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Modelling and analysis of a harvested prey–predator system incorporating a prey refuge

Tapan Kumar Kar

Department of Mathematics, B. E. College (a Deemed University), Howrah 711103, India

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Abstract

The present paper deals with a prey–predator model incorporating a prey-refuge and independent harvesting in either species. Our study shows that, using the harvesting efforts as controls, it is possible to break the cyclic behaviour of the system and drive it to a required state. The possibility of existence of bionomic equilibria has been considered. The problem of optimal harvest policy is then solved by using Pontryagin's maximal principle.

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1. Introduction

Economic progress and ecological balance always have conflicting interests. Catering to the necessities and comforts of human beings invariably robs the ecological structure of the nature. This, more often than not, leads to the extinction of a species of life. Often it is possible to prevent such extinction by proper planning. Such a planning has to be either by force or dissensive. For example, if a particular activity by individuals of a region is causing severe damage of the ecosystem of that region and if the activity is inevitable then the governing authority of the region should plan a regulating policy which would keep the damage of the ecosystem minimal. One such activity is harvesting, which has a strong impact on the dynamic evolution of a population subjected to it. Reasonable harvesting policies is indisputably one of

E-mail addresses: tkar@math.becs.ac.in, t_k_kar@yahoo.com.

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the major and interesting problems from ecological and economical point of view. The exploitation of biological resources and harvest of population species are commonly practiced in fishery, forestry and wildlife management. A management of multispecies fisheries which is needed to maintain an ecological balance, that is disrupted due to over exploitation of many conventional fish stocks and growing interest in harvesting new kinds of food from the sea.

The problem of predator–prey interactions under constant rate of harvesting or constant quota of harvesting of either species or both species simultaneously have been studied by some authors. For example, Brauer and Soudak [2–5] studied a class of predator–prey models under constant rate of harvesting and under constant quota of harvesting of both species simultaneously. They showed how to classify the possibilities of the quantitative behaviour of the solutions to locate the set of initial values in which the trajectories of the solutions approach to either an asymptotic stable equilibrium or an asymptotically stable limit cycle. Recently, Dai and Tang [9] studied the following predator–prey model in which two ecological interacting species are harvested independently with constant rates:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{k}\right) - a\phi(x)y - \mu, \\ \frac{dy}{dt} &= y(-d + ca\phi(x)) - h_1.\end{aligned}\tag{1.1}$$

They showed that system (1.1) possesses very complicated dynamics.

In this paper we consider the following set of prey–predator system:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x \left(1 - \frac{x}{k}\right) - \frac{\beta(1-m)xy}{1+a(1-m)x} - q_1 E_1 x, \\ \frac{dy}{dt} &= -\gamma y + \frac{c\beta(1-m)xy}{1+a(1-m)x} - q_2 E_2 y,\end{aligned}\tag{1.2}$$

where x and y denote the prey and predator population, respectively, at any time t . $\alpha > 0$ represents the intrinsic growth rate of the prey, k is the carrying capacity of the prey in the absence of predator and harvesting. The term $\beta x/(1 + \alpha x)$ denotes the functional response of the predator, which is known as Holling type II response function [13]. $c > 0$ is the conversion factor denoting the number of newly born predators for each captured prey. $\gamma > 0$ is the death rate of the predator. $E_1 \geq 0$, $E_2 \geq 0$ denote the harvesting efforts for the prey and predator, respectively. $q_1 E_1 x$ and $q_2 E_2 y$ represent the catch of the respective species, where q_1 and q_2 represents the catchability coefficients of the prey and predator, respectively. The model incorporates a refuge protecting mx of the prey, where $m \in [0, 1)$ is constant. This leaves $(1 - m)x$ of the prey available to the predator.

Mite prey–predator interactions often exhibit spatial refugia which afford the prey some degree of protection from predation and reduce the chance of extinction due to predation. Most of the theoretical and empirical prey–predator studies in ecology has focused mainly in the analysis of predator behaviour, a relatively small proportion of the ecological literature has addressed prey behaviour (see [17,11,20,8]). Refuge of prey being a natural phenomena, we have taken it into consideration.

