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Mean Convergence of Hermite Interpolation

PAUL NEVAI*

Department of Mathematics,
The Ohio State University, Columbus, Ohio 43210-1174

AND

YUAN XU†

Department of Mathematics,
University of Oregon, Eugene, Oregon 97403-1222

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We study weighted mean convergence properties of Hermite interpolation based on the zeros of certain generalized Jacobi polynomials such as necessary and sufficient conditions for such convergence for all continuously differentiable functions, convergence with given order $E_{2n-2}(f')/n$, and convergence of the differentiated Hermite interpolation process. © 1994 Academic Press, Inc.

1. INTRODUCTION

The main purpose of this paper is to investigate weighted L^p $(0<p<\infty)$ convergence of Hermite interpolating processes based on the zeros of generalized Jacobi polynomials and the rate of convergence as well. Let

$$-1<x_{nn}(w)<x_{n-1,n}(w)< \cdots <x_{2n}(w)<x_{1n}(w)<1 \tag{1.1}$$

be the zeros of the *generalized Jacobi polynomials* $p_n(w)$ which are orthonormal with respect to a *generalized Jacobi weight* w $(w\in \mathrm{GJ})$. In this paper, we write $w\in \mathrm{GJ}$ if

$$w(x)=\begin{cases} g(x)(1-x)^a\,(1+x)^b, & |x|\leqslant 1, \\ 0, & |x|>1, \end{cases} \tag{1.2}$$

where $g>0$, $g'\in \mathrm{Lip}\,1$ in $[-1,1]$, and $a>-1$, $b>-1$. When $g(x)\equiv 1$, w is called a Jacobi weight. Given a positive integer n and a bounded function

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† E-mail address: yuan@math.uoregon.edu.

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f, the Hermite interpolating polynomial $F_n(w,f)$ is defined to be the unique polynomial of degree at most $2n-1$ satisfying

$$\begin{aligned} &F_n(w,f,x_{kn}(w))=f(x_{kn}(w)) \quad \text{and} \\ &F_n'(w,f,x_{kn}(w))=f'(x_{kn}(w)), \quad k=1,2,...,n. \end{aligned} \tag{1.3}$$

In what follows, all the functions, weight functions, and spaces of functions under consideration are defined in the interval $[-1,1]$. Given a positive integer n and a function f, we denote by $E_n(f)=E_{n,\infty}(f)$ the degree of best uniform approximation of f by algebraic polynomials of degree at most n.

Paper [18] deals with mean convergence of $F_n(w,f)$ in the very special case of interpolation based on the zeros of the Chebyshev polynomials. It is proved there that $F_n(w,f)$ converges in weighted L^p norms to $f \in C^1$ at the rate of $E_{2n-2}(f')/n$.

The Hermite–Fejér interpolating polynomial $H_n(w,f)$ is a polynomial of degree at most $2n-1$ that agrees with f at the interpolation points, and whose first derivative vanishes there. It is closely related to the Hermite interpolating polynomial $F_n(w,f)$. Weighted mean convergence of $H_n(w,f)$ was studied in detail in [14, 20, 7]. In [14], necessary and sufficient conditions were found for weighted mean convergence of $H_n(w,f)$. Subsequently, [20] gave sufficient conditions for the convergence of $H_n(w,f)$ in weighted L^p spaces at the Jackson rate. The latter conditions are also necessary as shown in [7], where necessary conditions were proved for very general weight functions. The relationship between $H_n(f)$ and $F_n(w,f)$ enables us to apply the methods developed in [10, 11] and the above mentioned papers to investigate Hermite interpolation. Although our results show that there is certain similarity between $F_n'(w,f)$ and Lagrange interpolation, we do not have to use Hilbert transforms (a technique developed in [10]) to investigate $F_n'(w,f)$ as done in [21]; weighted Bernstein–Markov inequalities of [12] are all that we need in this case.

In what follows, given a non-negative measurable function u and $0<p<\infty$, the weighted L^p "norm" is defined by

$$\|f\|_{u,p}=\left(\int_{-1}^{1} |f(t)|^p\, u(t)\, dt\right)^{1/p}.$$

With this notation, the main results of this paper are as follows.

THEOREM 1. *Let* $0<p<\infty$, $w \in \mathrm{GJ}$, *and let* u *be an integrable Jacobi weight. Then*

(i) $\lim_{n\to\infty} \|F_n(w,f)-f\|_{u,p}=0$ $(\forall f\in C^1) \Leftrightarrow$ (ii) $(1-x^2)\,u(x)\leqslant Cw^p(x)$

for some constant $C>0$ *depending only on* p, w, *and* u.

Remark 1. It has been known for some time that, given $w \in \mathrm{GJ}$ and a closed interval $\Delta \subset (-1, 1)$, we have[1] $\lim_{n\to\infty} \|H_n(w,f)-f\|_{1_\Delta,\infty}=0$ for all $f \in C$ (cf. [15, Thm. 2.2, p. 80]), and then, using some technical estimates such as [15, formula (4.23) in Lemma 4.6, p. 88] or Lemma 2.2 of this paper, $\lim_{n\to\infty} \|F_n(w,f)-f\|_{1_\Delta,\infty}=0$ follows for all $f \in C^1$ as well.

THEOREM 2. *Let $0<p<\infty$, $w \in \mathrm{GJ}$, $w_1(x)=w(x)\sqrt{1-x^2}$, and let u be an integrable Jacobi weight. Then there is a constant $K>0$ depending only on p, w, and u such that*

$$\text{(i)}\quad \sup_{n\in\mathbb{N}} [n\,\|F_n(w,f)-f\|_{u,p}] \leqslant KE_{2n-2}(f')\ (\forall f \in C^1) \Leftrightarrow \text{(ii)}\quad w_1^{-p}u \in L^1.$$

Remark 2. Just as in the case of Hermite–Fejér interpolation, the implication (i)$\Rightarrow$(ii) in the above theorem holds for much more general weights (cf. [7]). The proof of this fact also follows from the results in [7]. More precisely, the implication (i)$\Rightarrow$(ii) in Theorem 2 remains valid if we replace $w \in \mathrm{GJ}$ by weight functions from either the Szegő class or from a certain subset of the Erdős class (see [7] for the definitions). Unlike in Remark 1, and with the notation of Remark 1, there does not seem to exist a constant $K>0$ such that the inequality $\sup_{n\in\mathbb{N}} [n\,\|F_n(w,f)-f\|_{1_\Delta,\infty}] \leqslant K\,E_{2n-2}(f')$ holds for all $f \in C^1$. According to (a hand corrected version of) [2, formula (3.6) in Thm. 3.1, p. 114], inequality (i) in Theorem 2 holds for more general functions u, namely, for $u \in (L\log^+ L)^p$, though the argument given on [2, p. 126] does not appear to be complete.

THEOREM 3. *Let $0<p<\infty$, $w \in \mathrm{GJ}$, $w_2(x)=w(x)(1-x^2)$, and let u be a Jacobi weight such that $(1-x^2)^{-p/2}u \in L^1$. Then there is a constant $K>0$ depending only on p, w, and u such that*

$$\text{(i)}\quad \sup_{n\in\mathbb{N}} \|F_n'(w,f)-f'\|_{u,p} \leqslant KE_{2n-2}(f')\ (\forall f \in C^1) \Leftrightarrow \text{(ii)}\quad w_2^{-p}u \in L^1.$$

Remark 3. The first two theorems state that the conditions for the best rate of convergence are stronger than those for just convergence. Interestingly, condition (ii) in Theorem 2 is exactly the same as that for the Hermite–Fejér interpolation (cf. [20, Thm. 2.1] and [7, Thm. 1]), but condition (ii) in Theorem 1 is slightly weaker than the corresponding condition $w^{-p}u \in L^1$ for Hermite–Fejér interpolation (cf. [14, Thm. 5, p. 55]). On the other hand, the conditions for the convergence of the derivative of the Hermite interpolating polynomial and the conditions for

[1] Here and in what follows, the characteristic function of a set $\mathscr{E}$ is denoted by $1_{\mathscr{E}}$.

the best rate of convergence are the same. This is very much like the case of the Lagrange interpolation (cf. [10, Thm. 6, p. 695]).

