A Partial Order in Completely Regular Semigroups

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1. INTRODUCTION

Burgess and Raphael [4] have drawn attention to the class of weakly separative semigroups $S$, i.e., those in which, for all $a, b \in S$, one has

$$asa = ash = has = bsa = bsh$$

for all $s \in S$ only if $a = b$.

Extending work of Conrad [8] for rings, they showed [4, p. 134, Proposition 3] that Conrad's relation, denoted here by $\mathcal{C}$ and defined by

$$a \mathcal{C} b \iff asa = ash = has$$

is a partial order on $S$ (qua set) iff $S$ is weakly separative. Moreover, weakly separative semigroups occur quite widely: for example, every semigroup regular in the sense of von Neumann is weakly separative, as is every semigroup with a "proper involution" and every semiprime ring.

Another widely applicable natural partial order (denoted here by $\mathcal{N}$) has recently been introduced by Nambooripad [12] and independently by Hartwig [11], who define $a \mathcal{N} b$ for given $a, b \in S$ iff there is some $x \in S$ such that

$$a = axa = axb = bxa,$$

and who prove [12, p. 249, Proposition 1.1; 11, p. 3, Theorem 1] that $\mathcal{N}$ is a partial order on $S$ (qua set) iff $S$ is regular (in which case obviously $\mathcal{C}$ implies $\mathcal{N}$ on $S$).

Even before these first studies of $\mathcal{C}$, $\mathcal{N}$, Sussman [14, p. 327, Theorem 2.6] (see also [1, 6]) showed that, in certain rings without non-zero nilpotent elements, the relation $a^2 = ab$ is a partial order, and that (in these rings) $a^2 = ab$ implies $a^4 = ab = ba$. In the present article we adapt
and extend this idea by introducing, for consideration in arbitrary semigroups, the relation $\mathcal{P}$ defined by

$$a\mathcal{P}b \text{ iff } a^2 = ab = ba.$$ 

After also introducing a closely connected new semigroup property (somewhat stronger than weak separativity) which we call "quasi-separativity," we show in Section 2 that $\mathcal{P}$ is a partial order on a given finite semigroup $S$ (qua set) iff $S$ is completely regular, and in Section 3 that $\mathcal{C} \Rightarrow \mathcal{P} \Rightarrow \mathcal{N}$ on all such $S$. On commutative completely regular semigroups, the three partial orders coincide; however (see Example 2), there is a completely regular semigroup, of order 4, on which $\mathcal{C}$, $\mathcal{P}$, $\mathcal{N}$ are distinct partial orders.

In Section 4 we explore the logical implications holding among the various species of separativity and certain other familiar semigroup properties, thereby obtaining a useful hierarchical classification of the weakly separative semigroups (see Figs. 3 and 4).

All of our arguments in this article are very brief and elementary, even to the extent that, taken separately, none of the results deserves to be described as a theorem. However, some of our "propositions" are surprising, and collectively they may provide a useful basis for deeper investigations.

2. Properties of $\mathcal{P}$

We recall [13, p. 51] that a semigroup $S$ is called separative iff

(i) $a^2 = ab$ and $ba = b^2$ together imply $a = b$

and

(ii) $a^2 = ba$ and $ab = b^2$ together imply $a = b$.

Just as weak separativity and regularity are, respectively, exactly what is needed to ensure that $\mathcal{C}$, $\mathcal{N}$ are partial orders, we now note a corresponding property (formally very much like separativity) of a semigroup $S$ which is (at least in the finite case) necessary and sufficient for $\mathcal{P}$ to be a partial order on the set $S$: we call $S$ quasi-separative iff, for all $a, b \in S$, we have that

$$a^2 = ab = ba = b^2 \text{ only if } a = b.$$ 

Obviously separativity implies quasi-separativity (and, by taking $b = 0$ in the definition of quasi-separativity, clearly separativity and quasi-separativity are the same for the multiplicative semigroup of any associative ring), but, in view of the left zero semigroup of order 2 (or Example 1), the two properties do not in general coincide, even on bands.
Twelve years before Petrich's use of the term, Clifford and Preston [7, p. 136, Exercise 7] had already used "separativity" in another (asymmetrically defined) sense, and we have chosen here to resolve this terminological ambiguity in favor of Petrich, who (using work of Buhmistrovich [5] and others) presented [13, Sect. II.6 and Corollary III.7.6, pp. 50-54, 97] several general structural results for the noncommutative case. However, there is no need to introduce any further word or phrase to refer to the property discussed by Clifford and Preston, since, as we show next, in fact their property is equivalent to quasi-separativity:

**Proposition 1.** If $S$ is any quasi-separative semigroup, then, for all $x, y \in S$, we have that

$$x^2 = xy = y^2 \iff x = y$$

(and, of course, conversely).

**Proof.** If $x^2 = xy = y^2$, then

$$(xy)^2 = (x^2)^2 = x^4,$$

$$(xy)(yx) = xy^2x = xx^2x = x^4,$$

$$(yx)(xy) = yx^2y = yy^2y = (y^2)^2 = (x^2)^2 = x^4,$$

and

$$(yx)^2 = y \cdot xy \cdot x = yy^2x = y^2yx = x^2yx = x \cdot x^2 \cdot x = x^4.$$  

Thus $(xy)^2 = (xy)(yx) = (yx)(xy) = (yx)^2$, whence, by quasi-separativity, $xy = yx$, so we have $x^2 = xy = yx = y^2$, whence, by quasi-separativity again, in fact $x = y$.]

In any case, the previous use of the word "separativity" in two different senses can hardly have caused serious misunderstanding, since, in their discussions involving separativity (i.e., quasi-separativity in our terminology), Clifford and Preston [7] were, except in their Exercise 7 on p. 136, concerned only with commutative semigroups (on which the properties of separativity and quasi-separativity obviously coincide). Any extension, beyond the previously cited material to be found in [13], of the results of [7, pp. 132-136, 198-200, 206] to the noncommutative case (e.g., by using either separativity or quasi-separativity) would be of considerable interest.

**Proposition 2.** If $S$ is a quasi-separative semigroup, then $S$ is also weakly separative.
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Proof. If \(a, b \in S\) satisfy \(asa = ash = bsa = bsh\) for all \(s \in S\), then

\[(as)^2 = as \cdot s = asb \cdot s = (as)(bs),\]

and similarly \((as)^2 = (bs)(as)\) and \((as)^2 = (bs)^2\). Hence, by quasi-separativity, \(as = bs\) for all \(s \in S\); in particular, the choices \(s = a, b\) yield \(a^2 = ba, ab = b^2\), whence also, by left-right symmetry, \(a^2 = ab\) and \(ba = b^2\). Thus in fact \(a^2 = ab = ba = b^2\), and so \(a = b\) by quasi-separativity. 

We recall that a semigroup \(S\) is called completely regular (see, e.g., [13, p. 104]) iff \(a \in \langle a^2S \rangle \cap \langle Sa^2 \rangle\) for every \(a \in S\). Besides its relevance to semigroup theory, this property has also been studied and widely applied (under the name "strong regularity") from the point of view of ring theory (see, e.g., [2, Sect. 3, pp. 462–464; 3]).

PROPOSITION 3. Every completely regular semigroup is quasi-separative.

Proof. Let \(a, b \in S\) satisfy \(a^2 = ab = ba = b^2\). If \(S\) is completely regular, then \(a \in Sa^2\) and \(b \in b^2S\), i.e., there exist \(x, y \in S\) such that \(a = xa^2\) and \(b = b^2y\). Also obviously \(ab^2 = ab \cdot b = a^2b\), and so (cf. [10, Proposition 6.14])

\[
a = xa^2 = x \cdot ab = xa \cdot b = xa \cdot b^2y = x \cdot ab^2 \cdot y = x \cdot a^2b \cdot y = xa^2 \cdot by = aby = b^2y = b.
\]

For finite semigroups, the converse of Proposition 3 holds. Indeed, we next prove this for the wider class of strongly \(\pi\)-regular semigroups, where we recall that \(S\) is called strongly \(\pi\)-regular iff, for each \(a \in S\), there exist positive integers \(m, n\) such that

\[a^m \in a^{m+1}S \quad \text{and} \quad a^n \in Sa^{n+1}.
\]

Of course complete regularity is just the special case of this where one may take \(m = n = 1\) for every \(a \in S\).