To find models that represent stable limit cycle, an attracting self-sustained oscillation, is one of the main and primary problem in modern mathematical ecology. If a model has to describe some particular ecological system, structurally stable features (like limit cycles), which are common in real life systems, should be visible in the model. Hence, the necessity for finding conditions that guarantee the uniqueness

of a limit cycle arises. Quite a good number of papers have appeared in the literature on uniqueness of limit cycles in predator–prey system (see [6,15,12] and the references therein). Also a great deal of works have already been performed on harvested predator–prey system (see [14,16,19,10] and the references therein). Unfortunately, most of the results on harvested predator–prey models deal only with the existence or non-existence of periodic solutions and to the best of our knowledge no work has yet been done on the problem of uniqueness of limit cycles in harvested predator–prey systems. The second objective of this paper is to find out optimal harvesting strategies.

The content of this paper is as follows. In Section 2, we have proved that the system is uniformly bounded which, in turn, implies that the system is biologically well behaved. In Section 3 we study the existence of the equilibria and their dependence on the harvesting efforts E_1 and E_2 . We have concentrated more on the interior equilibrium of the system as we are interested in the coexistence of the species. In Section 4, we study the stability and instability properties of the equilibria, and existence of limit cycles for system (1.2). We also study the possibility of compressing an existing limit cycle to a point or introducing a limit cycle into the system when it is not present in it, using the harvesting efforts E_1 and E_2 as controls. It has also been shown that hiding factor m also has a stabilizing or destabilizing effect on the system. In Section 5, taking simple economic consideration into account, we discuss the possibilities of existence of a bionomic equilibrium. In Section 6, the optimal policy of exploitation is derived by using Pontryagin’s maximal principle. The problem ends with a brief concluding remarks.

2. Boundedness of the system

Theorem 1. *All the solutions of system (1.2) which start in R_+^2 are uniformly bounded.*

Proof. We define the function $w = x + (1/c)y$. Therefore, time derivative

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{1}{c} \frac{dy}{dt} = \alpha x \left(1 - \frac{x}{k}\right) - \frac{\beta(1-m)xy}{1+a(1-m)x} - q_1 E_1 x - \frac{\gamma}{c} y + \frac{\beta(1-m)xy}{1+a(1-m)y} - \frac{q_2 E_2}{c} y.$$

Now for each $v > 0$, we have

$$\frac{dw}{dt} + vw \leq \frac{k}{4\alpha} (v + \alpha - q_1 E_1)^2 - \frac{1}{c} (\gamma + q_2 E_2 - v).$$

Let us choose $v > \gamma + q_2 E_2$, then the right-hand side is positive. As we assume that both E_1 and E_2 are bounded, the right-hand side is bounded for all $(x, y) \in R_+^2$.

Thus we choose a $\mu > 0$ such that $dw/dt + vw < \mu$.

Applying the theory of differential inequality [1], we obtain

$$0 < w(x, y) < \frac{\mu}{v} (1 - e^{-vt}) + w(x(0), y(0))e^{-vt},$$

which upon letting $t \rightarrow \infty$, yields $0 < w < (\mu/v)$. So, we have, that all the solution of system (1.2) that start in R_+^2 are confined to the region B , where

$$B = \left\{ (x, y) \in R_+^2 : w = \frac{\mu}{v} + \varepsilon \text{ for and } \varepsilon > 0 \right\}.$$

3. Equilibrium analysis

We now study the existence of equilibrium of system (1.2). Particularly, we are interested in the interior or positive equilibrium of the system. To begin with we list all possible equilibria:

- (i) the trivial equilibrium $P_0(0, 0)$,
- (ii) equilibrium in the absence of predator ($y = 0$) $P_1((k/\alpha)(\alpha - q_1 E_1), 0)$,
- (iii) the interior (positive) equilibrium $P_2(x^*, y^*)$,

where

$$x^* = \frac{\gamma + q_2 E_2}{[c\beta - a(\gamma + q_2 E_2)](1 - m)},$$

$$y^* = \frac{\alpha c}{k} \left\{ \frac{k[c\beta - (\gamma + q_2 E_2)a](1 - m) - (\gamma + q_2 E_2)}{[(c\beta - a(\gamma + q_2 E_2)a](1 - m))^2} \right\} - \frac{cq_1 E_1}{[c\beta - (\gamma + q_2 E_2)a](1 - m)}.$$

From the expression for (x^*, y^*) , it is clear that a nontrivial (interior) equilibrium point exists for system (1.2) only if the harvesting rates E_1 and E_2 satisfy

$$c\beta - (\gamma + q_2 E_2)a > 0, \quad (3.1)$$

$$(\gamma + q_2 E_2)[\alpha + a(1 - m)k(\alpha - q_1 E_1)] - c\beta(1 - m)k(\alpha - q_1 E_1) < 0. \quad (3.2)$$

We see that an increase in fishing effort directed to the predator will increase x^* , which is natural as an increase in E_2 decreases the predator population and hence enhancing the survival rate of the prey. We also observe that as m increases x^* increases. Again y^* decreases when E_1 increases and this is happened due to loss of food.

Now differentiating y^* with respect to E_2 we observe that, for a fixed

$$E_1 \in \left[0, \frac{\alpha}{q_1} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \right]. \quad (3.3)$$

y^* attains its maximum value at

$$E_2 = \frac{1}{q_2} \left\{ \frac{c\beta [ka(1 - m)(\alpha - q_1 E_1) - \alpha]}{a [ka(1 - m)(\alpha - q_1 E_1) + \alpha]} - \gamma \right\} \quad (3.4)$$

and it decreases with further increases in E_2 . This maximum value of y^* , for fixed E_1 satisfying (3.3) is given by

$$\frac{1}{4k\alpha} \{ka(\alpha + q_1 E_1) + \alpha - 2ca\alpha q_1 E_1\} [ka(1 - m)(\alpha - q_1 E_1) + \alpha].$$

We observe that if $E_1 > \alpha/q_1$, then as m increases, E_2 decreases whereas if $E_1 < \alpha/q_1$, then as m increases, E_2 increases. Also we see that as m increases x^* increases by y^* decreases, as loss of food for the predator causing a slump in its survival rate.

4. Dynamic behaviour

In this section we study the stability properties of the equilibria P_0 , P_1 and P_2 . We also study the possibility of transition of an interior equilibrium from a stable state to an unstable state and vice versa using harvesting efforts E_1 and E_2 as controls. We also find the conditions under which the system exhibits a unique globally stable limit cycle.

The Jacobian of the system about the equilibrium point $P_0(0, 0)$ is given by

$$\begin{pmatrix} \alpha - q_1 E_1 & 0 \\ 0 & -(\gamma + q_2 E_2) \end{pmatrix}.$$

We find that the eigenvalues for the steady state $(0, 0)$ are $\alpha - q_1 E_1$ and $-(\gamma + q_2 E_2)$, which is always negative. Now we see that $\alpha - q_1 E_1$ is positive or negative according as the prey biotechnical productivity (BTP) α/q_1 is greater or less than the effort. Thus if the prey BTP exceeds the effort, the origin is saddle point, otherwise the origin is a locally asymptotically stable node.

The Jacobian of the system about the equilibrium point $P_1((k/\alpha)(\alpha - q_1 E_1), 0)$ is given by

$$\begin{pmatrix} -(\alpha - q_1 E_1) & \frac{-\beta(1 - m)k(\alpha - q_1 E_1)}{\alpha + ka(1 - m)(\alpha - q_1 E_1)} \\ 0 & -(\gamma + q_2 E_2) + \frac{c\beta(1 - m)(\alpha - q_1 E_1)k}{\alpha + ka(1 - m)(\alpha - q_1 E_1)} \end{pmatrix}. \tag{4.1}$$