Since our intentions are to concentrate on proving conditions which are simultaneously both necessary and sufficient, all our weights u in $\|F_n(w,f)-f\|_{u,p}$ and $\|F'_n(w,f)-f'\|_{u,p}$ are (integrable) Jacobi weights. We thank one of the referees for pointing out that generalizations of the implications (i) $\Leftarrow$ (ii) in Theorems 2 and 3 for weights u of the form $u(x)=\prod_{k=1}^{m}|x-t_k|^{\gamma_k}$ have been discussed in (a hand corrected version of) [2].

For the proof of Theorem 3, we will need the following theorem which we state and prove in a more general setting. As before and in what follows, the characteristic function of a set $\mathscr{E}$ is denoted by $1_{\mathscr{E}}$.

THEOREM 4. *Let*

$$U(x)=\prod_{k=1}^{m}|x-y_k|^{A_k} \tag{1.4}$$

and[2]

$$W(x)=g(x)\prod_{k=1}^{m}|x-y_k|^{B_k}, \tag{1.5}$$

where $y_k\in[-1,1]$, $A_k>-1$, $B_k>-1$, *and* g *is a non-negative function. If either* (i) $r\geqslant 0$, $p\geqslant 1$, *and* $0<c_1\leqslant g(x)\leqslant c_2<\infty$ *for* $x\in[-1,1]$, *or* (ii) $r\geqslant 0$, $p>0$, $0<c_1\leqslant g(x)\leqslant c_2<\infty$ *for* $x\in[-1,1]$, *and* $\int_0^1\omega(g,t)\,t^{-1}\,dt<\infty$,[3] *then there is a constant* $C>0$ *such that*[4]

$$\int_{-1}^{1}\frac{1_\Delta(x)\,U(x)\,dx}{W(x)^{(r+p)/2}\,(1-x^2)^{(r+3p)/4}}$$
$$\leqslant C\liminf_{n\to\infty}\frac{1}{n^p}\int_{-1}^{1}|p_n(W,x)|^r\,|p'_n(W,x)|^p\,1_\Delta(x)\,U(x)\,dx \tag{1.6}$$

for every interval $\Delta\subseteq[-1,1]$.

A weaker version of Theorem 4 is given in [13, formula (6.3)].

[2] The weight function w in (1.2) is a special case of W in (1.5). Frequently, but not in this paper, the latter are also called *generalized Jacobi weights*.

[3] Here ω denotes the modulus of continuity of g.

[4] Here $p_n(W)$ denote the orthonormal polynomials associated with W.

2. LEMMAS

The Hermite interpolating polynomial (1.3) is given by

$$F_n(w, f, x) = H_n(w, f, x) + G_n(w, f', x)$$
$$= \sum_{k=1}^{n} f(x_{kn}(w))\, h_{kn}(w, x) + \sum_{k=1}^{n} f'(x_{kn}(w))\, g_{kn}(w, x) \qquad (2.1)$$

with

$$h_{kn}(w, x) = \left[1 + \frac{\lambda_n'(w, x_{kn}(w))}{\lambda_n(w, x_{kn}(w))} (x - x_{kn}(w)) \right] l_{kn}^2(w, x) \qquad (2.2)$$

and

$$g_{kn}(w, x) = (x - x_{kn}(w))\, l_{kn}^2(w, x) \qquad (2.3)$$

(cf. [5, p. 113]) where

$$l_{kn}(w, x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \lambda_{kn}(w)\, p_{n-1}(w, x_{kn}(w)) \frac{p_n(w, x)}{x - x_{kn}(w)}, \qquad (2.4)$$

$\lambda_n(w)$ is the nth Christoffel function, $\lambda_{kn}(w) = \lambda_n(w, x_{kn}(w))$, and $\gamma_n(w)$ is the leading coefficient of $p_n(w)$ (cf. [14, p. 43]). If P is a polynomial of degree at most $2n-1$, then

$$P = H_n(w, P) + G_n(w, P'). \qquad (2.5)$$

Given a positive integer n and a function f, the Lagrange interpolating polynomial $L_n(w, f)$ is defined to be the unique polynomial of degree at most $n-1$ that agrees with f at the points $x_{kn}(w)$ for $k = 1, 2, ..., n$. Hence,

$$L_n(w, f, x) = \sum_{k=1}^{n} f(x_{kn}(w))\, l_{kn}(w, x). \qquad (2.6)$$

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers. Furthermore, K denotes positive constants which are independent of variables and indices under consideration. We write $A \sim B$ if $|A^{-1}B|^{\pm 1} \leqslant K$. We will omit w in many formulas when it is clear what the weight function is in the given context .

The following lemma is a summary of some properties of generalized Jacobi polynomials which will be used to prove our main results.

LEMMA 2.1. *Let* $w \in \mathrm{GJ}$, *and let* $x_{kn}(w) = \cos\theta_{kn}(w)$ *with* $0 \leqslant \theta_{kn} \leqslant \pi$ *where* $x_{0n} = 1$ *and* $x_{n+1,n} = -1$. *Then*

$$\theta_{k+1,n}(w) - \theta_{kn}(w) \sim n^{-1} \tag{2.7}$$

uniformly for $n \in \mathbb{N}$ *and* $0 \leqslant k \leqslant n$ (*cf.* [11, *Thm.* 9.22, *p.* 166]),

$$\lambda_{kn}(w) \sim n^{-1} w(x_{kn}) \sqrt{1 - x_{kn}^2}, \tag{2.8}$$

uniformly for $n \in \mathbb{N}$ *and* $1 \leqslant k \leqslant n$ (*cf.* [11, *Thm.* 6.3.28, *p.* 120]),

$$|p_{n-1}(w, x_{kn})| \sim w(x_{kn})^{-1/2} (1 - x_{kn}^2)^{1/4}, \tag{2.9}$$

uniformly for $n \in \mathbb{N}$ *and* $1 \leqslant k \leqslant n$ (*cf.* [11, *Thm.* 9.31, *p.* 170]). *For every fixed* $0 < \sigma < 1$,

$$|p_n(w, x)| \leqslant K[w(x)\sqrt{1 - x^2}]^{-1/2}, \qquad |x| \leqslant 1 - \sigma n^{-2}, \tag{2.10}$$

uniformly for $n \in \mathbb{N}$ (*this follows from Korous' theorem and well-known estimates for the Jacobi polynomials, cf.* [17, *Thm.* 7.1.3, *p.* 162, *and Thm.* 7.32.2, *p.* 169]), *and*

$$|p_n(w, x)| \sim \begin{cases} n\,|x - x_{mn}|\,[w(x)(1 - x^2)^{3/2}]^{-1/2}, & -1 + x_{nn} \leqslant 2x \leqslant 1 + x_{1n} \\ \sqrt{n}\,[w(1 - n^{-2})]^{-1/2}, & 1 + x_{1n} \leqslant 2x \leqslant 2 \\ \sqrt{n}\,[w(-1 + n^{-2})]^{-1/2}, & -2 \leqslant 2x \leqslant -1 + x_{nn}, \end{cases} \tag{2.11}$$

uniformly for $n \in \mathbb{N}$, *where* m *is the index of that zero* x_{kn} ($k = 1, 2, \ldots, n$) *which is* (*one of*) *the closest to* x (*cf.* [11, *Thm.* 9.33, *p.* 171]). *In addition,*

$$0 < \lim_{n \to \infty} 2^{-n} \gamma_n(w) < \infty \tag{2.12}$$

(*cf.* [17, *Thm.* 12.7.1, *p.* 309]).

