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A study on pentagonal fuzzy number and its corresponding matrices

Apurba Panda*, Madhumangal Pal

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, 721102, India

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ABSTRACT

In this article, the notion of pentagonal fuzzy number (PFN) is introduced in a generalized way. A few articles have been published based on this topic, but they have some ambiguities in defining this type of fuzzy number. Here, we proposed the logical definition in developing a pentagonal fuzzy number, along with its arithmetic operations. Based on PFN, the structure of pentagonal fuzzy matrices (PFMs) is studied, together with their basic properties. Some special type of PFMs and their algebraic natures (trace of PFM, adjoint of PFM, determinant of PFM, etc.) are discussed in this article. Finally, the notion of nilpotent PFM, comparable PFM, and constant PFMs, with their many properties, are highlighted in this article.

1. Introduction

Decision making problems in the real world are very often uncertain or vague in most cases. Fuzzy numbers are used in various fields, namely, fuzzy process modelling, control theory, decision making, expert system reasoning and so forth. Previous authors' studies on fuzzy numbers highlighted the arithmetic and algebraic structure based on triangular fuzzy numbers and trapezoidal fuzzy numbers. Fuzzy systems, including fuzzy set theory (Zadeh, 1965) and fuzzy logic, have a variety of successful applications. Fuzzy set theoretic approaches have been applied to various areas, from fuzzy topological spaces to medicine and so on. However, it is easy to handle the matrix formulation to study the various mathematical models. Due to the presence of uncertainty in many mathematical formulations in different branches of science and technology, we introduced the concept of pentagonal fuzzy number (PFN) and corresponding pentagonal fuzzy matrices (PFMs). Several authors have presented results of the properties of a determinant, adjoint of fuzzy matrices, and convergence of the power sequence of fuzzy matrices. A brief review on fuzzy matrices is given below.

The concept of fuzzy matrices was introduced for the first time by Thomason (Thomason, 1977) in the article entitled convergence of power of fuzzy matrix; later, Hashimoto (Hashimoto, 1983a) studied the fuzzy transitive matrix. The theoretical development of the fuzzy matrix was influenced through an article on some properties of the determinant and adjoint of a square fuzzy matrix proposed by Ragab et al. (Ragab and Eman, 1994). Moreover, some important results of the determinant of a fuzzy matrix were proposed by Kim (Kim et al., 1989). Several authors studied the canonical form and generalized fuzzy matrix (Hashimoto, 1983b; Kim and Roush, 1980), application of fuzzy matrices in a system of linear fuzzy equations (Buckley, 1991, 2001), etc. Some of the interesting arithmetic works on fuzzy numbers can be found in (Bhowmik et al., 2008; Dubois and Prade, 1979; Dubois and Prade, 1980). Conversely, some other articles studied different types of fuzzy numbers, namely, L-R type fuzzy number, triangular fuzzy number, and trapezoidal fuzzy number (Bansal, 2010). Thereafter, these types of fuzzy numbers were applied as a mathematical tool in the various fields of applied mathematics. The notion of a triangular fuzzy matrix was proposed for the first time by Shyamal and Pal (Shayamal and Pal, 2007) and was made familiar through introducing some new operators on triangular fuzzy matrices (Shayamal and Pal, 2004). The progression of fuzzy numbers became so fruitful that it spread into intuitionistic fuzzy matrices (Adak et al., 2012a; Adak et al., 2012b; Bhowmik and Pal, 2012; Bhowmik and Pal, 2008; Mondal and Pal, 2014; Pal, 2001; Pradhan and Pal, 2014a; Pradhan and Pal, 2014b; Pradhan and Pal, 2012; Shayamal and Pal, 2002) and interval valued fuzzy set theory (Mondal and Pal, 2015; Pal and Khan, 2005; Shayamal and Pal, 2006).

In this article, we introduce the notion of pentagonal fuzzy number in a well-defined manner by generalizing some other types

* Corresponding author.
E-mail addresses: apurbavu@gmail.com, mmpalvu@gmail.com (A. Panda).
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of fuzzy numbers and studied the basic arithmetic and algebraic properties of the pentagonal fuzzy number. In Section 2, several preliminaries regarding the fuzzy number are presented. In Section 3, fundamentals of the pentagonal fuzzy number are established. Based on the pentagonal fuzzy number, the concept of pentagonal fuzzy matrix (PFM) is presented in Section 4. Some works related to nilpotent PFMs, comparable PFMs, and constant PFMs are studied in the remaining sections.

2. Preliminaries

We first recapitulate some underlying definitions and basic results of fuzzy numbers.

Definition 1. ***Fuzzy set.*** *A fuzzy set is characterized by its membership function, taking values from the domain, space or universe of discourse mapped into the unit interval [0,1]. A fuzzy set A in the universal set X is defined as* $A = (x,\mu(x);x\in X)$. *Here,* $\mu_A:A\to[0,1]$ *is the grade of the membership function and* $\mu_A(x)$ *is the grade value of* $x\in X$ *in the fuzzy set A.*

Definition 2. ***Normal fuzzy set.*** *A fuzzy set A is called normal if there exists an element* $x\in X$ *whose membership value is one, i.e.,* $\mu_A(x) = 1$.

Definition 3. ***Fuzzy number.*** *A fuzzy number A is a subset of real line R, with the membership function* μ_A *satisfying the following properties:*

(i) $\mu_A(x)$ is piecewise continuous in its domain.
(ii) A is normal, i.e., there is a $x_0\in A$ such that $\mu_A(x_0) = 1$.
(iii) A is convex, i.e., $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1),\mu_A(x_2))$. $\forall$ x_1,x_2 in X.

Due to wide applications of the fuzzy number, two types of fuzzy number, namely, triangular fuzzy number and trapezoidal fuzzy number, are introduced in the field of fuzzy algebra.

Definition 4. ***Triangular fuzzy number.*** *A fuzzy number* $A = (a,b,c)$ *is said to be a triangular fuzzy number if it has the following membership function*

Thus, the triplet (a,b,c) forms a triangular fuzzy number under this membership function. Graphically, its membership function looks like a triangle, which is depicted in Fig. 1.

Definition 5. ***Trapezoidal fuzzy number.*** *A fuzzy number* $A = (a,b,c,d)$ *is called a trapezoidal fuzzy number if it possesses the following membership function*

Graphically, the trapezoidal fuzzy number has a trapezoidal shape with four vertices (a,b,c,d), as depicted in Fig. 2.

However, real-life problems are sometimes concerned with more than four parameters. To resolve those problems, we propose another concept of the fuzzy number, called pentagonal fuzzy number (PFN). We discuss PFN in the next section.

$$\mu_A(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x \leq b \\ 1, & x = b \\ \dfrac{c-x}{c-b}, & b < x \leq c \\ 0, & x \geq c. \end{cases}$$

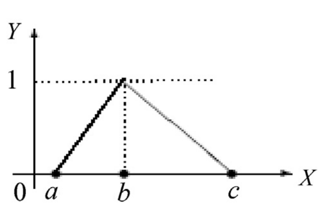

Fig. 1. Triangular fuzzy number.

$$\mu_A(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{c-x}{d-c}, & c < x \leq d \\ 0, & x \geq d \end{cases}$$

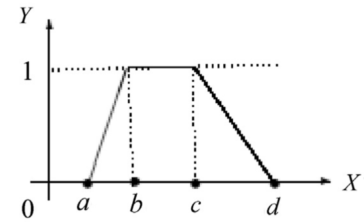

Fig. 2. Trapezoidal fuzzy number.

3. Pentagonal fuzzy number

Due to error in measuring technique, instrumental faultiness, etc., some data in our observation cannot be precisely or accurately determined. Let us consider that we measure the weather temperature and humidity simultaneously. The temperature is approximately 35°C with normal humidity, i.e., the temperature is not perfect either more or less than 35°C, which affects normal humidity in the atmosphere. Thus, variation in temperature also affects the percentage of humidity. This phenomenon happens in general. This concept of variation leads to a new type of fuzzy number called the pentagonal fuzzy number (PFN). Generally, a pentagonal fuzzy number is a 5-tuple subset of a real number R having five parameters.

A pentagonal fuzzy number A is denoted as $A = (a_1,a_2,a_3,a_4,a_5)$, where a_3 is the middle point and (a_1, a_2) and (a_4, a_5) are the left and right side points of a_3, respectively. Now, we construct the mathematical definition of a pentagonal fuzzy number.