PROPOSITION 4. Every quasi-separative, strongly \(\pi\)-regular semigroup is completely regular.

Proof. Let \(S\) be any strongly \(\pi\)-regular semigroup. As was shown in [3, p. 37, Theorem 3] (cf. also [9, p. 510, Theorem 4]), for any \(a \in S\), if \(a^m \in a^{m+1}S\) and \(a^n \in Sa^{n+1}\), then there exists \(x \in S\) such that

\[ax = xa \quad \text{and} \quad a^k = a^{k+1}x,
\]

where \(k = \max(m, n)\).
Given \((l_k)\) for \(a, x\) and any fixed \(k > 1\), we can apply quasi-separativity to deduce the corresponding statement for \(a, x, k - 1\). Explicitly, since \(ax = xa\), we have
\[
(a^k x)^2 = a^{k-1} \cdot a^{k+1} x \cdot x = a^{k-1} \cdot a^k \cdot x = a^{k-1} (a^k x),
\]
while also
\[
a^{k-1} (a^k x) = a^{k-2} \cdot a^{k+1} x = a^{k-2} \cdot a^k = (a^{k-1})^2,
\]
so \((a^k x)^2 = a^{k-1} (a^k x) = (a^k x) a^{k-1} = (a^{k-1})^2\), and, by quasi-separativity, consequently \(a^{k-1} = a^k x\), i.e., \((l_{k-1})\) holds.

Hence, by downward induction on \(k\), statement \((l_k)\) must hold for \(k = 1\), i.e., we have \(ax = xa\) and \(a = a^2 x\), so \(u \in (U'S) \cap (Sa^2)\) and \(S\) is completely regular.

**Corollary 1.** For strongly \(\pi\)-regular (e.g., finite) semigroups, the properties of complete regularity and quasi-separativity coincide.

For a given finite semigroup \(S\), it is usually a somewhat laborious procedure (without electronic aids) to test directly for complete regularity, whereas quasi-separativity (largely because it is quadratic, and involves no existential quantifier) can be decided rather quickly by inspection of the multiplication table of \(S\). Possibly there is no entirely satisfactory alternative description of the class of quasi-separative semigroups without finiteness conditions, but Propositions 3 and 4 at least make a start in this direction (and certainly not every quasi-separative semigroup is strongly \(\pi\)-regular or completely regular—consider the multiplicative semigroup \(\mathbb{Z}\) of integers).

**Proposition 5.** If \(S\) is any completely regular semigroup, then \(\mathcal{F}\) is a partial order on the set \(S\).

**Proof.** Obviously \(\mathcal{F}\) is always reflexive, and the quasi-separativity of \(S\) merely states the antisymmetry of \(\mathcal{F}\). Hence, by Proposition 3, we need only show that \(\mathcal{F}\) is transitive on every completely regular semigroup \(S\).

So let \(a, b, c \in S\) satisfy \(a \mathcal{F} b \mathcal{F} c\), i.e.,
\[
a^2 = ab = ba, \quad b^2 = bc = cb,
\]
so that \(a^4 - (a^2)^2 - (ba)^2 = b^2 a^2 = cb \cdot a^2 = c \cdot ba \cdot a = ca^3\).

Also, if \(S\) is completely regular, there exists \(x \in S\) such that \(a = a^2 x\), and so \(a = a \cdot a^2 x \cdot x = a^3 x^2\), whence \(a^2 = a^4 x^2 = ca^3 x^2 = ca\), and similarly \(a = ac\), i.e., \(a \mathcal{F} c\).
Corollary 2. If $S$ is strongly $\pi$-regular, then $\mathcal{S}$ is a partial order on the set $S$ iff $S$ is completely regular iff $S$ is quasi-separative.