LEMMA 2.2. *Let* $w \in \mathrm{GJ}$, *and let* $0 < \sigma < 1$ *be fixed. Then there is a constant* $K > 0$ *such that*

$$\sum_{k=1}^{n} |x - x_{kn}(w)|\, l_{kn}^2(w, x) \leqslant K\left[\frac{\log n}{n}\sqrt{1 - x^2} + \frac{1}{n w(x)\sqrt{1 - x^2}}\right], \tag{2.13}$$

uniformly for $n = 2, 3, \ldots$ *and* $|x| \leqslant 1 - \sigma n^{-2}$.

The sharp estimate in this inequality is significant for our purpose (cf. [14, Lemma 4, p. 40] for a weaker form), and, therefore, we will give a brief proof even though it goes along routine estimates.

Proof of Lemma 2.2. In what follows we assume $0 \leqslant x \leqslant 1 - \sigma n^{-2}$. Write $x = \cos\theta$ and $x_{kn} = \cos\theta_{kn}$ where $\theta, \theta_{kn} \in [0, \pi]$. By (2.4), (2.8), (2.9), and (2.12), we have

$$\sum_{k=1}^{n} |x - x_{kn}|\, l_{kn}^2(x) \leqslant \frac{K}{n^2} \sum_{k=1}^{n} w(x_{kn})(1 - x_{kn}^2)^{3/2} \frac{p_n^2(x)}{|x - x_{kn}|}.$$

Let m be the index of that zero x_{kn} $(k = 1, 2, ..., n)$ which is (one of) the closest to x. By (2.7) and (2.11) we have

$$\frac{1}{n^2} w(x_{mn})(1 - x_{mn}^2)^{3/2} \frac{p_n^2(x)}{|x - x_{mn}|} \leqslant K\, |x - x_{mn}| \leqslant \frac{K}{n} \sqrt{1 - x^2}.$$

Write now $w(x) = g(x)(1 - x)^a (1 + x)^b$. Then

$$\begin{aligned}
&\frac{1}{n^2} \sum_{k \neq m} w(x_{kn})(1 - x_{kn}^2)^{3/2} \frac{p_n^2(x)}{|x - x_{kn}|} \\
&\leqslant \frac{K}{n^2} \sum_{k \neq m} \frac{w(x_{kn})(1 - x_{kn}^2)^{3/2}}{w(x_{mn})(1 - x_{mn}^2)^{1/2}\, |x - x_{kn}|} \\
&\sim \frac{1}{n^2} \sum_{\substack{k=1 \\ k \neq m}}^{[(n+m)/2]} \left(\frac{k}{m}\right)^{2a+1} \frac{k^2}{|k - m|\, |k + m|} \\
&\quad + \sum_{k = [(n+m)/2]}^{n} \frac{(n - k + 1)^{2b+3}}{m^{2a+1}} \cdot \frac{1}{n^{2b-2a+4}} \\
&:= I_1 + I_2.
\end{aligned}$$

Splitting I_1, into three sums with corresponding index sets given by $1 \leqslant k \leqslant [m/2]$, $[m/2] < k \neq m \leqslant 2m$, and $2m < k \leqslant [(m+n)/2]$, we can estimate these sums individually to obtain

$$I_1 \sim \frac{m}{n^2} \log m + \frac{1}{n} \left(\frac{n}{m}\right)^{2a+1} \leqslant K \left[\frac{\log n}{n} \sqrt{1 - x^2} + \frac{1}{nw(x)\sqrt{1 - x^2}}\right].$$

In addition,

$$I_2 \sim \frac{n^{2b+4}}{n^{2b-2a+4} \cdot m^{2a+1}} \sim \frac{1}{n} \left(\frac{n}{m}\right)^{2a+1} \leqslant \frac{K}{nw(x)\sqrt{1 - x^2}}.$$

Putting these two estimates together, (2.13) follows when $0 \leqslant x \leqslant 1-\sigma n^{-2}$. For $-1+\sigma n^{-2} \leqslant x \leqslant 0$, the proof of (2.13) is analogous. ∎

LEMMA 2.3. *Let $w \in \mathrm{GJ}$, and let $0<\sigma<1$ be fixed. Then there is a constant $K>0$ such that*

$$\sum_{k=1}^{n} |h_{kn}(x)| \leqslant K\left[1+\frac{\log n}{n} \frac{1}{w(x)\sqrt{1-x^2}}\right] \tag{2.14}$$

uniformly for $n=2, 3, \ldots$ and $|x| \leqslant 1-\sigma n^{-2}$.

For Jacobi weights, this lemma has been proved in [19, Lemma 3.2, p. 374], and it can be proved similarly to Lemma 2.2, provided that one uses the estimate of λ'_{kn} as given in [14, p. 36].

The following result is a special case of a more general theorem proved in [10, Thm. 1, p. 680].

LEMMA 2.4. *Let $w \in \mathrm{GJ}$ and $0<p<\infty$. Let u be a Jacobi weight such that $w^{-p}u \in L^1$. Then there is a constant $K>0$ such that for every bounded function f,*

$$\|L_n(F)\, w_1^{-1/2}\|_{u,p} \leqslant K \|f\|_\infty, \qquad n=1, 2, \ldots,$$

where F and w_1 are given by $F(x)=f(x)\, w^{1/2}(x)(1-x^2)^{3/4}$ and $w_1(x)=w(x)\sqrt{1-x^2}$, respectively.

LEMMA 2.5 [14, Thm. 5, p. 55]. *Let $w \in \mathrm{GJ}$, $0<p<\infty$, and let u be an integrable Jacobi weight such that $w^{-p}u \in L^1$. Then there is a constant $K>0$ such that*

$$\|H_n(f)\|_{u,p} \leqslant K \|f\|_\infty, \qquad n=1, 2, \ldots,$$

holds for every bounded function f.

In what follows we will need that for every $0<p<\infty$ and for every integrable Jacobi weight u, there exists a constant $0<\sigma=\sigma(p, u)<1$ such that for every polynomial R_{2n} of degree at most $2n$

$$\int_{-1}^{1} |R_{2n}(t)|^p\, u(t)\, dt \leqslant 2 \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |R_{2n}(t)|^p\, u(t)\, dt \tag{2.15}$$

(cf. [11, Thm. 6.3.14, p. 113])[5]

[5] As a matter of fact, σ in (2.15) can be an arbitrary positive constant but then the constant 2 in front of the right-hand side integral may need to be increased (cf. [3, Thm. 8.4.8, p. 108; 4, Thm. 9, p. 247]).

LEMMA 2.6. *Let $w \in \mathrm{GJ}$, $0 < p < \infty$, and let u be an integrable Jacobi weight such that $(1-x^2)\,u(x) \leqslant Cw^p(x)$ for some constant $C > 0$. Then there is a constant $K > 0$ such that*

$$\|F_n(f)\|_{u,p} \leqslant K[\log n \, \|f\|_\infty + \|f'\|_\infty], \qquad n = 1, 2, ..., \tag{2.16}$$

for every differentiable function f.

