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Measure in the 2D Regge quantum gravity

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Abstract

We propose a version of the 2D Regge calculus, in which the areas of all triangles are equal to each other. In this discretization Lund–Regge measure over link lengths is simplified considerably. Contrary to the usual Regge models with Lund–Regge measure, where this measure is nonlocal and rather complicated, the models based on our approach can be investigated using the numerical simulations in a rather simple way.

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One of the main difficulties in quantization of gravity is the wideness of its gauge group, that is the group of general coordinate transformations. Due to this wideness it is not possible to put the theory into a regular lattice and express it in terms of metrics without loss of gauge invariance. At the present moment the only gauge invariant discretization of quantum gravity is Regge calculus. However, the price for the explicit gauge invariance is the problems with the definition of measure. Here instead of metrics, which was local field in continuum, the lengths of the links are fundamental gauge invariant dynamical variables. Metrics is composed of these variables in a nonlocal way. Therefore, measure is also expected to be nonlocal. Although in [1] and related publications there were argued that certain choices of local measures over link lengths can survive as appropriate, the point of view of the author of the present Letter is different. Namely, we accept that the correct measure over link lengths must be constructed in the spirit of construction of the so-called Lund–Regge measure (see, for example, [2], where it is also called DeWitt like measure). As was mentioned above, this measure in general case is nonlocal. Actually, its form is so complicated, that it seems not possible to use it in real numerical simulations.

However, in the present Letter we suggest the way to overcome this difficulty, at least, in two dimensions. In order to do this we accept the compromise decision. Namely, we start discretization of gravity from the partially gauge fixed version of the theory, where the conformal mode is fixed via using diffeomorphism invariance. As a result, we propose a version of the Regge calculus, in which areas of all triangles are kept equal to each other.

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Within this discretization of gravity Lund–Regge measure is simplified considerably, which allows to perform real numerical investigation of the correspondent models using Monte Carlo methods.

To begin the description of our approach let us remind main facts about the measure in quantum gravity theories. First of all, what is the original space of continuous geometries? We suppose here that our basic space to be discretized is the set of all compact D -dimensional Riemannian manifolds of certain implied smoothness properties. There is the following metrics on this set.

$$\|\delta g\|^2 = \frac{1}{2} \int d^D x \sqrt{|g|} (g^{\mu\nu} g^{\rho\eta} + g^{\rho\nu} g^{\mu\eta} + C g^{\mu\rho} g^{\nu\eta}) \delta g_{\mu\rho} \delta g_{\nu\eta}. \quad (1)$$

It is well known that in finite-dimensional case any metrics (on the compact space) produces the only measure μ . In a few words the procedure of its construction works as follows. First we set a certain ϵ . Next, we put balls of radius ϵ into the space. These balls should be putted in such a way that they intersect each other only by their boundaries. They should be arranged is such a way that their total number is maximal. After that for any given set $\Omega \subset \Lambda$ (we denote the entire space by Λ) we count the number of balls situated inside Ω . Let us denote it as $n_\epsilon(\Omega)$. Finally

$$\mu(\Omega) = \lim_{\epsilon \rightarrow 0} \frac{n_\epsilon(\Omega)}{n_\epsilon(\Lambda)}. \quad (2)$$

In principle this procedure might be generalized to infinite-dimensional case. The spaces could be represented as a limit of finite-dimensional ones.¹ Namely, each Riemannian manifold could be thought of as a limit of a sequence of piecewise linear manifolds constructed of larger and larger number of simplices. This procedure is known as Regge discretization. The set of all Riemannian spaces is then thought of as a limit of the sequence of finite-dimensional ones (each that space corresponds to the given fixed triangulation with varying link lengths). Then the continuum measure should be the limit of measures on these finite-dimensional approximating spaces. Our aim now is to construct these finite-dimensional measures in such a way that the result corresponds to the norm (1).