Definition 6. ***Pentagonal fuzzy number.*** *A fuzzy number* $A = (a_1,a_2,a_3,a_4,a_5)$ *is called a pentagonal fuzzy number when the membership function has the form*

where the middle point a_3 has the grade of membership 1 and w_1,w_2 are the respective grades of points a_2,a_4. Note that every PFN is associated with two weights w_1 and w_2. To avoid confusion, we use the notation w_{iA} for $i = 1,2$ to represent w_1 and w_2 as the weights of the PFN A.

3.1. Geometrical representation

From Fig. 3, it is clear that $\mu_A(x)$ has a piecewise continuous graph consisting of five points in its domain, forming a pentagonal shape. As chosen, the points in the domain have the ordering $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$; $a_1,a_2,a_3,a_4,a_5\in R$. We have to choose the value of the membership function at a_2,a_4 in such a way that $w_1 \geq \frac{a_2-a_1}{a_3-a_1}$ and $w_2 \geq \frac{a_1-a_5}{a_3-a_5}$. Otherwise, the convexity properties of the fuzzy number fail for the pentagonal fuzzy number.

Remark 1. *We define a pentagonal fuzzy number in a generalized way so that we can easily visualize two special fuzzy numbers, namely, triangular fuzzy number and trapezoidal fuzzy number, as follows:*

Case I When $w_1 = w_2 = 0$, then the pentagonal fuzzy number is reduced to a triangular fuzzy number, i.e., $\tilde{A} = (a_1, a_2, a_3, a_4, a_5) \cong (a_2,a_3,a_4)$; in this case

$$\mu_A(x) = \begin{cases} 0, & x \leq a_2 \\ 1 - \dfrac{a_2 - x}{a_2 - a_3}, & a_2 < x \leq a_3 \\ 1, & x = a_3 \\ 1 - \dfrac{a_4 - x}{a_4 - a_3}, & a_3 < x \leq a_4 \\ 0 & x \geq a_4 \end{cases}$$

$$\mu_A(x;w_1,w_2)=\begin{cases} w_1\dfrac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2\\ 1-(1-w_1)\dfrac{x-a_2}{a_3-a_2}, & a_2\le x\le a_3\\ 1, & x=a_3\\ 1-(1-w_2)\dfrac{x-a_3}{a_4-a_3}, & a_3\le x\le a_4\\ w_2\dfrac{x-a_5}{a_4-a_5}, & a_4\le x\le a_5\\ 0, & x>a_5\end{cases}$$

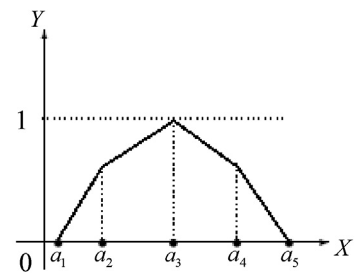

Fig. 3. Pentagonal fuzzy number.

Case II When $w_1 = w_2 = 1$, then the pentagonal fuzzy number becomes a trapezoidal fuzzy number, i.e., $A=(a_1,a_2,a_3,a_4,a_5)\cong(a_1,a_2,a_4,a_5)$,

and then $$\mu_A(x)=\begin{cases} 0 & for\quad x\le a_1\\ \dfrac{x-a_1}{a_2-a_4} & for\quad a_1<x\le a_2\\ 1 & for\quad a_2\le x\le a_4\\ \dfrac{a_4-x}{a_5-a_4} & for\quad a_4<x\le a_5\\ 0 & for\quad x>x_5\end{cases}$$

3.2. Arithmetic operations of PFN

Formation of an arithmetic operation is crucial in the study of fuzzy numbers; the author tries to establish some basic arithmetic operations of PFN. Note that every PFN is associated with two weights: w_1 and w_2. To avoid confusion, we use the notation w_{iA} for $i = 1, 2$ to represent w_1 and w_2 as the weights of the PFN A.

(1) **Addition:** Let $A = (a_1,a_2,a_3,a_4,a_5)$ and $B = (b_1,b_2,b_3,b_4,b_5)$ be two PFNs; then,

$$A+B=(a_1+b_1,a_2+b_2,a_3+b_3,a_4+b_4,a_5+b_5),$$
$$\text{with}\ \ w_{i(A+B)}\ge\max(w_{iA},w_{iB})\ \ \text{for}\ i=1,2.$$

(2) **Subtraction:** We define the subtraction of two PFNs $A = (a_1,a_2,a_3,a_4,a_5)$ and $B = (b_1,b_2,b_3,b_4,b_5)$ as

$$A-B=(a_1-b_1,a_2-b_2,a_3-b_3,a_4-b_4,a_5-b_5),$$
$$\text{with}\ \ w_{i(A-B)}\ge\max(w_{iA},w_{iB})\ \ \text{for}\ i=1,2.$$

(3) **Scalar Multiplication:** Let $A = (a_1,a_2,a_3,a_4,a_5)$ be a PFN and $k \in R$ be any scalar. If $k \ge 0, kA = (ka_1,ka_2,ka_3,ka_4,ka_5)$

$k\le 0, kA=(ka_5,ka_4,ka_3,ka_2,ka_1)$

(4) **Multiplication:** Let $A = (a_1,a_2,a_3,a_4,a_5)$ and $B = (b_1,b_2,b_3,b_4,b_5)$ be two PFNs; then,

$$AB=(a_1b_1,a_2b_2,a_3b_3,a_4b_4,a_5b_5)$$
$$\text{with}\ \ w_{i(AB)}\ge\max(w_{iA},w_{iB}),\ \ i=1,2.$$

(5) **Inverse:** We define the inverse of a PFN when all its components are non-zero. Suppose $A = (a_1,a_2,a_3,a_4,a_5)$ is a PFN; then,

$$A^{-1}\approx\frac{1}{A}\approx\left(\frac{1}{a_5},\frac{1}{a_4},\frac{1}{a_3},\frac{1}{a_2},\frac{1}{a_1}\right).$$

If one of the components of a PFN becomes zero, then we cannot find its inverse.

(6) **Division:** The division of two PFNs $A = (a_1,a_2,a_3,a_4,a_5)$ and $B = (b_1,b_2,b_3,b_4,b_5)$ is approximated as the multiplication with inverse.

$$\frac{A}{B}\approx AB^{-1}\approx\left(\frac{a_1}{b_5},\frac{a_2}{b_4},\frac{a_3}{b_3},\frac{a_4}{b_2},\frac{a_5}{b_1}\right).$$

Note that a PFN A is divisible by B only when B is a non-null PFN having non-zero components.

(7) **Exponent:** The exponent of a PFN $A = (a_1,a_2,a_3,a_4,a_5)$ is defined as the power of its components. $A^n\approx(a_1^n,a_2^n,a_3^n,a_4^n,a_5^n)$, with n being a real number.

Remark 2. *We choose the "max" relation of the weights of PFNs in all the above arithmetic operations because otherwise addition, subtraction, multiplication, division, etc., between two PFNs cannot be closed under these operations, i.e., the operations between two PFNs never produce another PFN. To verify, we assume that* $A = (-1,0,1,2,5)$ *and* $B = (1,2,4,5,6)$*, with* $w_{1A} \ge 0.5$, $w_{2A} \ge 0.7$ *and* $w_{1B} \ge 0.3$, $w_{2B} \ge 0.5$*; then, the value of* $w_{1(A+B)}$, $w_{1(A-B)}$, $w_{1(AB)}$ *are all greater than* $\min(0.5,0.3) = 0.3$*, which violates the convexity condition of the pentagonal fuzzy number.*

Definition 7. ***Positive PFN.*** *A PFN* $A = (a_1,a_2,a_3,a_4,a_5)$ *is said to be positive if all its entries are positive. Similarly,* $A = (a_1,a_2,a_3,a_4,a_5)$ *is negative if all of its entries are negative.*

Definition 8. ***Null PFN.*** *A PFN A is called a Null PFN if all of its entries are zero, i.e.,* $A = (0,0,0,0,0)$.

Definition 9. ***Null equivalent PFN.*** *A PFN* $A = (a_1,a_2,a_3,a_4,a_5)$ *is said to be null equivalent if its middle entry is at the point 0, i.e., of the form* $(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$*, where* $\delta_1\cdot\varepsilon_1\neq 0$, $\delta_2\cdot\varepsilon_2\neq 0$*. It is denoted by* $\tilde{0}$.