Proof. Using (2.15) for the polynomial $G_n(f')$ and applying Lemma 2.2, it follows that

$$\begin{aligned} \|G_n(f')\|_{u,p}^p &\leqslant K \, \|f'\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left[\sum_{k=1}^n |x - x_{kn}| \, l_{kn}^2(x)\right]^p u(x)\,dx \\ &\leqslant K \, \|f'\|_\infty^p \left\{\left(\frac{\log n}{n}\right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (1-x^2)^{p/2}\, u(x)\,dx \right. \\ &\quad \left. + \left(\frac{1}{n}\right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} [w(x)\sqrt{1-x^2}]^{-p}\, u(x)\,dx \right\}. \end{aligned}$$

Since $(1-x^2)\,u(x) \leqslant Cw^p(x)$, we have

$$\begin{aligned} &\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} [w(x)\sqrt{1-x^2}]^{-p}\, u(x)\,dx \\ &\qquad \leqslant K \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (1-x^2)^{-(p/2)-1}\,dx \leqslant Kn^p, \end{aligned}$$

and, therefore,

$$\|G_n(f')\|_{u,p}^p \leqslant K \, \|f'\|_\infty^p \left\{\left(\frac{\log n}{n}\right)^p + 1\right\} \leqslant K \, \|f'\|_\infty^p.$$

Next, using (2.15) for the polynomial $H_n(f)$, it follows from Lemma 2.3 that

$$\begin{aligned} \|H_n(f)\|_{u,p}^p &\leqslant K \, \|f\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left[1 + \left(\frac{\log n}{n}\right)^p [w(x)\sqrt{1-x^2}]^{-p}\right] u(x)\,dx \\ &\leqslant K \, \|f\|_\infty^p \left[1 + \left(\frac{\log n}{n}\right)^p n^p\right] \leqslant K(\log n)^p \, \|f\|_\infty^p. \end{aligned}$$

Now (2.16) follows from (2.1) and the inequalities in the last two displayed formulas. ∎

LEMMA 2.7. *Let $w \in \mathrm{GJ}$, $0 < p < \infty$, and let u be an integrable Jacobi weight such that $w_1^{-p} u \in L^1$ where $w_1(x) = w(x)\sqrt{1-x^2}$. Then there is a constant $K > 0$ such that*

$$\|F_n(f)\|_{u,p} \leqslant K\left(\|f\|_\infty + \frac{1}{n}\|f'\|_\infty\right), \qquad n = 1, 2, ...,$$

for every differentiable function f.

Proof. Since $w_1^{-p} u \in L^1$ implies $w^{-p} u \in L^1$, in view of Lemma 2.5 and (2.1) we need to estimate $\|G_n(f')\|_{u,p}$ only. By (2.3) and (2.4),

$$G_n(f', x) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} f'(x_{kn})\, \lambda_{kn} p_{n-1}(x_{kn})\, p_n(x)\, l_{kn}(x). \tag{2.17}$$

By (2.8), (2.9), and (2.12), we have

$$\frac{\gamma_{n-1}}{\gamma_n} \lambda_{kn} p_{n-1}(x_{kn}) = c_{kn} n^{-1} w^{1/2}(x_{kn})(1 - x_{kn}^2)^{3/4},$$

where c_{kn} are certain constants uniformly bounded for $n \in \mathbb{N}$ and $k = 1, 2, ..., n$. Given $n \in \mathbb{N}$, define the continuous function c_n such that $c_n(x_{kn}) = c_{kn}$ for $k = 1, 2, ..., n$, and $c_n(x)$ is uniformly bounded for $x \in [-1, 1]$ independently of n. If the function q_n is defined by $q_n = c_n f'/n$, then

$$\|q_n\|_\infty \leqslant K n^{-1} \|f'\|_\infty, \tag{2.18}$$

and, according to (2.6), formula (2.17) can be rewritten as $G_n(f') = p_n L_n(v q_n)$ where $v(x) = w^{1/2}(x)(1 - x^2)^{3/4}$. Hence, by (2.10) and (2.15),

$$\begin{aligned} \|G_n(f')\|_{u,p}^p &\leqslant K \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |L_n(v q_n, x)|^p\, w_1^{-p/2}(x)\, u(x)\, dx \\ &\leqslant K\, \|L_n(v q_n)\, w_1^{-1/2}\|_{u,p}^p, \end{aligned}$$

and, therefore, by Lemma 2.4 and (2.18),

$$\|G_n(f')\|_{u,p}^p \leqslant K\, \|q_n\|_\infty \leqslant K n^{-1} \|f'\|_\infty.$$

The proof of Lemma 2.7 is complete. ∎

We will also use the following Bernstein–Markov type inequality proved in [12, Thm. 5, p. 242].

LEMMA 2.8. *Given $0<p<\infty$ and u an integrable Jacobi weight, there is a constant $K>0$ such that given $n\in\mathbb{N}$, the inequality*

$$\|R_n'\sqrt{1-x^2}\|_{u,p}\leqslant Kn\,\|R_n\|_{u,p}$$

holds for every polynomial R_n of degree at most n.

The following result played an important role in establishing necessary conditions for mean convergence of Lagrange interpolation in [11] and Hermite–Fejér interpolation in [7] (cf. [11, the proof of Lemma 9.9, p. 159; 7, Lemma 5, p. 322]).

LEMMA 2.9.[6] *For every positive Borel measure α supported in $[-1,1]$ we have*[7]

$$\frac{2^{2n-4}}{n\gamma_{n-1}^2(\alpha)}\leqslant\sum_{k=1}^{n}\lambda_{kn}^2(\alpha)\,p_{n-1}^2(\alpha,x_{kn}(\alpha)),\qquad n=2,3,\ldots.$$

Now we construct a certain cubic Hermite spline function s_n defined in $[-1,1]$ which will assist us in proving the necessary conditions in our theorems. Let x_{kn} denote the zeros of $p_n(w)$ for $k=1,2,\ldots,n$, and let $x_{0,n}=1$ and $x_{n+1,n}=-1$. Given $n\in\mathbb{N}$, define s_n as a cubic polynomial on each interval $[x_{k+1,n},x_{kn}]$ for $k=0,1,2,\ldots,n$, which is uniquely determined by the conditions

$$s_n(x_{jn})=0\qquad\text{and}\qquad s_n'(x_{jn})=1,\qquad j=k,k+1.\tag{2.19}$$

It is easy to write s_n explicitly on each $[x_{k+1,n},x_{kn}]$ as

$$s_n(x)=\frac{(x-x_{kn})(x-x_{k+1,n})}{(x_{k+1,n}-x_{kn})^2}\,[2x-(x_{kn}+x_{k+1,n})],\qquad x\in[x_{k+1,n},x_{kn}].$$

From this formula we get

$$|s_n(x)|\leqslant 3^{-3/2}\,|x_{k+1,n}-x_{kn}|\leqslant\frac{K}{n},\qquad -1\leqslant x\leqslant 1,\tag{2.20}$$

and

$$|s_n'(x)|\leqslant 1,\qquad -1\leqslant x\leqslant 1.\tag{2.21}$$

[6] A stronger version of this lemma can be found in [16, Thm. 5].

[7] Here $p_n(\alpha)$ denotes the orthonormal polynomial associated with α, $\gamma_n(\alpha)$ is its leading coefficient, $\{x_{kn}(\alpha)\}_{k=1}^n$ are its zeros, and $\{\lambda_{kn}(\alpha)\}_{k=1}^n$ are the corresponding Cotes numbers. We realize the mistake in using simultaneously the notations $p_n(w)$ (with weights) and $p_n(\alpha)$ (with measures), and we hope that this is not going to confuse the reader. Apparently, both notations have happily coexisted in the past despite occasional objections.