First of all we note that (1) induces the so-called Lund–Regge norm on space of Regge skeletons with fixed triangulation [2]. To calculate it explicitly let us equip each simplex $[\Gamma_0 \Gamma_1 \dots \Gamma_D]$ with the basis $\{\frac{\Gamma_0 \Gamma_1}{|\Gamma_0 \Gamma_1|}, \frac{\Gamma_0 \Gamma_2}{|\Gamma_0 \Gamma_2|}, \dots, \frac{\Gamma_0 \Gamma_D}{|\Gamma_0 \Gamma_D|}\}$. In this basis metrics is

$$g_{ij} = \frac{1}{2} (|\Gamma_0 \Gamma_i|^2 + |\Gamma_0 \Gamma_j|^2 - |\Gamma_i \Gamma_j|^2). \quad (3)$$

Here $|\Gamma_i \Gamma_j|$ is the distance between points Γ_i and Γ_j . We can substitute (3) into (1) and obtain the form quadratic in variations of link lengths (we denote link lengths as a_i).

Before doing so let us rewrite (1) in the following way:

$$\begin{aligned} \|\delta g\|^2 &= \frac{1}{2} \int d^D x \sqrt{|g|} (g^{\mu\nu} g^{\rho\eta} + g^{\rho\nu} g^{\mu\eta} + C g^{\mu\rho} g^{\nu\eta}) \delta g_{\mu\rho} \delta g_{\nu\eta} \\ &= \int d^D x \sqrt{|g|} \left(-\delta g^{\mu\rho} \delta g_{\mu\rho} + \frac{C}{2} [\delta \log |g|]^2 \right). \end{aligned} \quad (4)$$

We used here, that $\delta |g| = |g| g^{ik} \delta g_{ik}$ and $g_{ji} \delta g^{ik} = -g^{ik} \delta g_{ji}$.

Let us represent g^{ij} as

$$g^{kl} = \frac{1}{(D-1)! g} \epsilon^{kab\dots d} \epsilon^{lvg\dots s} g_{av} g_{bg} \dots g_{ds}. \quad (5)$$

¹ This construction is, however, much more complicated and is not yet elaborated in sufficient details for most cases of interest (at least the author of the present Letter is not aware of the correspondent research).

Then, we substitute (5) into (4) and obtain:

$$\|\delta g\|^2 = \int d^D x \left[-\frac{\epsilon^{kab\dots d} \epsilon^{lv g\dots s} \delta g_{kl} \delta g_{av} g_{bg} \dots g_{ds}}{\sqrt{|g|} (D-2)!} + \frac{C+2}{2} (\delta \log |g|)^2 \sqrt{|g|} \right]. \tag{6}$$

Now let us introduce new variable $\hat{g}_{ij} = |g|^{-1/D} g_{ij}$. The determinant $|\hat{g}|$ is equal to unity by construction. The original field variable g_{ij} is, therefore, represented as a product of this new variable and the conformal factor $|g|^{1/D}$. As a result (6) can be rewritten as

$$\|\delta g\|^2 = \int d^D x \left[-\frac{\epsilon^{kab\dots d} \epsilon^{lv g\dots s} \delta \hat{g}_{kl} \delta \hat{g}_{av} \hat{g}_{bg} \dots \hat{g}_{ds}}{(D-2)!} + \frac{C + \frac{2}{D}}{2} (\delta \log |g|)^2 \right] \sqrt{|g|}. \tag{7}$$

Here we can see, that the conformal mode $|g|$ is orthogonal to the field \hat{g}_{ij} .

On the simplicial manifold expression (6) has the form:

$$\|\delta a\|^2 = \sum_{\text{simplices}} \left[-\frac{\epsilon^{kab\dots d} \epsilon^{lv g\dots s} \delta g_{kl} \delta g_{av} g_{bg} \dots g_{ds}}{\sqrt{|g|} (D-2)!} + \frac{C+2}{2} (\delta \log |g|)^2 \sqrt{|g|} \right]. \tag{8}$$

Here² at each simplex g_{ij} is expressed through link lengths a_m .

