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Structure of the Malvenuto–Reutenauer Hopf algebra of permutations

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Abstract

We analyze the structure of the Malvenuto–Reutenauer Hopf algebra $\mathfrak{S}\mathrm{Sym}$ of permutations in detail. We give explicit formulas for its antipode, prove that it is a cofree coalgebra, determine its primitive elements and its coradical filtration, and show that it decomposes as a crossed product over the Hopf algebra of quasi-symmetric functions. In addition, we describe the structure constants of the multiplication as a certain number of facets of the permutahedron. As a consequence we obtain a new interpretation of the product of monomial quasi-symmetric functions in terms of the facial structure of the cube. The Hopf algebra of Malvenuto and Reutenauer has a linear basis indexed by permutations. Our results are obtained from a combinatorial description of the Hopf algebraic structure with respect to a new basis for this algebra, related to the original one via Möbius inversion on the weak order on the symmetric groups. This is in analogy with the relationship between the monomial and fundamental bases of the algebra of quasi-symmetric functions. Our results reveal a close relationship between the structure of the Malvenuto–Reutenauer Hopf algebra and the weak order on the symmetric groups.

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Introduction

Malvenuto [22] introduced the Hopf algebra $\mathfrak{S}$Sym of permutations, which has a linear basis $\{\mathcal{F}_u \mid u \in \mathfrak{S}_n, n \geqslant 0\}$ indexed by permutations in all symmetric groups $\mathfrak{S}_n$. The Hopf algebra $\mathfrak{S}$Sym is non-commutative, non-cocommutative, self-dual, and graded. Among its sub-, quotient-, and subquotient-Hopf algebras are many algebras central to algebraic combinatorics. These include the algebra of symmetric functions [21,33], Gessel's algebra $\mathcal{Q}$Sym of quasi-symmetric functions [13], the algebra of non-commutative symmetric functions [12], the Loday–Ronco algebra of planar binary trees [19], Stembridge's algebra of peaks [34], the Billera–Liu algebra of Eulerian enumeration [2], and others. The structure of these combinatorial Hopf algebras with respect to certain distinguished bases has been an important theme in algebraic combinatorics, with applications to the combinatorial problems these algebras were created to study. Here, we obtain a detailed understanding of the structure of $\mathfrak{S}$Sym, both in algebraic and combinatorial terms.

Our main tool is a new basis $\{\mathcal{M}_u \mid u \in \mathfrak{S}_n,\ n \geqslant 0\}$ for $\mathfrak{S}$Sym related to the original basis by Möbius inversion on the weak order on the symmetric groups. These bases $\{\mathcal{M}_u\}$ and $\{\mathcal{F}_u\}$ are analogous to the monomial basis and the fundamental basis of $\mathcal{Q}$Sym, which are related via Möbius inversion on their index sets, the Boolean posets $\mathcal{Q}_n$. We refer to them as the monomial basis and the fundamental basis of $\mathfrak{S}$Sym.

We give enumerative-combinatorial descriptions of the product, coproduct, and antipode of $\mathfrak{S}$Sym with respect to the monomial basis $\{\mathcal{M}_u\}$. In Section 3, we show that the coproduct is obtained by splitting a permutation at certain special positions that we call global descents. Descents and global descents are left adjoint and right adjoint to a natural map $\mathcal{Q}_n \to \mathfrak{S}_n$. These results rely on some non-trivial properties of the weak order developed in Section 2.

The product is studied in Section 4. The structure constants are non-negative integers with the following geometric-combinatorial description. The 1-skeleton of the permutahedron Π_{n-1} is the Hasse diagram of the weak order on $\mathfrak{S}_n$. The facets of the permutahedron are canonically isomorphic to products of lower dimensional permutahedra. Say that a facet isomorphic to $\Pi_{p-1} \times \Pi_{q-1}$ has type (p, q). Given $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, such a facet has a distinguished vertex corresponding to (u, v) under the canonical isomorphism. Then, for $w \in \mathfrak{S}_{p+q}$, the coefficient of $\mathcal{M}_w$ in $\mathcal{M}_u \cdot \mathcal{M}_v$ is the number of facets of the permutahedron Π_{p+q-1} of type (p, q) with the property that the distinguished vertex is below w (in the weak order) and closer to w than any other vertex in the facet.

In Section 5 we find explicit formulas for the antipode with respect to both bases. The structure constants with respect to the monomial basis have constant sign, as for $\mathcal{Q}$Sym. The situation is more complicated for the fundamental basis, which may explain why no such explicit formulas were previously known.

Elucidating the elementary structure of $\mathfrak{S}$Sym with respect to the monomial basis reveals further algebraic structures of $\mathfrak{S}$Sym. In Section 6, we show that $\mathfrak{S}$Sym is a cofree graded coalgebra. A consequence is that its coradical filtration (a filtration encapsulating the complexity of iterated coproducts) is the algebraic counterpart of a filtration of the symmetric groups by certain lower order ideals. In particular, we show that the space of primitive elements is spanned by the set $\{\mathcal{M}_u \mid u \text{ has no global descents}\}$. Cofreenes was shown by Poirier and Reutenauer [28] in dual form, through the introduction of a different basis. The study of primitive elements was pursued from this point of view by Duchamp et al. [8]. The generating function for the graded space of primitive elements is

$$1 - \frac{1}{\sum_{n\geqslant 0} n!\, x^n}.$$

Comtet essentially studied the combinatorics of global descents [6, Exercise VI.14]. These results add an algebraic perspective to the pure combinatorics he studied.

There is a well-known morphism of Hopf algebras $\mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$ that maps one fundamental basis onto the other, by associating to a permutation u its descent set $\mathrm{Des}(u)$. In Section 7, we describe this map on the monomial bases and then derive a new geometric description for the product of monomial quasi-symmetric functions in which the role of the permutahedron is played by the cube.

In Section 8 we show that $\mathfrak{S}\mathrm{Sym}$ decomposes as a crossed product over $\mathcal{Q}\mathrm{Sym}$. This construction from the theory of Hopf algebras is a generalization of the notion of group extensions. We provide a combinatorial description for the Hopf kernel of the map $\mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$, which is a subalgebra of $\mathfrak{S}\mathrm{Sym}$.

We study the self-duality of $\mathfrak{S}\mathrm{Sym}$ in Section 9 and its enumerative consequences. For instance, a result of Foata and Schützenberger [11] on the numbers

$$d(\mathsf{S},\mathsf{T}) \coloneqq \#\{w \in \mathfrak{S}_n \mid \mathrm{Des}(w) = \mathsf{S},\ \mathrm{Des}(w^{-1}) = \mathsf{T}\}$$

follows directly from this self-duality and we obtain analogous results for the numbers

$$\theta(u,v) \coloneqq \#\{w \in \mathfrak{S}_n \mid w \leqslant u,\ w^{-1} \leqslant v\}.$$

Most of the order-theoretic properties that underlie these algebraic results are presented in Section 2. Central to these are the existence of two Galois connections (involving descents and global descents) between the posets of permutations of $[n]$ and of subsets of $[n-1]$, as well as the order properties of the decomposition of $\mathfrak{S}_n$ into cosets of $\mathfrak{S}_p \times \mathfrak{S}_q$.

1. Basic definitions and results

We use only elementary properties of Hopf algebras, as given in the book [26]. Our Hopf algebras H will be graded connected Hopf algebras over $\mathbb{Q}$. Thus the $\mathbb{Q}$-algebra H is the direct sum $\oplus\{H_n \mid n = 0, 1, \ldots\}$ of its homogeneous components H_n, with $H_0 = \mathbb{Q}$, the product and coproduct respect the grading, and the counit is projection onto H_0.

Throughout, n is a non-negative integer and $[n]$ denotes the set $\{1,2,\dots,n\}$. A *composition* α of n is a sequence $\alpha=(\alpha_1,\dots,\alpha_k)$ of positive integers with $n=\alpha_1+\alpha_2+\cdots+\alpha_k$. To a composition α of n, we associate the set $I(\alpha) := \{\alpha_1,\alpha_1+\alpha_2,\dots,\alpha_1+\cdots+\alpha_{k-1}\}$. This gives a bijection between compositions of n and subsets of $[n-1]$. Compositions of n are partially ordered by *refinement*. The cover relations are of the form

$$(\alpha_1,\dots,\alpha_i+\alpha_{i+1},\dots,\alpha_k) \lessdot (\alpha_1,\dots,\alpha_k).$$

Under the association $\alpha\leftrightarrow I(\alpha)$, refinement corresponds to set inclusion, so we simply identify the poset of compositions with the Boolean poset $\mathcal{Q}_n$ of subsets of $[n-1]$.

Let $\mathfrak{S}_n$ be the group of permutations of $[n]$. We use one-line notation for permutations, writing $u=(u_1,u_2,\dots,u_n)$ where $u_i=u(i)$. Sometimes we may omit the commas and write $u=u_1\dots u_n$. A permutation u has a *descent* at a position p if $u_p>u_{p+1}$. An *inversion* in a permutation $u\in\mathfrak{S}_n$ is a pair of positions $1\leqslant i<j\leqslant n$ with $u_i>u_j$. The set of descents and inversions are denoted by $\mathrm{Des}(u)$ and $\mathrm{Inv}(u)$, respectively. The length of a permutation u is $\ell(u)=\#\mathrm{Inv}(u)$.

Given $p,q\geqslant 0$, we consider the product $\mathfrak{S}_p\times\mathfrak{S}_q$ to be a subgroup of $\mathfrak{S}_{p+q}$, where $\mathfrak{S}_p$ permutes $[p]$ and $\mathfrak{S}_q$ permutes $\{p+1,\dots,p+q\}$. For $u\in\mathfrak{S}_p$ and $v\in\mathfrak{S}_q$, write $u\times v$ for the permutation in $\mathfrak{S}_{p+q}$ corresponding to $(u,v)\in\mathfrak{S}_p\times\mathfrak{S}_q$ under this embedding.

More generally, given a subset $\mathsf{S}=\{p_1<\cdots<p_k\}$ of $[n-1]$, we have the (standard) parabolic or Young subgroup

$$\mathfrak{S}_{\mathsf{S}} := \mathfrak{S}_{p_1}\times\mathfrak{S}_{p_2-p_1}\times\cdots\times\mathfrak{S}_{n-p_k} \subseteq \mathfrak{S}_n.$$

The notation $\mathfrak{S}_{\mathsf{S}}$ suppresses the dependence on n, which will either be understood or will be made explicit when this is used.

Lastly, we use $\coprod$ to denote disjoint union.

1.1. The Hopf algebra of permutations of Malvenuto and Reutenauer

Let $\mathfrak{S}\mathrm{Sym}$ be the graded vector space over $\mathbb{Q}$ with basis $\coprod_{n\geqslant 0}\mathfrak{S}_n$, graded by n. This vector space has a graded Hopf algebra structure first considered in Malvenuto's thesis [22, Section 5.2] and in her work with Reutenauer [23]. (In [8], it is called the algebra of free quasi-symmetric functions.) Write $\mathcal{F}_u$ for the basis element corresponding to $u\in\mathfrak{S}_n$ for $n>0$ and 1 for the basis element of degree 0.

The product of two basis elements is obtained by shuffling the corresponding permutations, as in the following example.

$$\begin{aligned}\mathcal{F}_{12}\cdot\mathcal{F}_{312} &= \mathcal{F}_{12534}+\mathcal{F}_{15234}+\mathcal{F}_{15324}+\mathcal{F}_{15342}+\mathcal{F}_{51234}\\ &\quad+\mathcal{F}_{51324}+\mathcal{F}_{51342}+\mathcal{F}_{53124}+\mathcal{F}_{53142}+\mathcal{F}_{53412}.\end{aligned}$$

More precisely, for $p, q > 0$, set

$$\begin{aligned}\mathfrak{S}^{(p,q)} &:= \{\zeta \in \mathfrak{S}_{p+q} \mid \zeta \text{ has at most one descent, at position } p\} \\ &= \{\zeta \in \mathfrak{S}_{p+q} \mid \zeta_1 < \cdots < \zeta_p,\ \zeta_{p+1} < \cdots < \zeta_n\}.\end{aligned}$$

This is the collection of minimal (in length) representatives of left cosets of $\mathfrak{S}_p \times \mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$. In the literature, they are sometimes referred to as (p,q)-shuffles, but sometimes it is the inverses of these permutations that carry that name. We will refer to them as Grassmannian permutations. With these definitions, we describe the product. For $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, set

$$\mathcal{F}_u \cdot \mathcal{F}_v = \sum_{\zeta \in \mathfrak{S}^{(p,q)}} \mathcal{F}_{(u \times v)\cdot \zeta^{-1}}. \tag{1.1}$$

This endows $\mathfrak{S}$Sym with the structure of a graded algebra with unit 1.

The algebra $\mathfrak{S}$Sym is also a graded coalgebra with coproduct given by all ways of splitting a permutation. For a sequence $(a_1, \ldots, a_p)$ of distinct integers, let its *standard permutation*[2] $\mathrm{st}(a_1, \ldots, a_p) \in \mathfrak{S}_p$ be the permutation u defined by

$$u_i < u_j \iff a_i < a_j. \tag{1.2}$$

For instance, $\mathrm{st}(625) = 312$. The coproduct $\Delta : \mathfrak{S}\mathrm{Sym} \to \mathfrak{S}\mathrm{Sym} \otimes \mathfrak{S}\mathrm{Sym}$ is defined by

$$\Delta(\mathcal{F}_u) = \sum_{p=0}^{n} \mathcal{F}_{\mathrm{st}(u_1, \ldots, u_p)} \otimes \mathcal{F}_{\mathrm{st}(u_{p+1}, \ldots, u_n)}, \tag{1.3}$$

when $u \in \mathfrak{S}_n$. For instance, $\Delta(\mathcal{F}_{42531})$ is

$$1 \otimes \mathcal{F}_{42531} + \mathcal{F}_1 \otimes \mathcal{F}_{2431} + \mathcal{F}_{21} \otimes \mathcal{F}_{321} + \mathcal{F}_{213} \otimes \mathcal{F}_{21} + \mathcal{F}_{3142} \otimes \mathcal{F}_1 + \mathcal{F}_{42531} \otimes 1.$$

$\mathfrak{S}$Sym is a graded connected Hopf algebra [22, Théorème 5.4].

We refer to the set $\{\mathcal{F}_u\}$ as the *fundamental* basis of $\mathfrak{S}$Sym. The main goal of this paper is to obtain a detailed description of the Hopf algebra structure of $\mathfrak{S}$Sym. To this end, the definition of a second basis for $\mathfrak{S}$Sym (in Section 1.3) will prove crucial.

This Hopf algebra $\mathfrak{S}$Sym of Malvenuto and Reutenauer has been an object of recent interest [7,8,16,19,20,23,27–29]. We remark that sometimes it is the dual Hopf algebra that is considered. To compare results, one may use that $\mathfrak{S}$Sym is self-dual under the map $\mathcal{F}_u \mapsto \mathcal{F}^*_{u^{-1}}$, where $\mathcal{F}^*_{u^{-1}}$ is the element of the dual basis that is dual to $\mathcal{F}_{u^{-1}}$. We explore this further in Section 9.

[2] Some authors call this flattening.

1.2. The Hopf algebra of quasi-symmetric functions

Basic references for quasi-symmetric functions are [29, 9.4] and [33, Section 7.19]; however, everything we need will be reviewed here.

The algebra $\mathcal{Q}\mathrm{Sym}$ of quasi-symmetric functions is a subalgebra of the algebra of formal power series in countably many variables $x_1, x_2, \ldots$. It has a basis of *monomial quasi-symmetric functions* M_α indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$, where

$$M_\alpha := \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The product of these monomial functions is given by quasi-shuffles of their indices. A *quasi-shuffle* of compositions α and β is a shuffle of the components of α and β, where in addition we may replace any number of pairs of consecutive components (α_i, β_j) in the shuffle by $\alpha_i + \beta_j$. Then we have

$$M_\alpha \cdot M_\beta = \sum_\gamma M_\gamma, \tag{1.4}$$

where the sum is over all quasi-shuffles γ of the compositions α and β. For instance,

$$M_{(2)} \cdot M_{(1,1)} = M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)} + M_{(1,3)} + M_{(3,1)}. \tag{1.5}$$

The unit element is indexed by the empty composition $1 = M_{(\,)}$.

Let X and Y be two countable ordered sets and $X \coprod Y$ its disjoint union, totally ordered by $X < Y$. Then $\Delta : f(X) \mapsto f(X \coprod Y)$ gives $\mathcal{Q}\mathrm{Sym}$ the structure of a coalgebra. In terms of the monomial quasi-symmetric functions, we have

$$\Delta(M_{(\alpha_1,\ldots,\alpha_k)}) = \sum_{p=0}^{k} M_{(\alpha_1,\ldots,\alpha_p)} \otimes M_{(\alpha_{p+1},\ldots,\alpha_k)}. \tag{1.6}$$

For instance, $\Delta(M_{(2,1)}) = 1 \otimes M_{(2,1)} + M_{(2)} \otimes M_{(1)} + M_{(2,1)} \otimes 1$.

The algebra of quasi-symmetric functions was introduced by Gessel [13]. Its Hopf algebra structure was introduced by Malvenuto [22, Section 4.1]. The description of the product in terms of quasi-shuffles can be found in [15] and is equivalent to [10, Lemma 3.3]. A q-version of this construction appears in [15] and in [36, Section 5].

The algebra $\mathcal{Q}\mathrm{Sym}$ is a graded connected Hopf algebra whose component in degree n is spanned by those M_α with α a composition of n. Malvenuto [22, Corollaire 4.20] and Ehrenborg [10, Proposition 3.4] independently gave an explicit formula for the antipode

$$S(M_\alpha) = (-1)^{c(\alpha)} \sum_{\beta \leqslant \alpha} M_{\widetilde{\beta}}. \tag{1.7}$$

Here, $c(\alpha)$ is the number of components of α, and if $\beta = (\beta_1, \beta_2, \dots, \beta_t)$ then $\widetilde{\beta}$ is β written in reverse order $(\beta_t, \dots, \beta_2, \beta_1)$.

Gessel's *fundamental* quasi-symmetric function F_α is defined by

$$F_\alpha = \sum_{\alpha \leqslant \beta} M_\beta.$$

By Möbius inversion, we have

$$M_\alpha = \sum_{\alpha \leqslant \beta} (-1)^{c(\beta)-c(\alpha)} F_\beta.$$

Thus the set $\{F_\alpha\}$ forms another basis of $\mathcal{Q}\mathrm{Sym}$.

It is sometimes advantageous to index these monomial and fundamental quasi-symmetric functions by subsets of $[n-1]$. Accordingly, given a composition α of n with $\mathsf{S} = I(\alpha)$, we define

$$F_{\mathsf{S}} := F_\alpha \quad \text{and} \quad M_{\mathsf{S}} := M_\alpha.$$

The notation F_{S} suppresses the dependence on n, which will be usually understood from the context; otherwise it will be made explicit by writing $F_{\mathsf{S},n}$.

In terms of power series,

$$F_{\mathsf{S}} = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ p \in \mathsf{S} \Rightarrow i_p < i_{p+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{1.8}$$

We mention that there is an analogous realization of the Malvenuto–Reutenauer Hopf algebra as a subalgebra of an algebra of non-commutative power series, due to Duchamp, Hivert, and Thibon. To this end, one defines

$$\mathcal{F}_u = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ p \in \mathrm{Des}(u) \Rightarrow i_p < i_{p+1}}} x_{i_{u^{-1}(1)}} x_{i_{u^{-1}(2)}} \cdots x_{i_{u^{-1}(n)}}. \tag{1.9}$$

This is discussed in [8, Section 3.1], in slightly different terms. In this realization, the coproduct of $\mathfrak{S}\mathrm{Sym}$ is induced by the ordinal sum of commuting alphabets [8, Proposition 3.4].

1.3. The monomial basis of the Malvenuto–Reutenauer Hopf algebra

The descent set of a permutation $u \in \mathfrak{S}_n$ is the subset of $[n-1]$ recording the descents of u

$$\mathrm{Des}(u) := \{p \in [n-1] \mid u_p > u_{p+1}\}. \tag{1.10}$$

Thus $\mathrm{Des}(46512837) = \{2, 3, 6\}$. Malvenuto [22, Théorèmes 5.12, 5.13, and 5.18] shows that there is a morphism of Hopf algebras

$$\begin{array}{rcccl} \mathcal{D} & : & \mathfrak{S}\mathrm{Sym} & \to & \mathcal{Q}\mathrm{Sym} \\ & & \mathcal{F}_u & \mapsto & F_{\mathrm{Des}(u)}. \end{array} \tag{1.11}$$

(This is equivalent to Theorem 3.3 in [23].) This explains our name and notation for the fundamental basis of $\mathfrak{S}\mathrm{Sym}$. This map extends to power series, where it is simply the abelianization: there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}\mathrm{Sym} & \hookrightarrow & k\langle x_1, x_2, \ldots \rangle \\ \mathcal{D}\downarrow & & \downarrow ab \\ \mathcal{Q}\mathrm{Sym} & \hookrightarrow & k[x_1, x_2, \ldots] \end{array}$$

This is evident from (1.8) and (1.9). It is easy to see, however, that $\mathcal{D}$ is *not* the abelianization of $\mathfrak{S}\mathrm{Sym}$.

In analogy to the basis of monomial quasi-symmetric functions, we define a new *monomial* basis $\{\mathcal{M}_u\}$ for the Malvenuto-Reutenauer Hopf algebra. For each $n \geqslant 0$ and $u \in \mathfrak{S}_n$, let

$$\mathcal{M}_u := \sum_{u \leqslant v} \mu_{\mathfrak{S}_n}(u, v) \cdot \mathcal{F}_v, \tag{1.12}$$

where $u \leqslant v$ in the weak order in $\mathfrak{S}_n$ (described in Section 2) and $\mu_{\mathfrak{S}_n}$ is the Möbius function of this partial order. By Möbius inversion,

$$\mathcal{F}_u := \sum_{u \leqslant v} \mathcal{M}_v, \tag{1.13}$$

so these elements $\mathcal{M}_u$ indeed form a basis of $\mathfrak{S}\mathrm{Sym}$. For instance,

$$\mathcal{M}_{4123} = \mathcal{F}_{4123} - \mathcal{F}_{4132} - \mathcal{F}_{4213} + \mathcal{F}_{4321}.$$

We will show that $\mathcal{M}_u$ maps either to $M_{\mathrm{Des}(u)}$ or to 0 under the map $\mathcal{D}: \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$.