It is easy to verify that, in any strongly $\pi$-regular semigroup $S$, the arguments for Propositions 3, 4, and 5 extend to show that $\mathcal{S}$ is a partial order on the set of all completely regular elements (i.e., those for which we may take $m = n = 1$). It would be of interest to obtain an analogous partial order applying throughout strongly $\pi$-regular semigroups, but this seems to be impossible. In particular, although it is easy to see that, in any completely regular semigroup, $ab \in \mathcal{S}$ iff $(b^* a =) a^* b = a^* a = a a^* b = b a^* (= a b^*)$, where $a^*$ denotes the “group inverse” of $a$, and although one can, formally, extend this relation to arbitrary strongly $\pi$-regular semigroups by replacing $a^*$ by the pseudo-inverse $a'$ (see [9]), unfortunately this does not in general yield a partial order (since antisymmetry fails, e.g., on nonzero nilpotent elements).

As was noted by Burgess and Raphael [4, p. 134, Proposition 31], Conrad’s relation $\mathcal{C}$ always admits multiplication, i.e., $a \mathcal{C} b$ implies $(ua) \mathcal{C} (ub)$ and $(av) \mathcal{C} (bv)$ for all $u, v \in S$. However, even on completely regular semigroups, the corresponding statement about $\mathcal{S}$ is false:

Example 1. If $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $E^2 = EF = E$, $FE = F^2 = F$, so that $\{I_2, E, F\}$ is a band (hence, by Propositions 3 and 2, both quasi-separative and weakly separative), in which $E \mathcal{S} I_2$, but $(EF) \mathcal{S} (I_2 F)$ is false. (Incidentally, since $\mathcal{S} = \mathcal{N}$ on this semigroup, the same example shows that $\mathcal{N}$ need not be compatible with multiplication.)

3. Connections between $\mathcal{C}$, $\mathcal{S}$, $\mathcal{N}$

We have already noted that $\mathcal{C}$ implies $\mathcal{N}$ in regular semigroups $S$ (i.e., for all $a, b \in S$, we have that $a \mathcal{C} b$ implies $a \mathcal{N} b$). We show next that, for completely regular $S$, one can interpolate $\mathcal{S}$ between $\mathcal{C}$ and $\mathcal{N}$:

Proposition 6. Let $S$ be any completely regular semigroup. Then $\mathcal{C}$ implies $\mathcal{S}$ on $S$, and $\mathcal{S}$ implies $\mathcal{N}$.

Proof. (i) Let $a, b \in S$ satisfy $a \mathcal{C} b$, i.e., $asa = ash = bsa$ for all $s \in S$. As in the proof of Proposition 4, since $S$ is completely regular, there exists $x \in S$ such that $a = axa$ and $ax = xa$. Hence (with $s = x$) we have

$$a^2 = a \cdot axa = a \cdot axb = a \cdot xa \cdot b = ab,$$

and similarly $a^2 = ba$, i.e. $a \mathcal{S} b$. 
(ii) Let $a, b \in S$ satisfy $aSb$, i.e., $a^2 = ab = ba$. Then, with $x$ as in (i), we have $axa = a$, $axb = xab = xa^2 = axa = a$, and similarly $bxa = a$, so that $a[N b$.

For completely regular $S$, it is easy to see that $\mathcal{C} = \mathcal{S} = \mathcal{N}$ whenever $S$ is also commutative. However, for noncommutative completely regular $S$, we have already seen (in Example 1) that $\mathcal{C}, \mathcal{S}$ need not coincide (even on bands).

**Example 2.** The completely regular $2 \times 2$ integer matrix semigroup

$$\{I_2, D, E, F\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right\},$$

with Hasse diagrams as shown in Fig. 1, provides a case where also $\mathcal{S} \neq \mathcal{N}$.

It is easy to see that $\mathcal{S} = \mathcal{N}$ on every band, and on every completely regular inverse semigroup; also, for every completely regular ring $R$, the representation of $R$ as a subdirect product of division rings immediately yields $\mathcal{C} = \mathcal{S} = \mathcal{N}$ on $R$.