Collecting all terms quadratic in δa_m one might obtain the final result

$$\|\delta a\|^2 = \sum_{i,j \in C_1} O_{ij}[a] \delta(a_i^2) \delta(a_j^2), \tag{9}$$

with the matrix O , which (in general) depends on link lengths in rather complicated way. Further we shall show, however, that in two dimensions this expression can be greatly simplified if we impose rather restrictive but very natural constraint on the approximating Regge skeleton.

From (9) it follows that the resulting measure is given by³ [1,2]:

$$Da = \text{Det}^{\frac{1}{2}} O[a] \prod_{i \in C_1} da_i^2. \tag{10}$$

Expression (10) was called in [2] DeWitt like measure. However, we guess it more appropriate to call it Lund–Regge measure, as it is derived from the expression for metrics, which was given first by Lund and Regge in their unpublished preprint. As we already told, unfortunately, in general case, (10) is not suitable for practical calculations due to its nonlocal and rather complicated form. The main purpose of this Letter is to show that in two dimensions the considerable simplification may be achieved, which allows to use Lund–Regge measure in real computer simulations.

So, let us turn to the two-dimensional case. In order to set up the model, let us first come back to the continuum formulation. Our point is that before discretizing the theory we perform partial gauge fixing. The gauge condition is that the conformal mode $|g|$ has a value, which does not depend upon the point of the surface:

$$\sqrt{|g(x)|} = \sqrt{|g(x_0)|}, \tag{11}$$

² It is worth mentioning that in a similar way expression (7) can be transferred to the simplicial manifold. However, there the main property of (7) is lost. Namely, on the lattice again we may introduce new variables (instead of the link lengths). These new variables are triangle areas and the remaining angle variables. Naively one would expect, that in analogy with \hat{g}_{ij} and $|g|$ the subspaces spanned on these two sets of variables are orthogonal. But this is not true, because now the induced \hat{g}_{ij} on the simplices functionally depend both upon mentioned angle variables and on the areas of the triangles.

³ In this Letter we denote the set of links of the lattice as C_1 and the set of triangles as C_2 . The number of sites is denoted as N_0 , the number of triangles is denoted as N_2 , the number of links is denoted as N_1 .

where $g(x)$ is the metric tensor, while x_0 is the fixed point of the surface. The Faddeev–Popov procedure (when applying to the partition function of the model with the action $S[g]$) gives

$$\begin{aligned} Z &= \int Dg \exp(-S[g]) \int Df \delta\left(\log \sqrt{|g^f|} - \log \sqrt{|g^f(x_0)|}\right) \Delta_{\text{FP}}[g] \\ &= \int Dg^{f^{-1}} Df \exp(-S[g^{f^{-1}}]) \delta\left(\log \sqrt{|g|} - \log \sqrt{|g(x_0)|}\right) \Delta_{\text{FP}}[g^{f^{-1}}] \\ &= \int Dg \exp(-S[g]) \delta\left(\log \sqrt{|g|} - \log \sqrt{|g(x_0)|}\right) \Delta_{\text{FP}}[g]. \end{aligned} \quad (12)$$

Here⁴ the reparametrization is denoted as f , and its action on metrics is denoted as g^f . We take into account that the action $S[g]$, the measure Dg , and the Faddeev–Popov determinant $\Delta_{\text{FP}}[g]$ are reparametrization invariant.

