with the maximum length of the cycles in $A$ equal to $m \geq 1$. Then $A$ requires unique inertia if and only if all the terms in $E_m(B)$ have the same sign for any $B \in Q(A)$. In this case, $A$ requires rank $m$.

**Proof.** The sufficiency is clear. Conversely, if $A$ requires unique inertia, we then have $\text{smr}(A) = \text{SMR}(A) = \text{MR}(A) = m$. By Theorem 3.5, $E_m(B)$ has the same sign for all $B = B^T \in Q(A)$. Since the only simple cycles in $A$ are 1- and 2-cycles, for any composite cycle $\gamma$ of length $m$ in $A$, by symmetrically emphasizing the entries on $\gamma$, we can obtain a $B = B^T \in Q(A)$ so that $E_m(B)$ has the same sign as $(\text{sgn } \gamma)\gamma$. It follows that all terms in $E_m(B)$ have the same sign for any $B \in Q(A)$. Thus each
matrix $B \in Q(A)$ has $m$ nonzero eigenvalues, and hence, $mr(A) \geq m$. It is then clear that $A$ requires rank $m$. □

The second situation that we consider in this section is the case where $G(A)$ is a cycle. This case is easy to analyze.

**Theorem 4.6.** Let $A \in Q_n$ be a symmetric sign pattern with all diagonal entries equal to 0, and suppose $G(A)$ is a simple cycle of length $n$. If $n$ is odd, then $A$ is sign nonsingular and hence $A$ requires unique inertia. If $n$ is even, then $A$ requires unique inertia if and only if $A$ is sign nonsingular. More specifically, for even $n$, $A$ requires unique inertia if and only if $\frac{1}{2}n$ is odd (respectively, even) and the number of $-$ entries on a simple $n$-cycle in $A$ is even (respectively, odd).

**Proof.** The case when $n$ is odd is essentially discussed in the proof of Theorem 2.2. If $n$ is even, then $A$ is bipartite with maximal rank $n$. It follows from the comment after Proposition 4.1 that if $n$ is even, then $A$ requires unique inertia if and only if $A$ is sign nonsingular. Now, for even $n$, a cycle of length $n$ in $A$ consists of one $n$-cycle or $\frac{1}{2}n$ 2-cycles (resulting in a $(-)^{n/2}$ term in $\det A$). Hence, $A$ requires unique inertia if and only if $\frac{1}{2}n$ is odd (even) and the number of $-$ entries on a simple $n$-cycle in $A$ is even (odd); this unique inertia is $(\frac{1}{2}n, \frac{1}{2}n, 0)$. □

For example, the sign pattern

$$
\begin{pmatrix}
0 & + & + \\
+ & 0 & + \\
+ & + & 0 \\
\end{pmatrix}
$$

requires unique inertia. By replacing each $+$ by 1 in the sign pattern, we see that the unique inertia is $(1, 2, 0)$. On the other hand, the sign pattern

$$
\begin{pmatrix}
0 & + & 0 & + \\
+ & 0 & + & 0 \\
0 & + & 0 & + \\
+ & 0 & + & 0 \\
\end{pmatrix}
$$

is not sign nonsingular, and hence, does not require unique inertia.

Other situations where the graphs $G(A)$ are familiar graphs can be considered. Generally speaking, the denser $A$ is, the more difficult it is to analyze $\iota(A)$. For example, the inertia set of the $n \times n$ ($n \geq 4$) sign pattern $A$, all of whose off-diagonal entries are $+$ and all diagonal entries are 0, is unknown. However, it is clear that $A$ does not require unique inertia. Indeed, by making all off-diagonal entries to be 1, we obtain a symmetric matrix $B \in Q(A)$ with $\iota(B) = (1, n - 1, 0)$; while by symmetrically emphasizing $k$ (where $2 \leq k \leq n/2$) disjoint simple 2-cycles, we can obtain a symmetric $B \in Q(A)$ with at least $k$ positive eigenvalues.
5. Some generalizations

Complex sign pattern matrices are discussed in [4]. We now consider \( n \times n \) Hermitian complex sign patterns, namely complex sign patterns of the form \( A = A_1 + iA_2 \), where \( A_1 \) is symmetric and \( A_2 \) is skew-symmetric. In this case, we define the inertia set to be \( i(A) = \{ i(B) : B = B^H \in Q(A) \} \). If \( i(A) \) consists of only one inertia triple, we say that \( A \) requires unique inertia.