2. The weak order on the symmetric group

Let $\mathrm{Inv}(u)$ be the set of inversions of a permutation $u \in \mathfrak{S}_n$,

$$\mathrm{Inv}(u) := \{(i,j) \in [n] \times [n] \mid i < j \quad \text{and} \quad u_i > u_j\}.$$

The inversion set determines the permutation. Given u and $v \in \mathfrak{S}_n$, we write $u \leqslant v$ if $\mathrm{Inv}(u) \subseteq \mathrm{Inv}(v)$. This defines the *left weak order* on $\mathfrak{S}_n$. Fig. 1 shows the (left) weak

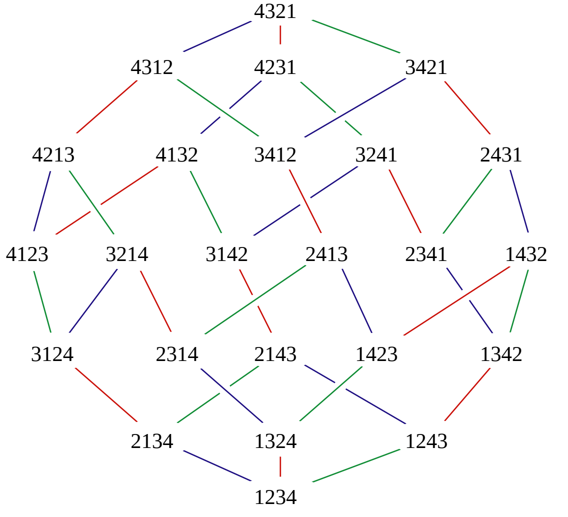

Fig. 1. The weak order on $\mathfrak{S}_4$.

order on $\mathfrak{S}_4$. The weak order has another characterization

$$u \leqslant v \quad \Leftrightarrow \quad \exists w \in \mathfrak{S}_n \quad \text{such that } v = wu \text{ and } \ell(v) = \ell(w) + \ell(u),$$

where $\ell(u)$ is the number of inversions of u. The cover relations $u \lessdot v$ occur when w is an adjacent transposition. Thus, $u \lessdot v$ precisely when v is obtained from u by transposing a pair of consecutive values of u; a pair (u_i, u_j) such that $i < j$ and $u_j = u_i + 1$. The identity permutation 1_n is the minimum element of $\mathfrak{S}_n$ and $\omega_n = (n, \ldots, 2, 1)$ is the maximum.

This weak order is a lattice [14], whose structure we describe. First, a set J is the inversion set of a permutation in $\mathfrak{S}_n$ if and only if both J and its complement $\text{Inv}(\omega_n) - J$ are *transitively closed* ($(i,j) \in J$ and $(j,k) \in J$ imply $(i,k) \in J$, and the same for its complement). The join (least upper bound) of two permutations u and $v \in \mathfrak{S}_n$ is the permutation $u \vee v$ whose inversion set is the transitive closure of the union of the inversion sets of u and v

$$\{(i,j) \mid \exists \text{ chain } i = k_0 < \cdots < k_s = j \ s.t.\ \forall r,\ (k_{r-1}, k_r) \in \text{Inv}(u) \cup \text{Inv}(v)\}. \tag{2.1}$$

Similarly, the meet (greatest lower bound) of u and v is the permutation $u \wedge v$ whose inversion set is

$$\{(i,j) \mid \forall \text{ chains } i = k_0 < \cdots < k_s = j,\ \exists r \text{ s.t. } (k_{r-1}, k_r) \in \text{Inv}(u) \cap \text{Inv}(v)\}. \tag{2.2}$$

The Möbius function of the weak order takes values in $\{-1, 0, 1\}$. Explicit descriptions can be found in [3, Corollary 3] or [9, Theorem 1.2]. We will not need that description, but will use several basic facts on the weak order that we develop here.

2.1. Grassmannian permutations and the weak order

In Section 1, we defined $\mathfrak{S}^{(p,q)}$ to be the set of minimal (in length) representatives of (left) cosets of $\mathfrak{S}_p \times \mathfrak{S}_q$ in the symmetric group $\mathfrak{S}_{p+q}$. Thus the map

$$\begin{array}{ccccc} \lambda & : & \mathfrak{S}^{(p,q)} \times \mathfrak{S}_p \times \mathfrak{S}_q & \to & \mathfrak{S}_{p+q} \\ & & (\zeta, u, v) & \mapsto & \zeta \cdot (u \times v) \end{array}$$

is a bijection. We leave the following description of the inverse to the reader.

Lemma 2.1. *Let* $w \in \mathfrak{S}_{p+q}$, *and set* $\zeta := w \cdot (\mathrm{st}(w_1, \ldots, w_p) \times \mathrm{st}(w_{p+1}, \ldots, w_{p+q}))^{-1}$. *Then* $\zeta \in \mathfrak{S}^{(p,q)}$ *and* $\lambda^{-1}(w) = (\zeta,\ \mathrm{st}(w_1, \ldots, w_p),\ \mathrm{st}(w_{p+1}, \ldots, w_{p+q}))$.

We describe the order-theoretic properties of this decomposition into cosets. The first step is to characterize the inversion sets of Grassmannian permutations. A subset J of $[p] \times [q]$ is *cornered* if $(h,k) \in J$ implies that $(i,j) \in J$ whenever $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. The reason for this definition is that a set I is the inversion set of a Grassmannian permutation $\zeta \in \mathfrak{S}^{(p,q)}$ if and only if

$$\begin{array}{ll} \text{(i)} & I \subseteq \{1, \ldots, p\} \times \{p+1, \ldots, p+q\}, \text{ and} \\ \text{(ii)} & \text{the shifted set } \{(p+1-i, j-p) \mid (i,j) \in I\} \subseteq [p] \times [q] \text{ is cornered.} \end{array} \tag{2.3}$$

Given an arbitrary subset J of $[p] \times [q]$, let $cr(J)$ denote the smallest cornered subset containing J. Denote the obvious action of $(u,v) \in \mathfrak{S}_p \times \mathfrak{S}_q$ on a subset J of $[p] \times [q]$ by $(u,v)(J)$.

Lemma 2.2. *Let J be a cornered subset of $[p] \times [q]$ and $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$ any permutations. Then*

$$J \subseteq \mathrm{cr}((u,v)(J)).$$

Proof. Let $(i,j) \in J$. The set $\{u(h) \mid 1 \leqslant h \leqslant i\}$ has i elements. Hence there is a number h such that $1 \leqslant h \leqslant i$ and $u(h) \geqslant i$. Similarly there is number k such that $1 \leqslant k \leqslant j$ and $v(k) \geqslant j$. Since J is cornered, $(h,k) \in J$. Hence $(u(h), v(k)) \in (u,v)(J)$. By construction, $i \leqslant u(h)$ and $j \leqslant v(k)$, so $(i,j) \in \mathrm{cr}((u,v)(J))$, as needed. $\square$

Denote the diagonal action of $w \in \mathfrak{S}_n$ on a subset I of $[n] \times [n]$ by $w(I)$. Suppose $w = u \times v \in \mathfrak{S}_p \times \mathfrak{S}_q$ and $I \subseteq \{1, \ldots, p\} \times \{p+1, \ldots, p+q\}$. Let J be the result of shifting I, as in (2.3)(ii). It is easy to see that the result of shifting $(u \times v)(I)$ is $(\tilde{u}, v)(J)$, where $\tilde{u}(i) = p + 1 - u(p+1-i)$.

Corollary 2.3. *Let ζ and $\zeta' \in \mathfrak{S}^{(p,q)}$ be Grassmannian permutations, and $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$ be permutations. If $(u \times v)(\mathrm{Inv}(\zeta)) \subseteq \mathrm{Inv}(\zeta')$ then $\zeta \leqslant \zeta'$.*

Proof. We show that $\mathrm{Inv}(\zeta)\subseteq\mathrm{Inv}(\zeta')$. Let J and J' be the corresponding shifted sets. According to the previous discussion and the hypothesis, $(\tilde{u},v)(J)\subseteq J'$. Hence $\mathrm{cr}((\tilde{u},v)(J))\subseteq J'$, since J' is cornered. By Lemma 2.2, $J\subseteq\mathrm{cr}((\tilde{u},v)(J))$, so $J\subseteq J'$. This implies the inclusion of inversion sets, as needed. $\square$

The following lemma is straightforward.

Lemma 2.4. *Let $\zeta\in\mathfrak{S}^{(p,q)}$, $u\in\mathfrak{S}_p$, $v\in\mathfrak{S}_q$ and $w:=\zeta\cdot(u\times v)\in\mathfrak{S}_{p+q}$. There is a decomposition of* $\mathrm{Inv}(w)$ *into disjoint subsets*

$$\mathrm{Inv}(w)=\mathrm{Inv}(u)\coprod((p,p)+\mathrm{Inv}(v))\coprod(u^{-1}\times v^{-1})(\mathrm{Inv}(\zeta)).$$

We deduce some order-theoretic properties of the decomposition into left cosets. Define $\zeta_{p,q}$ to be the permutation of maximal length in $\mathfrak{S}^{(p,q)}$, so that

$$\zeta_{p,q}:=(q+1,\ q+2,\ \dots,\ q+p,\ 1,\ 2,\dots,q).$$

Proposition 2.5. *Let $\lambda:\mathfrak{S}^{(p,q)}\times\mathfrak{S}_p\times\mathfrak{S}_q\to\mathfrak{S}_{p+q}$ be the bijection*

$$\lambda(\zeta,u,v)=\zeta\cdot(u\times v).$$

Then

(i) λ^{-1} *is order preserving. That is,*

$$\zeta\cdot(u\times v)\leqslant\zeta'\cdot(u'\times v')\quad\Rightarrow\quad\zeta\leqslant\zeta',\ u\leqslant u',\ \text{and}\ v\leqslant v'.$$

(ii) λ *is order preserving when restricted to any of the following sets*

$$\{\zeta_{p,q}\}\times\mathfrak{S}_p\times\mathfrak{S}_q,\ \{1_{p+q}\}\times\mathfrak{S}_p\times\mathfrak{S}_q,\ \text{or}\ \mathfrak{S}^{(p,q)}\times\{(u,v)\},$$

for any $u\in\mathfrak{S}_p$, $v\in\mathfrak{S}_q$.

Proof. Let $w=\zeta\cdot(u\times v)$ and $w'=\zeta'\cdot(u'\times v')$. Suppose $w\leqslant w'$, so that $\mathrm{Inv}(w)\subseteq\mathrm{Inv}(w')$. By Lemma 2.4, we have $\mathrm{Inv}(u)\subseteq\mathrm{Inv}(u')$, $\mathrm{Inv}(v)\subseteq\mathrm{Inv}(v')$, and $(u''\times v'')(\mathrm{Inv}(\zeta))\subseteq\mathrm{Inv}(\zeta')$, where $u'':=u'u^{-1}$ and $v'':=v'v^{-1}$. Therefore, $u\leqslant u'$, $v\leqslant v'$, and by Corollary 2.3, $\zeta\leqslant\zeta'$. This proves (i).

Statement (ii) follows by a similar application of Lemma 2.4, noting that $\mathrm{Inv}(\zeta_{p,q})=\{1,\dots,p\}\times\{p+1,\dots,n\}$ and $\mathrm{Inv}(1_{p+q})=\emptyset$ are invariant under any permutation in $\mathfrak{S}_p\times\mathfrak{S}_q$. $\square$

Since Grassmannian permutations in $\mathfrak{S}^{(p,q)}$ are left coset representatives of $\mathfrak{S}_p\times\mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$, their inverses are right coset representatives. We discuss order-theoretic properties of this decomposition into right cosets.

Given a subset J of $[n] \times [n]$, let

$$\widetilde{J} = \{(j,i) \mid (i,j) \in J\}.$$

We have the following key observation about the diagonal action of $\mathfrak{S}_n$ on subsets of $[n] \times [n]$.

Lemma 2.6. *For any $u \in \mathfrak{S}_n$, we have $u(\widetilde{\mathrm{Inv}(u)}) = \mathrm{Inv}(u^{-1})$.*

Proof. Note that $u^{-1}(u_i) = i$. Thus $\mathrm{Inv}(u^{-1}) = \{u_h < u_k \mid h > k\}$. Then

$$u^{-1}(\mathrm{Inv}(u^{-1})) = \{(h,k) \mid k < h \quad \text{and} \quad u_k > u_h\} = \widetilde{\mathrm{Inv}(u)}. \qquad \square$$

Proposition 2.7. *Fix $\zeta \in \mathfrak{S}^{(p,q)}$ and consider the map $\rho_\zeta : \mathfrak{S}_p \times \mathfrak{S}_q \to \mathfrak{S}_{p+q}$ given by*

$$\rho_\zeta(u,v) = (u \times v) \cdot \zeta^{-1}.$$

Then ρ_ζ is a convex embedding in the sense that

(a) *ρ_ζ is injective;*
(b) *ρ_ζ is order-preserving:* $u \leqslant u'$ and $v \leqslant v' \;\Leftrightarrow\; \rho_\zeta(u,v) \leqslant \rho_\zeta(u',v')$;
(c) *ρ_ζ is convex: If $\rho_\zeta(u,v) \leqslant w \leqslant \rho_\zeta(u',v')$, for some $u,u' \in \mathfrak{S}_p$ and $v,v' \in \mathfrak{S}_q$, then there are $u'' \in \mathfrak{S}_p$ and $v'' \in \mathfrak{S}_q$ with $w = \rho_\zeta(u'',v'')$.*

It follows that

(d) *ρ_ζ preserves meets and joins.*

Proof. Assertion (a) is immediate. Set $w := (u \times v) \cdot \zeta^{-1} = \rho_\zeta(u,v)$. Then $w^{-1} = \zeta \cdot (u^{-1} \times v^{-1})$. By Lemmas 2.4 and 2.6, we have

$$\begin{aligned}
\mathrm{Inv}(w) &= w^{-1}(\widetilde{\mathrm{Inv}(w^{-1})}) \\
&= \zeta \cdot (u^{-1} \times v^{-1})\Big(\widetilde{\mathrm{Inv}(u^{-1})} \cup \Big((p,p) + \widetilde{\mathrm{Inv}(v^{-1})}\Big) \cup (u \times v)\Big(\widetilde{\mathrm{Inv}(\zeta)}\Big)\Big) \\
&= \zeta\Big(\mathrm{Inv}(u) \cup ((p,p) + \mathrm{Inv}(v)) \cup \widetilde{\mathrm{Inv}(\zeta)}\Big).
\end{aligned}$$

Assertion (b) follows from this and the characterization of the weak order in terms of inversion sets.

For (c), decompose $w = (u'' \times v'') \cdot \xi^{-1}$. By assumption,

$$\zeta\Big(\widetilde{\mathrm{Inv}(\zeta)}\Big) \subseteq \xi\Big(\widetilde{\mathrm{Inv}(\xi)}\Big) \subseteq \zeta\Big(\widetilde{\mathrm{Inv}(\zeta)}\Big).$$

Then $\zeta = \xi$ by Lemma 2.6, so $w = \rho_\zeta(u'',v'')$ as needed. $\square$

2.2. Cosets of parabolic subgroups and the weak order

Write a subset S of $[n-1]$ as $\mathsf{S} = \{p_1 < \cdots < p_k\}$. In Section 1, we defined the parabolic subgroup

$$\mathfrak{S}_{\mathsf{S}} = \mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2 - p_1} \times \cdots \times \mathfrak{S}_{n-p_k} \subseteq \mathfrak{S}_n.$$

Let $\mathfrak{S}^{\mathsf{S}}$ be the set of minimal (in length) representatives of left cosets of $\mathfrak{S}_{\mathsf{S}}$ in $\mathfrak{S}_n$,

$$\mathfrak{S}^{\mathsf{S}} = \{\zeta \in \mathfrak{S}_n \mid \mathrm{Des}(\zeta) \subseteq \mathsf{S}\}.$$

Grassmannian permutations are the special case $\mathfrak{S}^{(p,n-p)} = \mathfrak{S}^{\{p\}}$.

Let ζ_{S} be the permutation of maximal length in $\mathfrak{S}^{\mathsf{S}}$,

$$\zeta_{\mathsf{S}} := (n - p_1 + 1, \ldots, n,\ n - p_2 + 1, \ldots, n - p_1, \ldots,\ 1, \ldots, n - p_k). \tag{2.4}$$

We record the following facts about these coset representatives:

Lemma 2.8. *$\mathfrak{S}^{\mathsf{S}}$ is an interval in the weak order of $\mathfrak{S}_n$. The minimum element is the identity 1_n and the maximum is ζ_{S}.*

Our proofs rely upon the following basic fact. Suppose p, q are positive integers and T is a subset of $[p-1]$. Define the subset S of $[p+q-1]$ to be $\mathsf{T} \cup \{p\}$. Then $(\zeta, \zeta') \mapsto \zeta \cdot (\zeta' \times 1_q)$ defines a bijection

$$\mathfrak{S}^{(p,q)} \times \mathfrak{S}^{\mathsf{T}} \to \mathfrak{S}^{\mathsf{S}}. \tag{2.5}$$

The maximum elements are preserved under this map

$$\zeta_{p,q} \cdot (\zeta_{\mathsf{T}} \times 1_q) = \zeta_{\mathsf{S}}. \tag{2.6}$$

The analog of Proposition 2.5 for this decomposition of $\mathfrak{S}_n$ into left cosets of $\mathfrak{S}_{\mathsf{S}}$ follows from Proposition 2.5 by induction using (2.5) and (2.6).

Proposition 2.9. *Suppose S is a subset of $[n-1]$. Let $\lambda : \mathfrak{S}^{\mathsf{S}} \times \mathfrak{S}_{\mathsf{S}} \to \mathfrak{S}_n$ be the bijection*

$$\lambda(\zeta, u) = \zeta \cdot u.$$

Then λ^{-1} is order preserving, while λ is order preserving when restricted to any of the following sets:

$$\{\zeta_{\mathsf{S}}\} \times \mathfrak{S}_{\mathsf{S}},\ \{1_n\} \times \mathfrak{S}_{\mathsf{S}},\ \text{or}\ \mathfrak{S}^{\mathsf{S}} \times \{u\},\ \textit{for any}\ u \in \mathfrak{S}_{\mathsf{S}}.$$

We state the analog of Proposition 2.7.

Proposition 2.10. *Let* S *be a subset of* $[n-1]$. *Fix* $\zeta \in \mathfrak{S}^{\mathsf{S}}$ *and consider the map* $\rho_\zeta : \mathfrak{S}_{\mathsf{S}} \to \mathfrak{S}_n$ *given by*

$$\rho_\zeta(u) = u \cdot \zeta^{-1}.$$

Then ρ_ζ *is a convex embedding. In particular, it preserves meets and joins.*

2.3. Descents

Let $\mathcal{Q}_n$ denote the Boolean poset of subsets of $[n-1]$, which we identify with the poset of compositions of n. We have the descent map $\mathrm{Des} : \mathfrak{S}_n \to \mathcal{Q}_n$ given by $u \mapsto \mathrm{Des}(u)$, the descent set (1.10) of u. Let $Z : \mathcal{Q}_n \to \mathfrak{S}_n$ be the map defined by $\mathsf{S} \mapsto \zeta_{\mathsf{S}}$, the maximum left coset representative of $\mathfrak{S}_{\mathsf{S}}$ as in (2.4).

A *Galois connection* between posets P and Q is a pair (f, g) of order preserving maps $f : P \to Q$ and $g : Q \to P$ such that for any $x \in P$ and $y \in Q$,

$$f(x) \leqslant y \iff x \leqslant g(y). \tag{2.7}$$

Proposition 2.11. *The pair of maps* $(\mathrm{Des}, Z) : \mathfrak{S}_n \rightleftarrows \mathcal{Q}_n$ *is a Galois connection.* (See Fig. 2.)

Proof. We verify that

(a) $\mathrm{Des} : \mathfrak{S}_n \to \mathcal{Q}_n$ is order preserving;
(b) $Z : \mathcal{Q}_n \to \mathfrak{S}_n$ is order preserving;
(c) $\mathrm{Des} \circ Z = id_{\mathcal{Q}_n}$;
(d) $Z(\mathsf{S}) = \max\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) = \mathsf{S}\}$.

First of all, the map Des is order preserving simply because p is a descent of u if and only if $(p, p+1) \in \mathrm{Inv}(u)$. This is (a). The remaining assertions follow immediately from

$$\begin{aligned}\zeta_{\mathsf{S}} &= \max\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) \subseteq \mathsf{S}\} \\ &= \max\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) = \mathsf{S}\},\end{aligned}$$

which we know from Lemma 2.3.

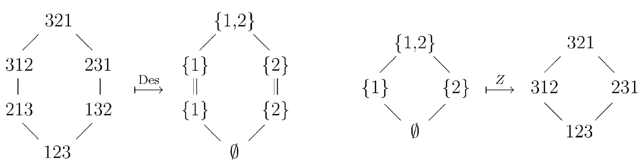

Fig. 2. The Galois connection $\mathfrak{S}_3 \rightleftarrows \mathcal{Q}_3$.