Definition 10. ***Unit equivalent PFN.*** *A PFN A is said to be a unit equivalent PFN when its middle entry is at 1, i.e., of the form* $(\delta_1,\varepsilon_1,1,\varepsilon_2,\delta_2)$*, where* $\delta_1\cdot\varepsilon_1\neq 0$, $\delta_2\cdot\varepsilon_2\neq 0$.

From our previous arithmetic operations of PFN, we observed that subtraction of two PFNs with a common middle entry produces a null equivalent PFN, while their division yields another unit equivalent PFN. Additionally, we have the basic operations, i.e., addition and multiplication of PFNs are both commutative and associative, while multiplication is also distributive over addition.

Now, we construct a pentagonal fuzzy matrix whose elements are considered as pentagonal fuzzy numbers. This type of fuzzy matrix plays a vital role in fuzzy algebra.

4. Pentagonal fuzzy matrix

Definition 11. **Pentagonal fuzzy matrix.** *A fuzzy matrix $A=(a_{ij})_{m\times n}$ of order $m \times n$ is called a pentagonal fuzzy matrix if the elements of the matrix are pentagonal fuzzy numbers, i.e., of the form $(a_{1ij},a_{2ij},a_{3ij},a_{4ij},a_{5ij})$.*

Through classical matrix algebra, we achieve some algebraic operations of PFM. Let $A=(a_{ij})$ and $B=(b_{ij})$ be two PFMs of the same order; then, we have the following results:

(i) $A+B=(a_{ij}+b_{ij})$
(ii) $A-B=(a_{ij}-b_{ij})$
(iii) for $A=(a_{ij})_{m\times r}$ and $B=(b_{ij})_{r\times n}$, we have $A\cdot B=(c_{ij})_{m\times n}$, where $(c_{ij})=\sum_{k=1}^{n}a_{ik}b_{kj}$ for $i=1,2,\ldots,m$, $j=1,2,\ldots,n$.
(iv) $A^T=(a_{ji})$, the transpose of A.
(v) $kA=(ka_{ij})$, where k is any scalar.

Some special types of pentagonal fuzzy matrices corresponding to classical matrices are now introduced in this section. However, in fuzzy matrix algebra, we define some other types of pentagonal fuzzy matrices and their algebraic properties.

Definition 12. ***Pure null PFM.*** *A PFM is said to be a pure null PFM if all its entries are null PFNs, i.e., all the elements are (0,0,0,0,0). It is denoted by O.*

Definition 13. ***Null equivalent PFM.*** *A PFM $A=(a_{ij})$ is said to be a null equivalent PFM if all its elements are of the form $a_{ij}=(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$, where $\delta_1\cdot\varepsilon_1\neq 0$, $\delta_2\cdot\varepsilon_2\neq 0$. It is denoted as $\tilde{O}$.*

Definition 14. ***Pure unit PFM.*** *A square PFM $A=(a_{ij})$ is said to be a pure unit PFM if $a_{ii}=(0,0,1,0,0)$ and $a_{ij}=\tilde{0}$, $i\neq j$ for all $i,j=1,2,\ldots,n$. It is denoted by I.*

Definition 15. ***Unit equivalent PFM.*** *A square PFM $A=(a_{ij})$ is said to be a unit equivalent PFM if $a_{ii}=(\delta_1,\varepsilon_1,1,\varepsilon_2,\delta_2)$ and $a_{ij}=\tilde{0}$, $i\neq j$, where $\delta_1\cdot\varepsilon_1\neq 0$, $\delta_2\cdot\varepsilon_2\neq 0$ for all $i,j=1,2,\ldots,n$.*

Definition 16. ***Pure triangular PFM.*** *A square PFM $A=(a_{ij})$ is called a pure triangular PFM if either $a_{ij}=0$ for $i>j$ or $a_{ij}=0$ for $i<j$. $\forall\ i,j=1,2,\ldots,n$.*

When $a_{ij}=0$ for $i>j$, then it is said to be a pure upper triangular PFM. Otherwise, for $a_{ij}=0$, $i<j$, it is called a pure lower triangular PFM, $i,j=1,2,\ldots,n$.

Definition 17. ***Fuzzy triangular PFM.*** *A square PFM $A=a_{ij}$ is called a fuzzy triangular PFM if either $a_{ij}=(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$ for $i>j$ or $a_{ij}=(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$ for $i<j$, where $\delta_1\cdot\varepsilon_1\neq 0,\delta_2\cdot\varepsilon_2\neq 0$.*

Definition 18. ***Strictly fuzzy triangular PFM.*** *A square PFM $A=(a_{ij})$ is called a strictly fuzzy triangular PFM if either $a_{ij}=\tilde{0}$ for $i\geq j$ or $a_{ij}=\tilde{0}$ for $i\leq j$, $\tilde{0}$ being the null equivalent PFN.*

Definition 19. ***Symmetric PFM.*** *A square PFM $A=(a_{ij})$ is called a symmetric PFM if $A=A^T$, i.e., $a_{ij}=a_{ji}$.*

Definition 20. ***Pure skew symmetric PFM.*** *A square PFM $A=(a_{ij})$ is called a pure skew symmetric PFM if $A=-A^T$.*

Definition 21. ***Fuzzy skew symmetric PFM.*** *A square PFM $A=(a_{ij})$ is called a fuzzy skew symmetric PFM if $A=-A^T$ and $a_{ii}=(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$ for all $i=1,2,\ldots,n$, i.e., $a_{ij}=-a_{ji}$ and $a_{ii}=(\delta_1,\varepsilon_1,0,\varepsilon_2,\delta_2)$, $\delta_1\cdot\varepsilon_1\neq 0$, $\delta_2\cdot\varepsilon_2\neq 0$. $\forall\ i,j=1,2,\ldots,n$.*

5. Fundamental properties of PFM

Here, we introduce some fundamental properties of pentagonal fuzzy matrices. Here, we furnish the commutative and associative laws, which are well defined, for PFM under the arithmetic operations addition and multiplication.

Property 1. *For any three square PFMs P,Q,R of the same order $s\times n$, we have the following results:*

(i) $P+Q=Q+P$.
(ii) $P+(Q+R)=(P+Q)+R$.
(iii) $P+P=2P$.
(iv) $P-P=\tilde{O}$, a null equivalent PFM.
(v) $P+O=P-O=P$.

Property 2. *Let P and Q be any two PFNs of the same order and s,t be any two scalars. Then,*

(i) $s(tP)=(st)P$.
(ii) $s(P+Q)=sP+sQ$.
(iii) $(s+t)P=sP+tP$, $\forall s,t\neq 0$.
(iv) $s(P-Q)=sP-sQ$.

Property 3. *Let P and Q be any two PFMs such that $P+Q$ and $P\cdot Q$ are well defined. Then,*

(i) $(P^T)^T=P$.
(ii) $(P+Q)^T=P^T+Q^T$.
(iii) $(P\cdot Q)^T=Q^T\cdot P^T$.

Property 4. *Let P and Q be any two PFNs of the same order and s,t be any two scalars. Then,*

(i) $(sP)^T=sP^T$.
(ii) $(sP+tQ)^T=sP^T+tQ^T$.

Property 5. *Let, P be any square PFM. Then,*

(i) PP^T and P^TP are both symmetric.
(ii) $P+P^T$ is a fuzzy symmetric PFM.
(iii) $P-P^T$ is a fuzzy skew-symmetric PFM.

6. Trace of a PFM

Definition 22. ***Trace of a PFM.*** *The trace of a square PFM $A=(a_{ij})$ is defined as the sum of the elements of the principle diagonal. It is denoted by tr(A), i.e., $tr(A)=\sum_{i=1}^{n}a_{ii}$.*

Property 6. *Let $P=(p_{ij})$ and $Q=(q_{ij})$ be two square PFMs of the same order m; then, the following holds well.*

(i) $tr(P+Q)=tr(P)+tr(Q)$.
(ii) $tr(P)=tr(P^T)$.
(iii) $tr(P\cdot Q)=tr(Q\cdot P)$.