We also need the following inequalities concerning simultaneous approximation according to which for every continuously differentiable function f in $[-1, 1]$ and for every $n \in \mathbb{N}$ there exists a polynomial R_{2n-1} of degree at most $2n-1$ such that

$$\|R_{2n-1} - f\|_\infty \leqslant K \frac{E_{2n-2}(f')}{n} \quad \text{and} \qquad (2.22)$$
$$\|R'_{2n-1} - f'\|_\infty \leqslant K E_{2n-2}(f')$$

with a suitable constant K independent of f and n (cf. [6, Thm. 2, p. 172]).

3. Proof of Theorem 4

In this section α is a positive Borel measure with finite moments and infinite support, and then $p_n(\alpha)$ denote the corresponding orthonormal polynomials. First we need two lemmas.

Lemma 3.1. *Let* $\operatorname{supp}(\alpha) = [-1, 1]$ *and let* $\alpha'(x) > 0$ *a.e. in* $[-1, 1]$. *Let* $\{f_n\}_{n=1}^\infty$ *be a bounded and almost everywhere convergent sequence of Lebesgue-measurable functions in* $[-1, 1]$, *and let* $f = \lim_{n\to\infty} f_n$. *Then*

$$\lim_{n\to\infty} \int_{-1}^1 f_n(t)\, p_n^2(\alpha, t)\, \alpha'(t)\, dt = \frac{1}{\pi} \int_{-1}^1 \frac{f(t)\, dt}{\sqrt{1-t^2}}. \qquad (3.1)$$

Remark. This lemma generalizes [9, formula (11.4) in Thm. 11.1, p. 271] where the case $f_n \equiv f$ was proved. The proof of this lemma is based on results from [9].

Proof of Lemma 3.1. Fix $0 < \varepsilon < 1$. Define Ω_n by

$$\Omega_n(\alpha, x) = p_n^2(\alpha, x) - 2x p_n(\alpha, x)\, p_{n-1}(\alpha, x) + p_{n-1}^2(\alpha, x).$$

Then[8]

$$(1 - x^2)\, p_n^2(\alpha, x) \leqslant \Omega_n(\alpha, x), \qquad x \in (-1, 1), \qquad (3.2)$$

and

$$\lim_{n\to\infty} \int_{-1}^1 \left| \Omega_n(\alpha, t)\, \alpha'(t) - \frac{2}{\pi}\sqrt{1-t^2} \right| dt = 0 \qquad (3.3)$$

[8] To see (3.2), complete the square.

(cf. [9, Thm. 10.1 and formula (10.3), p. 268]). By (3.2) and (3.3)

$$\int_{-1+\varepsilon}^{1-\varepsilon} |f_n(t)-f(t)|\; p_n^2(\alpha, t)\, \alpha'(t)\, dt$$

$$\leqslant \int_{-1+\varepsilon}^{1-\varepsilon} \frac{|f_n(t)-f(t)|}{1-t^2}\, \Omega_n(\alpha, t)\, \alpha'(t)\, dt$$

$$\leqslant \frac{1}{\varepsilon^2} \int_{-1}^{1} |f_n(t)-f(t)|\; \Omega_n(\alpha, t)\, \alpha'(t)\, dt$$

$$\leqslant \sup_{t\in[-1,1]} \{|f_n(t)|+|f(t)|\}\, \frac{1}{\varepsilon^2} \int_{-1}^{1} \left|\Omega_n(\alpha, t)\, \alpha'(t) - \frac{2}{\pi}\sqrt{1-t^2}\right| dt$$

$$+\frac{2}{\pi\varepsilon^2} \int_{-1}^{1} |f_n(t)-f(t)|\, \sqrt{1-t^2}\, dt$$

so that

$$\lim_{n\to\infty} \int_{-1+\varepsilon}^{1-\varepsilon} |f_n(t)-f(t)|\; p_n^2(\alpha, t)\, \alpha'(t)\, dt = 0. \tag{3.4}$$

On the other hand,

$$\int_{1-\varepsilon\leqslant|t|\leqslant 1} |f_n(t)-f(t)|\; p_n^2(\alpha, t)\, \alpha'(t)\, dt$$

$$\leqslant \sup_{t\in[-1,1]} \{|f_n(t)|+|f(t)|\} \int_{1-\varepsilon\leqslant|t|\leqslant 1} p_n^2(\alpha, t)\, \alpha'(t)\, dt,$$

and by [9, formula (11.4) in Thm. 11.1, p. 271] applied with the characteristic function of the set $\{t\colon 1-\varepsilon\leqslant|t|\leqslant 1\}$ we obtain

$$\limsup_{n\to\infty} \int_{1-\varepsilon\leqslant|t|\leqslant 1} |f_n(t)-f(t)|\; p_n^2(\alpha, t)\, \alpha'(t)\, dt$$

$$\leqslant \sup_{t\in[-1,1]} \{|f_n(t)|+|f(t)|\}\, \frac{1}{\pi} \int_{1-\varepsilon\leqslant|t|\leqslant 1} \frac{dt}{\sqrt{1-t^2}}. \tag{3.5}$$

Since $0<\varepsilon<1$ can be made arbitrarily small, we can combine (3.4) and (3.5) to obtain

$$\lim_{n\to\infty} \int_{-1}^{1} |f_n(t)-f(t)|\; p_n^2(\alpha, t)\, \alpha'(t)\, dt = 0.$$

Therefore, Lemma 3.1 follows from the previously proved case of $f_n \equiv f$ (cf. [9, formula (11.4) in Thm. 11.1, p. 271]). ∎

LEMMA 3.2. *Let* $\operatorname{supp}(\alpha)=[-1, 1]$, $\alpha'(x)>0$ *a.e. in* $[-1, 1]$, *and let* $0<p\leqslant\infty$. *Then for every sequence* $\{f_n\}$ *of non-negative Lebesgue-measurable functions in* $[-1, 1]$,

$$\left[\int_{-1}^{1}\left(\frac{\liminf_{n\to\infty} f_n(t)}{\sqrt{\alpha'(t)\sqrt{1-t^2}}}\right)^p dt\right]^{1/p}$$
$$\leqslant\sqrt{\pi}\,2^{\max\{(1/p)-(1/2),\,0\}}\liminf_{n\to\infty}\left[\int_{-1}^{1}|f_n(t)\,p_n(\alpha, t)|^p\,dt\right]^{1/p}. \tag{3.6}$$

In particular, if $\liminf_{n\to\infty}[\int_{-1}^{1}|f_n(t)\,p_n(\alpha, t)|^p\,dt]^{1/p}=0$ *then* $\liminf_{n\to\infty} f_n=0$ *a.e.*

Remark. This lemma generalizes [8, Thm. 2, p. 317] where the case $f_n\equiv f$ was proved. That theorem plays a crucial role in proving this lemma. The best constant in (3.6) is still unknown, even for the case $f_n\equiv f$. The second part of Lemma 3.2 is probably true under much more general conditions (cf. [16, Coro. 8]).

Proof of Lemma 3.2. Without loss of generality we can assume that $0<p<\infty$ since once (3.6) is proved for $0<p<\infty$ then the case $p=\infty$ follows by letting $p\uparrow\infty$.

A second assumption we can make is that $\lim_{n\to\infty} f_n(x)=f(x)$ exists almost everywhere. Whether or not $\{f_n\}$ converges, we can first prove the lemma for $F_n=\inf_{m\geqslant n}\{f_m\}$. Since $F_n(x)\leqslant f_n(x)$ for every $x\in[-1, 1]$, we have

$$\liminf_{n\to\infty}\left[\int_{-1}^{1}|F_n(t)\,p_n(\alpha, t)|^p\,dt\right]^{1/p}\leqslant\liminf_{n\to\infty}\left[\int_{-1}^{1}|f_n(t)\,p_n(\alpha, t)|^p\,dt\right]^{1/p}.$$

On the other hand, $\lim_{n\to\infty} F_n(x)=\liminf_{n\to\infty} f_n(x)$ a.e. in $[-1, 1]$. Therefore, it is sufficient to prove the lemma for convergent sequences.