Condition (2.7) follows formally. In fact, suppose $\mathsf{T} = \mathrm{Des}(u) \subseteq \mathsf{S}$. Then by (d), $u \leqslant Z(\mathsf{T})$, and by (b), $Z(\mathsf{T}) \leqslant Z(\mathsf{S})$, so $u \leqslant Z(\mathsf{S})$. Conversely, suppose $u \leqslant Z(\mathsf{S})$. Then by (a) and (c), $\mathrm{Des}(u) \subseteq \mathrm{Des}(Z(\mathsf{S})) = \mathsf{S}$. $\square$

The Galois connection is why the monomial basis of $\mathfrak{S}\mathrm{Sym}$ is truly analogous to that of $\mathcal{Q}\mathrm{Sym}$, and explains why we consider the weak order rather than any other order on $\mathfrak{S}_n$. The connection between the monomial bases of these two algebras will be elucidated in Theorem 7.3 using the previous result.

2.4. Global descents

Definition 2.12. A permutation $u \in \mathfrak{S}_n$ has a *global descent* at a position $p \in [n-1]$ if

$$\forall\ i \leqslant p \quad \text{and} \quad j \geqslant p+1\,,\ u_i > u_j.$$

Equivalently, if $\{u_1, \ldots, u_p\} = \{n, n-1, \ldots, n-p+1\}$. Let $\mathrm{GDes}(u) \subseteq [n-1]$ be the set of global descents of u. Note that $\mathrm{GDes}(u) \subseteq \mathrm{Des}(u)$, but these are not equal in general.

In Section 2.3 we showed that the descent map $\mathrm{Des} : \mathfrak{S}_n \to \mathcal{Q}_n$ is left adjoint to the map $Z : \mathcal{Q}_n \to \mathfrak{S}_n$, in the sense that the pair (Des, Z) forms a Galois connection, as in Proposition 2.11. That is, for $u \in \mathfrak{S}_n$ and $\mathsf{S} \in \mathcal{Q}_n$,

$$\mathrm{Des}(u)\ \subseteq\ \mathsf{S} \quad \Leftrightarrow \quad u\ \leqslant\ Z(\mathsf{S}) = \zeta_{\mathsf{S}}. \tag{2.8}$$

The notion of global descents is a very natural companion of that of (ordinary) descents, in that the map $\mathrm{GDes} : \mathfrak{S}_n \to \mathcal{Q}_n$ is *right* adjoint to $Z : \mathcal{Q}_n \to \mathfrak{S}_n$.

Proposition 2.13. *The pair of maps* $(Z, \mathrm{GDes}) : \mathcal{Q}_n \rightleftarrows \mathfrak{S}_n$ *is a Galois connection.*

Proof. We already know that Z is order preserving. So is GDes, because p is a global descent of a permutation u if and only if $(i,j) \in \mathrm{Inv}(u)$ for every $i \leqslant p$, $j \geqslant p+1$. It remains to check that

$$\zeta_{\mathsf{S}}\ \leqslant\ u\ \Leftrightarrow\ \mathsf{S}\ \subseteq\ \mathrm{GDes}(u). \tag{2.9}$$

As in the proof of Proposition 2.11, this follows from

$$\begin{aligned} \zeta_{\mathsf{S}} &= \min\{u \in \mathfrak{S}_n \mid \mathrm{GDes}(u) \subseteq \mathsf{S}\} \\ &= \min\{u \in \mathfrak{S}_n \mid \mathrm{GDes}(u) = \mathsf{S}\}, \end{aligned}$$

which is clear from the definition of ζ_{S}. $\square$

We turn to properties of the decomposition of $\mathfrak{S}_n$ into left cosets of $\mathfrak{S}_{\mathsf{S}}$ related to the notion of global descents. Recall that $\mathfrak{S}^{(p,q)}$ is a set of representatives for the left cosets of $\mathfrak{S}_p \times \mathfrak{S}_q$ in $\mathfrak{S}_{p+q}$, and that $\zeta_{p,q} = (q+1, q+2, \ldots, q+p,\ 1, 2, \ldots, q)$.

Lemma 2.14. *Suppose p, q are non-negative integers and let $w \in \mathfrak{S}_{p+q}$. Then*

$$p \in \mathrm{GDes}(w) \iff w \equiv \zeta_{p,q} \bmod \mathfrak{S}_p \times \mathfrak{S}_q \iff w \geqslant \zeta_{p,q}.$$

Proof. First suppose that $w \in \mathfrak{S}_{p+q}$ is in the same left coset of $\mathfrak{S}_p \times \mathfrak{S}_q$ as is $\zeta_{p,q}$. Thus, there are permutations $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$ such that

$$w = \zeta_{p,q} \cdot (u \times v).$$

If $i \leqslant p$, then $u_i \in \{1, \ldots, p\}$ and thus $w_i \in \{q+1, \ldots, q+p\}$, so p is a global descent of w as needed.

For the other direction, suppose p is a global descent of w and set

$$\bar{w} \coloneqq \zeta_{p,q}^{-1} \cdot w = (p+1, p+2, \ldots, p+q\,, 1, 2, \ldots, p) \cdot w.$$

Let $1 \leqslant i \leqslant p$. By assumption, $w_i \in \{q+1, \ldots, q+p\}$. Hence $\bar{w}_i \in \{1, \ldots, p\}$, which means that $\bar{w} = u \times v$ for some $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, as needed.

Noting that $\zeta_{p,q}$ is a minimal coset representative and that the map λ^{-1} is order preserving (Proposition 2.5(a)) proves the second equivalence. $\square$

For any subset S of $[n-1]$, we have the left coset map $\lambda: \mathfrak{S}^{\mathsf{S}} \times \mathfrak{S}_{\mathsf{S}} \to \mathfrak{S}_n$ of Section 2.2. Given a permutation $u \in \mathfrak{S}_n$, consider its 'projection' u_{S} to $\mathfrak{S}_{\mathsf{S}}$, which is defined to be the second component of $\lambda^{-1}(u)$. That is, $\lambda^{-1}(u) = (\zeta, u_{\mathsf{S}})$ for some permutation $\zeta \in \mathfrak{S}^{\mathsf{S}}$. If $\mathsf{S} = \{p_1 < p_2 < \cdots < p_k\}$, then by Lemma 2.1,

$$u_{\mathsf{S}} = \mathrm{st}(u_1, \ldots, u_{p_1}) \times \mathrm{st}(u_{p_1+1}, \ldots, u_{p_2}) \times \cdots \times \mathrm{st}(u_{p_k+1}, \ldots, u_n). \tag{2.10}$$

In particular, $u_{\emptyset} = u$ and $u_{[n-1]} = 1_n$. We relate this projection to the order and lattice structure of $\mathfrak{S}_n$. For $i < j$, let $[i,j) \coloneqq \{i, i+1, \ldots, j-1\}$.

Lemma 2.15. *For any $u \in \mathfrak{S}_n$ and subset S of $[n-1]$,*

$$\mathrm{Inv}(u_{\mathsf{S}}) = \{(i,j) \in \mathrm{Inv}(u) \mid [i,j) \cap \mathsf{S} = \emptyset\}.$$

In particular, $u_{\mathsf{S}} \leqslant u$.

Proof. Let $i < j$ be integers in $[n]$. Suppose that there is an element $p \in \mathsf{S}$ with $i \leqslant p < j$. Since $\mathfrak{S}_{\mathsf{S}} \subseteq \mathfrak{S}_p \times \mathfrak{S}_{n-p}$, we have $u_{\mathsf{S}} \in \mathfrak{S}_p \times \mathfrak{S}_{n-p}$, and so $u_{\mathsf{S}}(i) < u_{\mathsf{S}}(j)$. Thus $(i,j) \notin \mathrm{Inv}(u_{\mathsf{S}})$. Suppose now that $[i,j) \cap \mathsf{S} = \emptyset$. Then there are consecutive elements

p and q of S such that $p<i<j\leqslant q$. By (2.10),

$$u_{\mathsf{S}}(i) = p + \mathrm{st}(u_{p+1},\ldots,u_q)(i) \quad \text{and} \quad u_{\mathsf{S}}(j) = p + \mathrm{st}(u_{p+1},\ldots,u_q)(j).$$

By (1.2), this implies that

$$u_{\mathsf{S}}(i) > u_{\mathsf{S}}(j) \;\Leftrightarrow\; u(i) > u(j),$$

and thus $(i,j)\in\mathrm{Inv}(u_{\mathsf{S}}) \;\Leftrightarrow\; (i,j)\in\mathrm{Inv}(u)$. This completes the proof. $\square$

Proposition 2.16. *Let $u,\ v\in\mathfrak{S}_n$ and S,T be subsets of $[n-1]$. Then*

(i) *If $u\leqslant v$ then $u_{\mathsf{S}}\leqslant v_{\mathsf{S}}$ and if $\mathsf{T}\subseteq\mathsf{S}$ then $u_{\mathsf{T}}\geqslant u_{\mathsf{S}}$.*
(ii) $u_{\mathsf{S}}\wedge v_{\mathsf{T}} = (u\wedge v)_{\mathsf{S}\cup\mathsf{T}}$,
(iii) *If $\mathsf{S}\subseteq\mathrm{GDes}(v)$ and $\mathsf{T}\subseteq\mathrm{GDes}(u)$, then $u_{\mathsf{S}}\vee v_{\mathsf{T}} = (u\vee v)_{\mathsf{S}\cap\mathsf{T}}$.*

Proof. The first statement is an immediate consequence of Lemma 2.15. For the second, we use (2.2) to show that $\mathrm{Inv}(u_{\mathsf{S}}\wedge v_{\mathsf{T}}) = \mathrm{Inv}((u\wedge v)_{\mathsf{S}\cup\mathsf{T}})$.

First, suppose $(i,j)\in\mathrm{Inv}((u\wedge v)_{\mathsf{S}\cup\mathsf{T}})$. Then by Lemma 2.15 and (2.2), we have $[i,j)\cap(\mathsf{S}\cup\mathsf{T})=\emptyset$, and given a chain $i=k_0<\cdots<k_s=j$, there is an index r such that $(k_{r-1},k_r)\in\mathrm{Inv}(u)\cap\mathrm{Inv}(v)$. Hence we also have $[k_{r-1},k_r)\cap(\mathsf{S}\cup\mathsf{T})=\emptyset$, and thus $(k_{r-1},k_r)\in\mathrm{Inv}(u_{\mathsf{S}})\cap\mathrm{Inv}(v_{\mathsf{T}})$. Thus $(i,j)\in\mathrm{Inv}(u_{\mathsf{S}}\wedge v_{\mathsf{T}})$.

We show the other inclusion. Let $(i,j)\in\mathrm{Inv}(u_{\mathsf{S}}\wedge v_{\mathsf{T}})$. Considering the chain $i<j$, we must have $(i,j)\in\mathrm{Inv}(u_{\mathsf{S}})\cap\mathrm{Inv}(v_{\mathsf{T}})$. In particular, $[i,j)\cap(\mathsf{S}\cup\mathsf{T})=\emptyset$. On the other hand, for any chain $i=k_0<\cdots<k_s=j$ there is an index r such that $(k_{r-1},k_r)\in\mathrm{Inv}(u_{\mathsf{S}})\cap\mathrm{Inv}(v_{\mathsf{T}})$. Since this is a subset of $\mathrm{Inv}(u)\cap\mathrm{Inv}(v)$, we have $(i,j)\in\mathrm{Inv}(u\wedge v)$. Together with $[i,j)\cap\mathsf{S}\cup\mathsf{T}=\emptyset$, we see that $(i,j)\in\mathrm{Inv}((u\wedge v)_{\mathsf{S}\cup\mathsf{T}})$, proving the second statement.

For the third statement, first note that statement (i) implies that $u_{\mathsf{S}}\leqslant(u\vee v)_{\mathsf{S}}\leqslant(u\vee v)_{\mathsf{S}\cap\mathsf{T}}$ and similarly $v_{\mathsf{T}}\leqslant(u\vee v)_{\mathsf{S}\cap\mathsf{T}}$. Thus we have $u_{\mathsf{S}}\vee v_{\mathsf{T}}\leqslant(u\vee v)_{\mathsf{S}\cap\mathsf{T}}$. To show the other inequality, we need the assumptions on S and T. With those assumptions, we show $\mathrm{Inv}((u\vee v)_{\mathsf{S}\cap\mathsf{T}})\subseteq\mathrm{Inv}(u_{\mathsf{S}}\vee v_{\mathsf{T}})$.

Suppose that $\mathsf{S}\subseteq\mathrm{GDes}(v)$ and $\mathsf{T}\subseteq\mathrm{GDes}(u)$, so that S consists of global descents of v and T consists of global descents of u. Let $(i,j)\in\mathrm{Inv}((u\vee v)_{\mathsf{S}\cap\mathsf{T}})$. Then, by Lemma 2.15 and (2.1), $[i,j)\cap\mathsf{S}\cap\mathsf{T}=\emptyset$ and there is a chain $i=k_0<\cdots<k_s=j$ such that for every r, $(k_{r-1},k_r)\in\mathrm{Inv}(u)\cup\mathrm{Inv}(v)$. We refine this chain so that every pair of consecutive elements belongs to $\mathrm{Inv}(u_{\mathsf{S}})\cup\mathrm{Inv}(v_{\mathsf{T}})$.

If $[k_{r-1},k_r)\cap(\mathsf{S}\cup\mathsf{T})=\emptyset$ then, by Lemma 2.15, $(k_{r-1},k_r)\in\mathrm{Inv}(u_{\mathsf{S}})\cup\mathrm{Inv}(v_{\mathsf{T}})$ and this interval need not be refined. If however the intersection is not empty, then choose any refinement

$$k_{r-1} = k_0^{(r)} < k_1^{(r)} < \cdots < k_{s_r}^{(r)} = k_r,$$

with the property that each interval $[k_{t-1}^{(r)},k_t^{(r)})$ contains exactly *one* element of S or T, but not an element of both. This is possible because $[i,j)\cap\mathsf{S}\cap\mathsf{T}=\emptyset$. We claim that each pair $(k_{t-1}^{(r)},k_t^{(r)})$ is in $\mathrm{Inv}(u_{\mathsf{S}})\cup\mathrm{Inv}(v_{\mathsf{T}})$. In fact, if $[k_{t-1}^{(r)},k_t^{(r)})$ contains an element $p \in \mathsf{S}$, then that is a global descent of v, so $(k_{t-1}^{(r)}, k_t^{(r)}) \in \mathrm{Inv}(v)$. Thus $(k_{t-1}^{(r)}, k_t^{(r)}) \in \mathrm{Inv}(v_\mathsf{T})$, since $[k_{t-1}^{(r)}, k_t^{(r)}) \cap \mathsf{T} = \emptyset$ by our construction of the refinement. Similarly, if $[k_{t-1}^{(r)}, k_t^{(r)})$ contains an element of T, then $(k_{t-1}^{(r)}, k_t^{(r)}) \in \mathrm{Inv}(u_\mathsf{S})$. We have thus constructed a chain from i to j with the required property, which shows that $(i,j) \in \mathrm{Inv}(u_\mathsf{S} \vee v_\mathsf{T})$ and completes the proof. □

We calculate the descents and global descents of some particular permutations. The straightforward proof is left to the reader.

Lemma 2.17. *Let $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$. Then*

(i) $\mathrm{Des}(u \times v) = \mathrm{Des}(u) \cup (p + \mathrm{Des}(v))$,
(ii) $\mathrm{GDes}(u \times v) = p + \mathrm{GDes}(v)$,
(iii) $\mathrm{Des}(\zeta_{p,q} \cdot (u \times v)) = \mathrm{Des}(u) \cup \{p\} \cup (p + \mathrm{Des}(v))$,
(iv) $\mathrm{GDes}(\zeta_{p,q} \cdot (u \times v)) = \mathrm{GDes}(u) \cup \{p\} \cup (p + \mathrm{GDes}(v))$.

More generally, let $u_{(i)} \in \mathfrak{S}_{p_i}$, $i = 1, \dots, k$, $\mathsf{S} = \{p_1, p_1 + p_2, \dots, p_1 + \cdots + p_{k-1}\} \subseteq [n-1]$. Then

(v) $\mathrm{Des}(u_{(1)} \times \cdots \times u_{(k)}) = \bigcup_{i=1}^k (p_1 + \cdots + p_{i-1} + \mathrm{Des}(u_{(i)}))$,
(vi) $\mathrm{GDes}(u_{(1)} \times \cdots \times u_{(k)}) = p_1 + \cdots + p_{k-1} + \mathrm{GDes}(u_{(k)})$,
(vii) $\mathrm{Des}(\zeta_\mathsf{S} \cdot (u_{(1)} \times \cdots \times u_{(k)})) = \mathsf{S} \cup \bigcup_{i=1}^k (p_1 + \cdots + p_{i-1} + \mathrm{Des}(u_{(i)}))$,
(viii) $\mathrm{GDes}(\zeta_\mathsf{S} \cdot (u_{(1)} \times \cdots \times u_{(k)})) = \mathsf{S} \cup \bigcup_{i=1}^k (p_1 + \cdots + p_{i-1} + \mathrm{GDes}(u_{(i)}))$.

Lemma 2.18. *Let $u \in \mathfrak{S}_n$ and $\mathsf{S} \subseteq [n-1]$. Then*

$$\mathsf{S} \subseteq \mathrm{GDes}(u) \iff u = \zeta_\mathsf{S} u_\mathsf{S}.$$

Proof. The reverse implication follows from Lemma 2.17(viii). The other follows by induction from Lemma 2.14 and (2.6). □

3. The coproduct of $\mathfrak{S}$Sym

The coproduct of $\mathfrak{S}$Sym (1.3) takes a simple form on the monomial basis. We derive this formula using some results of Section 2. For a permutation $u \in \mathfrak{S}_n$, define $\overline{\mathrm{GDes}}(u)$ to be $\mathrm{GDes}(u) \cup \{0, n\}$.

Theorem 3.1. *Let $u \in \mathfrak{S}_n$. Then*

$$\Delta(\mathcal{M}_u) = \sum_{p \in \overline{\mathrm{GDes}}(u)} \mathcal{M}_{\mathrm{st}(u_1, \dots, u_p)} \otimes \mathcal{M}_{\mathrm{st}(u_{p+1}, \dots, u_n)}. \tag{3.1}$$

Proof. Let $\Delta' : \mathfrak{S}\mathrm{Sym} \to \mathfrak{S}\mathrm{Sym} \otimes \mathfrak{S}\mathrm{Sym}$ be the map whose action on the monomial basis is defined by the sum (3.1). We show that Δ' is the coproduct Δ, as defined by (1.3). We use the following notation. For $w \in \mathfrak{S}_n$ and $0 \leqslant p \leqslant n$, let $w^p_{(1)} \coloneqq \mathrm{st}(w_1, \ldots, w_p)$ and $w^p_{(2)} \coloneqq \mathrm{st}(w_{p+1}, \ldots, w_n)$. By virtue of Lemmas 2.1 and 2.14, we have

$$v = \zeta_{p,n-p} \cdot (v^p_{(1)} \times v^p_{(2)}) \;\Leftrightarrow\; p \in \overline{\mathrm{GDes}}(v).$$

Therefore,

$$\begin{aligned}\Delta'(\mathcal{F}_u) &= \sum_{u \leqslant v} \Delta'(\mathcal{M}_v) = \sum_{u \leqslant v} \sum_{p \in \overline{\mathrm{GDes}}(v)} \mathcal{M}_{v^p_{(1)}} \otimes \mathcal{M}_{v^p_{(2)}} \\ &= \sum_{p=0}^{n} \sum_{\substack{u \leqslant v \\ v = \zeta_{p,n-p} \cdot (v^p_{(1)} \times v^p_{(2)})}} \mathcal{M}_{v^p_{(1)}} \otimes \mathcal{M}_{v^p_{(2)}} = \sum_{p=0}^{n} \sum_{\substack{v_1,\, v_2 \\ u \leqslant \zeta_{p,n-p} \cdot (v_1 \times v_2)}} \mathcal{M}_{v_1} \otimes \mathcal{M}_{v_2}.\end{aligned}$$

Write $u = \zeta \cdot (u^p_{(1)} \times u^p_{(2)})$ for some $\zeta \in \mathfrak{S}^{(p,n-p)}$ which depends on p. By Proposition 2.5,

$$\zeta \cdot (u^p_{(1)} \times u^p_{(2)}) \leqslant \zeta_{p,n-p} \cdot (v_1 \times v_2) \;\Leftrightarrow\; u^p_{(1)} \leqslant v_1 \text{ and } u^p_{(2)} \leqslant v_2.$$

Therefore,

$$\begin{aligned}\Delta'(\mathcal{F}_u) &= \sum_{p=0}^{n} \sum_{\substack{v_1, v_2 \\ u^p_{(1)} \leqslant v_1,\, u^p_{(2)} \leqslant v_2}} \mathcal{M}_{v_1} \otimes \mathcal{M}_{v_2} = \sum_{p=0}^{n} \sum_{u^p_{(1)} \leqslant v_1} \mathcal{M}_{v_1} \otimes \sum_{u^p_{(2)} \leqslant v_2} \mathcal{M}_{v_2} \\ &= \sum_{p=0}^{n} \mathcal{F}_{u^p_{(1)}} \otimes \mathcal{F}_{u^p_{(2)}} = \Delta(\mathcal{F}_u).\end{aligned}$$

Remark 3.2. The action of the coproduct of $\mathfrak{S}\mathrm{Sym}$ on the fundamental basis can also be expressed in terms of the weak order. To see this, let $u \in \mathfrak{S}_n$ and $0 \leqslant p \leqslant n$ and write $u = \zeta \cdot (u^p_{(1)} \times u^p_{(2)})$. By Proposition 2.5, $u^p_{(1)} \times u^p_{(2)} \leqslant u \leqslant \zeta_{p,n-p} \cdot (u^p_{(1)} \times u^p_{(2)})$. Moreover, $u^p_{(1)}$ and $u^p_{(2)}$ are the only permutations in $\mathfrak{S}_p$ and $\mathfrak{S}_{n-p}$ with this property, again by Proposition 2.5. Therefore, equation (1.3) is also described by $\Delta(\mathcal{F}_u) = \sum \mathcal{F}_v \otimes \mathcal{F}_w$, where the sum is over all p from 0 to n and all permutations $v \in \mathfrak{S}_p$ and $w \in \mathfrak{S}_{n-p}$ such that $v \times w \leqslant u \leqslant \zeta_{p,n-p} \cdot (v \times w)$. This fact (in its dual form) is due to Loday and Ronco [20, Theorem 4.1], who were the first to point out the relevance of the weak order to the Hopf algebra structure of $\mathfrak{S}\mathrm{Sym}$.