The Faddeev–Popov determinant is expressed as (the reparametrization is written here as $x \rightarrow x - y(x)$):

$$\begin{aligned} \Delta_{\text{FP}}^{-1}[g]_{\sqrt{|g(x)|}=\sqrt{|g(x_0)|}} &= \int Dy \delta\left(\log \frac{|g(x - y(x))|^{1/2} (1 + \frac{1}{2} \partial_\mu y^\mu(x))}{|g(x_0 - y(x_0))|^{1/2} (1 + \frac{1}{2} \partial_\mu y^\mu(x_0))}\right) \\ &= \int Dy \delta\left(\log \frac{1 + \frac{1}{2} \partial_\mu y^\mu(x)}{1 + \frac{1}{2} \partial_\mu y^\mu(x_0)}\right) = \text{const.} \end{aligned} \quad (13)$$

Thus, $\Delta_{\text{FP}}(g)$ does not depend upon metrics.

So, we start discretization of quantum gravity from the model with the partition function

$$Z = \int Dg \exp(-S[g]) \delta\left(\log \sqrt{|g(x)|} - \log \sqrt{|g(x_0)|}\right). \quad (14)$$

In Regge discretization of (14) we use Lund–Regge measure (10) instead of Dg . Instead of the delta function $\delta(\log \sqrt{|g(x)|} - \log \sqrt{|g(x_0)|})$ we use its discretized version:

$$\Theta[a] = \prod_{J \in C_2, J \neq J_0} \delta(\log A^J - \log A^{J_0}), \quad (15)$$

where the product is over the triangles (all but the given triangle J_0) of the simplicial manifold and A^J is the area of the triangle. Equivalently, we can represent the last expression in the following form:

$$\Theta[a] = \int \prod_{J \in C_2} \delta(\log A^J - \log(V/N_2)) \frac{dV}{V}, \quad (16)$$

where V is the overall invariant volume.

So, the partition function of the discretized model is

$$Z = \int Da \exp(-S[a]), \quad (17)$$

where $S[a]$ is the discretized action and Da is the measure:

$$Da = \left\{ \text{Det}^{\frac{1}{2}} O[a] \frac{dV}{V} \right\} \left\{ \prod_{i \in C_1} da_i^2 \right\} \left\{ \prod_{J \in C_2} \delta(\log A^J - \log(V/N_2)) \right\}. \quad (18)$$

In (18) the additional constraint on the link variables has appeared. We keep areas of all triangles equal to each other. Initially we had N_1 degrees of freedom. There are N_2 additional conditions. At each site there are also 2

⁴ It is important to note, that the definition of delta function on the Riemannian manifold requires the background metric g_0 to be concretized. Namely, this function can be represented as $\delta(h(x)) = \lim_{t \rightarrow \infty} \exp(-t \int h^2(x) \sqrt{g_0(x)} d^2x)$. In our formulas we imply that a certain background metric is chosen. And this metric has nothing to do with our main dynamical variable g . (Of course, in the same way the measure over reparametrizations Df is defined with respect to this background metrics.)

local redundant degrees of freedom. It is most simple to understand their nature if we consider flat surface. Then local movements of each lattice site within this surface live the approximated manifold unchanged. Thus total number of local redundant degrees of freedom is $2N_0 = N_2 + 4(1 - h)$, where h is genus of the surface. For the case of torus the number of additional conditions is equal to the number of local redundant degrees of freedom. In case of sphere the number of additional conditions is less than the number of redundant degrees of freedom. For $h > 1$, however, our constraint may become too restrictive. Therefore, for the surface of genus $h > 1$ the number of additional conditions must be reduced. Further we restrict ourselves with the cases $h = 0, 1$.

The main simplification is achieved in the expression for the determinant $\text{Det } O[a]$ in the special case $C = -2$ when the areas of all triangles are equal to each other. In this case (8) has the form:

$$\|\delta a\|^2 = -\frac{N_2}{2V} \sum_{J \in C_2} \epsilon^{ka} \epsilon^{lv} \delta g_{kl}^J \delta g_{av}^J. \tag{19}$$

This expression can be easily rewritten in terms of the link lengths a_i :

$$\|\delta a\|^2 = \frac{1}{V} \sum_{ij} U^{ij} \delta(a_i^2) \delta(a_j^2), \tag{20}$$

where the matrix U depends upon the triangulation and does not depend on the link lengths. So, in this case $\text{Det } O[a] \sim 1/V^{N_1}$.