In addition, we can also assume that there is a constant M such that

$$\frac{f_n(x)}{\sqrt{\alpha'(x)\sqrt{1-x^2}}}\leqslant M \quad\text{and}\quad \frac{f_n(x)}{[\alpha'(x)]^{p/2}}\leqslant M \quad\text{for a.e.}\quad x\in[-1, 1]. \tag{3.7}$$

Otherwise, we can set

$$f_{n,M}(x)=\min\{f_n(x), M\sqrt{\alpha'(x)\sqrt{1-x^2}}, M\alpha'(x)^{p/2}\},$$

and then once (3.6) is proved for the sequence $\{f_{n,M}\}$, the general case follows by applying (3.6) with $\{f_{n,M}\}$ and $f_{\infty,M}=\lim_{n\to\infty} f_{n,M}$, and by letting $M \uparrow \infty$ while using $f_{\infty, M} \uparrow f$ a.e. in $[-1, 1]$ and Lebesgue's *Monotone Convergence Theorem* on the left-hand side of (3.6).

The following arguments are valid if all three above assumptions hold. If $0 < p \leqslant 2$ then by Hölder's inequality

$$\int_{-1}^{1} |f_n(t) - f(t)| \, |p_n(\alpha, t)|^p \, dt$$
$$\leqslant 2^{1-(p/2)} \left[\int_{-1}^{1} \left(\frac{|f_n(t) - f(t)|}{[\alpha'(t)]^{p/2}} \right)^{2/p} p_n^2(\alpha, t) \, \alpha'(t) \, dt \right]^{p/2}$$

so that by (3.7) and Lemma 3.1

$$\lim_{n \to \infty} \int_{-1}^{1} |f_n(t) - f(t)| \, |p_n(\alpha, t)|^p \, dt = 0.$$

Hence, by (3.6) applied with $f_n \equiv f$, that is by [8, Thm. 2, p. 317] the lemma follows for $0 < p \leqslant 2$.

The case when $2 < p < \infty$ is reduced to the case $p = 2$ by using Hölder's inequality to obtain

$$\int_{-1}^{1} \left(\frac{f_n(t)}{\sqrt{\alpha'(t)} \sqrt{1-t^2}} \right)^p p_n^2(\alpha, t) \, \alpha'(t) \sqrt{1-t^2} \, dt$$
$$= \int_{-1}^{1} \left(\frac{f_n(t)}{\sqrt{\alpha'(t)} \sqrt{1-t^2}} \right)^{p-2} f_n^2(t) \, p_n^2(\alpha, t) \, dt$$
$$\leqslant \left[\int_{-1}^{1} \left(\frac{f_n(t)}{\sqrt{\alpha'(t)} \sqrt{1-t^2}} \right)^p dt \right]^{(p-2)/p}$$
$$\times \left[\int_{-1}^{1} |f_n(t) \, p_n(\alpha, t)|^p \, dt \right]^{2/p}. \tag{3.8}$$

In view of (3.7) and by Lebesgue's *Dominated Convergence Theorem*, for the first term on the right-hand side of (3.8) we have

$$\lim_{n \to \infty} \left[\int_{-1}^{1} \left(\frac{f_n(t)}{\sqrt{\alpha'(t)} \sqrt{1-t^2}} \right)^p dt \right]^{(p-2)/p}$$
$$= \left[\int_{-1}^{1} \left(\frac{f_n(t)}{\sqrt{\alpha'(t)} \sqrt{1-t^2}} \right)^p dt \right]^{(p-2)/p}.$$

Therefore, applying Lemma 3.1 to the left-hand side of (3.8), inequality (3.6) follows immediately. ∎

Proof of Theorem 4. In what follows, for the sake of simplicity in the notations, we will write p_n for $p_n(W)$. Let $\{x_{jn}\}_{j=1}^{n}$ denote the zeros of p_n in decreasing order, and let $x_{0n}=1$ and $x_{n+1,n}=-1$. Then

$$\sup_{x\in[x_{k+1,n},\,x_{kn}]}\left[\frac{x_{kn}-x_{k+1,n}}{((\sqrt{1-x^2})/n)+(1/n^2)}\right]^{\pm 1}<\infty, \qquad k=0,1,...,n, \tag{3.9}$$

(cf. [11, Thm. 9.22, p. 166]). Let U_n be defined by

$$U_n(x)=\prod_{|y_k|<1}\left[|x-y_k|+\frac{1}{n}\right]^{A_k}\times\prod_{|y_k|=1}\left[|x-y_k|+\frac{1}{n^2}\right]^{A_k}. \tag{3.10}$$

Given $a>0$, define the set $\mathscr{E}_n(a)$ by

$$\mathscr{E}_n(a)=[-1,1]\Big\backslash\left\{\left[\bigcup_{|y_k|<1}\left[y_k-\frac{a}{n},y_k+\frac{a}{n}\right]\right]\cup\left[-1,-1+\frac{a}{n^2}\right]\cup\left[1-\frac{a}{n^2},1\right]\right\}. \tag{3.11}$$

Then, by (3.9), for every sufficiently small $0<a<a_0=a_0(U)$

$$\sup_{x\in\mathscr{E}_n(a)}\left[\frac{U(x)}{U_n(x)}\right]^{\pm 1}<\infty \quad\text{and}\quad \sup_{x,\,y\in[x_{k+1,n},\,x_{k-1,n}]}\left[\frac{U_n(x)}{U_n(y)}\right]<\infty. \tag{3.12}$$

In what follows we fix $0<a<1$ so that (3.12) holds. Even though it is natural to assume that $A_k>-1$ for $k=1,2,...,m$, in (1.4), the proof of (1.6) does not require this condition. As a matter of fact, in Part (ii) we will temporarily relax this assumption.

Part (i). First assume that $r\geqslant 0$, $p\geqslant 1$, and $0<c_1\leqslant g(x)\leqslant c_2<\infty$ for $x\in[-1,1]$. Then, since p_n vanishes at x_{kn}, we have

$$\begin{aligned}|p_n(x)|^{1+(r/p)}&\leqslant\left|\left(1+\frac{r}{p}\right)\int_{x_{kn}}^{x}|p_n(t)|^{r/p}\,|p_n'(t)|\,dt\right|\\&\leqslant\left(1+\frac{r}{p}\right)\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^{r/p}\,|p_n'(t)|\,dt\end{aligned}$$

for $x\in[x_{k+1,n},x_{k-1,n}]$, $k=1,2,...,n$. Thus, by Hölder's inequality

$$|p_n(x)|^{1+(r/p)}\leqslant\left(1+\frac{r}{p}\right)[x_{k-1,n}-x_{k+1,n}]^{1/q}\left[\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^{r}\,|p_n'(t)|^{p}\,dt\right]^{1/p},$$

where $(1/p)+(1/q)=1$; that is,

$$|p_n(x)|^{r+p}\,[x_{k-1,n}-x_{k+1,n}]^{1-p}\leqslant\left(1+\frac{r}{p}\right)^p\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^r\,|p_n'(t)|^p\,dt$$

for all $x_{k+1,n}\leqslant x\leqslant x_{k-1,n}$. Therefore, multiplying both sides by U_n and using the second inequality in (3.12), we obtain