4. The product of $\mathfrak{S}$Sym

We give an explicit formula for the product of $\mathfrak{S}$Sym in terms of its monomial basis and a geometric interpretation for the structure constants. Remarkably, these are still non-negative integers. For instance,

$$\begin{aligned}\mathcal{M}_{12}\cdot\mathcal{M}_{21} = {} & \mathcal{M}_{4312}+\mathcal{M}_{4231}+\mathcal{M}_{3421}+\mathcal{M}_{4123}+\mathcal{M}_{2341}\\ & + \mathcal{M}_{1243}+\mathcal{M}_{1423}+\mathcal{M}_{1342}+3\mathcal{M}_{1432}+2\mathcal{M}_{2431}+2\mathcal{M}_{4132}.\end{aligned} \tag{4.1}$$

The structure constants count special ways of shuffling two permutations, according to certain conditions involving the weak order. Specifically, for $u\in\mathfrak{S}_p$, $v\in\mathfrak{S}_q$ and $w\in\mathfrak{S}_{p+q}$, define $A^w_{u,v}\subseteq\mathfrak{S}^{(p,q)}$ to be those $\zeta\in\mathfrak{S}^{(p,q)}$ satisfying

$$\begin{aligned}&\text{(i)}\quad (u\times v)\cdot\zeta^{-1}\leqslant w,\ \text{ and}\\ &\text{(ii)}\quad \text{if } u\leqslant u' \text{ and } v\leqslant v' \text{ satisfy } (u'\times v')\cdot\zeta^{-1}\leqslant w,\\ &\qquad\ \text{then } u=u' \text{ and } v=v'.\end{aligned} \tag{4.2}$$

Set $\alpha^w_{u,v} := \#A^w_{u,v}$. We will prove the following theorem.

Theorem 4.1. *For any $u\in\mathfrak{S}_p$ and $v\in\mathfrak{S}_q$, we have*

$$\mathcal{M}_u\cdot\mathcal{M}_v = \sum_{w\in\mathfrak{S}_{p+q}} \alpha^w_{u,v}\,\mathcal{M}_w. \tag{4.3}$$

For instance, in (4.1) the coefficient of $\mathcal{M}_{2431}$ in $\mathcal{M}_{12}\cdot\mathcal{M}_{21}$ is 2 because among the six permutations in $\mathfrak{S}^{(2,2)}$,

$$1234,\ 1324,\ 1423,\ 2314,\ 2413,\ 3412,$$

only the first two satisfy conditions (i) and (ii) of (4.2). In fact, 2314, 2413 and 3412 do not satisfy (i), while 1423 satisfies (i) but not (ii).

The structure constants $\alpha^w_{u,v}$ admit a geometric-combinatorial description in terms of the permutahedron. To derive it, recall the convex embeddings of Proposition 2.7.

$$\rho_\zeta : \mathfrak{S}_p\times\mathfrak{S}_q \to \mathfrak{S}_{p+q},\quad \rho_\zeta(u,v) := (u\times v)\cdot\zeta^{-1}.$$

Since ρ_ζ preserves joins, we may further rewrite definition (4.2) of $A^w_{u,v}$ as

$$A^w_{u,v} = \{\zeta\in\mathfrak{S}^{(p,q)} \mid (u,v) = \max\rho_\zeta^{-1}[1,w]\}, \tag{4.4}$$

where $[w,w'] := \{w'' \mid w\leqslant w''\leqslant w'\}$ denotes the interval between w and w'.

The vertices of the $(n-1)$-dimensional permutahedron can be indexed by the elements of $\mathfrak{S}_n$ so that its 1-skeleton is the Hasse diagram of the weak order (see Fig. 1). Facets of the permutahedron are products of two lower dimensional permutahedra, and the image of ρ_ζ is the set of vertices in a facet. Moreover, every

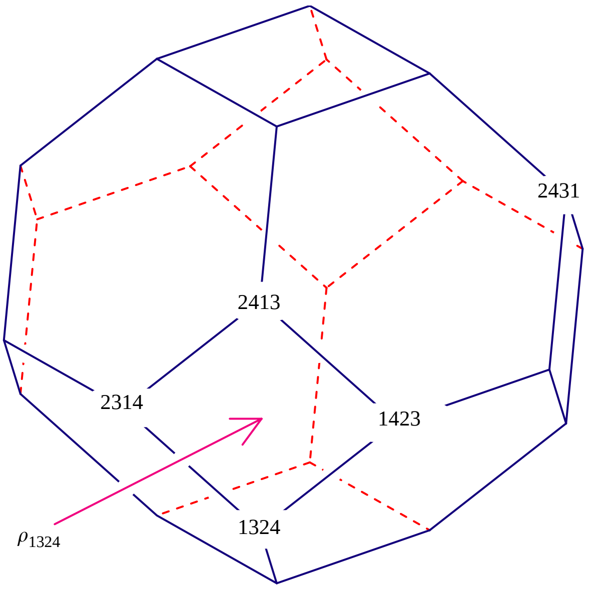

Fig. 3. The facet ρ_{1324} of type $(2,2)$ and $w = 2431$.

facet arises in this way for a unique triple (p, q, ζ) with $p + q = n$ and $\zeta \in \mathfrak{S}^{(p,q)}$; see [24, Lemma 4.2], or [4, Exercise 2.9], or [18, Proposition A.1]. Let us say that such a facet has *type* (p, q). Fig. 3 displays the image of ρ_{1324}, a facet of the 3-permutahedron of type $(2,2)$, and the permutation 2431.

The description (4.4) of $A^w_{u,v}$ (and hence of $\alpha^w_{u,v}$) can be interpreted as follows: Given $u \in \mathfrak{S}_p$, $v \in \mathfrak{S}_q$, and $w \in \mathfrak{S}_{p+q}$, the structure constant $\alpha^w_{u,v}$ counts the number of facets of type (p, q) of the $(p + q - 1)$-permutahedron such that the vertex $\rho_\zeta(u, v)$ is below w and it is the maximum vertex in that facet below w.

For instance, the facet ρ_{1324} contributes to the structure constant $\alpha^{2431}_{12,21}$ because the vertex $\rho_{1324}(12, 21) = 1423$ satisfies the required properties in relation to the vertex $w = 2431$, as shown in Fig. 3.

This description of the product of $\mathfrak{S}$Sym has an analog for $\mathcal{Q}$Sym that we present in Section 7.

Proof of Theorem 4.1. Expand the product $\mathcal{M}_u \cdot \mathcal{M}_v$ in the fundamental basis and then use Formula (1.1) to obtain

$$\begin{aligned}\mathcal{M}_u \cdot \mathcal{M}_v &= \sum_{\substack{u \leqslant u' \\ v \leqslant v'}} \mu(u, u')\mu(v, v')\, \mathcal{F}_{u'} \cdot \mathcal{F}_{v'} \\ &= \sum_{\zeta \in \mathfrak{S}^{(p,q)}} \sum_{\substack{u \leqslant u' \\ v \leqslant v'}} \mu(u, u')\mu(v, v')\mathcal{F}_{(u' \times v')\cdot \zeta^{-1}}.\end{aligned}$$

Expressing this result in terms of the monomial basis and collecting like terms gives

$$\begin{aligned}\mathcal{M}_u \cdot \mathcal{M}_v &= \sum_{\zeta \in \mathfrak{S}^{(p,q)}} \sum_{\substack{u \leqslant u',\ v \leqslant v' \\ (u' \times v') \cdot \zeta^{-1} \leqslant w}} \mu(u,u')\mu(v,v')\mathcal{M}_w \\ &= \sum_w \sum_{\substack{u \leqslant u' \\ v \leqslant v'}} \mu(u,u')\mu(v,v')\beta^w_{u',v'}\mathcal{M}_w,\end{aligned}$$

where $\beta^w_{u',v'}$ is the number of permutations in the set

$$B^w_{u',v'} := \{\zeta \in \mathfrak{S}^{(p,q)} \mid (u' \times v') \cdot \zeta^{-1} \leqslant w\}.$$

The theorem will follow once we show that

$$\alpha^w_{u,v} = \sum_{u \leqslant u',\ v \leqslant v'} \mu(u,u')\mu(v,v')\beta^w_{u',v'},$$

or equivalently, by Möbius inversion on $\mathfrak{S}_p \times \mathfrak{S}_q$,

$$\beta^w_{u,v} = \sum_{u \leqslant u',\ v \leqslant v'} \alpha^w_{u',v'}.$$

We prove this last equality by showing that

$$B^w_{u,v} = \coprod_{u \leqslant u',\ v \leqslant v'} A^w_{u',v'},$$

where the union is disjoint.

To see this, first suppose $\zeta \in A^w_{u,v} \cap A^w_{u',v'}$. Then, by condition (i) of (4.2),

$$(u \times v) \cdot \zeta^{-1} \leqslant w \quad \text{and} \quad (u' \times v') \cdot \zeta^{-1} \leqslant w.$$

By Proposition 2.7(d),

$$((u \vee u') \times (v \vee v')) \cdot \zeta^{-1} \leqslant w.$$

But then, by condition (ii) of (4.2),

$$u = u \vee u' = u' \quad \text{and} \quad v = v \vee v' = v',$$

so the union is disjoint.

Next, suppose that $\zeta \in A^w_{u',v'}$ for some $u \leqslant u'$ and $v \leqslant v'$. Then, by condition (i) of (4.2), $(u' \times v') \cdot \zeta^{-1} \leqslant w$. By Proposition 2.7(c) we have, $(u \times v) \cdot \zeta^{-1} \leqslant w$, so $\zeta \in B^w_{u,v}$. This proves one inclusion.

For the other inclusion, suppose that $\zeta \in B^w_{u,v}$. Define

$$(\bar{u}, \bar{v}) := \bigvee \{(u', v') \mid u \leqslant u',\ v \leqslant v',\ \text{and}\ (u' \times v') \cdot \zeta^{-1} \leqslant w\}.$$

Then $\zeta \in A^{w}_{\bar{u},\bar{v}}$: condition (i) is satisfied because ρ_ζ preserves joins, and (ii) simply by definition. This completes the proof. $\square$

5. The antipode of $\mathfrak{S}$Sym

Malvenuto left open the problem of an explicit formula for the antipode of $\mathfrak{S}$Sym [22, pp. 59–60]. We solve that problem, giving formulas that identify the coefficients of the antipode in terms of both the fundamental and monomial basis in explicit combinatorial terms.

We first review a general formula for the antipode of a connected Hopf algebra H, due to Takeuchi [35, Lemma 14] (see also [25]). Let H be an arbitrary bialgebra with structure maps: multiplication $m{:}H\otimes H\to H$, unit $u{:}\mathbb{Q}\to H$, comultiplication $\Delta{:}H\to H\otimes H$, and counit $\varepsilon{:}H\to\mathbb{Q}$. Set $m^{(1)}=m$, $\Delta^{(1)}=\Delta$, and for any $k\geqslant 2$,

$$m^{(k)} = m(m^{(k-1)}\otimes \mathrm{id}) \,:\, H^{\otimes k+1}\to H, \quad \text{and}$$

$$\Delta^{(k)} = (\Delta^{(k-1)}\otimes \mathrm{id})\Delta \,:\, H\to H^{\otimes k+1}.$$

These are the higher or iterated products and coproducts. We also set

$$m^{(-1)} = u \,:\, \mathbb{Q}\to H,$$

$$\Delta^{(-1)} = \varepsilon \,:\, H\to\mathbb{Q}, \quad \text{and}$$

$$m^{(0)} = \Delta^{(0)} = \mathrm{id} \,:\, H\to H.$$

If $f : H\to H$ is any linear map, the convolution powers of f are, for any $k\geqslant 0$,

$$f^{*k} = m^{(k-1)} f^{\otimes k} \Delta^{(k-1)}.$$

In particular, $f^{*0}=u\varepsilon$ and $f^{*1}=f$.

We set $\pi := \mathrm{id} - u\varepsilon$. If π is locally nilpotent with respect to convolution, then $\mathrm{id} = u\varepsilon + \pi$ is invertible with respect to convolution, with inverse

$$S = \sum_{k\geqslant 0} (-\pi)^{*k} = \sum_{k\geqslant 0} (-1)^k m^{(k-1)} \pi^{\otimes k} \Delta^{(k-1)}. \tag{5.1}$$

This is certainly the case if H is a graded connected bialgebra, in which case π annihilates the component of degree 0 (and hence π^{*k} annihilates components of degree $<k$). Thus (5.1) is a general formula for the antipode of a graded connected Hopf algebra.

We will make use of this formula to find explicit formulas for the antipode of $\mathfrak{S}$Sym. The first task is to describe the higher products and coproducts explicitly. We begin with the higher coproducts in terms of the fundamental and monomial bases.

Proposition 5.1. *Let* $v \in \mathfrak{S}_n$, $n \geqslant 0$, *and* $k \geqslant 1$. *Then*

(i) $\Delta^{(k)}(\mathcal{F}_v) = \sum_{0 \leqslant p_1 \leqslant \cdots \leqslant p_k \leqslant n} \mathcal{F}_{\mathrm{st}(v_1,\,\ldots,\,v_{p_1})} \otimes \cdots \otimes \mathcal{F}_{\mathrm{st}(v_{p_k+1},\,\ldots,\,v_n)}$, *and*

(ii) $\Delta^{(k)}(\mathcal{M}_v) = \sum_{\substack{0 \leqslant p_1 \leqslant \ldots \leqslant p_k \leqslant n \\ p_i \in \overline{\mathrm{GDes}}(v)}} \mathcal{M}_{\mathrm{st}(v_1,\,\ldots,\,v_{p_1})} \otimes \cdots \otimes \mathcal{M}_{\mathrm{st}(v_{p_k+1},\,\ldots,\,v_n)}$.

Proof. Both formulas follow by induction from the corresponding descriptions of the coproduct, Eqs. (1.3) and (3.1). $\square$

We describe higher products in terms of minimal coset representatives $\mathfrak{S}^{\mathsf{S}}$ of parabolic subgroups, whose basic properties were discussed in Section 2.2. Recall that for a subset $\mathsf{S} = \{p_1 < p_2 < \cdots < p_k\}$ of $[n-1]$, we have $\mathfrak{S}^{\mathsf{S}} = \{\zeta \in \mathfrak{S}_n \mid \mathrm{Des}(\zeta) \subseteq \mathsf{S}\}$. Analogously to (4.2), given permutations $v_{(1)} \in \mathfrak{S}_{p_1},\ v_{(2)} \in \mathfrak{S}_{p_2-p_1}, \ldots, v_{(k+1)} \in \mathfrak{S}_{n-p_k}$, define $A^w_{v_{(1)},v_{(2)},\ldots,v_{(k+1)}} \subseteq \mathfrak{S}^{\mathsf{S}}$ to be those $\zeta \in \mathfrak{S}^{\mathsf{S}}$ satisfying

$$\begin{array}{ll} \text{(i)} & (v_{(1)} \times v_{(2)} \times \cdots \times v_{(k+1)}) \cdot \zeta^{-1} \leqslant w, \text{ and} \\ \text{(ii)} & \text{if } v_{(i)} \leqslant v'_{(i)}\ \forall i \text{ and } (v'_{(1)} \times v'_{(2)} \times \cdots \times v'_{(k+1)}) \cdot \zeta^{-1} \leqslant w, \\ & \text{then } v_{(i)} = v'_{(i)},\ \forall i. \end{array} \tag{5.2}$$

Set $\alpha^w_{v_{(1)},v_{(2)},\ldots,v_{(k+1)}} := \# A^w_{v_{(1)},v_{(2)},\ldots,v_{(k+1)}}$.

Proposition 5.2. *Let* S *and* $v_{(1)}, \ldots, v_{(k+1)}$ *be as in the preceding paragraph. Then*

(i) $\mathcal{F}_{v_{(1)}} \cdot \mathcal{F}_{v_{(2)}} \cdots \mathcal{F}_{v_{(k+1)}} = \sum_{\zeta \in \mathfrak{S}^{\mathsf{S}}} \mathcal{F}_{(v_{(1)} \times v_{(2)} \times \cdots \times v_{(k+1)}) \cdot \zeta^{-1}}$ *and*

(ii) $\mathcal{M}_{v_{(1)}} \cdot \mathcal{M}_{v_{(2)}} \cdots \mathcal{M}_{v_{(k+1)}} = \sum_{w \in \mathfrak{S}_n} \alpha^w_{v_{(1)},v_{(2)},\ldots,v_{(k+1)}} \mathcal{M}_w$.

Proof. The first formula follows immediately by induction from (1.1) (the case $k=2$), using (2.5). The second formula can be deduced from (i) in the same way as in the proof of Theorem 4.1. $\square$

The structure constants for the iterated product admit a geometric description similar to that of the product. The image of the map

$$\rho_\zeta : \mathfrak{S}_{\mathsf{S}} \to \mathfrak{S}_n, \quad (v_{(1)} \times \cdots \times v_{(k+1)}) \mapsto (v_{(1)} \times \cdots \times v_{(k+1)}) \cdot \zeta^{-1},$$

consists of the vertices of a face of codimension k in the $(n-1)$-permutahedron, and every such face arises in this way for a unique pair (S, ζ) with $\mathsf{S} \subseteq [n-1]$ having k elements and $\zeta \in \mathfrak{S}^{\mathsf{S}}$. Let us say that such a face has *type* S. The structure constant $\alpha^w_{v_{(1)},\ldots,v_{(k+1)}}$ counts the number of faces of type S with the property that the vertex $\rho_\zeta(v_{(1)}, \dots, v_{(k+1)})$ is below w and it is the maximum vertex in its face below w.

We next determine the convolution powers of the projection $\pi = \mathrm{id} - u\varepsilon$. Recall that for any subset $\mathsf{S} = \{p_1 < p_2 < \cdots < p_k\} \subseteq [n-1]$ and $v \in \mathfrak{S}_n$ we have

$$v_{\mathsf{S}} := \mathrm{st}(v_1, \dots, v_{p_1}) \times \mathrm{st}(v_{p_1+1}, \dots, v_{p_2}) \times \cdots \times \mathrm{st}(v_{p_k+1}, \dots, v_n) \ \in \ \mathfrak{S}_n,$$

as given by (2.10). We slightly amend our notation in order to simplify some subsequent statements. For $v, w \in \mathfrak{S}_n$ and $\mathsf{S} \subseteq [n-1]$, set $A_{\mathsf{S}}(v, w) := A^w_{v_{(1)}, \dots, v_{(k+1)}}$, where $v_{(1)}, \dots, v_{(k+1)}$ are the factors of v_{S} in the definition above. Comparing with (5.2), we see that $A_{\mathsf{S}}(v, w) \subseteq \mathfrak{S}^{\mathsf{S}}$ consists of those $\zeta \in \mathfrak{S}^{\mathsf{S}}$ satisfying

$$\begin{aligned} &\text{(i)} \quad v_{\mathsf{S}}\zeta^{-1} \leqslant w, \ \text{and} \\ &\text{(ii)} \quad \text{if } v \leqslant v' \text{ and } v'_{\mathsf{S}}\zeta^{-1} \leqslant w \text{ then } v = v'. \end{aligned} \tag{5.3}$$

Similarly, we define $\alpha_{\mathsf{S}}(v, w) := \# A_{\mathsf{S}}(v, w)$. If $v_{(1)}, \dots, v_{(k+1)}$ are the factors in the definition of v_{S}, then

$$\alpha_{\mathsf{S}}(v, w) = \alpha^w_{v_{(1)}, \dots, v_{(k+1)}}. \tag{5.4}$$

Let $\binom{[n-1]}{k-1}$ be the collection of subsets of $[n-1]$ of size $k-1$.