It is natural to ask whether any of the three partial orders $\mathcal{C}, \mathcal{S}, \mathcal{N}$ yields a lattice structure. However, since clearly every invertible element is maximal under each of $\mathcal{C}, \mathcal{S}, \mathcal{N}$, obviously no binary union (i.e. least upper bound) operation is available in any monoid having an invertible element other than 1.

**Example 3.** The $3 \times 3$ integer matrix semigroup

$$\{0, I, A, B, C\} = \left\{ 0, I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

![Hasse diagrams](image)

*Fig. 1. Hasse diagrams for $\mathcal{C}, \mathcal{S}, \mathcal{N}$ for Example 2.*
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Fig. 2. Hasse diagrams (for all of $\mathcal{C}$, $\mathcal{F}$, $\mathcal{N}$) for Example 3 and (for $\mathcal{F}$, $\mathcal{N}$) for Example 4.

(cf. [10, Sect. 10]), with Hasse diagram as in Fig. 2, shows also that no intersection (i.e., greatest lower bound) operation is generally definable (for any of $\mathcal{C}$, $\mathcal{F}$, $\mathcal{N}$) even on commutative completely regular semigroups.

It is easy to see that, in any commutative band, the operation $(a, b) \to ab$ satisfies the requirements for an intersection operation with respect to each of $\mathcal{C}$, $\mathcal{F}$, $\mathcal{N}$. However, without commutativity, even a band need not be a lower $\mathcal{F}$-semilattice:

**Example 4.** The $4 \times 4$ integer matrix band

$$\{0, J, K, L, M\} = \left\{0, \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix} \right\},$$

where $E, F$ are as in Examples 1 and 2, has the Hasse diagram for $\mathcal{F}$ (and $\mathcal{N}$) shown in Fig. 2.

It would also be of interest to obtain results about the Finiteness and/or Schröder-Bernstein properties of $\mathcal{C}$, $\mathcal{F}$, $\mathcal{N}$ with respect to Green's relations (see [10, Sects. 12 and 13], [12, Sect. 2, pp. 253–255]).

4. Connections between Quasi-separativity and Other Semigroup Properties

We have, above, discussed a variety of semigroup properties, and, in Propositions 1, 2, 3, and 4, have already noted certain implications between them. A knowledge of such implications is helpful in suggesting appropriate ways to try to sharpen preliminary results, and tends to enhance the viability of each individual property involved. In this con-
cluding section we shall briefly note some further implications, beginning with one already remarked on by Burgess and Raphael [4, p. 134]:

**Proposition 7.** Every regular semigroup is weakly separative.

*Proof.* Let $S$ be any regular semigroup, and let $a, b \in S$, say with $a = axa$, $b = byb$. Then, if $asa = asb = bsa = bsb$ for every $s \in S$, we have in particular $a = axa = axb$ and $ayb = byb = b$, so that

$$a = axb = ax \cdot byb = axb \cdot yb = ayb = b.$$

Proposition 7, together with other known facts, provides the implications shown in Fig. 3. In detail, the three implications proceeding up the left side of Fig. 3 are respectively established in Propositions 6.14, 6.13(5) and 7.1 of [10], while the three implications going vertically up the middle are respectively obvious, [2, p. 463, Theorem 3.21, and Proposition 7, and the three implications proceeding up the right side are, respectively, an immediate consequence of Proposition 3, obvious, and Proposition 2. This leaves only the two north-eastward-pointing arrows at the middle level of the figure, of which the left arrow is obvious, and the other simply restates Proposition 3.