$$\begin{aligned}&|p_n(x)|^{r+p}\,U_n(x)[x_{k-1,n}-x_{k+1,n}]^{1-p}\\&\quad\leqslant C_1\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^r\,|p_n'(t)|^p\,U_n(t)\,dt\end{aligned}\tag{3.13}$$

for all $x_{k+1,n}\leqslant x\leqslant x_{k-1,n}$ where $C_1=C_1(r,p,U)$. Now, by (3.9),

$$\begin{aligned}&[x_{k-1,n}-x_{k+1,n}]\frac{|p_n(x)|^{r+p}\,U_n(x)}{[((\sqrt{1-x^2})/n)+(1/n^2)]^p}\\&\quad\leqslant C_2\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^r\,|p_n'(t)|^p\,U_n(t)\,dt\end{aligned}\tag{3.14}$$

for all $x_{k+1,n}\leqslant x\leqslant x_{k-1,n}$ where $C_2=C_2(r,p,U,W)$. Integrating the latter over $[x_{k+1,n},x_{k-1,n}]$, we obtain

$$\begin{aligned}&\int_{x_{k+1,n}}^{x_{k-1,n}}\frac{|p_n(x)|^{r+p}\,U_n(x)\,dx}{[((\sqrt{1-x^2})/n)+(1/n^2)]^p}\\&\quad\leqslant C_2\int_{x_{k+1,n}}^{x_{k-1,n}}|p_n(t)|^r\,|p_n'(t)|^p\,U_n(t)\,dt\end{aligned}\tag{3.15}$$

for $k=1,2,\ldots,n$. Let $1_{\mathscr{E}}$ denote the characteristic function of the set $\mathscr{E}$, let $\Delta\subset[-1,1]$ be a fixed interval, and let $D_n(a)$ be the set defined by

$$D_n(a)=\bigcup\{[x_{k+1,n},x_{k-1,n}]:[x_{k+1,n},x_{k-1,n}]\subseteq\mathscr{E}_n(a)\}.$$

Then, adding together the inequalities (3.15) for all k such that $[x_{k-1,n},x_{k+1,n}]\subseteq D_n(a)\cap\Delta$, we obtain

$$\begin{aligned}&\int_{-1}^{1}\frac{|p_n(x)|^{r+p}\,1_{D_n(a)\cap\Delta}(x)\,U_n(x)\,dx}{[((\sqrt{1-x^2})/n)+(1/n^2)]^p}\\&\quad\leqslant C_2\int_{D_n(a)\cap\Delta}|p_n(t)|^r\,|p_n'(t)|^p\,U_n(t)\,dt.\end{aligned}\tag{3.16}$$

Since $D_n(a) \subseteq \mathscr{E}_n(a)$, we can use the first relation in (3.12) to replace U_n by U in the right-hand side of (3.16). Therefore,

$$\int_{-1}^{1} \frac{|p_n(x)|^{r+p} 1_{D_n(a) \cap \Delta}(x) U_n(x) \, dx}{[((\sqrt{1-x^2})/n) + (1/n^2)]^p} \leqslant C_3 \int_{-1}^{1} |p_n(t)|^r |p_n'(t)|^p 1_{\Delta(t)} U(t) \, dt, \tag{3.17}$$

where $C_3 = C_3(r, p, U, W)$. Note that

$$\lim_{n \to \infty} \frac{1_{D_n(a) \cap \Delta}(x) U_n(x)}{[\sqrt{1-x^2} + 1/n]^p} = \frac{1_\Delta(x) U(x)}{(1-x^2)^{p/2}} \qquad \text{for a.e.} \quad x \in [-1, 1].$$

Hence, by (3.17) and Lemma 3.2, inequality (1.6) follows when $r \geqslant 0$, $p \geqslant 1$, and $0 < c_1 \leqslant g(x) \leqslant c_2 < \infty$ for $x \in [-1, 1]$.

Part (ii). If $r \geqslant 0$, $p > 0$, $0 < c_1 \leqslant g(x) \leqslant c_2 < \infty$ for $x \in [-1, 1]$, and $\int_0^1 \omega(g, t) t^{-1} \, dt < \infty$, then the proof goes along the same lines except that one also needs a pointwise estimate for p_n' which requires the assumption concerning the modulus of continuity of g in (1.5). We start with (3.13) applied with $p = 1$, that is, with

$$|p_n(x)|^{r+1} U_n(x) \leqslant C_4 \int_{x_{k+1,n}}^{x_{k-1,n}} |p_n(t)|^r |p_n'(t)| U_n(t) \, dt \tag{3.18}$$

for all $x_{k+1,n} \leqslant x \leqslant x_{k-1,n}$, $k = 1, 2, ..., n$, where $C_4 = C_4(r, U)$. Given $0 < p < 1$, the next step is to apply (3.18) with U^* instead of U where

$$U^*(x) = U(x)(1-x^2)^{3(1-p)/4} W(x)^{(1-p)/2}.$$

As mentioned before in Part (i), at this point it is not necessary to assume $U^* \in L_1([-1, 1])$. We obtain

$$|p_n(x)|^{r+1} U_n^*(x) \leqslant C_4 \int_{x_{k+1,n}}^{x_{k-1,n}} |p_n(t)|^r |p_n'(t)|^p |p_n'(t)|^{1-p} U_n^*(t) \, dt \tag{3.19}$$

for all $x_{k+1,n} \leqslant x \leqslant x_{k-1,n}$, $k = 1, 2, ..., n$. Now we eliminate $|p_n'(t)|^{1-p}$ in (3.19) by using the estimate

$$|p_n'(t)|^{1-p} U_n^*(t) \leqslant C_5 n^{1-p} U_n(t), \qquad -1 \leqslant t \leqslant 1, \tag{3.20}$$

where $C_5 = C_5(p, W)$. Inequality (3.20) follows from Badkov's pointwise estimates for generalized Jacobi polynomials [1, Thm. 1.1, p. 226] (cf. [11, Lemma 9.29 and Thm. 3.33, p. 170–171]) and from some weighted

Bernstein–Markov type inequalities for generalized Jacobi weights (cf. [11, Thm. 9.19, p. 164]). Applying (3.20) to (3.19) leads to

$$|p_n(x)|^{r+1}\, U_n^*(x) \leqslant C_6 n^{1-p} \int_{x_{k+1,n}}^{x_{k-1,n}} |p_n(t)|^r\, |p_n'(t)|^p\, U_n(t)\, dt \qquad (3.21)$$

for all $x_{k+1,n} \leqslant x \leqslant x_{k-1,n}$, $k = 1, 2, ..., n$, where $C_6 = C_6(r, U, W)$. Inequality (3.21) is analogous to (3.13) except that this time $0<p<1$ as opposed to $p \geqslant 1$ in (3.13). For the case $0<p<1$ the rest of the proof is now completely analogous to how we proved (1.6) from (3.13) for $p \geqslant 1$ in Part (i). We do not elaborate on the details. ∎

4. Proofs of Theorems 1–3

Proof of Theorem 1. First we will prove (ii) ⇒ (i). Let R_{2n-1} be the polynomial of degree at most $2n-1$ that satisfies (2.22). Then

$$\|F_n(f)-f\|_{u,p} \leqslant 2^{1/p}\, \|F_n(f-R_{2n-1})\|_{u,p} + 2^{1/p}\, \|R_{2n-1}-f\|_{u,p}.$$

In view of Lemma 2.6, we get

$$\|F_n(f-R_{2n-1})\|_{u,p} \leqslant K \log n\, \|f-R_{2n-1}\|_\infty + K\, \|f'-R'_{2n-1}\|_\infty$$
$$\leqslant K\left(\frac{\log n}{n}+1\right) E_{2n-2}(f') \xrightarrow[n\to\infty]{} 0$$

which gives (i).