Proposition 5.3. *Let $n, k \geqslant 1$ and $v \in \mathfrak{S}_n$. Then*

(i) $\pi^{*k}(\mathcal{F}_v) = \sum_{w \in \mathfrak{S}_n} \sum_{\substack{\mathsf{S} \in \binom{[n-1]}{k-1} \\ \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S}}} \mathcal{F}_w$, *and*

(ii) $\pi^{*k}(\mathcal{M}_v) = \sum_{w \in \mathfrak{S}_n} \sum_{\mathsf{S} \in \binom{\mathrm{GDes}(v)}{k-1}} \alpha_{\mathsf{S}}(v, w)\, \mathcal{M}_w.$

Proof. By Proposition 5.1(i),

$$\Delta^{(k-1)}(\mathcal{F}_v) = \sum_{0 \leqslant p_1 \leqslant \cdots \leqslant p_{k-1} \leqslant n} \mathcal{F}_{\mathrm{st}(v_1, \dots, v_{p_1})} \otimes \mathcal{F}_{\mathrm{st}(v_{p_1+1}, \dots, v_{p_2})} \otimes \cdots \otimes \mathcal{F}_{\mathrm{st}(v_{p_{k-1}+1}, \dots, v_n)}.$$

Suppose that an equality $p_i = p_{i+1}$ occurs (where we define $p_0 = 0$ and $p_k = n$). The corresponding permutation $\mathrm{st}(v_{p_i+1}, \dots, v_{p_{i+1}})$ is then simply the unique permutation in $\mathfrak{S}_0$, which indexes the element $1 \in \ker(\pi)$. Therefore,

$$\begin{aligned}
\pi^{*k}(\mathcal{F}_v) &= m^{(k-1)}\pi^{\otimes k}\Delta^{(k-1)}(\mathcal{F}_v) \\
&= \sum_{0<p_1<p_2<\cdots<p_{k-1}<n} \mathcal{F}_{\mathrm{st}(v_1,\dots,v_{p_1})} \cdot \mathcal{F}_{\mathrm{st}(v_{p_1+1},\dots,v_{p_2})} \cdots \mathcal{F}_{\mathrm{st}(v_{p_{k-1}+1},\dots,v_n)} \\
&= \sum_{0<p_1<p_2<\cdots<p_{k-1}<n} \; \sum_{\zeta \in \mathfrak{S}^{\{p_1,p_2,\dots,p_{k-1}\}}} \mathcal{F}_{(\mathrm{st}(u_1,\dots,u_{p_1})\times\cdots\times\mathrm{st}(u_{p_{k-1}+1},\dots,u_n))\cdot\zeta^{-1}},
\end{aligned}$$

the last equality by the formula of Proposition 5.2(i) for the iterated product. Changing the index of summation in the first sum to $\mathsf{S} \in \binom{[n-1]}{k-1}$ and using the definition of v_{S} gives

$$\pi^{*k}(\mathcal{F}_v) = \sum_{\mathsf{S} \in \binom{[n-1]}{k-1}} \; \sum_{\zeta \in \mathfrak{S}^{\mathsf{S}}} \mathcal{F}_{v_{\mathsf{S}}\zeta^{-1}}.$$

Again reindexing the sum and using that $\mathfrak{S}^{\mathsf{S}}$ consists of permutations whose descent set is a subset of S, we obtain

$$\pi^{*k}(\mathcal{F}_v) = \sum_{w \in \mathfrak{S}_n} \; \sum_{\substack{\mathsf{S} \in \binom{[n-1]}{k-1} \\ w^{-1}v_{\mathsf{S}} \in \mathfrak{S}^{\mathsf{S}}}} \mathcal{F}_w = \sum_{w \in \mathfrak{S}_n} \; \sum_{\substack{\mathsf{S} \in \binom{[n-1]}{k-1} \\ \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S}}} \mathcal{F}_w,$$

establishing (i).

The second formula in terms of the monomial basis follows in exactly the same manner from Propositions 5.1(i) and 5.2(ii) for the higher coproducts and products in terms of the monomial basis, using (5.4). $\square$

We derive explicit formulas for the antipode on both bases. The formula for the fundamental basis is immediate from Proposition 5.3(i) and (5.1).

Theorem 5.4. *For $v, w \in \mathfrak{S}_n$ set*

$$\begin{aligned}
\lambda(v,w) := \; & \#\{\mathsf{S} \subseteq [n-1] \mid \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is odd}\} \\
& - \#\{\mathsf{S} \subseteq [n-1] \mid \mathrm{Des}(w^{-1}v_{\mathsf{S}}) \subseteq \mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is even}\}.
\end{aligned}$$

Then

$$S(\mathcal{F}_v) = \sum_{w \in \mathfrak{S}_n} \lambda(v,w)\, \mathcal{F}_w. \tag{5.5}$$

The coefficients of the antipode on the fundamental basis may indeed be positive or negative. For instance

$$S(\mathcal{F}_{231}) = \mathcal{F}_{132} - \mathcal{F}_{213} - 2\mathcal{F}_{231} + \mathcal{F}_{312}.$$

The coefficient of $\mathcal{F}_{312}$ is 1 because $\{1\}$, $\{2\}$, and $\{1,2\}$ are the subsets S of $\{1,2\}$ which satisfy $\mathrm{Des}((312)^{-1}(231)_{\mathsf{S}}) \subseteq \mathsf{S}$.

Our description of these coefficients is semi-combinatorial, in the sense that it involves a difference of cardinalities of sets. On the monomial basis the situation is different. The sign of the coefficients of $S(\mathcal{M}_v)$ only depends on the number of global descents of v. We provide a fully combinatorial description of these coefficients. Let $v, w \in \mathfrak{S}_n$ and suppose $\mathsf{S} \subseteq \mathrm{GDes}(v)$. Define $C_{\mathsf{S}}(v,w) \subseteq \mathfrak{S}^{\mathsf{S}}$ to be those $\zeta \in \mathfrak{S}^{\mathsf{S}}$ satisfying

$$\begin{aligned} &\text{(i)} \quad v_{\mathsf{S}}\zeta^{-1} \leqslant w, \\ &\text{(ii)} \quad \text{if } v \leqslant v' \text{ and } v'_{\mathsf{S}}\zeta^{-1} \leqslant w \text{ then } v = v', \text{ and} \\ &\text{(iii)} \quad \text{if } \mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq \mathsf{S} \text{ and } v_{\mathsf{R}}\zeta^{-1} \leqslant w \text{ then } \mathsf{R} = \mathsf{S}. \end{aligned} \tag{5.6}$$

Set $\kappa(v,w) := \# C_{\mathrm{GDes}(v)}(v,w)$.

Theorem 5.5. *For $v, w \in \mathfrak{S}_n$, we have*

$$S(\mathcal{M}_v) = (-1)^{\#\mathrm{GDes}(v)+1} \sum_{w \in \mathfrak{S}_n} \kappa(v,w)\, \mathcal{M}_w. \tag{5.7}$$

For instance,

$$\begin{aligned} S(\mathcal{M}_{3412}) = {} & \mathcal{M}_{1234} + 2\mathcal{M}_{1324} + \mathcal{M}_{1342} + \mathcal{M}_{1423} \\ & + \mathcal{M}_{2314} + \mathcal{M}_{2413} + \mathcal{M}_{3124} + \mathcal{M}_{3142} + \mathcal{M}_{3412}. \end{aligned}$$

Consider the coefficient of $\mathcal{M}_{3412}$. In this case, $\mathsf{S} = \mathrm{GDes}(3412) = \{2\}$, so

$$\mathfrak{S}^{\mathsf{S}} = \{1234,\ 1324,\ 1423,\ 2314,\ 2413,\ 3412\}.$$

Then 1234 satisfies (i) and (ii) of (5.6) but not (iii), 1324 satisfies (i) and (iii) but not (ii), 1423, 2314 and 2413 do not satisfy (i), and 3412 is the only element of $\mathfrak{S}^{\{2\}}$ that satisfies all three conditions of (5.6). Therefore $C_{\mathsf{S}}(3412, 3412) = \{3412\}$ and the coefficient is $\kappa(3412, 3412) = 1$.

Remark 5.6. The antipode of $\mathfrak{S}\mathrm{Sym}$ has infinite order. In fact, one may verify by induction that

$$S^{2m}(\mathcal{M}_{231}) = \mathcal{M}_{231} + 2m(\mathcal{M}_{213} - \mathcal{M}_{132}) \quad \forall\ m \in \mathbb{Z}.$$

Proof of Theorem 5.5. By formula (5.1) and Proposition 5.3(ii), we have

$$S(\mathcal{M}_v) = \sum_{w \in \mathfrak{S}_n} \sum_{\mathsf{S} \subseteq \mathrm{GDes}(v)} (-1)^{\#\mathsf{S}+1} \alpha_{\mathsf{S}}(v,w)\, \mathcal{M}_w.$$

For any $\mathsf{T}\subseteq \mathrm{GDes}(v)$, define

$$\gamma_{\mathsf{T}}(v,w) := \sum_{\mathsf{S}\subseteq\mathsf{T}} (-1)^{\#\mathsf{T}\setminus\mathsf{S}}\alpha_{\mathsf{S}}(v,w) = \sum_{\mathsf{S}\subseteq\mathsf{T}} \mu(\mathsf{S},\mathsf{T})\alpha_{\mathsf{S}}(v,w), \tag{5.8}$$

where $\mu(\cdot,\cdot)$ is the Möbius function of the Boolean poset $\mathcal{Q}_n$. We then have

$$S(\mathcal{M}_v) = (-1)^{\#\mathrm{GDes}(u)+1} \sum_{w\in\mathfrak{S}_n} \gamma_{\mathrm{GDes}(v)}(v,w)\mathcal{M}_w.$$

We complete the proof by showing that $\kappa(v,w) = \gamma_{\mathrm{GDes}(v)}(v,w)$, and more generally that $\gamma_{\mathsf{S}}(v,w) = \#C_{\mathsf{S}}(v,w)$, where $C_{\mathsf{S}}(v,w)$ is defined in (5.6).

Möbius inversion using the definition (5.8) of $\gamma_{\mathsf{T}}(v,w)$ gives

$$\alpha_{\mathsf{T}}(v,w) = \sum_{\mathsf{S}\subseteq\mathsf{T}} \gamma_{\mathsf{S}}(v,w).$$

We prove this last equality by showing that

$$A_{\mathsf{T}}(v,w) = \coprod_{\mathsf{S}\subseteq\mathsf{T}} C_{\mathsf{S}}(v,w), \tag{5.9}$$

where the union is disjoint. This implies that $\gamma_{\mathsf{S}}(v,w) = \#C_{\mathsf{S}}(v,w)$, which will complete the proof. We argue that this is a disjoint union in several steps.

Claim 1. *If* $\mathsf{S}\subseteq\mathsf{T}\subseteq\mathrm{GDes}(v)$ *then* $A_{\mathsf{S}}(v,w)\subseteq A_{\mathsf{T}}(v,w)$.

Let $\zeta\in A_{\mathsf{S}}(v,w)$. First of all, $\zeta\in\mathfrak{S}^{\mathsf{S}}\subseteq\mathfrak{S}^{\mathsf{T}}$, as $\mathfrak{S}^{\mathsf{S}}$ is the set of permutations with descent set a subset of S. By condition (i) of (5.6), $v_{\mathsf{S}}\zeta^{-1}\leqslant w$. On the other hand, Proposition 2.16(i) implies that $u_{\mathsf{T}}\leqslant u_{\mathsf{S}}$ and both permutations are elements of the parabolic subgroup $\mathfrak{S}_{\mathsf{S}}$. Hence by Proposition 2.10, $u_{\mathsf{T}}\zeta^{-1}\leqslant u_{\mathsf{S}}\zeta^{-1}$. Thus $u_{\mathsf{T}}\zeta^{-1}\leqslant w$, which establishes condition (i) of (5.6) for ζ to be in $A_{\mathsf{T}}(v,w)$.

Now suppose that $v\leqslant v'$ with $v'_{\mathsf{T}}\zeta^{-1}\leqslant w$. Since $v_{\mathsf{S}}\zeta^{-1}\leqslant w$, we deduce that

$$w \geqslant (v_{\mathsf{S}}\zeta^{-1})\vee(v'_{\mathsf{T}}\zeta^{-1}) = (v_{\mathsf{S}}\vee v'_{\mathsf{T}})\zeta^{-1} = (v\vee v')_{\mathsf{S}\cap\mathsf{T}}\zeta^{-1} = v_{\mathsf{S}}\zeta^{-1}.$$

The first equality is because ρ_ζ is a convex embedding and hence preserves joins by Proposition 2.10, and the second follows from Proposition 2.16(iii) as $\mathsf{S},\mathsf{T}\subseteq\mathrm{GDes}(v)\subseteq\mathrm{GDes}(v')$. Hence, by condition (ii) for $A_{\mathsf{S}}(v,w)$, we have $v=v'$. This establishes (ii) for ζ to be in $A_{\mathsf{T}}(v,w)$ and completes the proof of Claim 1.

Claim 2. *If* $\mathsf{S},\mathsf{T}\subseteq\mathrm{GDes}(v)$, *then* $A_{\mathsf{S}}(v,w)\cap A_{\mathsf{T}}(v,w) = A_{\mathsf{S}\cap\mathsf{T}}(v,w)$.

The inclusion $A_{\mathsf{S}\cap\mathsf{T}}(v,w)\subseteq A_{\mathsf{S}}(v,w)\cap A_{\mathsf{T}}(v,w)$ is a consequence of Claim 1. To prove the converse, let $\zeta\in A_{\mathsf{S}}(v,w)\cap A_{\mathsf{T}}(v,w)$. Note that $\zeta\in\mathfrak{S}^{\mathsf{S}}\cap\mathfrak{S}^{\mathsf{T}}$, which equals $\mathfrak{S}^{\mathsf{S}\cap\mathsf{T}}$.

By condition (i) for $\zeta \in A_{\mathsf{S}}(v,w)$ and for $\zeta \in A_{\mathsf{T}}(v,w)$, we have $v_{\mathsf{S}}\zeta^{-1} \leqslant w$ and $v_{\mathsf{T}}\zeta^{-1} \leqslant w$. Therefore,

$$w \geqslant (v_{\mathsf{S}}\zeta^{-1}) \vee (v_{\mathsf{T}}\zeta^{-1}) = (v_{\mathsf{S}} \vee v_{\mathsf{T}})\zeta^{-1} = v_{\mathsf{S}\cap\mathsf{T}}\zeta^{-1}.$$

As before, this uses Proposition 2.16(iii), which applies as $\mathsf{S},\mathsf{T} \subseteq \mathrm{GDes}(u)$. This proves condition (i) of (5.6) for ζ to be in $A_{\mathsf{S}\cap\mathsf{T}}(v,w)$.

Now suppose that $v \leqslant v'$ with $v'_{\mathsf{S}\cap\mathsf{T}}\zeta^{-1} \leqslant w$. By Proposition 2.16(i), $v'_{\mathsf{S}} \leqslant v'_{\mathsf{S}\cap\mathsf{T}}$. Then by Proposition 2.10, $v'_{\mathsf{S}}\zeta^{-1} \leqslant v'_{\mathsf{S}\cap\mathsf{T}}\zeta^{-1}$. Thus $v'_{\mathsf{S}}\zeta^{-1} \leqslant w$ and by condition (ii) for $A_{\mathsf{S}}(v,w)$ we deduce that $v = v'$. This proves condition (ii) for ζ to be in $A_{\mathsf{S}\cap\mathsf{T}}(v,w)$, and establishes Claim 2.

We complete the proof by showing that for $\mathsf{T} \subseteq \mathrm{GDes}(v)$ we have the decomposition (5.9) of $A_{\mathsf{T}}(v,w)$ into disjoint subsets $C_{\mathsf{S}}(v,w)$. Comparing the definitions (5.3) and (5.6), we see that $C_{\mathsf{S}}(v,w) \subseteq A_{\mathsf{S}}(v,w)$. Together with Claim 1 this implies that the right-hand side of (5.9) is contained in the left hand side.

We show the union is disjoint. Suppose there is a permutation $\zeta \in C_{\mathsf{S}}(v,w) \cap C_{\mathsf{S}'}(v,w)$. Then $\zeta \in A_{\mathsf{S}}(v,w) \cap A_{\mathsf{S}'}(v,w)$ which equals $A_{\mathsf{S}\cap\mathsf{S}'}(v,w)$, by Claim 2. Hence, by condition (i) for ζ to be in $A_{\mathsf{S}\cap\mathsf{S}'}(v,w)$, we have $v_{\mathsf{S}\cap\mathsf{S}'}\zeta^{-1} \leqslant w$. But then, from condition (iii) for $C_{\mathsf{S}}(v,w)$ and for $C_{\mathsf{S}'}(v,w)$, we deduce that $\mathsf{S} = \mathsf{S}\cap\mathsf{S}' = \mathsf{S}'$, proving the union is disjoint.

We show that $A_{\mathsf{T}}(v,w)$ is contained in the union in (5.9). Let $\zeta \in A_{\mathsf{T}}(v,w)$ and set

$$\mathsf{S} := \bigcap \{\mathsf{R} \mid \mathsf{R} \subseteq \mathsf{T},\ \zeta \in A_{\mathsf{R}}(v,w)\}. \tag{5.10}$$

By Claim 2,

$$A_{\mathsf{S}}(v,w) = \bigcap \{A_{\mathsf{R}}(v,w) \mid \mathsf{R} \subseteq \mathsf{T},\ \zeta \in A_{\mathsf{R}}(v,w)\},$$

so $\zeta \in A_{\mathsf{S}}(v,w)$. To show that $\zeta \in C_{\mathsf{S}}(v,w)$, we must verify condition (iii) of (5.6).

Suppose $\mathrm{Des}(\zeta) \subseteq \mathsf{R} \subseteq \mathsf{S}$ and $v_{\mathsf{R}}\zeta^{-1} \leqslant w$. We need to show that $\mathsf{S} \subseteq \mathsf{R}$. By the definition (5.10) of S, it suffices to show that $\zeta \in A_{\mathsf{R}}(v,w)$. By our assumption that $v_{\mathsf{R}}\zeta^{-1} \leqslant w$, condition (i) for ζ to be in $A_{\mathsf{R}}(v,w)$ holds. We show that condition (ii) also holds. Suppose $v \leqslant v'$ and $v'_{\mathsf{R}}\zeta^{-1} \leqslant w$. By Proposition 2.16(i) we have $v'_{\mathsf{S}} \leqslant v'_{\mathsf{R}}$, and so by Proposition 2.10, $v'_{\mathsf{S}}\zeta^{-1} \leqslant v'_{\mathsf{R}}\zeta^{-1}$. Thus $v'_{\mathsf{S}}\zeta^{-1} \leqslant w$, and by condition (ii) for ζ to be in $A_{\mathsf{S}}(v,w)$, we have $v = v'$. This establishes condition (ii) for ζ to be in $A_{\mathsf{R}}(v,w)$. Thus, $\zeta \in A_{\mathsf{R}}(u,w)$, and as explained above, shows that (5.9) is a disjoint union and completes the proof of the theorem. $\square$

6. Cofreeness, primitive elements, and the coradical filtration of $\mathfrak{S}$Sym

The monomial basis reveals the existence of a second coalgebra grading on $\mathfrak{S}$Sym, given by the number of global descents of the indexing permutations. We show that with respect to this grading, $\mathfrak{S}$Sym is a cofree graded coalgebra. We deduce an elegant description of the coradical filtration: it corresponds to a filtration of the symmetric groups by certain lower order ideals determined by the number of global descents. In particular, the space of primitive elements is spanned by those $\mathcal{M}_u$ where u has no global descents.

We review the notion of cofree graded coalgebras. Let V be a vector space and set

$$Q(V) := \bigoplus_{k \geqslant 0} V^{\otimes k}.$$

The space $Q(V)$, graded by k, becomes a graded coalgebra with the *deconcatenation* coproduct

$$\Delta(v_1 \otimes \ldots \otimes v_k) = \sum_{i=0}^{k} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k),$$

and counit $\varepsilon(v_1 \otimes \cdots \otimes v_k) = 0$ for $k \geqslant 1$. $Q(V)$ is connected, in the sense that the component of degree 0 is identified with the base field via ε.

We call $Q(V)$ the cofree graded coalgebra cogenerated by V. The canonical projection $\pi : Q(V) \to V$ satisfies the following universal property. Given a graded coalgebra $C = \oplus_{k \geqslant 0} C^k$ and a linear map $\varphi : C \to V$ where $\varphi(C^k) = 0$ when $k \neq 1$, there is a unique morphism of graded coalgebras $\hat{\varphi} : C \to Q(V)$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[rr, dashed, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & & Q(V) \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Explicitly, $\hat{\varphi}$ is defined by

$$\hat{\varphi}_{|_{C^k}} = \varphi^{\otimes k} \Delta^{(k-1)}. \tag{6.1}$$

In particular, $\hat{\varphi}_{|_{C^0}} = \varepsilon$, $\hat{\varphi}_{|_{C^1}} = \varphi$, and $\hat{\varphi}_{|_{C^2}} = (\varphi \otimes \varphi)\Delta$.

We establish the cofreeness of $\mathfrak{S}$Sym by first defining a second coalgebra grading. Let $\mathfrak{S}^0 := \mathfrak{S}_0$, and for $k \geqslant 1$, let

$$\mathfrak{S}^k_n := \{u \in \mathfrak{S}_n \mid u \text{ has exactly } k-1 \text{ global descents}\}, \text{ and}$$

$$\mathfrak{S}^k := \coprod_{n \geqslant 0} \mathfrak{S}^k_n.$$

For instance,

$$\mathfrak{S}^1 = \{1\} \cup \{12\} \cup \{123, 213, 132\} \cup \{1234, 2134, 1324, 1243, 3124, \\ 2314, 2143, 1423, 1342, 3214, 3142, 2413, 1432\} \cup \cdots.$$

Let $(\mathfrak{S}\mathrm{Sym})^k$ be the vector subspace of $\mathfrak{S}$Sym spanned by $\{\mathcal{M}_u \mid u \in \mathfrak{S}^k\}$.

Theorem 6.1. *The decomposition* $\mathfrak{S}\mathrm{Sym} = \oplus_{k\geqslant 0}(\mathfrak{S}\mathrm{Sym})^k$ *is a coalgebra grading. Moreover, endowed with this grading,* $\mathfrak{S}\mathrm{Sym}$ *is a cofree graded coalgebra.*

Proof. Let $u\in\mathfrak{S}_n^k$ and write $\mathrm{GDes}(u) = \{p_1<\cdots<p_{k-1}\}$. By Theorem 3.1,

$$\Delta(\mathcal{M}_u) = 1\otimes\mathcal{M}_u + \sum_{i=1}^{k-1}\mathcal{M}_{\mathrm{st}(u_1,\dots,u_{p_i})}\otimes\mathcal{M}_{\mathrm{st}(u_{p_i+1},\dots,u_n)} + \mathcal{M}_u\otimes 1.$$

Since $\mathrm{st}(u_1,\dots,u_{p_i})$ and $\mathrm{st}(u_{p_i+1},\dots,u_n)$ have $i-1$ and $k-1-i$ global descents, we have

$$\Delta((\mathfrak{S}\mathrm{Sym})^k) \subseteq \bigoplus_{i=0}^{k}(\mathfrak{S}\mathrm{Sym})^i\otimes(\mathfrak{S}\mathrm{Sym})^{k-i}.$$

Thus $\mathfrak{S}\mathrm{Sym} = \oplus_{k\geqslant 0}(\mathfrak{S}\mathrm{Sym})^k$ is a graded coalgebra.