Proof. *(i)* Let P and Q be two PFMs of order m, where $p_{ij}=(p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij})$ and $q_{ij}=(q_{1ij},q_{2ij},q_{3ij},q_{4ij},q_{5ij})$. Now, $tr(P)=$

$\sum_{i=1}^{m} p_{ii} = \sum_{i=1}^{m}(p_{1ii}, p_{2ii}, p_{3ii}, p_{4ii}, p_{5ii})$ and $tr(Q) = \sum_{i=1}^{m} q_{ii} = \sum_{i=1}^{m}(q_{1ii}, q_{2ii}, q_{3ii}, q_{4ii}, q_{5ii})$.

$$\text{Thus, } tr(P+Q) = \sum_{i=1}^{m}(p_{ii}+q_{ii}) = \sum_{i=1}^{m}(p_{ii}) + \sum_{i=1}^{m}(q_{ii})$$
$$= \sum_{i=1}^{m}(p_{1ii}, p_{2ii}, p_{3ii}, p_{4ii}, p_{5ii})$$
$$+ \sum_{i=1}^{m}(q_{1ii}, q_{2ii}, q_{3ii}, q_{4ii}, q_{5ii})$$
$$= tr(P) + tr(Q)$$

Hence the result.

(ii) We know that the principle diagonal of a PFM remains invariant under transposition. Hence, the proof is obvious.

(iii) We know that for any two PFM of the same order, their multiplication is well defined. Let $P=(p_{ij})$ and $Q=(q_{ij})$ be two PFMs of the same order m, where $(p_{ij}) = (p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij})$ and $(q_{ij}) = (q_{1ij},q_{2ij},q_{3ij},q_{4ij},q_{5ij})$. Again, let $C = (c_{ij})$, where $(c_{ij}) = \sum_{r=1}^{m} p_{ir}q_{rj}$, for $i,j = 1,2,\ldots,m$. Now, $tr(C) = \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m}(\sum_{r=1}^{n} p_{ir}q_{ri})$. Again, let $D = (d_{ij}) = Q{\cdot}P$, where $d_{ij} = \sum_{r=1}^{m} q_{ir}p_{rj}$, for $i,j = 1,2,\ldots,m$. Therefore, $tr(D) = tr(Q{\cdot}P) = \sum_{i=1}^{m} d_{ii}$

$$= \sum_{i=1}^{n}\left(\sum_{r=1}^{m} q_{ir}p_{ri}\right)$$
$$= \sum_{r=1}^{m}\left(\sum_{i=1}^{m} p_{ir}q_{ri}\right) \text{(interchanging the dummy indices i and r)}$$
$$= tr(P{\cdot}Q) = tr(C)$$

Hence the proof.

Property 7. *The product of two pure upper triangular PFMs of order $k \times k$ is a pure upper triangular PFM.*

Proof. Let $P = (p_{ij})$ and $Q = (q_{ij})$ be two pure upper triangular PFMs of the same order k, where $(p_{ij}) = (p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij})$ and $(q_{ij}) = (q_{1ij},q_{2ij},q_{3ij},q_{4ij},q_{5ij})$. Because P,Q are both upper triangular PFMs, $p_{ij} = (0,0,0,0,0)$ and $q_{ij} = (0,0,0,0,0)$ for $i > j$, $i,j = 1,2,\ldots,k$. Let $N = P{\cdot}Q = (n_{ij})$; then, $(p_{ij}) = \sum_{r=1}^{k} p_{ir}q_r\ j = \sum_{r=1}^{k}(p_{1ij}, p_{2ij}, p_{3ij}, p_{4ij}, p_{5ij})(q_{1ij}, q_{2ij}, q_{3ij}, q_{4ij}, q_{5ij})$. Now, it is enough to establish that $(n_{ij}) = (0,0,0,0,0)$ for $i > j$, $i,j = 1,2,\ldots,k$. For $i > r$, we have $p_{ir} = (0,0,0,0,0)$, $r = 1,2,\ldots,i-1$, and $q_{ir} = (0,0,0,0,0)$, $r = i,i+1,\ldots,k$

$$\text{Therefore, } (n_{ij}) = \sum_{r=1}^{k} p_{ir}q_{rj} = \sum_{r=1}^{i-1} p_{ir}q_{rj} + \sum_{r=i-1}^{k} p_{ir}q_{rj}$$
$$= (0,0,0,0,0).$$

$$\text{Now, } n_{ii} = \sum_{r=1}^{k} p_{ir}q_{ri}$$
$$= \sum_{r=1}^{i-1} p_{ir}q_{rj} + \sum_{r=i-1}^{k} p_{ir}q_{rj}$$
$$= p_{ii}{\cdot}q_{ii} = (0,0,0,0,0)$$

because $p_{ir} = (0,0,0,0,0)$, $r = 1,2,\ldots,i-1$ and $q_{ir} = (0,0,0,0,0)$, $r = i,i+1,\ldots,k$. Hence the result.

Property 8. *The product of two pure lower triangular PFMs is also a pure lower triangular PFM.*

Property 9. *Let P be any square PFM of order m.*

(i) If P is a pure upper triangular PFM, then P^T is a pure lower triangular PFM.
(ii) If P is a pure lower triangular PFM, then P^T is a pure upper triangular PFM.
(iii) If P and Q are both pure upper triangular PFMs, then $P{\cdot}Q$ and $Q{\cdot}P$ both hold well.

7. Determinant of a PFM

In this section, we introduce another important algebraic property, i.e., the determinant of a PFM, together with its several postulates. Also we mention the characteristic of an adjoint, cofactor, minor, etc., and classify their properties.

Definition 23. ***Determinant of a PFM.*** *The pentagonal fuzzy determinant of a pentagonal fuzzy matrix A of order $n \times n$ is denoted by $det(A)$ or $|A|$ and defined as $|A| = \sum_{\sigma\in S_n}(\text{sgn}\sigma{\cdot}\prod_{i=1}^{n} a_{i\sigma i})$, where $a_{i\sigma i} = (a_{1i\sigma i},a_{2i\sigma i},a_{3i\sigma i},a_{4i\sigma i},a_{5i\sigma i})$ are PFNs and S_n denotes the symmetric group of all permutation of indices 1,2,...,n. Additionally, sgn is the signature of the permutation, defined as sgn $\sigma = 1$ or -1 if the permutation is even or odd, respectively.*

There are several products and additions of PFNs in the computation of $det(A)$, and the value of PFNs generates another PFN. Thus, the determinant value of a PFM yields a pentagonal fuzzy number.

Definition 24. ***Minor.*** *Let A be a square PFM of order $n \times n$. The minor of an element a_{ij} in $det(A)$ is the determinant of order $(n-1) \times (n-1)$, which can be obtained by deleting the i^{th} row and j^{th} column from A. The minor of A is denoted by M_{ij}.*

Definition 25. ***Cofactor.*** *Let A be a square PFM of order $n \times n$. The cofactor of an element a_{ij} in A is denoted by A_{ij} and is defined by $A_{ij} = (-1)^{i+j}M_{ij}$.*

Definition 26. ***Adjoint of a PFM.*** *Let $A = (a_{ij})_{n\times n}$ be a square PFM. The adjoint of a PFM A is denoted by adj(A) and is defined as $b_{ij} = |A_{ji}|$, where $|A_{ji}|$ is the determinant of a $(n-1) \times (n-1)$ PFM formed by deleting row j and column i from A, i.e., B = adj(A).*

This can be defined as

$$adj(A) = B = b_{ij} = \sum_{\pi\in S_{n_i n_j}} \prod_{t\in n_j} a_{t\pi(t)}$$

where $n_j = 1,2,\ldots,n-j$ and π is an arbitrary permutation chosen from the set of all permutations $S_{ni\,nj}$ of set n_i over n_j.

Here, $det(A)$ contains $n!$ terms, out of which $\frac{n!}{2}$ are positive and the remaining same number of terms are negative. All these $n!$ terms contain n quantities at a time in product form, subject to the condition that from the n quantities in the product, exactly one is taken from each row and exactly one is taken from each column. Another way of representing the pentagonal fuzzy determinant of a PFM $A = (a_{ij})$ is to expand it to the form $\sum a_{ij}A_{ji}$, $i = (1,2,\ldots,n)$, where A_{ij} is the cofactor of a_{ij} in $det(A)$. Thus, the pentagonal fuzzy determinant is the sum of the product of the elements of any row (column) and the co-factors of the corresponding elements of the same row (column).

In classical matrix algebra, the value of a determinant can be computed using any one of the above-mentioned two processes, with both yielding the same result. The simplest way to determine the value of the pentagonal fuzzy determinant is given by the formula as in definition. We now study the important properties of pentagonal fuzzy matrices.