That is why we came to the following

Conclusion. In two dimensions (for $h = 0, 1$; $C = -2$) the measure over discretized geometries⁵ is

$$Da = \{V^{-1+N_1/2} dV\} \left\{ \prod_{i \in C_1} d\left[\frac{a_i^2}{V/N_2}\right] \right\} \left\{ \prod_{J \in C_2} \delta(\log A^J - \log(V/N_2)) \right\}. \tag{21}$$

The formula (21) can be considered as the main result of this Letter.

Now let us turn to the description of the particular example of the $2D$ model, which can be investigated in order to examine the measure (21). Namely, we are going to consider the R^2 discretized model. Its partition function has the form:

$$Z = \int Da \exp\left(-\beta \sum_{i \in C_0} \frac{\theta_i^2}{B_i[a]}\right). \tag{22}$$

The deficit angle at the point i is denoted as θ_i . Here we use the traditional definition of discretized squared curvature $\theta_i/B_i[a]$ (where $B_i[a] = \sum_{j \in J} \frac{1}{3} A^j$ is the sum over triangles incident at the point i).

Now let us consider the partition function for the fixed invariant volume (we rescaled a as $a \rightarrow \sqrt{2V/N_2}a$):

$$Z(V) = V^{N_1/2-1} \hat{Z}(\hat{V}), \tag{23}$$

where

$$\hat{Z}(\hat{V}) = \int \prod_{i \in C_1} da_i^2 \exp\left(-\frac{1}{\hat{V}} \frac{N_2}{2} \sum_{i \in C_0} \frac{\theta_i^2}{B_i[a]}\right) \prod_{J \in C_2} \delta(2A^J[a] - 1), \tag{24}$$

and $\hat{V} = \frac{V}{\beta}$.

⁵ Here and below we consider measure up to the factor, which depends upon the triangulation only.

In order to calculate $Z(V)$ and extract the string susceptibility we have to investigate the correlator

$$F(V, \beta) = \frac{\partial \log Z}{\partial \log V} - \left(\frac{N_1}{2} - 1 \right) = \frac{N_2}{2\hat{Z}\hat{V}} \left\langle \sum_{i \in C_0} \frac{\theta_i^2}{B_i[a]} \right\rangle, \quad (25)$$

where the averaging is over the model defined by the partition function (24).

At $\hat{V} \gg 1$ the structure of $Z(V)$ is expected to match the following expression

$$Z(V) \sim V^{-3+\gamma} \exp(mV), \quad (26)$$

where γ is the so-called string susceptibility while m is the renormalized cosmological constant [3].

On the other hand, we expect that in the limit $N_2 \gg \hat{V} \gg 1$ the system in leading order behaves like a collection of N_d oscillators, where N_d is the effective number of degrees of freedom. Thus the leading part of (25) is

$$F(\hat{V}) = \frac{N_d}{2} + o(N_2). \quad (27)$$

Naively one would suppose, that $N_d = N_1 - N_2$. However, close to the continuum limit the number of degrees of freedom may be reduced due to the restored symmetry (which is lost in the discretized model). Nevertheless, we expect, that this change in the number of degrees of freedoms remains finite at $N_2 \rightarrow \infty$ because we have already eliminated most of redundant degrees of freedom via the constraint on the triangle areas. Under this assumption we can represent γ as (we used here, that $N_1 = \frac{3}{2}N_2$):

$$\gamma = N_2 + \gamma_{\text{fin}} + O\left(\frac{1}{N_2}\right), \quad (28)$$

where γ_{fin} does not depend upon N_2 . From (28) it follows, that, strictly speaking, the string susceptibility is infinite. Moreover, this statement would remain valid even if the number of degrees of freedom is reduced by a number, that depends on N_2 . That is why we came to the conclusion that value of string susceptibility in our model contradicts with the one given by the KPZ formula [3]:

$$\gamma = 2 - \frac{5}{2}(1 - h). \quad (29)$$

Thus from our analysis it follows that the only possibility to compare KPZ result with the string susceptibility calculated via the discretized model given by (22) is to consider its finite part (which under mentioned above assumptions can be given, say, by γ_{fin} of (28)).