For a Hermitian complex sign pattern, the hermitian maximal rank can be strictly less than the maximal rank. For example, the complex sign pattern

\[
A = \begin{pmatrix}
0 & + & -i \\
+ & 0 & + \\
i & + & 0
\end{pmatrix}
\]

has maximal rank 3, but its Hermitian maximal rank is 2. However, it can be shown that results analogous to the results in Section 3 hold for Hermitian complex sign patterns. Two such results are as follows.

**Theorem 5.1.** A Hermitian complex sign pattern requires unique inertia iff the Hermitian minimal and Hermitian maximal ranks are the same.

**Theorem 5.2.** Let \( A \) be a Hermitian complex sign pattern, with the Hermitian maximal rank of \( A \) equal to \( m \geq 1 \). Then \( A \) requires unique inertia iff \( E_m(B) \) has the same sign for all Hermitian \( B \in Q(A) \).

It appears that characterizing sign pattern matrices \( A \) (symmetric or non-symmetric) which require unique inertia in the general sense (that is, \( i(B_1) = i(B_2) \), for any two real matrices \( B_1 \) and \( B_2 \) in \( Q(A) \)) is very difficult. Sign nonsingularity of \( A \) does not imply that \( A \) requires unique inertia in this sense, as shown in the second example after Theorem 3.5. However, we can show the following result, which gives two necessary conditions. To formulate the result, we need to make a definition. For a cycle \( \gamma \) (simple or composite) in \( A \in Q_n \), we define \( B_\gamma \) to be the \( n \times n \) \((1, -1, 0)\) matrix whose entries with the same positions as the entries of \( \gamma \) are signed the same as in \( \gamma \) and all other entries are zero.

**Theorem 5.3.** Let \( A \in Q_n \), and let \( m \) be the maximum length of the cycles in \( A \). Suppose that \( A \) requires unique inertia. Then all the terms in \( E_m(B) \) have the same sign, for any \( B \in Q(A) \). In particular, \( A \) requires a fixed number of zero eigenvalues. Further, for any two maximum length cycles \( \gamma_1 \) and \( \gamma_2 \) in \( A \) such that \( B_{\gamma_1} \) and \( B_{\gamma_2} \) have no nonzero pure imaginary eigenvalues, we have \( i(B_{\gamma_1}) = i(B_{\gamma_2}) \).

We give a sketch of the proof of Theorem 5.3. Recall that each term in \( E_m(B) \), \( B \in Q(A) \), corresponds to some cycle \( \gamma \) of length \( m \) in \( A \). For discussion of the signs of the terms in \( E_m(B) \), without loss of generality, we may assume that \( B \) is the unique
The $(1, -1, 0)$ matrix in $Q(A)$. The $m$ nonzero eigenvalues of $B_\gamma$ are located on the unit circle and are symmetric about the real axis. Any slight perturbation $C \in Q(A)$ of $B_\gamma$ also has $m$ nonzero eigenvalues with the sign of the product of the nonzero eigenvalues remaining the same, which is $(-)^{i-(C)} = (-)^{i-(B)}$. It follows that the term in $E_m(B)$ corresponding to $\gamma$ also has the sign $(-)^{i-(B)}$, which does not depend on $\gamma$. Thus, all the terms in $E_m(B)$ have the same sign for any $B \in Q(A)$.

The last statement of the theorem follows from a simple perturbation argument. The sign pattern
\[
\begin{pmatrix}
+ & + & 0 \\
0 & + & + \\
+ & 0 & 0
\end{pmatrix}
\]
shows that the necessary conditions in the above theorem are not sufficient.

Finally, we remark that in a future publication we shall more generally investigate the inertia sets of symmetric sign pattern matrices.

References