Property 10. *Let $P = (p_{ij})$ be a PFM of order $m \times m$.*

(i) If all the elements of a row (column) of P are (0,0,0,0,0), then $|P| = (0,0,0,0,0)$.
(ii) If a row (column) is multiplied by a scalar λ, then $det(P)$ is also multiplied by λ.

(iii) If P is a pure triangular PFM, then $|P| = \prod_{i=1}^{m}(p_{1ii}, p_{2ii}, p_{3ii}, p_{4ii}, p_{5ii})$.

Proof. *(i)*Let $P = (p_{ij})_{m\times m}$ be a square PFM, where $(p_{ij}) = (p_{1ii}, p_{2ii}, p_{3ii}, p_{4ii}, p_{5ii})$. We define the determinant in the following way: $E_i(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n}(a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij})A_{ij}$, where A_{ij} is the cofactor of p_{ij} in $det(P)$. Obviously, $E_1(P) = E_2(P) = \cdots = E_m(P) = |P|$. Now, assume that all the elements of the r^{th} row, $1 \le r \le m$, are pure null PFN. Then, $E_r(P) = (0,0,0,0,0)$. Because $p_{rj} = (0,0,0,0,0)$ for all $j = 1,2,...,m$, $|P| = E_r(P) = (0,0,0,0,0)$.

Hence the result.

*(ii)*If $\lambda = 0$, then P has a zero row (column). Thus, $|P| = (0,0,0,0,0)$. Thus, the result is obviously as follows.

Let $Q = (q_{ij})$ be a square PFM of order m obtained from P by multiplying its r^{th} row by a non-zero scalar λ. Then, clearly $q_{ij} = (q_{1ij}, q_{2ij}, q_{3ij}, q_{4ij}, q_{5ij}) = \lambda(p_{1ij}, p_{2ij}, p_{3ij}, p_{4ij}, p_{5ij}) = (\lambda p_{1ij}, \lambda p_{2ij}, \lambda p_{3ij}, \lambda p_{4ij}, \lambda p_{5ij})$ when λ is a positive scalar and $q_{ij} = (\lambda p_{5ij}, \lambda p_{4ij}, \lambda p_{3ij}, \lambda p_{2ij}, \lambda p_{1ij})$ when λ is a negative scalar.

$$\begin{aligned}\text{Now, } |Q| &= \sum_{\sigma\in S_m} \text{sgn}\sigma\Big(q_{1(1\sigma1)}, q_{2(1\sigma1)}, q_{3(1\sigma1)}, q_{4(1\sigma1)}, q_{5(1\sigma1)}\Big)\cdots \Big(q_{1(r\sigma r)}, q_{2(r\sigma r)}, q_{3(r\sigma r)}, q_{4(r\sigma r)}, q_{5(r\sigma r)}\Big)\cdots \Big(q_{1(n\sigma n)}, q_{2(m\sigma m)}, q_{3(n\sigma n)}, q_{4(n\sigma n)}, q_{5(m\sigma m)}\Big)\\ &= \sum_{\sigma\in S_m} \text{sgn}\sigma\Big(p_{1(1\sigma1)}, p_{2(1\sigma1)}, p_{3(1\sigma1)}, p_{4(1\sigma1)}, p_{5(1\sigma1)}\Big)\cdots \Big(\lambda p_{1(r\sigma r)}, \lambda p_{2(r\sigma r)}, \lambda p_{3(r\sigma r)}, \lambda p_{4(r\sigma r)}, \lambda p_{5(r\sigma r)}\Big)\cdots \Big(p_{1(m\sigma m)}, p_{2(n\sigma n)}, p_{3(m\sigma m)}, p_{4(m\sigma m)}, p_{5(m\sigma m)}\Big)\\ &= \lambda \sum_{\sigma\in S_m} \text{sgn}\sigma \cdot \prod_{i=1}^{m} p_{i\sigma i}\\ &= \lambda|P|\end{aligned}$$

When λ is a negative scalar,

$$\begin{aligned}|Q| &= \sum_{\sigma\in S_m} \text{sgn}\sigma\Big(q_{1(1\sigma1)}, q_{2(1\sigma1)}, q_{3(1\sigma1)}, q_{4(1\sigma1)}, q_{5(1\sigma1)}\Big)\cdots \Big(\lambda p_{5(r\sigma r)}, \lambda p_{4(r\sigma r)}, \lambda p_{3(r\sigma r)}, \lambda p_{2(r\sigma r)}, \lambda p_{1(r\sigma r)}\Big)\cdots \Big(p_{5(m\sigma m)}, p_{4(m\sigma m)}, p_{3(m\sigma m)}, p_{2(m\sigma m)}, p_{1(m\sigma m)}\Big)\\ &= \lambda\cdot \sum_{\sigma\in S_m} \text{sgn}\sigma\Big(p_{1(1\sigma1)}, p_{2(1\sigma1)}, p_{3(1\sigma1)}, p_{4(1\sigma1)}, p_{5(1\sigma1)}\Big)\cdots \Big(\lambda p_{1(r\sigma r)}, \lambda p_{2(r\sigma r)}, \lambda p_{3(r\sigma r)}, \lambda p_{4(r\sigma r)}, \lambda p_{5(r\sigma r)}\Big)\cdots \Big(p_{1(m\sigma m)}, p_{2(m\sigma m)}, p_{3(m\sigma m)}, p_{4(m\sigma m)}, p_{5(m\sigma m)}\Big)\\ &= \lambda \sum_{\sigma\in S_m} \text{sgn}\sigma \cdot \prod_{i=1}^{m} p_{i\sigma i}\\ &= \lambda|P|\end{aligned}$$

Hence the result.

*(iii)*Let $P = (a_{ij})_{m\times m}$ be an upper (lower) triangular PFM. Then, for $i \le j$, $p_{ij} = (0,0,0,0,0)$. Now consider a term t in $|P|$; then, $t = \prod_{i=1}^{m}(p_{1\ i\sigma i}, p_{2\ i\sigma i}, p_{3\ i\sigma i}, p_{4\ i\sigma i}, p_{5\ i\sigma i})$. Let $\sigma(1) \ne 1$, i.e., $1 \le \sigma(1)$, so that $p_{11\sigma1} = 0$, $p_{21\sigma1} = 0$, $p_{31\sigma1} = 0$, $p_{41\sigma1} = 0$, $p_{51\sigma1} = 0$. Consequently, $p_{i\sigma i} = 0$ for $i = 1$. Again, let $\sigma(1) = 1$ but $\sigma(2) \ne 2$; then, $p_{2\sigma2} = 0$. Hence, $t = (0,0,0,0,0)$. This means that for each term, $|P| = (0,0,0,0,0)$, if $\sigma(1) \ne 1$, $\sigma(2) \ne 2$. Preceding in this way, we have for $\sigma(i) \ne i$ $t = (0,0,0,0,0)$. Therefore, $|P| = \prod_{i=1}^{m}(p_{1ii}, p_{2ii}, p_{3ii}, p_{4ii}, p_{5ii})$.

This implies that the product of the diagonal entries is the value of the determinant for a triangular PFM.

Property 11. *The determinant of a diagonal PFM is the product of its diagonal entries.*

Property 12. *If any two rows (columns) of a square PFM A are interchanged, then only the sign of determinant |A| of A changes.*

Proof. Let $A = (a_{ij})$ be a square PFM of order $n \times n$. If $P = (p_{ij})$ is obtained from A by interchanging the r^{th} and s^{th} row $(r < s)$ of A, then it is clear that $p_{ij} = a_{ij}$, $i \ne r$, $i \ne s$ and $p_{rj} = a_{sj}$, $p_{sj} = a_{rj}$. Now, $|P| = \sum_{\sigma\in S_n} \text{sgn}\sigma(p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{r\sigma(r)}\cdots p_{s\sigma(s)}\cdots p_{n\sigma(n)}) = \sum_{\sigma\in S_n} \text{sgn}\sigma(a_{1\sigma(1)} a_{2\sigma(2)}\cdots a_{r\sigma(s)}\cdots a_{s\sigma(r)}\cdots a_{n\sigma(n)})$

$$\text{Let } \gamma = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n\\ 1 & 2 & \ldots & s & \ldots & r & \ldots & n\end{pmatrix}$$

Then, γ is a transposition of interchanging r and s. Thus, γ is an odd permutation; thus, $sgn\lambda = -1$. Let $\gamma\sigma = \delta$. As σ runs through all permutations on $(1,2,...,n)$, δ also runs over the same permutations because $\sigma_1\gamma = \sigma_2\gamma$ or $\sigma_1 = \sigma_2$.