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**Fig. 3.** Implications between certain semigroup properties.
In the class of all semigroups, it is easy to find examples to show that none of the implications in Figure 3 is an equivalence; in fact only three such examples are needed, namely the multiplicative semigroup $\mathbb{Z}$, the semigroup

$$M = \{ \{1 \}, 2, 2: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} = \{ 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \}.$$

and the left zero semigroup of order 2. With only one exception, to be dealt with just below, these three examples also suffice to show that, between the eight properties appearing in Fig. 3, no other implications hold besides those which are implicit by transitivity. The exception noted is that, to show that not every separative semigroup has a (proper) involution, we need one further counter-example:

**Example 5.** The multiplicative semigroup

$$S = \left\{ \begin{pmatrix} 2^k & 0 \\ s & 1 \end{pmatrix} : k, s = 0, 1, 2, \ldots \right\}$$

(cf. [7, p. 36, Exercise 11]) consists of nonsingular matrices, hence is a cancellation semigroup, and in particular certainly separative. However, this $S$ admits no anti-automorphism.

To see this, let $p = (\frac{2^2}{s}, 0)$, $q = (\frac{1}{s}, 1)$, and note first that

$$\begin{pmatrix} 2^k & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = p^k q^s.$$

Thus $p, q$ generate the semigroup $S$, with every element of $S$ uniquely represented in the form $p^k q^s$, while also

$$q^t p^k = \begin{pmatrix} 1 \\ t \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^k & 0 \\ 2^k t & 1 \end{pmatrix} = p^k q^{2k}. \quad (k, t = 0, 1, 2, \ldots).$$

Now suppose that $S$ has some anti-automorphism $*: S \to S$, say with $p^* = p^m q^n$, $q^* = p^q q^r$. Then on the one hand

$$(qp)^* = p^* q^* = p^{m+q} q^{n+r} = p^m + q q^{2n+q},$$

while on the other hand

$$(pq)^* = (pq^2)^* = (q^*)^2 p^* = (p^n q^r)^2 p^m q^n = p^m + 2n q^n + 2n(1 + 2^n)q^n.$$

By uniqueness, we should then have $n = v = 0$, i.e., $q^* = 1$, so that $q = 1$, a contradiction.

Thus $S$ has no anti-automorphism, and in particular no involution. \[\square\]
The above provides a complete picture of the situation for semigroups without finiteness conditions, and immediately yields the stronger situation shown in Fig. 4 (with 25 implications rather than only 18 as in Fig. 3) for the subclass of strongly \(\pi\)-regular (in particular, finite) semigroups. For in this class quasi-separativity and complete regularity coincide by Corollary 1 above, while every strongly \(\pi\)-regular proper \(*\)-semigroup is regular by [10, Proposition 8.1], and we also have

**Proposition 8.** *Every strongly \(\pi\)-regular separative semigroup is inverse.*

![Diagram](image)
Proof. Let $S$ be strongly $\pi$-regular and separative. Then $S$ is quasi-separative, hence completely regular by Proposition 4, and in particular regular. Also, by [13, p. 54, Ex. 4], the idempotents of $S$ commute pairwise, and so, by [13, p. 159, Lemma V.4.5], $S$ is inverse. 1

Again, Fig. 4 is complete, i.e., no arrow can be reversed, and no further arrow can be inserted. This can largely be verified by using two of the same four examples already noted in connection with Figure 3, namely $\mathcal{M}_5$ and the left zero semigroup of order 2. Of course Example 5 and the semigroup $\mathbb{Z}$ are no longer applicable, but we may now use instead any nonabelian finite group to show that not every strongly $\pi$-regular (or finite) separative semigroup is commutative, and the multiplicative semigroup of $2\times2$ matrices over the integers modulo 3, with transposition as involution, to show that not every strongly $\pi$-regular (or even finite) proper $*$-semigroup is inverse. To show that not every strongly $\pi$-regular (or finite) weakly separative semigroup is regular, consider

Example 6.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Of the eight properties appearing in Fig. 3 (and Fig. 4), some already have well-known structural characterizations. It would be of interest to obtain, for the remaining properties, structure results which are mutually compatible, in the sense that all the implications in Figs. 3 and 4 become obvious consequences of the eight structures involved.

References

5. I. E. Burmistrovich, Commutative bands of cancellative semigroups, Siberian Mat. Z. 6 (1965), 284-299. [Russian]