It is important to notice that usual Metropolis algorithm applied to the system (24) must be redefined in order to implement the constraints on the triangle areas. While updating link lengths the condition $A_i = 0.5$ is resolved as follows. When the link to update is chosen at random, we consider the two triangles having it as a common side. In each of these two triangles we choose the vertex, that does not belong to the given link. Next we consider the set of all triangles having one of these two points as one of their vertices. The resulting figure consists of two stars with the given two points as their centers.

If there are n_1 internal links in this figure (the links that do not belong to the boundary of the figure), then there are either $n_1 - 1$ or $n_1 - 2$ triangles that belong to the figure. So, keeping lengths of boundary links and the length of the first updated link, in both cases we might be able to resolve the constraints $A^J = 0.5$ within the figure. However, triangle inequalities may forbid these constraints to be resolved for certain choices of the first chosen link length. If so, we change it and repeat resolving constraints (at least for its initial value they may be resolved by the initial conditions).

In practice we use the constraint

$$|A^J - 0.5| < \delta \quad (30)$$

on each triangle. Here δ is chosen to be sufficiently small. (In principle it could be made as small as necessary. However, this may cause considerable increasing of the computer time. So we must adjust δ and, as a result, final accuracy of calculations in order to keep CPU time acceptable.)

In order to resolve the constraints, we make several steps over all links of the constructed figure that are allowed to be changed. Adjusting each link, we minimize the quantity $(A_1^2 - \frac{1}{4})^2 + (A_2^2 - \frac{1}{4})^2$, where A_1 and A_2 are the areas of the triangles having the given link as common. This minimization is achieved via direct solving of cubic equation. So this procedure is repeated iteratively until the requirement (30) is achieved at each triangle.

After this procedure is completed, the suggestion is formed, which includes the update of the initially chosen link and the calculated update of all other internal links of the constructed figure. Then, this suggestion is accepted or rejected with the usual probability ($p = 1$ if $\Delta S < 0$, and $p = e^{-\Delta S}$ otherwise, where ΔS is the correspondent change in the action).

To conclude, we have proposed in this Letter the modification of the 2D Regge calculus, in which the areas of all triangles are equal to each other. As a result the Lund–Regge measure over link lengths (at $C = -2$) is simplified considerably and becomes local. The discretized quantum gravity models with this measure can be investigated numerically in a relatively simple way contrary to the models with Lund–Regge measure defined on the conventional Regge lattice. As a result of brief consideration of the squared curvature model we argue, that the string susceptibility contains an infinite part. And this is reasonable to compare it with the KPZ result only modulo this infinite part. We also have described changes in the Metropolis algorithm necessary for implementing the constraints on the triangle areas.

It is important to note, that in order to obtain the mentioned simplification we were forced to start the discretization of gravity from the partially gauge-fixed version of the theory. Namely, we used the gauge condition, which makes the conformal mode $|g|$ of the metric field constant along the surface. The correspondent Faddeev–Popov determinant is shown to be independent of metric field. So, we lost a part of reparametrization invariance when coming to our version of Regge discretization. This was the price for the simplification of measure. We do not think this loss to be a pathology, keeping in mind that, say, Lorentz invariance is lost in any model defined on rectangular lattice. Nevertheless, it would be important to check if the mentioned part of reparametrization invariance is restored or not when a continuum limit of the discretized model is approached.

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