Now, $\delta = \sigma\gamma = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n\\ \sigma(1) & \sigma(2) & \ldots & \sigma(r) & \ldots & \sigma(s) & \ldots & n\end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n\\ 1 & 2 & \ldots & s & \ldots & r & \ldots & n\end{pmatrix}$. Therefore, $\sigma(i) = i$; $i \ne r,s$; $\sigma(r) = \sigma(s)$, $\sigma(s) = \sigma(r)$. Because γ is an odd permutation, δ is even or odd if σ is even or odd, i.e., $sgn\delta = -sgn\sigma$.

$$\begin{aligned}\text{Then, } |P| &= \sum_{\sigma\in S_n} \text{sgn}\sigma\left(\prod_{i=1}^{n} a_{i\sigma i}\right)\\ &= -\sum_{\delta\in S_n} \text{sgn}\delta\left(\prod_{i=1}^{n} a_{i\sigma i}\right)\\ &= -|A|.\end{aligned}$$

Hence the result.

Property 13. *If A is a square PFM, then the determinant value of A equals to that of its transpose, i.e.,* $|A| = |A^T|$.

Proof. Let $A = (a_{ij})$ be a square PFM of order n and let $P = A^T$ be the transpose of A. Then, by the definition of a pentagonal fuzzy determinant, we have $|P| = \sum_{\sigma\in S_n}(\text{sgn}\sigma \cdot \prod_{i=1}^{n} p_{i\sigma(i)}) = \sum_{\sigma\in S_n}(\text{sgn}\sigma \cdot \prod_{i=1}^{n} a_{\sigma(i)i})$. Let ϕ be a permutation on 1,2,...,n such that $\phi\sigma = I$, I being the identity permutation. Thus, $\phi = \sigma^{-1}$. Let $\sigma(i) = j$; then, $i = \sigma(j)^{-1}$ and $a_{\sigma(i)i} = a_{j\sigma(j)}$, $\forall\ i,j$.

$$\begin{aligned}\text{Therefore, } |P| &= \sum_{\sigma\in S_n}(\text{sgn}\sigma \prod_{i=1}^{n}\Big(p_{1\sigma(i)i}, p_{2\sigma(i)i}, p_{3\sigma(i)i}, p_{4\sigma(i)i}, p_{5\sigma(i)i}\Big)\\ &= \sum_{\sigma\in S_n} \text{sgn}\sigma \prod_{j=1}^{n}\Big(a_{1j\sigma(j)}, a_{2j\sigma(j)}, a_{3j\sigma(j)}, a_{4j\sigma(j)}, a_{5j\sigma(j)}\Big)\\ &= \sum_{\sigma\in S_n} \text{sgn}\sigma \prod_{i=1}^{n}\Big(a_{1i\sigma(i)}, a_{2i\sigma(i)}, a_{3i\sigma(i)}, a_{4i\sigma(i)}, a_{5i\sigma(i)}\Big)\\ &\quad [\textit{interchanging indices}] = |A|.\end{aligned}$$

Hence the result.

Property 14. *For a square PFM A of order n:*

(i) If A contains a zero row, then $adj(A)A$ is a null equivalent PFM.
(ii) $adj(A^T) = [adj(A)]^T$.

Proof. *(i)* Let $A = (a_{ij})$ be a square PFM of order $n \times n$, where $a_{ij} = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij})$. Let $B = adj(A)$; then, by the definition of the adjoint of a PFM, the $(ij)^{th}$ element of b_{ij} of B is $|A_{ij}|$, where A_{ij} is the sub matrix obtained from A by suppressing the i^{th} row and j^{th}

column, i.e., A_{ij} is the cofactor of a_{ij} in A. Without loss of generality, we assume that the k^{th} row of A is the zero row. Therefore, the elements of the k^{th} row are of the form $(\varepsilon_{1kj},\delta_{1kj},0,\delta_{2kj},\varepsilon_{2kj})$, $\delta_{1kj}\cdot\delta_{2kj}\neq 0$ and $\varepsilon_{1kj}\cdot\varepsilon_{2kj}\neq 0$ for all j. Then, all elements of $adj(A)$ are of the form $|A_{ij}| = (\varepsilon_{1kj},\delta_{1kj},0,\delta_{2kj},\varepsilon_{2kj})$, except $j\neq k$. Let $P = (adjA)A$. Then, the $(ij)^{th}$ element of P is of the form $p_{ij} = \sum_{m=1}^{n}|A_{im}|a_{mj} = \sum_{m\neq k}|A_{im}|a_{mj} + \sum_{m=1}^{n}|A_{ik}|a_{kj}$. Now all $|A_{im}|$, $m\neq k$, are of the form of the null equivalent PFN. Hence, p_{ij} is of the form of a null equivalent PFN for all $i,j = 1,2,...,n$. Thus, $(adjA)A$ is a null equivalent PFM.

(ii) The proof for this part obviously follows from the definition.

Property 15. *Let A be a square PFM of order n.*

(i) If A is a symmetric PFM, then $adj(A)$ is a symmetric PFM.
(ii) If A is a null equivalent PFM, then $adj(A)$ is also a null equivalent PFM.
(iii) If A is a pure unit PFM, then $adj(A)$ is a unit equivalent PFM.

Proof. *(i)* From Property 15, it is clear that the adjoint property for a PFM preserves transposition, i.e.,$adj(A^T) = [adj(A)]^T$. Because A is a symmetric PFM, $A^T = A$. Now $[adj(A)]^T = adj(A^T) = adj(A)$. Hence, $adj(A)$ is a symmetric PFM.

(ii) Let A be a null equivalent PFM of order n; then, all the elements of A are null equivalent PFNs, i.e., $a_{ij} = \tilde{0}$. Again, $adj(A)$ is the transpose of the cofactor matrix of A. Thus, $adj(A) = [A_{ij}]^T = (-1)^{i+j}M_{ji}$. Additionally, M_{ij} is the determinant of an $(n-1) \times (n-1)$ order matrix, deleting the i^{th} row and j^{th} column from A. Because each a_{ij} is a null equivalent PFN, the cofactors of the elements of A are null equivalent PFNs and hence its transpose. Finally, we conclude that $adj(A)$ is a null equivalent PFM.

(iii) Because A is a pure unit PFM of order n, its diagonal entries are of the form $a_{ii} = (0,0,1,0,0)$ and $a_{ij} = \tilde{0}$, $i\neq j$. It is now clear that the cofactors of diagonal elements are nothing but the determinant value of a pure unit PFM of order $(n-1) \times (n-1)$, which is a unit equivalent PFN and null equivalent PFN of non-diagonal elements. Hence, $adj(A)$ is a unit equivalent PFM.

8. Fuzzy comparable PFM

In this section, we newly introduce fuzzy comparable PFM to resolve the fuzzy order preference problems. Between any two matrices, there is an ordering relation: either they are equal or different. This deals with pairwise comparison of matrices under elementary-order priority. Pairwise comparison is applied whenever the decision maker is not sure regarding the evaluation of relative importance.

Here, we adopt the concept of fuzzy order relations between two elements of a fuzzy set, i.e., for a fuzzy set T and x, $y\in T$; the order relation denoted by order lattice "$\leq$" holds whenever $x\leq y$ or $y\leq x$.

Definition 27. ***Fuzzy comparable PFM.*** *Let P and Q be two PFMs of order $n \times n$. We say that P is comparable to Q if either $P \leq Q$ or $Q \leq P$, i.e., when $p_{ij} \leq q_{ij} \Rightarrow P \leq Q$ or $p_{ij} \geq q_{ij} \Rightarrow Q \leq P$. When both are equal, we called them equivalent PFM.*

Property 16. *Let P and Q be two PFMs of order $m \times n$. Then, we have the following:*

(i) For any PFM T of order $m \times n$, we have $P \leq Q$, which implies $P + T \leq Q + T$ and vice versa.
(ii) For any PFM R of order $n \times p$, we have $P \leq Q$, which implies $P\cdot R \leq Q\cdot R$.
(iii) If $P_1 \leq P_2$ and $Q_1 \leq Q_2$, then their product is also comparable, i.e., $P_1\cdot Q_1 \leq P_2\cdot Q_2$ for the compatible matrix product of $P_1\cdot Q_1$ and $P_2\cdot Q_2$.