Now we will prove (i) ⇒ (ii). Let s_n be the cubic spline function defined by (2.19). By the uniform boundedness principle, it follows from (i), (2.20), and (2.21) that

$$\sup_{n\in\mathbb{N}} \|F_n(s_n)\|_{u,p} \leqslant K < \infty.$$

By (2.1), (2.3), and (2.19),

$$\|F_n(s_n)\|_{u,p} = \|G_n(s_n')\|_{u,p} = \left\| \sum_{k=1}^{n} (x-x_{kn})\, l_{kn}(x)^2 \right\|_{u,p}.$$

Therefore, according to formula (74) in the proof of [14, Theorem 3, p. 52], Lemma 2.9, and (2.12), we get

$$\sup_{n\in\mathbb{N}} \int_{(1+x_{1n})/2}^{1} \left|\frac{p_n^2(x)}{n}\right|^p u(x)\, dx \leqslant K < \infty,$$

so that by (2.11)

$$\sup_{n \in \mathbb{N}} w(1-n^{-2})^{-p} \int_{(1+x_{1n})/2}^{1} u(x)\,dx \leqslant K < \infty.$$

Writing $w(x)=g(x)(1-x)^a(1+x)^b$ and $u(x)=(1-x)^c(1+x)^d$, by (2.7) the above inequality is equivalent to $n^{2(ap-c-1)} \leqslant K$ for $n=1, 2, \ldots$, so that $(1-x^2)\,u(x) \leqslant Kw^p(x)$ for $x \in [0, 1]$. The proof of the latter inequality for $x \in [-1, 0]$ is analogous. ▌

Proof of Theorem 2. First we will prove (ii) ⇒ (i). Obviously, we only need to consider $\|F_n(f-R_{2n-1})\|_{u,p}$ where R_{2n-1} satisfies (2.22). By Lemma 2.7, we have

$$\|F_n(f-R_{2n-1})\|_{u,p} \leqslant K\,\|f-R_{2n-1}\|_\infty + \frac{K}{n}\,\|f'-R'_{2n-1}\|_\infty \leqslant \frac{K}{n}\,E_{2n-2}(f').$$

Now we will prove (i) ⇒ (ii). Let s_n be the cubic spline function defined by (2.19), and let $f_1(x)=x$. Then by (i), (2.20) and (2.21),

$$\begin{aligned}
&\|f_1F_n(s_n)-F_n(f_1s_n)\|_{u,p}\\
&\quad\leqslant 2^{1/p}\,\|f_1(F_n(s_n)-s_n)\|_{u,p} + 2^{1/p}\,\|f_1s_n-F_n(f_1s_n)\|_{u,p}\\
&\quad\leqslant \frac{K}{n}\,E_{2n-2}(s_n') + \frac{K}{n}\,E_{2n-2}((f_1s_n)') \leqslant \frac{K}{n}\,\|s_n'\|_\infty + \frac{K}{n}\,\|(f_1s_n)'\|_\infty\\
&\quad\leqslant \frac{K}{n}\,\|s_n\|_\infty + \frac{K}{n}\,\|s_n'\|_\infty \leqslant \frac{K}{n}, \qquad n \in \mathbb{N}.
\end{aligned}$$

Hence, by (2.1) and (2.3), we have

$$\left\| \sum_{k=1}^{n} (x-x_{kn})^2\, l_{kn}^2(x) \right\|_{u,p} = \|f_1F_n(s_n)-F_n(f_1s_n)\|_{u,p} \leqslant \frac{K}{n}, \qquad n \in \mathbb{N},$$

and, by (2.4) and (2.12),

$$\sup_{n \in \mathbb{N}} n\,\|p_n^2\|_{u,p} \sum_{k=1}^{n} \lambda_{kn}^2\, p_{n-1}^2(x_{kn}) \leqslant K < \infty.$$

By Lemma 2.9, $\sup_{n \in \mathbb{N}} \|p_n^2\|_{u,p} \leqslant K$, and the integrability of $w_1^{-p}u$ follows from [8, Thm. 2, p. 317] (or, equivalently, from Lemma 3.2 applied with $f_n \equiv f$).[9] ▌

[9] Since the results in [7, 8] hold for more general weights, this also justifies the remark following Theorem 2.

Proof of Theorem 3. First we will prove (ii) ⇒ (i). By Lemmas 2.8 and 2.7 applied with $(1-x^2)^{-p/2}u$ in place of u, we have

$$\|F_n'(f)\|_{u,p} \leqslant Kn\,\|F_n(f)(1-x^2)^{-1/2}\|_{u,p} \leqslant Kn\,\|f\|_\infty + K\,\|f'\|_\infty, \qquad n \in \mathbb{N}.$$

Therefore, choosing R_{2n-1} as in (2.22),

$$\begin{aligned}\|F_n'(f-R_{2n-1})\|_{u,p} &\leqslant Kn\,\|f-R_{2n-1}\|_\infty + K\,\|f'-R_{2n-1}'\|_\infty\\ &\leqslant KE_{2n-2}(f'), \qquad n\in\mathbb{N},\end{aligned}$$

and (i) follows.

Now we will prove (i) ⇒ (ii). Let s_n be again given by (2.19), and let $f_1(x)=x$. Then by (2.1) and (2.3),

$$f_1(x)\,F_n(s_n,x) - F_n(f_1 s_n, x) = \sum_{k=1}^{n} (x-x_{kn})^2\, l_{kn}^2(x),$$

so that by (2.4)

$$f_1(x)\,F_n(s_n,x) - F_n(f_1 s_n, x) = \frac{\gamma_{n-1}^2}{\gamma_n^2}\, p_n^2(x) \sum_{k=1}^{n} \lambda_{kn}^2 p_{n-1}^2(x_{kn}).$$

Differentiating this formula, we obtain

$$\begin{aligned}&F_n(s_n,x) + f_1(x)\,F_n'(s_n,x) - F_n'(f_1 s_n, x)\\ &\quad = 2\,\frac{\gamma_{n-1}^2}{\gamma_n^2}\, p_n(x)\, p_n'(x) \sum_{k=1}^{n} \lambda_{kn}^2 p_{n-1}^2(x_{kn}).\end{aligned} \tag{4.1}$$

Let $\Delta \subset (-1,1)$ be a fixed closed interval. Since, by Lemma 2.2, $F_n(s_n)$ converges uniformly to 0 on Δ, we have

$$\lim_{n\to\infty} \|1_\Delta F_n(s_n)\|_{u,p} = 0. \tag{4.2}$$

Condition (i) in Theorem 3 and Eq. (2.21) imply that

$$\limsup_{n\to\infty} \|1_\Delta[f_1 F_n'(s_n) - F_n'(f_1 s_n)]\|_{u,p} \leqslant K \tag{4.3}$$

with a constant K independent of Δ. By Lemma 2.9 and (2.12),

$$\liminf_{n\to\infty} n \sum_{k=1}^{n} \lambda_{kn}^2 p_{n-1}^2(x_{kn}) > 0. \tag{4.4}$$

Putting together the pieces (4.1)–(4.4), we obtain

$$\limsup_{n \to \infty} n^{-1} \| 1_\Delta p_n p_n' \|_{u,p} \leqslant K,$$

so that by Theorem 4,

$$\| 1_\Delta w_2^{-p} \|_{u,p} \leqslant K$$

with a constant K independent of Δ. Applying this with $\Delta = \Delta_m = [-1 + m^{-1}, 1 - m^{-1}]$ and letting $m \to \infty$, the integrability of $w_2^{-p} u$ follows from Fatou's Lemma. ∎

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