Let $V = (\mathfrak{S}\mathrm{Sym})^1$ and $\varphi : \mathfrak{S}\mathrm{Sym}\to V$ the projection associated to the grading. Let $\hat\varphi : \mathfrak{S}\mathrm{Sym}\to Q(V)$ be the morphism of graded coalgebras into the cofree graded coalgebra on V. For u as above, Proposition 5.1 gives,

$$\Delta^{(k-1)}(\mathcal{M}_u) = \sum_{\substack{0\leqslant q_1\leqslant\cdots\leqslant q_{k-1}\leqslant n\\ q_i\in\overline{\mathrm{GDes}}(u)}}\mathcal{M}_{\mathrm{st}(u_1,\dots,u_{q_1})}\otimes\cdots\otimes\mathcal{M}_{\mathrm{st}(u_{q_{k-1}+1},\dots,u_n)}.$$

Among these chains $0\leqslant q_1\leqslant\cdots\leqslant q_{k-1}\leqslant n$ of global descents of u, there is the chain $0<p_1<\cdots<p_{k-1}<n$. In any other chain there must be at least one equality, say $q_i = q_{i+1}$. Then $\mathrm{st}(u_{q_i+1},\dots,u_{q_{i+1}})$ is the empty permutation and the corresponding term is just the identity 1, which is annihilated by φ. Therefore, by (6.1), $\hat\varphi$ is given by

$$\hat\varphi(\mathcal{M}_u) = \mathcal{M}_{\mathrm{st}(u_1,\dots,u_{p_1})}\otimes\cdots\otimes\mathcal{M}_{\mathrm{st}(u_{p_{k-1}+1},\dots,u_n)} \in V^{\otimes k}.$$

Consider the map $\psi : V^{\otimes k}\to(\mathfrak{S}\mathrm{Sym})^k$ that sends

$$\mathcal{M}_{v_{(1)}}\otimes\cdots\otimes\mathcal{M}_{v_{(k)}}\mapsto\mathcal{M}_{\zeta_{\mathsf{T}}\cdot(v_{(1)}\times\cdots\times v_{(k)})},$$

where each $v_{(i)}\in\mathfrak{S}_{q_i}$ and $\mathsf{T} = \{q_1, q_1+q_2,\dots,q_1+\cdots+q_{k-1}\}\subseteq[n-1]$.

Lemma 2.17 implies that $\mathrm{GDes}(\zeta_{\mathsf{T}}\cdot(v_{(1)}\times\cdots\times v_{(k)})) = \mathsf{T}$, since each $v_{(i)}$ has no global descents. Together with (2.10) this shows that $\hat\varphi\circ\psi = \mathrm{id}$.

On the other hand, letting $\mathsf{S} = \mathrm{GDes}(u)$, Lemma 2.18 implies that

$$u = \zeta_{\mathsf{S}}\cdot(\mathrm{st}(u_1,\dots,u_{p_1})\times\cdots\times\mathrm{st}(u_{p_{k-1}+1},\dots,u_n)).$$

This shows that $\psi\circ\hat\varphi = \mathrm{id}$. Thus $\hat\varphi$ is an isomorphism of graded coalgebras. $\square$

Remark 6.2. If V is finite dimensional then the graded dual of $Q(V)$ is simply the (free) tensor algebra $T(V^*)$. More generally, suppose $V = \oplus_{n\geqslant 1}V_n$ is a graded vector space for which each component V_n is finite dimensional. Then $Q(V)$ admits another grading, for which the elements of $V_{n_1} \otimes \cdots \otimes V_{n_k}$ have degree $n_1 + \cdots + n_k$ (with respect to the other grading, these elements have degree k). With respect to this new grading, the homogeneous components are finite dimensional, and the graded dual of $Q(V)$ is the tensor algebra on the graded dual of V (again a free algebra).

In our situation, $\mathfrak{S}\mathrm{Sym} = Q(V)$, with V graded by the size n of the indexing permutations $u \in \mathfrak{S}_n$. The corresponding grading on $\mathfrak{S}\mathrm{Sym}$ is the original one, for which $\mathcal{M}_u$ has degree n if $u \in \mathfrak{S}_n$. Its graded dual is therefore a free algebra. It is known that $\mathfrak{S}\mathrm{Sym}$ is self-dual with respect to this grading (see Section 9). It follows that $\mathfrak{S}\mathrm{Sym}$ is also a free algebra. This is a result of Poirier and Reutenauer [28] who construct a different set of algebra generators, not directly related to the monomial basis. (See Remark 6.5.)

Let C be a graded connected coalgebra. The coradical $C^{(0)}$ of C is the 1-dimensional component in degree 0 (identified with the base field via the counit). The primitive elements of C are

$$\mathrm{P}(C) := \{x \in C \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Set $C^{(1)} := C^{(0)} \oplus \mathrm{P}(C)$, the first level of the coradical filtration. More generally, the kth level of the coradical filtration is

$$C^{(k)} := (\Delta^{(k)})^{-1}\left(\sum_{i+j=k} C^{\otimes i} \otimes C^{(0)} \otimes C^{\otimes j}\right).$$

We have $C^{(0)} \subseteq C^{(1)} \subseteq C^{(2)} \subseteq \cdots \subseteq C = \bigcup_{k \geqslant 0} C^{(k)}$, and

$$\Delta(C^{(k)}) \subseteq \sum_{i+j=k} C^{(i)} \otimes C^{(j)}.$$

Thus, the coradical filtration measures the complexity of iterated coproducts.

Suppose now that C is a cofree graded coalgebra $Q(V)$. Then the space of primitive elements is just V, and the kth level of the coradical filtration is $\oplus_{i=0}^{k} V^{\otimes i}$. These are straightforward consequences of the definition of the deconcatenation coproduct.

Define

$$\mathfrak{S}_n^{(k)} := \coprod_{i=0}^{k} \mathfrak{S}_n^k \quad \text{and} \quad \mathfrak{S}^{(k)} := \coprod_{i=0}^{k} \mathfrak{S}^k.$$

In other words, $\mathfrak{S}^{(0)} = \mathfrak{S}_0$ and for $k \geqslant 1$,

$$\mathfrak{S}_n^{(k)} = \{u \in \mathfrak{S}_n \mid u \text{ has at most } k-1 \text{ global descents}\}.$$

In Proposition 2.13 we showed that $\mathrm{GDes} : \mathfrak{S}_n \to \mathcal{Q}_n$ is order-preserving. Since $\mathcal{Q}_n$ is ranked by the cardinality of a subset, it follows that $\mathfrak{S}_n^{(k)}$ is a lower order ideal of $\mathfrak{S}_n$, with $\mathfrak{S}_n^{(k)} \subseteq \mathfrak{S}_n^{(k+1)}$. The coradical filtration corresponds precisely to this filtration of the weak order on the symmetric groups by lower ideals.

Corollary 6.3. *A linear basis for the kth level of the coradical filtration of* $\mathfrak{S}\mathrm{Sym}$ *is*

$$\{\mathcal{M}_u \mid u \in \mathfrak{S}^{(k)}\}.$$

In particular, a linear basis for the space of primitive elements is

$$\{\mathcal{M}_u \mid u \text{ has no global descents}\}.$$

Proof. This follows from the preceding discussion. $\square$

The original grading of $\mathfrak{S}\mathrm{Sym} = \oplus_n \mathbb{Q}\mathfrak{S}_n$ yields a grading on the subspace $\mathrm{P}(\mathfrak{S}\mathrm{Sym})$ of primitive elements and on each $(\mathfrak{S}\mathrm{Sym})^k$. Let $G_1(t)$ denote the Hilbert series of the space of primitive elements, or equivalently, the generating function for the set of permutations in $\mathfrak{S}_n$ with no global descents,

$$G_1(t) := \sum_{n \geqslant 1} \dim_{\mathbb{Q}}(\mathrm{P}_n(\mathfrak{S}\mathrm{Sym}))\, t^n.$$

More generally, let $G_k(t)$ be the Hilbert series of $(\mathfrak{S}\mathrm{Sym})^k$, or equivalently, the generating function for permutations in $\mathfrak{S}_n$ with exactly $k-1$ global descents,

$$G_k(t) := \sum_{n \geqslant k} \dim_{\mathbb{Q}}((\mathfrak{S}\mathrm{Sym})_n^k)\, t^n.$$

For instance,

$$G_1(t) = t + t^2 + 3t^3 + 13t^4 + 71t^5 + 461t^6 + 3447t^7 + \cdots$$

$$G_2(t) = t^2 + 2t^3 + 7t^4 + 32t^5 + 177t^6 + 1142t^7 + \cdots$$

$$G_3(t) = t^3 + 3t^4 + 12t^5 + 58t^6 + 327t^7 + 2109t^8 + \cdots.$$

There are well-known relationships between the Hilbert series of a graded space V, its powers $V^{\otimes k}$ and their sum $Q(V)$. In our case, these give the following formulas.

Corollary 6.4. *We have*

(i)
$$\dim_{\mathbb{Q}}(\mathrm{P}_n(\mathfrak{S}\mathrm{Sym})) = (-1)^{n-1} \begin{vmatrix} 1! & 2! & \dots & \dots & n! \\ 1 & 1! & \dots & \dots & (n-1)! \\ 0 & 1 & 1! & \dots & (n-2)! \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1! \end{vmatrix}.$$

(ii) $G_1(t) = 1 - \dfrac{1}{\sum_{n\geqslant 0} n!\, t^n}$.

(iii) $G_k(t) = (G_1(t))^k$.

Remark 6.5. Formula (i) is analogous to a formula for ordinary descents in [32, Example 2.2.4]. Formulas (ii) and (iii) in Corollary 6.4 are due to Lentin [17, Section 6.3], see also Comtet [6, Exercise VI.14]. These references do not consider global descents, but rather the problem of decomposing a permutation $u \in \mathfrak{S}_n$ as a non-trivial product $u = v \times w$. This is equivalent to our study of global descents, as we may write $u = v \times w$ with $v \in \mathfrak{S}_p$ exactly when $n+1-p$ is a global descent of $u\omega_n$. For instance, $u = 563241$ has global descents $\{2, 5\}$ and $u\omega_6 = 142365 = 1 \times 312 \times 21$. See the Encyclopedia of Integer Sequences [31] (A003319 and A059438) for additional references in this connection.

Poirier and Reutenauer [28] showed that the elements of the dual basis $\{\mathcal{F}_u^*\}$ indexed by the connected permutations freely generate $(\mathfrak{S}\mathrm{Sym})^*$. Duchamp et al. dualize the resulting linear basis, giving a different basis than we do for the space of primitive elements [8, Proposition 3.6].

7. The descent map to quasi-symmetric functions

We study the effect of the morphism of Hopf algebras (1.11)

$$\mathcal{D} : \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}, \qquad \text{defined by} \quad \mathcal{F}_u \mapsto F_{\mathrm{Des}(u)}$$

on the monomial basis. Here, we use subsets S of $[n-1]$ to index monomial and fundamental quasi-symmetric functions of degree n, as discussed at the end of Section 1.2. Our main tool is the Galois connection $\mathfrak{S}_n \rightleftarrows \mathcal{Q}_n$ of Section 2.3.

When we have a Galois connection between posets P and Q given by a pair of maps $f : P \to Q$ and $g : Q \to P$ as in (2.7), a classical theorem of Rota [30, Theorem 1] states that the Möbius functions of P and Q are related by

$$\forall\ x \in P \text{ and } w \in Q, \qquad \sum_{\substack{y \in P \\ x \leqslant y,\, f(y)=w}} \mu_P(x, y) = \sum_{\substack{v \in Q \\ v \leqslant w,\, g(v)=x}} \mu_Q(v, w).$$

A conceptual proof of this simple but extremely useful result can be found in [1].

Definition 7.1. A permutation $u \in \mathfrak{S}_n$ is *closed* if it is of the form $u = \zeta_{\mathsf{T}}$ for some $\mathsf{T} \in \mathcal{Q}_n$.

Equivalently, in view of (2.8) and (2.9), u is closed if and only if $\mathrm{Des}(u) = \mathrm{GDes}(u)$.

From Proposition 2.11, we deduce the following fact about the Möbius function of the weak order.

Corollary 7.2. *Let $u \in \mathfrak{S}_n$ and $\mathsf{S} \in \mathcal{Q}_n$. Then*

$$\sum_{\substack{u \leqslant v \in \mathfrak{S}_n \\ \mathrm{Des}(v)=\mathsf{S}}} \mu_{\mathfrak{S}_n}(u,v) = \begin{cases} \mu_{\mathcal{Q}_n}(\mathrm{Des}(u),\mathsf{S}) & \text{if } u \text{ is closed}, \\ 0 & \text{if not}. \end{cases} \tag{7.1}$$

Proof. Rota's formula says in this case that

$$\sum_{\substack{u \leqslant v \in \mathfrak{S}_n \\ \mathrm{Des}(v)=\mathsf{S}}} \mu_{\mathfrak{S}_n}(u,v) = \sum_{\substack{\mathsf{T} \subseteq \mathsf{S} \in \mathcal{Q}_n \\ \zeta_T = u}} \mu_{\mathcal{Q}_n}(\mathsf{T},\mathsf{S}).$$

If u is not closed, then the index set on the right hand side is empty. If u is closed, then the index set consists only of the set $\mathsf{T} = \mathrm{Des}(u)$, by assertion (c) in the proof of Proposition 2.11. $\square$

While there are explicit formulas for the Möbius function of the weak order, it is precisely the above result that allows us to obtain the description of the map $\mathcal{D} : \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$ in terms of the monomial bases.

Theorem 7.3. *Let $u \in \mathfrak{S}_n$. Then*

$$\mathcal{D}(\mathcal{M}_u) = \begin{cases} M_{\mathrm{GDes}(u)} & \text{if } u \text{ is closed}, \\ 0 & \text{if not}. \end{cases}$$

Proof. By definition, $\mathcal{M}_u = \sum_{u \leqslant v} \mu_{\mathfrak{S}_n}(u,v)\mathcal{F}_v$, hence

$$\begin{aligned} \mathcal{D}(\mathcal{M}_u) &= \sum_{u \leqslant v} \mu_{\mathfrak{S}_n}(u,v) F_{\mathrm{Des}(v)} \\ &= \sum_{\mathsf{S}} \left(\sum_{\substack{u \leqslant v \\ \mathrm{Des}(v)=\mathsf{S}}} \mu_{\mathfrak{S}_n}(u,v) \right) F_{\mathsf{S}} \\ &= \begin{cases} \sum_{\mathsf{S}} \mu_{\mathcal{Q}_n}(\mathrm{Des}(u),\mathsf{S})\, F_{\mathsf{S}} & \text{if } u \text{ is closed} \\ 0 & \text{if not}. \end{cases} \end{aligned}$$

We complete the proof by noting that

$$M_{\mathrm{Des}(u)} = \sum_{\mathsf{S}} \mu_{\mathcal{Q}_n}(\mathrm{Des}(u),\mathsf{S})\, F_{\mathsf{S}}$$

by the definition of $M_{\mathrm{Des}(u)}$, and that since u is closed, $\mathrm{Des}(u) = \mathrm{GDes}(u)$. $\square$

Malvenuto shows that $\mathcal{D}$ is a morphism of Hopf algebras by comparing the structures on the fundamental bases of $\mathfrak{S}$Sym and $\mathcal{Q}$Sym. We do the same for the monomial bases of $\mathfrak{S}$Sym and $\mathcal{Q}$Sym.

To compare the coproducts, first note that for any subsets $\mathsf{S} \subseteq [p-1]$ and $\mathsf{T} \subseteq [q-1]$,

$$\zeta_{\mathsf{S} \cup \{p\} \cup \mathsf{T}} = \zeta_{p,q} \cdot (\zeta_{\mathsf{S}} \times \zeta_{\mathsf{T}}).$$

Therefore, if $u \in \mathfrak{S}_n$ and $p \in \overline{\mathrm{GDes}}(u)$, then

$$u \text{ is closed } \Leftrightarrow \text{ both } \mathrm{st}(u_1, \ldots, u_p) \text{ and } \mathrm{st}(u_{p+1}, \ldots, u_n) \text{ are closed.}$$

It follows that applying the map $\mathcal{D}: \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$ to formula (3.1) gives the usual formula (1.6) for the coproduct of monomial quasi-symmetric functions.

For instance, we compare formula (4.1) with (1.5). Since $\mathcal{D}(\mathcal{M}_{21}) = M_{(1,1)}$ and $\mathcal{D}(\mathcal{M}_{12}) = M_{(2)}$, applying $\mathcal{D}$ to (4.1) results in (1.5). Indeed, the indices u in the first row of (4.1) all are closed, while none in the second row are closed. It is easy to verify that the five terms on the right in the first row in (4.1) map to the five terms on the right in (1.5).

The situation is different for the products. The geometric description of the structure constants of the product on the monomial basis of $\mathfrak{S}$Sym (4.4) admits an analogue for $\mathcal{Q}$Sym, but this turns out to be very different from the known description in terms of quasi-shuffles (1.4). We present this new description of the structure constants for the product of monomial quasi-symmetric functions.

The role of the permutahedron is now played by the cube. Associating a subset S of $[n-1]$ to its characteristic function gives a bijection between subsets of $[n-1]$ and vertices of the $(n-1)$-dimensional cube $[0,1]^{n-1}$. Coordinatewise comparison corresponds to subset inclusion, and the 1-skeleton of the cube becomes the Hasse diagram of the Boolean poset $\mathcal{Q}_n$. In this way, we identify $\mathcal{Q}_n$ with the vertices of the $(n-1)$-dimensional cube.

For each Grassmannian permutation $\zeta \in \mathfrak{S}^{(p,q)}$, consider the map

$$r_\zeta : \mathcal{Q}_p \times \mathcal{Q}_q \to \mathcal{Q}_{p+q}, \quad (\mathsf{S}, \mathsf{T}) \mapsto \mathrm{Des}((\zeta_{\mathsf{S}} \times \zeta_{\mathsf{T}}) \cdot \zeta^{-1}).$$

We describe this map r_ζ in more detail. To that end, set

$$\mathrm{Cons}_p(\zeta) := \{ i \in [p+q-1] \mid \zeta^{-1}(i) + 1 = \zeta^{-1}(i+1) \quad \text{and} \quad \zeta^{-1}(i) \neq p \},$$

and recall that the vertices in a face of the cube are an interval in the Boolean poset, with every interval corresponding to a unique face.

Lemma 7.4. *Let p, q be positive integers and $\zeta \in \mathfrak{S}^{(p,q)}$. The image of r_ζ is the face*

$$[\mathrm{Des}(\zeta^{-1}),\ \mathrm{Des}(\zeta^{-1}) \coprod \mathrm{Cons}_p(\zeta)],$$

which is isomorphic to the Boolean poset of subsets of $\mathrm{Cons}_p(\zeta)$.

Proof. This is an immediate consequence of an alternative (and direct) description of $r_\zeta(\mathsf{S},\mathsf{T})$. For $\mathsf{T}\in\mathcal{Q}_q$, set $p+\mathsf{T} := \{p+t \mid t\in\mathsf{T}\}$. Then, for $(\mathsf{S},\mathsf{T})\in\mathcal{Q}_p\times\mathcal{Q}_q$, we have

$$r_\zeta(\mathsf{S},\mathsf{T}) = \mathrm{Des}(\zeta^{-1})\coprod(\mathrm{Cons}_p(\zeta)\cap\zeta(\mathsf{S}\cup(p+\mathsf{T}))). \tag{7.2}$$

Assuming this for a moment, we note that the association $(\mathsf{S},\mathsf{T})\mapsto\zeta(\mathsf{S}\cup(p+\mathsf{T}))$ is a bijection between $\mathcal{Q}_p\times\mathcal{Q}_q$ and subsets of $\{i \mid \zeta^{-1}(i)\neq p\}$. Intersecting with $\mathrm{Cons}_p(\zeta)$ we obtain a surjection onto subsets of $\mathrm{Cons}_p(\zeta)$, which yields the desired description of the image of r_ζ.

We prove (7.2). Let $(\mathsf{S},\mathsf{T})\in\mathcal{Q}_p\times\mathcal{Q}_q$ and set $w := (\zeta_\mathsf{S}\times\zeta_\mathsf{T})\cdot\zeta^{-1}$ so that $\mathrm{Des}(w) = r_\zeta(\mathsf{S},\mathsf{T})$. Note that $\mathrm{Des}(\zeta_\mathsf{S}\times\zeta_\mathsf{T}) = \mathsf{S}\cup(p+\mathsf{T})$ (this is a particular case of Lemma 2.17) and if $i\leqslant p<j$, then $(\zeta_\mathsf{S}\times\zeta_\mathsf{T})(i)\leqslant p<(\zeta_\mathsf{S}\times\zeta_\mathsf{T})(j)$.

Let $i\in[n-1]$. We consider whether or not i is a descent of w. First, suppose $i\in\mathrm{Des}(\zeta^{-1})$. Since the values $1,2,\dots,p$ and $p+1,p+2,\dots,p+q$ occur in order in the permutation ζ^{-1} (because $\zeta\in\mathfrak{S}^{(p,q)}$), we must have $\zeta^{-1}(i)>p\geqslant\zeta^{-1}(i+1)$ and so $w(i)>p\geqslant w(i+1)$, thus $i\in\mathrm{Des}(w)$.