Proof. *(i)* Because $P \leq Q$, then we have $p_{ij} \leq q_{ij}$, which implies $(p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij}) \leq (q_{1ij},q_{2ij},q_{3ij},q_{4ij},q_{5ij})$

$$\begin{aligned}\text{Now,}\ \ P + T = p_{ij} + t_{ij} &= \left(p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij}\right)\\ &\quad +(t_{1ij},t_{2ij},t_{3ij},t_{4ij},t_{5ij})\\ &\Rightarrow p_{ij}+t_{ij} \leq q_{ij}+t_{ij} [\text{because}\ \ P\leq Q]\\ &\Rightarrow P+T \leq Q+T\end{aligned}$$

Conversely, let $P + T \leq Q + T$. Then,

$$\begin{aligned}&p_{ij}+t_{ij} \leq q_{ij}+t_{ij}\\ &\Rightarrow \left(p_{1ij},p_{2ij},p_{3ij},p_{4ij},p_{5ij}\right) + (t_{1ij},t_{2ij},t_{3ij},t_{4ij},t_{5ij})\\ &\quad \leq \left(q_{1ij},q_{2ij},q_{3ij},q_{4ij},q_{5ij}\right) + (t_{1ij},t_{2ij},t_{3ij},t_{4ij},t_{5ij})\\ &\Rightarrow P \leq Q.\end{aligned}$$

(ii) Here, P, Q are comparable PFMs of order $m \times n$. Let R be any PFM of order $n \times p$.

Because $p_{ij} \leq q_{ij},\ (p_{ik}\cdot r_{kj}) \leq (q_{ik}\cdot r_{kj})\ \ \forall\ \ i,j,k.$ Therefore, $\sum_{k=1}^{n}(p_{ik}\cdot r_{kj}) \leq \sum_{k=1}^{n}(q_{ik}\cdot r_{kj})$, i.e., $P\cdot R \leq Q\cdot R$. Hence the result.

(iii) By a similar approach, we can get the result. It is observed that *(ii)* is the particular case of *(iii)* only when $Q_1 = Q_2$.

9. Some results of nilpotent PFM

Nilpotent matrices are of great importance in fuzzy algebra. Here, we define the nilpotent matrix in the fuzzy sense based on the pentagonal fuzzy matrix and study some properties that hold especially for pentagonal fuzzy matrices.

Definition 28. ***Nilpotent PFM.*** *Let $A = (a_{ij})_{n\times n}$ be a square PFM of order n. A is said to be nilpotent for the index λ if λ is the least positive integer such that $A^{\lambda} = \tilde{O}$.*

Property 17. *Let A and B be two nilpotent PFM of index m,n, respectively. Then, $A\cdot B$ and $A + B$ are both nilpotent whenever $A\cdot B = B\cdot A$.*

Proof. Let A and B be two nilpotent PFMs of index m,n, respectively. Then, $A^m = \tilde{O}$ and $B^n = \tilde{O}$. Again, let $k = lcm(m,n)$.

$$\begin{aligned}\text{Now,}\ \ (A\cdot B)^k &= (A\cdot B)(A\cdot B)(A\cdot B)\cdots k\ \text{ times.}\\ &= (AABB)[(A\cdot B)(A\cdot B)\cdots(k-2)\ \text{ times.}]\\ &= \left(A^2\cdot B^2\right)[(A\cdot B)(A\cdot B)\cdots(k-2)\ \text{ times.}]\\ &= \left(A^3\cdot B^3\right)[(A\cdot B)(A\cdot B)\cdots(k-3)\ \text{ times.}]\\ &= \left(A^k\cdot B^k\right),\ \ [\text{because}\ A\cdot B = B\cdot A]\\ &= \tilde{O}.\end{aligned}$$

Thus, the product of two nilpotent PFMs is also a nilpotent PFM.

Second part. The nilpotency of $A + B$ can be shown directly from the Binomial theorem.

Property 18. *Every strictly fuzzy triangular PFM of order n is nilpotent for index n.*

Proof. Let us consider an $n \times n$ strictly fuzzy upper triangular PFM $A = (\alpha_{ij})$, where $(\alpha_{ij}) = (\alpha_{1ij},\alpha_{2ij},\alpha_{3ij},\alpha_{4ij},\alpha_{5ij})$. Let A be of the form

$$A = \begin{pmatrix}\tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{0}\\ \alpha_{21} & \tilde{0} & \cdots & \cdots & \tilde{0}\\ \vdots & & \ddots & & \vdots\\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{n\,n-1} & \tilde{0}\end{pmatrix}$$

Let the entries of A^2 be β_{ij}; then, $(\beta_{ij}) = \sum_{k=1}^{n}\alpha_{ik}\alpha_{kj}$, for

$i,j = 1,2,...,n$. Because A is strictly upper triangular, $\alpha_{ij} = \tilde{0}, i \le j$. Thus, by looking at the entries of A, we have $\beta_{ij} = \tilde{0},\ \forall\ i \le j$. Now, for $j = i-1$, $\beta_{ii-1} = \sum_{k=1}^{n} \alpha_{ik}\alpha_{ki-1} = \alpha_{1k}\alpha_{1i-1} + \alpha_{i2}\alpha_{ki-1} + \cdots \alpha_{in}\alpha_{ni-1}$. Because each α_{in} or $\alpha_{i\,n-1}$ will lie on or above the principle diagonal of A, $\beta_{ii-1} = \tilde{0}$. $A^2 = \begin{pmatrix} \tilde{0} & \tilde{0} & \cdots & \cdots & & \tilde{0} \\ \tilde{0} & \tilde{0} & \cdots & \cdots & & \tilde{0} \\ \beta_{21} & \tilde{0} & \cdots & & & \tilde{0} \\ \vdots & \ddots & \vdots & & \ddots & \vdots \\ \beta_{n1} & \cdots & \cdots & \beta_{n\,n-2} & \tilde{0} & \tilde{0} \end{pmatrix}$

That is, $\beta_{ij} = \tilde{0}$, for $i-1 \le j$. We see that each time we raise A to its power, the next diagonal under the principle diagonal becomes zeros. Again, let us assume that this occurs for the power of A, i.e., for A^k, the k^{th} diagonals, including the main diagonal, become zeros. We assume the entries of A^k to be γ_{ij}; then, $\gamma_{ij} = \tilde{0}$ for $i-k+1 \le j$. Now it is sufficient to prove that the $(k+1)^{th}$ power of A is a strictly upper triangular PFM having the next k diagonals until the principle diagonal vanishes. Let the elements of A^{k+1} be δ_{ij} because $A^{k+1} = AA^k$. Thus, we have $\delta_{ij} = \sum_{k=1}^{n} \alpha_{ik}\gamma_{kj}$ because for all $i \le j$, $\alpha_{ij} = \tilde{0}$. Thus, it is clear that $\delta_{ij} = \tilde{0},\ \forall\ i \le j$ for $i,j = 1,2,...,n$. Again, $\delta_{ij} = \sum_{k=1}^{n} \alpha_{ik}\gamma_{kj}$. We see that for $i-k+1 \le j$, $\delta_{ii-k+1} = \alpha_{i1}\gamma_{1i-k+1} + \alpha_{i2}\gamma_{2i-k+1} + \cdots + \alpha_{in}\gamma_{ni-k+1}\ \ \forall\ i,j = 1,2,\ldots,n.$ From the above expression, we have α_{in} or γ_{ni-k+1} vanishes for $\gamma_{ij} = \tilde{0}$, $i-k+1 \le j$.

Therefore, A^{k+1} gives the PFM whose $(k+1)^{th}$ diagonals under the main diagonal are zero, i.e., of the form

$$A^{k+1} = \begin{pmatrix} \tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \tilde{0} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \cdots & \cdots & \cdots & \cdots & \\ \gamma_{k+1\,1} & \tilde{0} & \cdots & \cdots & \tilde{0} \\ \vdots & \ddots & & & \vdots \\ \gamma_{n\,1} & \cdots & \gamma_{n\,n-k+1} & \tilde{0}\cdots & \tilde{0} \end{pmatrix}$$

Hence, the result is true for $n = k+1$ when it is true for $n = k$. Additionally, the result is true for $n = 2$. Therefore, by mathematical induction, we conclude that the result is true for all n. Thus, A^n finally produces a pure null PFM, i.e., $A^n = \tilde{O}$.

Hence, a strictly fuzzy triangular PFM of order n is nilpotent for the index that is exactly n.

Property 19. *(i) If A and B are both strictly fuzzy triangular PFMs, then the block matrix* $\begin{pmatrix} A & C \\ \tilde{O} & B \end{pmatrix}$ *is nilpotent.*

(ii) Generally, $\begin{pmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{nn} \end{pmatrix}$ is nilpotent whenever A_{ii}'s are strictly fuzzy triangular PFMs.

Proof. *(i)* Let us introduce the concept of block matrix P of the form $P = \begin{pmatrix} A & C \\ \tilde{O} & B \end{pmatrix}$, where A,B are strictly fuzzy triangular PFMs of order m, n, respectively. Thus, based on the property, A and B are both nilpotent for index m and n, respectively.

$$\text{Now, } P^2 = \begin{pmatrix} A & C \\ \tilde{O} & B \end{pmatrix}\begin{pmatrix} A & C \\ \tilde{O} & B \end{pmatrix} = \begin{pmatrix} A^2 & AC+BC \\ \tilde{O} & B^2 \end{pmatrix}$$
$$P^3 = \begin{pmatrix} A^2 & AC+BC \\ \tilde{O} & B^2 \end{pmatrix}\begin{pmatrix} A & C \\ \tilde{O} & B \end{pmatrix} = \begin{pmatrix} A^3 & A^2C+B^2C+ABC \\ \tilde{O} & B^2 \end{pmatrix}$$

Thus, note that when we raise the power to P, the elements p_{11} and p_{22} increase their power that of element p_{12}, i.e., in general, we have $P^k = \begin{pmatrix} A^k & \mathring{a} \\ \tilde{O} & B^k \end{pmatrix}$, with k being a positive integer and assuming "$\mathring{a}$" as the value of the element p_{21} in P^k. A and B are nilpotent for index m,n, respectively; therefore, $A^m = \tilde{O}$, $B^n = \tilde{O}$. Taking $\lambda = lcm(m,n)$ (say), $(P^k)^\lambda = \begin{pmatrix} A^{k\lambda} & \mathring{a} \\ \tilde{O} & B^{k\lambda} \end{pmatrix} = \begin{pmatrix} \tilde{O} & \mathring{a} \\ \tilde{O} & \tilde{O} \end{pmatrix}$, which is a strictly fuzzy triangular PFM; hence, based on Property 18, P is a nilpotent PFM. Hence the proof.

Second Part. *(ii)* It follows from the previous properties.

10. Singular and constant PFM

Definition 29. ***Singular PFM.*** *A square PFM A is said to be singular if the determinant value is a pure null PFN, i.e.,* $|A| = (0,0,0,0,0)$.

Definition 30. ***Semi-singular PFM.*** *A square PFM is called semi-singular when its determinant value produces a null equivalent PFN, i.e.,* $|A| = (\delta_1, \varepsilon_1, 0, \varepsilon_2, \delta_2),\ \ \delta_1 \cdot \varepsilon_1 \ne 0,\ \ \delta_2 \cdot \varepsilon_2 \ne 0.\ \ \forall\ \ i,j = 1,2,\ldots,n.$

Definition 31. ***Constant PFM.*** *A square PFM* $A = (a_{ij})$ *of order* $n \times n$ *is called a constant PFM if all the rows are equal to each other, i.e.,* $(a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij}) = (a_{1rj}, a_{2rj}, a_{3rj}, a_{4rj}, a_{5rj})\ \ \forall\ \ i,r,j.$

Example 1. *For example, we consider a constant square PFM A of order 3 as*

$$A = \begin{pmatrix} (-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5) \\ (-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5) \\ (-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5) \end{pmatrix}$$

Property 20. *Let A and B be two constant PFMs of the same order. Then, the following holds.*

(i) $A + B$ is a constant PFM.
(ii) $A \cdot B$ is also a constant PFM.

Proof. *(i)* Let $A = (a_{ij})$ and $B = (b_{ij})$, where $(a_{ij}) = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij})$ and $b_{ij} = (b_{1ij}, b_{2ij}, b_{3ij}, b_{4ij}, b_{5ij})$ are two constant PFMs of order n. Then, $(a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij}) = (a_{1rj}, a_{2rj}, a_{3rj}, a_{4rj}, a_{5rj})$ and $(b_{1ij}, b_{2ij}, b_{3ij}, b_{4ij}, b_{5ij}) = (b_{1rj}, b_{2rj}, b_{3rj}, b_{4rj}, b_{5rj})$

$$\begin{aligned} \text{Let } C = (c_{ij}) &= (a_{ij} + b_{ij}) = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij}) + (b_{1ij}, b_{2ij}, b_{3ij}, b_{4ij}, b_{5ij}).\ \ i,r,j = 1,2,\ldots,n \\ &= (a_{1rj}, a_{2rj}, a_{3rj}, a_{4rj}, a_{5rj}) + (b_{1rj}, b_{2rj}, b_{3rj}, b_{4rj}, b_{5rj}) \\ &\quad [\text{because } A, B \text{ are constant.}] \\ &= (c_{1rj}, c_{2rj}, c_{3rj}, c_{4rj}, c_{5rj}) = c_{rj}\ \ \forall\ \ i,r,j. \end{aligned}$$

Thus, the rows of $(A + B)$ are similar to each other.
Hence the proof.
(ii) The proof for this part follows from the definition.

Property 21. *Let A be any square PFM; then, the following results hold:*

(i) $[adjA]^T$ is constant.
(ii) $A.adjA$ is constant.
(iii) $A.adj(A^T)$ is constant.
(iv) $(A^T \cdot adjA)^T$ is constant.
(v) $(A^T \cdot adjA)$ is constant.
(vi) The determinant value of a constant PFM is a null equivalent PFM.

Proof. *(i)* Because $A = (a_{ij})$ is a constant PFM of order $n \times n$, where $(a_{ij}) = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij})$, then $(a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij}) = (a_{1rj}, a_{2rj}, a_{3rj}, a_{4rj}, a_{5rj})$. Now, let $b_{ij} = adj(A)$.

Then,

$$b_{ij} = \sum_{\pi \in S_{n_i n_j}} \prod_{t \in n_j} a_{t\pi(t)}$$

and

$$b_{ik} = \sum_{\pi \in S_{n_i n_k}} \prod_{t \in n_k} a_{t\pi(t)}$$

It is obvious that $b_{ij} = b_{ik}$ because the numbers $\pi(t)$ of the columns cannot be changed in the two expansions of b_{ij} and b_{ik}. Thus, $[adjA]^T$ is constant.

(ii) Because A is a constant PFM, both $A_{jk} = A_{ik}$ and $det(A_{jk}) = det(A_{ik})$ hold for every $i, j \in \{1,2,...,n\}$. Again, let $C = A \cdot adjA$. Thus,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot \det\left(A_{jk}\right) = \sum_{k=1}^{n} a_{ik} \cdot \det(A_{ik})$$

Additionally, from the definition of the determinant of a PFM in terms of the adjoint, $\det(A) = \sum_{k=1}^{n} a_{ik} \cdot \det(A_{ik})$. Thus, $c_{ij} = det(A)$. Thus, similar to the fuzzy matrix, this result also holds well for a pentagonal fuzzy matrix.

(iii)→*(iv)* These proofs can be obtained via transpose operations and also by transposing of the constant PFM to remain constant.

(v) We earlier proved (Property 14) that the determinant of a PFM A having a zero row is a null equivalent PFN. Additionally, for a constant PFM A, the rows are equal to each other. Thus, one can get a zero row via the elementary row operation. Hence the result.

11. Conclusion

In this article, special attention is paid to the pentagonal fuzzy number (PFN) and the corresponding pentagonal fuzzy matrix (PFM), along with the related mathematical expressions. Based on applying elementary algebraic operations to the PFM, we studied various types of PFM and their properties (determinant, adjoint, trace, etc.). Second, this paper addresses the nature of nilpotent PFM and comparable PFM, with some interesting properties. There are several opportunities to develop the applications of such pentagonal fuzzy number. We are trying to investigate such applications.

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