Now suppose that i is not a descent of ζ^{-1}. If $\zeta^{-1}(i)+1<\zeta^{-1}(i+1)$, then we must have $\zeta^{-1}(i)\leqslant p<\zeta^{-1}(i+1)$, again because $\zeta\in\mathfrak{S}^{(p,q)}$. Hence $w(i)\leqslant p<w(i+1)$ and i is not a descent of w. If instead we have $\zeta^{-1}(i)+1=\zeta^{-1}(i+1)$, then there are two cases to consider. If $i=\zeta^{-1}(p)$, then this forces ζ to be 1_{p+q} so $w(i) = w(p)\leqslant p<w(i+1)$, and we conclude that i is not a descent of w. If $i\neq\zeta^{-1}(p)$, then $i\in\mathrm{Cons}_p(\zeta)$ and we see that i is a descent of w exactly when $\zeta^{-1}(i)\in\mathsf{S}\cup(p+\mathsf{T})$. This proves (7.2) and completes the proof of the lemma. $\square$

Unlike the case of the permutahedron, the image of r_ζ need not be a facet. Indeed, by Lemma 7.4, the image of r_ζ is a facet only if $\#\mathrm{Cons}_p(\zeta)=p+q-2$, and this occurs only when $\zeta=1_{p+q}$ or $\zeta=\zeta_{p,q}$. Fig. 4 displays the vertices of the 3-cube and Fig. 5 shows which faces occur as the image $r_\zeta(\mathcal{Q}_p\times\mathcal{Q}_q)$. Observe that while not all faces occur as images of some $r_\zeta(\mathcal{Q}_p\times\mathcal{Q}_q)$, any face that does occur is the image of a unique such map. This is the general case.

Lemma 7.5. *A face of $\mathcal{Q}_n$ is the image of $\mathcal{Q}_p\times\mathcal{Q}_{n-p}$ under a map r_ζ for at most one pair (ζ,p).*

Proof. Suppose $\zeta\in\mathfrak{S}^{(p,n-p)}$ for some $0<p<n$. We will observe that the pair of sets $\mathrm{Des}(\zeta^{-1})$ and $\mathrm{Cons}_p(\zeta)$ determines ζ and p uniquely by describing these sets.

Suppose first that $\zeta=1_n$. Then $\mathrm{Des}(\zeta^{-1})=\emptyset$ and $\mathrm{Cons}_p(\zeta)=[n-1]-\{p\}$.

Suppose now that $\zeta\in\mathfrak{S}^{(p,n-p)}$ is not the identity permutation. Then ζ determines p and $\mathrm{Des}(\zeta^{-1})\neq\emptyset$. Since the values $1,2,\dots,p$ and $p+1,\dots,n$ occur in order in ζ^{-1}, there exist numbers

$$0\leqslant b_0<a_1<b_1<\cdots<a_k<b_k\leqslant n$$

such that the values in $[p]$ occur in order in the intervals

$$[0, b_0], [a_1 + 1, b_1], \ldots, [a_k + 1, b_k],$$

and the values in $\{p+1, \ldots, n\}$ in the complementary set. Thus $\mathrm{Des}(\zeta^{-1}) = \{a_1, \ldots, a_k\}$ and $\mathrm{Cons}_p(\zeta) = [n-1] - \{b_0, a_1, b_1, a_2, \ldots, a_k, b_k\}$.

It follows that ζ and p determine and are determined by the sets $\mathrm{Des}(\zeta^{-1})$ and $\mathrm{Cons}_p(\zeta)$, which completes the proof of the lemma. $\square$

Theorem 7.6. *Suppose p, q are positive integers. Let $\mathsf{S} \subseteq [p-1]$, $\mathsf{T} \subseteq [q-1]$ and $\mathsf{R} \subseteq [p+q-1]$. The coefficient of $M_{p+q,\mathsf{R}}$ in $M_{p,\mathsf{S}} \cdot M_{q,\mathsf{T}}$ is*

$$\#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (\mathsf{S}, \mathsf{T}) = \max r_\zeta^{-1}[\emptyset, \mathsf{R}]\}. \tag{7.3}$$

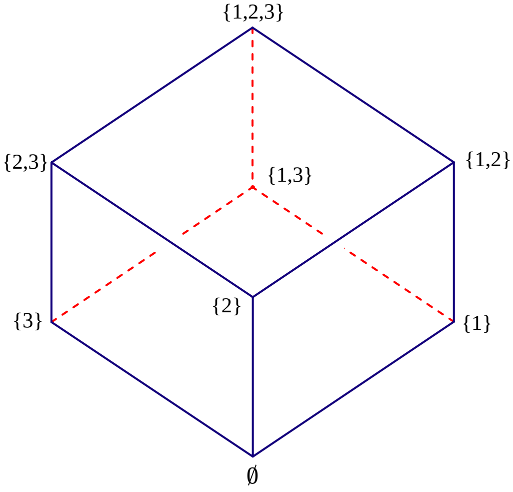

Fig. 4. Vertices of the cube.

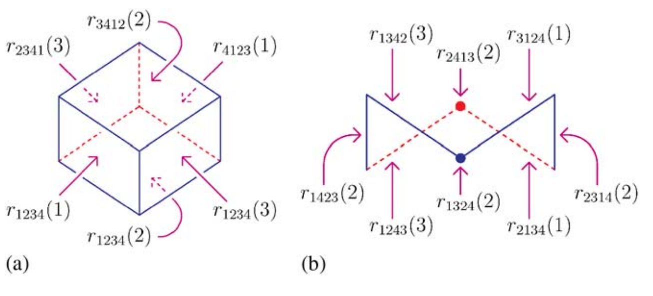

Fig. 5. (a) The facets of the cube: $r_{1234}(\mathcal{Q}_p \times \mathcal{Q}_q)$ and $r_{\zeta_{p,q}}(\mathcal{Q}_p \times \mathcal{Q}_q) = (r_{\zeta_{p,q}}(p))$. (b) The edges and vertices $r_\zeta(\mathcal{Q}_p \times \mathcal{Q}_q)$, $\zeta \neq 1234, \zeta_{p,q}$.

In other words, this coefficient counts the number of faces of the cube of type (p,q) with the property that the vertex $r_\zeta(\mathsf{S},\mathsf{T})$ is below R *and it is the maximal vertex in the face $r_\zeta(\mathcal{Q}_p \times \mathcal{Q}_q)$ below.*

Proof. By Theorem 7.3, $M_{p,\mathsf{S}} \cdot M_{q,\mathsf{T}} = \mathcal{D}(\mathcal{M}_{\zeta_\mathsf{S}} \cdot \mathcal{M}_{\zeta_\mathsf{T}})$. We expand the product using Theorem 4.1, and then apply the map $\mathcal{D}$ and Theorem 7.3 to obtain

$$M_{p,\mathsf{S}} \cdot M_{q,\mathsf{T}} = \mathcal{D}(\mathcal{M}_{\zeta_\mathsf{S}} \cdot \mathcal{M}_{\zeta_\mathsf{T}}) = \mathcal{D}\left(\sum_{w \in \mathfrak{S}_{p+q}} \alpha^{w}_{\zeta_\mathsf{S},\zeta_\mathsf{T}} \mathcal{M}_w\right) = \sum_{\mathsf{R} \in \mathcal{Q}_{p+q}} \alpha^{\zeta_R}_{\zeta_\mathsf{S},\zeta_\mathsf{T}} M_{p+q,\mathsf{R}}.$$

According to (4.4),

$$\alpha^{\zeta_\mathsf{R}}_{\zeta_\mathsf{S},\zeta_\mathsf{T}} = \#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (\zeta_\mathsf{S}, \zeta_\mathsf{T}) = \max \rho_\zeta^{-1}[1, \zeta_\mathsf{R}]\}.$$

By Proposition 2.11, for any S, T, and R we have

$$\mathrm{Des}((\zeta_\mathsf{S} \times \zeta_\mathsf{T}) \cdot \zeta^{-1}) \subseteq \mathsf{R} \Leftrightarrow (\zeta_\mathsf{S} \times \zeta_\mathsf{T}) \cdot \zeta^{-1} \leqslant \zeta_\mathsf{R}.$$

In other words,

$$r_\zeta(\mathsf{S},\mathsf{T}) \leqslant \mathsf{R} \Leftrightarrow \rho_\zeta(\zeta_\mathsf{S}, \zeta_\mathsf{T}) \leqslant \zeta_\mathsf{R}.$$

This implies that the structure constant $\alpha^{\zeta_R}_{\zeta_\mathsf{S},\zeta_\mathsf{T}}$ is as stated. $\square$

We give an example. Let $p=1$, $q=3$, $\mathsf{S}=\emptyset$ and $\mathsf{T}=\{1\}$. In terms of compositions, we have $M_{\emptyset,1} = M_{(1)}$, and $M_{\{1\},3} = M_{(1,2)}$. Eq. (1.4) gives

$$\begin{aligned} M_{\emptyset,1} \cdot M_{\{1\},3} = M_{(1)} \cdot M_{(1,2)} &= 2M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,2)} + M_{(1,3)} \\ &= 2M_{\{1,2\},4} + M_{\{1,3\},4} + M_{\{2\},4} + M_{\{1\},4}. \end{aligned}$$

On the other hand, (7.3) also predicts that the coefficient of $M_{\{1,2\}}$ is 2. Of the four possible faces of type $(1,3)$, only two satisfy the required condition. One corresponds to the shuffle 1234 (it is a facet) and the other to 2134 (it is an edge). They are shown in Fig. 6, together with the vertices $r_{1234}(\emptyset,\{1\}) = \{2\}$, $r_{2134}(\emptyset,\{1\}) = \{1\}$, and the vertex $\{1,2\}$.

8. $\mathfrak{S}$Sym is a crossed product over $\mathcal{Q}$Sym

We obtain a decomposition of the algebra structure of $\mathfrak{S}$Sym as a crossed product over the Hopf algebra $\mathcal{Q}$Sym. We refer the reader to [26, Section 7] for a review of this construction in the general Hopf algebraic setting. Let us only say that the

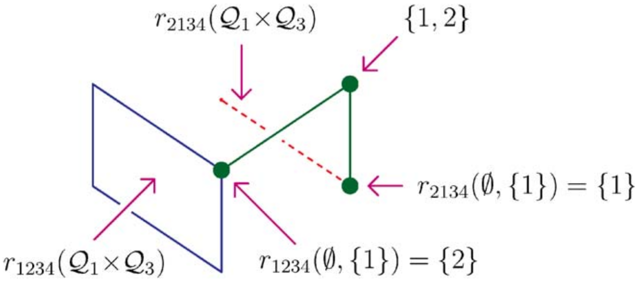

Fig. 6. The faces r_{1234} and r_{2134} of type (1,3), and the vertex $\{1,2\}$.

crossed product of a Hopf algebra K with an algebra A with respect to a Hopf cocycle $\sigma : K \otimes K \to A$ is a certain algebra structure on the space $A \otimes K$, denoted by $A\#_\sigma K$.

Theorem 8.1. *The map* $\mathcal{Z} : \mathcal{Q}\mathrm{Sym} \to \mathfrak{S}\mathrm{Sym}$, $M_{\mathsf{S}} \mapsto \mathcal{M}_{\zeta_{\mathsf{S}}}$, *is a morphism of coalgebras and a right inverse to the morphism of Hopf algebras* $\mathcal{D} : \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$.

Proof. This is immediate from Theorems 3.1 and 7.3. $\square$

In this situation, an important theorem of Blattner, Cohen, and Montgomery [5] applies. Namely, suppose $\pi : H \to K$ is a morphism of Hopf algebras that admits a coalgebra splitting (right inverse) $\gamma : K \to H$. Then there is a *crossed product* decomposition

$$H \;\cong\; A\#_\sigma K,$$

where A, a subalgebra of H, is the *left Hopf kernel* of π:

$$A = \{h \in H \mid \sum h_1 \otimes \pi(h_2) = h \otimes 1\}$$

and the *Hopf cocycle* $\sigma : K \otimes K \to A$ is

$$\sigma(k, k') = \sum \gamma(k_1)\gamma(k'_1)S\gamma(k_2k'_2). \tag{8.1}$$

This result, as well as some generalizations, can be found in [26, Section 7]. Note that if π and γ preserve gradings, then so does the rest of the structure.

Let A be the left Hopf kernel of $\mathcal{D} : \mathfrak{S}\mathrm{Sym} \to \mathcal{Q}\mathrm{Sym}$ and A_n its nth homogeneous component. Once again the monomial basis of $\mathfrak{S}\mathrm{Sym}$ proves useful in describing A.

Theorem 8.2. *A basis for* A_n *is the set* $\{\mathcal{M}_u\}$ *where* u *runs over all permutations of* n *that are not of the form*

$$* \cdots * \, 12 \ldots n-k \tag{$*$}$$

for any $k = 0, \ldots, n-1$. *In particular*,

$$\dim A_n = n! - \sum_{k=0}^{n-1} k!.$$

Proof. By the theorem of Blattner et al., $\mathfrak{S}\mathrm{Sym} \cong A \#_\sigma \mathcal{Q}\mathrm{Sym}$, in particular $\mathfrak{S}\mathrm{Sym} \cong A \otimes \mathcal{Q}\mathrm{Sym}$ as vector spaces. The generating functions for the dimensions of these algebras are therefore related by

$$\sum_{n\geqslant 0}^{\infty} n!t^n = \sum_{n\geqslant 0} a_n t^n \cdot \left(1 + \sum_{n\geqslant 1} 2^{n-1} t^n\right) = \sum_{n\geqslant 0} a_n t^n \cdot \frac{1}{1 - \sum_{n\geqslant 1} t^n}.$$

It follows that $a_n = n! - \sum_{k=0}^{n-1} k!$ as claimed.

Observe that a_n counts the permutations in $\mathfrak{S}_n$ that are not of the form $(*)$. Since the $\mathcal{M}_u$ are linearly independent, it suffices to show that if u is not of that form then $\mathcal{M}_u$ is in the Hopf kernel. Now, for any $u \in \mathfrak{S}_n$ and $p \in \mathrm{GDes}(u)$, we have that $\mathrm{st}(u_{p+1}, \ldots, u_n) = (u_{p+1}, \ldots, u_n)$. Hence, if u is not of the form $(*)$, the same is true of $\mathrm{st}(u_{p+1}, \ldots, u_n)$ and therefore this permutation is not closed. It follows from Theorems 3.1 and 7.3 that $(\mathrm{id} \otimes \mathcal{D})\Delta(\mathcal{M}_u) = \mathcal{M}_u \otimes 1$.

Remark 8.3. These results were motivated by a question of Nantel Bergeron, who asked (in dual form) if $\mathfrak{S}\mathrm{Sym}$ is cofree as right comodule over $\mathcal{Q}\mathrm{Sym}$. This is an immediate consequence of the crossed product decomposition.

Consider again the general situation of a morphism of Hopf algebras $\pi : H \to K$ with a coalgebra splitting $\gamma : K \to H$. This induces an exact sequence of Lie algebras

$$0 \to P(H) \cap A \to P(H) \xrightarrow{\pi} P(K) \to 0 \tag{8.2}$$

with a linear splitting $P(K) \xrightarrow{\gamma} P(H)$, where $P(H)$ denotes the space of primitive elements of H, viewed as a Lie algebra under the commutator bracket $[h, h'] = hh' - h'h$.

The Hopf cocycle restricts to a linear map $\sigma : P(K) \otimes P(K) \to P(H) \cap A$; in fact, for primitive elements k and k', (8.1) specializes to

$$\sigma(k, k') = S\gamma(kk') - \gamma(k')\gamma(k) \tag{8.3}$$

and a direct calculation shows that this element of H is primitive. Moreover, the Lie cocycle corresponding to (8.2) is the map $\tilde{\sigma} : P(K) \wedge P(K) \to P(H) \cap A$ given by

$$\tilde{\sigma}(k, k') = [\gamma(k), \gamma(k')] - \gamma([k, k']) = \sigma(k, k') - \sigma(k', k). \tag{8.4}$$

This map is a non-abelian Lie cocycle in the sense that the following conditions hold. For k, $k' \in P(K)$ and $a \in P(H) \cap A$,

$$k \cdot (k' \cdot a) - k' \cdot (k \cdot a) = [\tilde{\sigma}(k, k'), a] + [k, k'] \cdot a$$

$$k \cdot \tilde{\sigma}(k', k'') - k' \cdot \tilde{\sigma}(k, k'') + k'' \cdot \tilde{\sigma}(k, k') = \tilde{\sigma}([k, k'], k'') - \tilde{\sigma}([k, k''], k') + \tilde{\sigma}([k', k''], k)$$

where $k \cdot a = [\gamma(k), a]$.

Let us apply these considerations to the morphism $\mathfrak{S}\mathrm{Sym} \xrightarrow{\mathcal{D}} \mathcal{Q}\mathrm{Sym}$ and the coalgebra splitting $\mathcal{Q}\mathrm{Sym} \xrightarrow{\mathcal{Z}} \mathcal{Q}\mathrm{Sym}$. The structure constants of the Hopf cocycle σ do not have constant sign. However, its restriction to primitive elements of $\mathcal{Q}\mathrm{Sym}$ has non-negative structure constants on the monomial bases. They turn out to be particular structure constants of the product of $\mathfrak{S}\mathrm{Sym}$.

Recall that these structure constants $\alpha_{u,v}^{w}$ are defined for $u \in \mathfrak{S}_p$, $v \in \mathfrak{S}_q$ and $w \in \mathfrak{S}_{p+q}$ by the identity

$$\mathcal{M}_u \cdot \mathcal{M}_v = \sum_{w \in \mathfrak{S}_{p+q}} \alpha_{u,v}^{w} \mathcal{M}_w.$$

The combinatorial description of these constants showing their non-negativity is given by (4.2).

Lemma 8.4. *For $p, q \geqslant 1$, and $w \in \mathfrak{S}_{p+q}$ closed, we have $\alpha_{1_p,1_q}^{w} = 0$ except in the following cases*

$$\alpha_{1_p,1_q}^{1_{p+q}} = 1, \quad \alpha_{1_p,1_p}^{\zeta_{p,p}} = 2 \quad \textit{and if } p \neq q, \textit{ then } \quad \alpha_{1_p,1_q}^{\zeta_{p,q}} = 1.$$

Proof. Apply the map $\mathcal{D}$ to the product

$$\sum_{w \in \mathfrak{S}_{p+q}} \alpha_{1_p,1_q}^{w} \mathcal{M}_w = \mathcal{M}_{1_p} \cdot \mathcal{M}_{1_q},$$

to obtain (using (1.4))

$$\sum_{w \in \mathfrak{S}_{p+q}} \alpha_{1_p,1_q}^{w} \mathcal{D}(\mathcal{M}_w) = M_{(p)} \cdot M_{(q)} = M_{(p,q)} + M_{(q,p)} + M_{(p+q)}.$$

The result is immediate, as $\mathcal{D}(\mathcal{M}_w) = 0$ unless w is closed, and we have $\mathcal{D}(\mathcal{M}_{\zeta_{p,q}}) = M_{(p,q)}$ and $\mathcal{D}(\mathcal{M}_{1_{p+q}}) = M_{(p+q)}$. $\square$

We use this lemma to give a combinatorial description of σ and the Lie cocycle $\tilde{\sigma}$ on primitive elements. By (1.6), $\{M_{(n)}\}_{n \geqslant 1}$ is a linear basis for the space of primitive elements of $\mathcal{Q}\mathrm{Sym}$. Thus $\mathrm{P}(\mathcal{Q}\mathrm{Sym})$ is an abelian Lie algebra with each homogeneous component of dimension 1. Recall that $\{\mathcal{M}_u \mid u$ has no global descents$\}$ is a basis of the primitive elements of $\mathfrak{S}$Sym, and thus $A \cap \mathrm{P}(\mathfrak{S}\mathrm{Sym})$ has a basis given by those $\mathcal{M}_u$ where u has no global descents and u is not an identity permutation, 1_n.

Theorem 8.5. *For any* $p, q \geqslant 1$,

$$\sigma(M_{(p)}, M_{(q)}) = \sum_{w \neq \zeta_{p,q},\, \zeta_{q,p},\, 1_{p+q}} \alpha^{w}_{1_q,1_p} \mathcal{M}_w$$

$$\tilde{\sigma}(M_{(p)}, M_{(q)}) = \sum_{w} (\alpha^{w}_{1_q,1_p} - \alpha^{w}_{1_p,1_q}) \mathcal{M}_w.$$

Proof. Since $M_{(p)} \cdot M_{(q)} = M_{(p,q)} + M_{(q,p)} + M_{(p+q)}$, (8.3) gives

$$\begin{aligned}\sigma(M_{(p)}, M_{(q)}) &= S\mathcal{Z}(M_{(p,q)} + M_{(q,p)} + M_{(p+q)}) - \mathcal{Z}(M_{(q)}) \cdot \mathcal{Z}(M_{(p)}) \\ &= S(\mathcal{M}_{\zeta_{p,q}} + \mathcal{M}_{\zeta_{q,p}} + \mathcal{M}_{1_{p+q}}) - \mathcal{M}_{1_q} \cdot \mathcal{M}_{1_p}.\end{aligned}$$

Using (5.1) and (3.1), we compute $S(\mathcal{M}_{\zeta_{p,q}}) = \mathcal{M}_{1_p} \cdot \mathcal{M}_{1_q} - \mathcal{M}_{\zeta_{p,q}}$ and $S(\mathcal{M}_{1_{p+q}}) = -\mathcal{M}_{1_{p+q}}$. Therefore,

$$\sigma(M_{(p)}, M_{(q)}) = \mathcal{M}_{1_p} \cdot \mathcal{M}_{1_q} - \mathcal{M}_{\zeta_{p,q}} - \mathcal{M}_{\zeta_{q,p}} - \mathcal{M}_{1_{p+q}}.$$

The formula for $\sigma(M_{(p)}, M_{(q)})$ follows by expanding the product and using Lemma 8.4. The expression for $\tilde{\sigma}$ follows immediately from (8.4). $\square$

9. Self-duality of $\mathfrak{S}$Sym and applications

The Hopf algebra $\mathfrak{S}$Sym is self-dual. This appears in [16; 22, Section 5.2; 23, Theorem 3.3]. We provide a proof below, for completeness. We investigate the combinatorial implications of this self-duality, particularly when expressed in terms of the monomial basis. We explain how a result of Foata and Schützenberger on the numbers

$$d(\mathsf{S}, \mathsf{T}) = \#\{x \in \mathfrak{S}_n \mid \mathrm{Des}(x) = \mathsf{S},\ \mathrm{Des}(x^{-1}) = \mathsf{T}\}$$

is a consequence of self-duality of $\mathfrak{S}$Sym and obtain analogous results for the numbers

$$\theta(u, v) := \#\{x \in \mathfrak{S}_n \mid x \leqslant u,\ x^{-1} \leqslant v\}.$$

The Hopf algebra $\mathfrak{S}$Sym is connected and graded with each homogeneous component finite dimensional. We consider its *graded dual* $(\mathfrak{S}\mathrm{Sym})^*$ whose homogeneous component in degree n is the linear dual of the homogeneous component in degree n of $\mathfrak{S}$Sym. Let $\{\mathcal{F}^*_u \mid u \in \mathfrak{S}_n,\ n \geqslant 0\}$ and $\{\mathcal{M}^*_u \mid u \in \mathfrak{S}_n,\ n \geqslant 0\}$ be the bases of $(\mathfrak{S}\mathrm{Sym})^*$ dual to the fundamental and monomial bases of $\mathfrak{S}\mathrm{Sym}$, respectively. $(\mathfrak{S}\mathrm{Sym})^*$ is another graded connected Hopf algebra.

Theorem 9.1. *The map*

$$\Theta : (\mathfrak{S}\mathrm{Sym})^* \to \mathfrak{S}\mathrm{Sym}, \quad \mathcal{F}^*_u \mapsto \mathcal{F}_{u^{-1}} \tag{9.1}$$

is an isomorphism of Hopf algebras. On the monomial basis it is given by

$$\Theta(\mathcal{M}^*_u) = \sum_v \theta(u,v)\mathcal{M}_v. \tag{9.2}$$

Proof. Note that $\Theta^* = \Theta$. Therefore, it suffices to show that Θ is a morphism of coalgebras. We rewrite the product (1.1) of $\mathfrak{S}\mathrm{Sym}$. Let $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$. Then

$$\mathcal{F}_u \cdot \mathcal{F}_v = \sum_{w \in \mathfrak{S}_{p+q}} \#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (u \times v) \cdot \zeta^{-1} = w\}\, \mathcal{F}_w.$$

Therefore the (dual) coproduct of $(\mathfrak{S}\mathrm{Sym})^*$ is

$$\Delta(\mathcal{F}^*_w) = \sum_{p+q=n} \sum_{u \in \mathfrak{S}_p, v \in \mathfrak{S}_q} \#\{\zeta \in \mathfrak{S}^{(p,q)} \mid (u \times v) \cdot \zeta^{-1} = w\} \mathcal{F}^*_u \otimes \mathcal{F}^*_v.$$

On the other hand, as observed in (3.2), the coproduct of $\mathfrak{S}\mathrm{Sym}$ can be written as

$$\Delta(\mathcal{F}_w) = \sum_{p+q=n} \sum_{u \in \mathfrak{S}_p, v \in \mathfrak{S}_q} \#\{\zeta \in \mathfrak{S}^{(p,q)} \mid \zeta \cdot (u \times v) = w\}\; \mathcal{F}_u \otimes \mathcal{F}_v.$$

It follows that Θ is a morphism of coalgebras because

$$w = \zeta \cdot (u \times v) \;\Leftrightarrow\; w^{-1} = (u^{-1} \times v^{-1}) \cdot \zeta^{-1}.$$

Since $\mathcal{F}_u = \sum_{u \leqslant x} \mathcal{M}_x$, we have $\mathcal{M}^*_u = \sum_{x \leqslant u} \mathcal{F}^*_x$. Therefore,

$$\Theta(\mathcal{M}^*_u) = \sum_{x \leqslant u} \mathcal{F}_{x^{-1}} = \sum_{x \leqslant u} \sum_{x^{-1} \leqslant v} \mathcal{M}_v = \sum_v \theta(u,v)\mathcal{M}_v. \qquad \square$$

Formula (9.2) for the morphism Θ of Hopf algebras has combinatorial implications which we develop. Recall that $\alpha^w(u,v)$ and $\kappa(u,w)$ denote the structure constants of the product and antipode of $\mathfrak{S}\mathrm{Sym}$ in terms of the monomial basis. These integers were described in Theorems 4.1 and 5.5. Consider θ, α^w, and κ to be matrices with rows and columns indexed by elements of $\mathfrak{S}_n$.

Theorem 9.2. *For any $u \in \mathfrak{S}_p$, $v \in \mathfrak{S}_q$, and $w \in \mathfrak{S}_{p+q}$, we have*

(i) $(\theta\alpha^w\theta)(u,v) = \theta(\zeta_{p,q} \cdot (u \times v),\, w)$,
(ii) $\kappa^t\theta = \theta\kappa$.

Proof. By Lemma 2.14, the coproduct of $\mathfrak{S}$Sym (3.1) can be written as

$$\Delta(\mathcal{M}_w) = \sum_{p+q=n} \sum_{\substack{u\in\mathfrak{S}_p, v\in\mathfrak{S}_q \\ \zeta_{p,q}\cdot(u\times v)=w}} \mathcal{M}_u \otimes \mathcal{M}_v.$$

Therefore, the dual product is

$$\mathcal{M}^*_u \cdot \mathcal{M}^*_v = \mathcal{M}^*_{\zeta_{p,q}\cdot(u\times v)}.$$

Thus, the right-hand side of (i) is the coefficient of $\mathcal{M}_w$ in $\Theta(\mathcal{M}^*_u \cdot \mathcal{M}^*_v)$. On the other hand, since $\theta(u,v) = \theta(v,u)$, we have

$$(\theta\alpha^w\theta)(u,v) = \sum_{x,y\in\mathfrak{S}_n} \theta(u,x)\alpha^w(x,y)\theta(y,v) = \sum_{x,y\in\mathfrak{S}_n} \theta(u,x)\theta(v,y)\alpha^w(x,y).$$

Thus the left-hand side of (i) is the coefficient of $\mathcal{M}_w$ in $\Theta(\mathcal{M}^*_u)\cdot\Theta(\mathcal{M}^*_v)$. Since Θ is a morphism of algebras, (i) holds.

The second formula directly expresses that Θ preserves antipodes, since the antipode of $(\mathfrak{S}\mathrm{Sym})^*$ is the dual of the antipode of $\mathfrak{S}$Sym. $\square$

One may view Theorem 9.2(i) as a recursion reducing the computation of $\theta(u,v)$ to the case when u and v have no global descents, by virtue of Lemma 2.14. On the other hand, since θ is an invertible matrix, this provides another semi-combinatorial description of the structure constants $\alpha^w(u,v)$.

One may also impose the condition that Θ preserves coproducts, but this leads again to (i) of Theorem 9.2. On the other hand, the equivalent of (ii) of Theorem 9.2 for the fundamental basis leads to the following non-trivial identity.

Proposition 9.3. *For any u and* $v\in\mathfrak{S}_n$,

$$\begin{aligned}
&\#\{\mathsf{S}\subseteq[n-1] \mid \mathrm{Des}(vu_{\mathsf{S}})\subseteq\mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is odd}\}\\
&\quad+\#\{\mathsf{S}\subseteq[n-1] \mid \mathrm{Des}(uv_{\mathsf{S}})\subseteq\mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is even}\}\\
&\quad=\#\{\mathsf{S}\subseteq[n-1] \mid \mathrm{Des}(vu_{\mathsf{S}})\subseteq\mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is even}\}\\
&\quad+\#\{\mathsf{S}\subseteq[n-1] \mid \mathrm{Des}(uv_{\mathsf{S}})\subseteq\mathsf{S} \textit{ and } \#\mathsf{S} \textit{ is odd}\}.
\end{aligned}$$

Proof. The formula above is equivalent to

$$\lambda(u,v^{-1}) = \lambda(v,u^{-1}), \tag{9.3}$$

where $\lambda(\cdot,\cdot)$ is the structure constant for the antipode with respect to the fundamental basis, as proven in Theorem 5.4. But (9.3) expresses that Θ preserves antipodes (on the fundamental basis and its dual). $\square$

We turn now to quasi-symmetric functions. The dual $\mathcal{Q}\mathrm{Sym}^*$ of $\mathcal{Q}\mathrm{Sym}$ is the Hopf algebra of *non-commutative symmetric functions* of Gelfand et al. [12]. It is the free associative algebra with generators $\{M^*_{\emptyset,n} \mid n \geqslant 0\}$. This statement is dual to formula (1.6) for the coproduct of $\mathcal{Q}\mathrm{Sym}$.

Define numbers

$$b(\mathsf{S},\mathsf{T}) := \#\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) \subseteq \mathsf{T}\},$$

$$c(\mathsf{S},\mathsf{T}) := \#\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) \supseteq \mathsf{T}\},$$

$$d(\mathsf{S},\mathsf{T}) := \#\{u \in \mathfrak{S}_n \mid \mathrm{Des}(u) = \mathsf{S},\ \mathrm{Des}(u^{-1}) = \mathsf{T}\}.$$

Let Φ denote the composite

$$\mathcal{Q}\mathrm{Sym}^* \xrightarrow{\mathcal{D}^*} (\mathfrak{S}\mathrm{Sym})^* \xrightarrow{\Theta} \mathfrak{S}\mathrm{Sym} \xrightarrow{\mathcal{D}} \mathcal{Q}\mathrm{Sym}.$$

Proposition 9.4. *The morphism $\Phi : \mathcal{Q}\mathrm{Sym}^* \to \mathcal{Q}\mathrm{Sym}$ sends*

$$F^*_{\mathsf{S}} \mapsto \sum_{\mathsf{T} \in \mathcal{Q}_n} d(\mathsf{S},\mathsf{T}) F_{\mathsf{T}} \text{ and } M^*_{\mathsf{S}} \mapsto \sum_{\mathsf{T} \in \mathcal{Q}_n} b(\mathsf{S},\mathsf{T}) M_{\mathsf{T}},$$

for $\mathsf{S} \in \mathcal{Q}_n$.

Proof. Since $\mathcal{D}(\mathcal{F}_u) = F_{\mathrm{Des}(u)}$, the dual map satisfies $\mathcal{D}^*(F^*_{\mathsf{S}}) = \sum_{\mathrm{Des}(u)=\mathsf{S}} \mathcal{F}^*_u$. Also, Theorem 7.3 dualizes to $\mathcal{D}^*(M^*_{\mathsf{S}}) = \mathcal{M}^*_{\zeta_{\mathsf{S}}}$. The descriptions of the composite above follow now from those for Θ in (9.1) and (9.2), plus that $\theta(\zeta_{\mathsf{S}}, \zeta_{\mathsf{T}}) = b(\mathsf{S},\mathsf{T})$, which in turn follows from (2.8). $\square$

We now use the fact that $\Phi : \mathcal{Q}\mathrm{Sym}^* \to \mathcal{Q}\mathrm{Sym}$ is a morphism of Hopf algebras. The image of Φ is precisely the subalgebra of $\mathcal{Q}\mathrm{Sym}$ consisting of *symmetric* functions. Since $\mathcal{Q}\mathrm{Sym}^*$ is generated by $\{M^*_{\emptyset,n} \mid n \geqslant 0\}$, its image $\Phi(\mathcal{Q}\mathrm{Sym}^*)$ is generated by $\Phi(M^*_{\emptyset,n})$, for $n \geqslant 0$. Observe that $b(\emptyset,\mathsf{T}) = 1$ for every $\mathsf{T} \in \mathcal{Q}_n$ as 1_n is the only permutation u in $\mathfrak{S}_n$ with $\mathrm{Des}(u) \subseteq \emptyset$ and $\emptyset = \mathrm{Des}(1_n^{-1}) \subseteq \mathsf{T}$. Thus

$$\Phi(M^*_{\emptyset,n}) = \sum_{\mathsf{T} \in \mathcal{Q}_n} M_{\mathsf{T}} = F_{\emptyset,n}.$$

Formula (1.8) shows that $F_{\emptyset,n}$ is the complete homogeneous symmetric function of degree n. These generate the algebra of symmetric functions [21,33]. Thus, Φ is the *abelianization* map from non-commutative to commutative symmetric functions. We will not use this, but rather the explicit expression of Φ of Proposition 9.4.

Let $a^{\mathsf{R}}(\mathsf{S},\mathsf{T})$ denote the structure constants of the product of $\mathcal{Q}\mathrm{Sym}$ with respect to its monomial basis. These integers are combinatorially described by (1.4) or (7.3). The following analog of Theorem 9.2 provides a recursion for computing the numbers $b(\mathsf{S},\mathsf{T})$ in terms of the structure constants $a^{\mathsf{R}}(\mathsf{S},\mathsf{T})$. We view a^{R} and b as matrices with entries indexed by subsets of $[n-1]$.

Proposition 9.5. *For any* $\mathsf{S}\subseteq[p-1]$, $\mathsf{T}\subseteq[q-1]$, *and* $\mathsf{R}\subseteq[p+q-1]$,

$$(ba^{\mathsf{R}}b)(\mathsf{S},\mathsf{T}) = b(\mathsf{S}\cup\{p\}\cup(p+\mathsf{T}),\mathsf{R}). \tag{9.4}$$

Proof. The dual of the coproduct of $\mathcal{Q}$Sym (1.6) is

$$M^*_{\mathsf{S}}\cdot M^*_{\mathsf{T}} = M^*_{\mathsf{S}\cup\{p\}\cup(p+\mathsf{T})}.$$

Thus, the right-hand side of (9.4) is the coefficient of M_{R} in $\Phi(M^*_{\mathsf{S}}\cdot M^*_{\mathsf{T}})$. On the other hand, since $b(\mathsf{S},\mathsf{T}) = b(\mathsf{T},\mathsf{S})$, we have

$$(ba^{\mathsf{R}}b)(\mathsf{S},\mathsf{T}) = \sum_{\mathsf{S}',\mathsf{T}'} b(\mathsf{S},\mathsf{S}')a^{\mathsf{R}}(\mathsf{S}',\mathsf{T}')b(\mathsf{T}',\mathsf{T}) = \sum_{\mathsf{S}',\mathsf{T}'} b(\mathsf{S},\mathsf{S}')b(\mathsf{T},\mathsf{T}')a^{\mathsf{R}}(\mathsf{S}',\mathsf{T}').$$

Thus the left hand side of (9.4) is the coefficient of M_{R} in $\Phi(M^*_{\mathsf{S}})\cdot\Phi(M^*_{\mathsf{T}})$. Since Φ is a morphism of algebras, (9.4) holds. $\square$

Expressing that Θ preserves the antipode in terms of the fundamental basis and its dual gives a result of Foata and Schützenberger [11], which Gessel obtained in his original work on quasi-symmetric functions by other means [13, Corollary 6] (Eq. (iv) in the following corollary). For $\mathsf{S}\subseteq[n-1]$, define

$$\mathsf{S}^c = \{i\in[n-1] \mid i\notin\mathsf{S}\}$$

$$\widetilde{\mathsf{S}} = \{i\in[n-1] \mid n-i\in\mathsf{S}\}.$$

Corollary 9.6. *For* $\mathsf{S},\mathsf{T}\subseteq[n-1]$, *the numbers* $d(\mathsf{S},\mathsf{T})$ *satisfy*

(i) $d(\mathsf{S},\mathsf{T}) = d(\mathsf{T},\mathsf{S})$,
(ii) $d(\mathsf{S},\mathsf{T}) = d(\widetilde{\mathsf{S}},\widetilde{\mathsf{T}})$,
(iii) $d(\mathsf{S},\mathsf{T}) = d(\mathsf{S}^c,\mathsf{T}^c)$, *and*
(iv) $d(\mathsf{S},\mathsf{T}) = d(\widetilde{\mathsf{S}},\mathsf{T})$.

Proof. The symmetry (i) follows by considering the bijection $u\mapsto u^{-1}$. Similarly, (ii) follows by considering the bijection $u\mapsto\omega_n u\omega_n^{-1}$, where $\omega_n = (n,\ldots,2,1)$, as it is easy to see that $\mathrm{Des}(\omega_n u\omega_n^{-1}) = \widetilde{\mathrm{Des}(u)}$.

The antipode of $\mathcal{Q}$Sym is [22, Corollaire 4.20]

$$S(F_{\mathsf{T}}) = (-1)^n F_{\widetilde{\mathsf{T}^c}}.$$

Since Φ preserves antipodes, its explicit description in Proposition 9.4 implies that $d(\widetilde{\mathsf{S}}^c,\mathsf{T}) = d(\mathsf{S},\widetilde{\mathsf{T}}^c)$. Together with (ii) this yields (iii).

Finally, to deduce (iv), consider the bijection $u \mapsto \omega_n u$. Note that $\mathrm{Des}(\omega_n u) = \mathrm{Des}(u)^c$. Therefore

$$\mathrm{Des}((\omega_n u)^{-1}) = \mathrm{Des}(\omega_n \omega_n u^{-1} \omega_n^{-1}) = \mathrm{Des}(\omega_n u \omega_n^{-1})^c = \widetilde{\mathrm{Des}(u^{-1})}^c.$$

This shows that $d(\mathsf{S}, \mathsf{T}) = d(\mathsf{S}^c, \widetilde{\mathsf{T}}^c)$. Together with (ii) and (iii) this gives (iv). $\square$

Expressing the preservation of the antipode under Φ in terms of monomial quasi-symmetric functions and their duals gives further, similar results.

Proposition 9.7. *The map* $S\Phi = \Phi S^*: \mathcal{Q}\mathrm{Sym}^* \to \mathcal{Q}\mathrm{Sym}$ *sends*

$$M_{\mathsf{S}}^* \;\mapsto\; (-1)^n \sum_{\mathsf{R}} c(\mathsf{S}, \widetilde{\mathsf{R}}^c) M_{\mathsf{R}} = (-1)^n \sum_{\mathsf{R}} c(\mathsf{R}, \widetilde{\mathsf{S}}^c) M_{\mathsf{R}}.$$

Therefore,

$$c(\mathsf{S}, \widetilde{\mathsf{R}}^c) = c(\mathsf{R}, \widetilde{\mathsf{S}}^c).$$

Proof. We will show that $S\Phi(M_{\mathsf{S}}^*) = (-1)^n \sum_{\mathsf{R}} c(\mathsf{S}, \widetilde{\mathsf{R}}^c) M_{\mathsf{R}}$. One shows similarly that $\Phi S^*(M_{\mathsf{S}}^*) = (-1)^n \sum_{\mathsf{R}} c(\mathsf{R}, \widetilde{\mathsf{S}}^c) M_{\mathsf{R}}$.

As mentioned in (1.7), the antipode of $\mathcal{Q}\mathrm{Sym}$ is

$$S(M_{\mathsf{T}}) = (-1)^{\#\mathsf{T}+1} \sum_{\mathsf{R} \subseteq \mathsf{T}} M_{\widetilde{\mathsf{R}}}.$$

Combining this with Proposition 9.4 shows that $S\Phi$ sends

$$M_{\mathsf{S}}^* \;\mapsto\; \sum_{\mathsf{T}} b(\mathsf{S}, \mathsf{T}) (-1)^{\#\mathsf{T}+1} \sum_{\mathsf{R} \subseteq \mathsf{T}} M_{\widetilde{\mathsf{R}}}.$$

Thus, we have to show that for each S and R,

$$\sum_{\mathsf{R} \subseteq \mathsf{T}} (-1)^{\#\mathsf{T}+1} b(\mathsf{S}, \mathsf{T}) = (-1)^n c(\mathsf{S}, \mathsf{R}^c).$$

Now,

$$\begin{aligned}
\sum_{\mathsf{R} \subseteq \mathsf{T}} (-1)^{\#\mathsf{T}+1} b(\mathsf{S}, \mathsf{T}) &= \sum_{\mathsf{R} \subseteq \mathsf{T}} \sum_{\mathsf{T}' \subseteq \mathsf{T}} (-1)^{\#\mathsf{T}+1} \#\{u \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) = \mathsf{T}'\} \\
&= \sum_{\mathsf{T}'} \#\{u \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) = \mathsf{T}'\} \sum_{\mathsf{R} \cup \mathsf{T}' \subseteq \mathsf{T}} (-1)^{\#\mathsf{T}+1} \\
&= \sum_{\mathsf{T}' : \mathsf{R} \cup \mathsf{T}' = [n-1]} (-1)^n \#\{u \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) = \mathsf{T}'\}
\end{aligned}$$

$$= (-1)^n \#\{u \mid \mathrm{Des}(u) \subseteq \mathsf{S},\ \mathrm{Des}(u^{-1}) \cup \mathsf{R} = [n-1]\}$$

$$= (-1)^n \#\{u \mid \mathrm{Des}(u) \subseteq \mathsf{S}, \mathrm{Des}(u^{-1}) \supseteq \mathsf{R}^c\}$$

$$= (-1)^n c(\mathsf{S}, \mathsf{R}^c).$$

For completeness, we include the consequences on the numbers b and c that follow. Note that these also follow directly from

$$b(\mathsf{S},\mathsf{T}) = \sum_{\substack{S' \subseteq \mathsf{S} \\ T' \subseteq \mathsf{T}}} d(\mathsf{S},\mathsf{T}) \quad \text{and} \quad c(\mathsf{S},\mathsf{T}) = \sum_{\substack{S' \subseteq \mathsf{S} \\ T' \supseteq \mathsf{T}}} d(\mathsf{S},\mathsf{T}).$$

Corollary 9.8. *For any* S, $\mathsf{T} \subseteq [n-1]$,

(i) $b(\mathsf{S},\mathsf{T}) = b(\mathsf{T},\mathsf{S})$,
(ii) $b(\mathsf{S},\mathsf{T}) = b(\widetilde{\mathsf{S}},\widetilde{\mathsf{T}})$, *and* $c(\mathsf{S},\mathsf{T}) = c(\widetilde{\mathsf{S}},\widetilde{\mathsf{T}})$,
(iii) $c(\mathsf{S},\mathsf{T}) = c(\mathsf{T}^c,\mathsf{S}^c)$,
(iv) $b(\mathsf{S},\mathsf{T}) = b(\widetilde{\mathsf{S}},\mathsf{T})$, *and* $c(\mathsf{S},\mathsf{T}) = c(\widetilde{\mathsf{S}},\mathsf{T})$.

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