Eigenvalues of matrix (4.1) are

$$(\alpha - q_1 E_1) \quad \text{and} \quad -(\gamma + q_2 E_2) + \frac{c\beta(1 - m)k(\alpha - q_1 E_1)}{1 + ak(1 - m)\alpha - q_1 E_1}.$$

Now, if $E_2 \geq 0$ and

$$\frac{\alpha}{q_1} \left[1 - \frac{\gamma + q_2 E_2}{k\{c\beta - (\gamma + q_2 E_2)a\}(1 - m)} \right] < E_1 < \frac{\alpha}{q_1}$$

then both the eigenvalues are negative and hence P_1 is locally asymptotically stable.

Jacobian matrix for P_2 is given by

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned} X &= \alpha - \frac{2\alpha}{k} \frac{(\gamma + q_2 E_2)}{[c\beta - (\gamma + q_2 E_2)a](1 - m)} - \frac{1}{kc\beta(1 - m)} \{ \alpha[k(c\beta - (\gamma + q_2 E_2)a)(1 - m) \\ &\quad - (\gamma + q_2 E_2)] - kq_1 E_1 [c\beta - (\gamma + q_2 E_2)a](1 - m) \} - q_1 E_1, \\ Y &= -\frac{\gamma + q_2 E_2}{c}, \\ Z &= \frac{1}{k\beta(1 - m)} \{ \alpha[k(c\beta - (\gamma + q_2 E_2)a)(1 - m) - (\gamma + q_2 E_2)] \\ &\quad - kq_1 E_1 [c\beta - (\gamma + q_2 E_2)a](1 - m) \}. \end{aligned}$$

It can be shown that all the eigenvalues of matrix (4.2) will be negative if

$$E_2 \notin \left(-\frac{\gamma}{q_2}, \frac{c\beta}{q_2a} \left[\frac{ka(1-m)(\alpha - q_1E_1) - \alpha}{ka(1-m)(\alpha - q_1E_1) + \alpha} - \gamma \right] \right).$$

The second term in the above parenthesis will be positive if

$$E_1 \in \left[0, \frac{\alpha}{q_1} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \right].$$

Therefore, interior equilibrium $P_2(x^*, y^*)$ will be asymptotically stable if

$$E_1 \in \left[0, \frac{\alpha}{q_1} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \right], \quad (4.3)$$

$$E_2 > \frac{c\beta}{q_2a} \left[\frac{ka(1-m)(\alpha - q_1E_1)}{ka(1-m)(\alpha - q_1E_1)} \right] - \frac{\gamma}{q_2}. \quad (4.4)$$

and unstable if

$$E_1 \in \left[0, \frac{\alpha}{q_1} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \right],$$

and

$$E_2 < \frac{c\beta}{q_2a} \left[\frac{ka(1-m)(\alpha - q_1E_1)}{ka(1-m)(\alpha - q_1E_1)} \right] - \frac{\gamma}{q_2}.$$

If

$$E_1 > \frac{\alpha}{q_1} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \quad (4.5)$$

then the interior equilibrium will be asymptotically stable for all $E_2 \geq 0$.

4.1. Uniqueness of limit cycles

It is known that for prey–predator systems existence and stability of a limit cycle is related to the existence and stability of a positive equilibrium. If the limit cycles do not exist, in this case the equilibrium is globally asymptotically stable. On the other hand if the positive equilibrium exists and unstable, there must occur at least one limit cycle.

Let us consider system (1.2) in the form

$$\begin{aligned} \frac{dx}{dt} &= xg(x) - yp(x), \quad x(0) > 0, \\ \frac{dy}{dt} &= y(-\gamma - q_2E_2 + q(x)), \quad y(0) > 0, \end{aligned} \quad (4.6)$$

where

$$g(x) = \alpha \left(1 - \frac{x}{k} \right) - q_1E_1, \quad p(x) = \frac{\beta(1-m)x}{1+a(1-m)x}, \quad q(x) = \frac{c\beta(1-m)x}{1+a(1-m)x}.$$

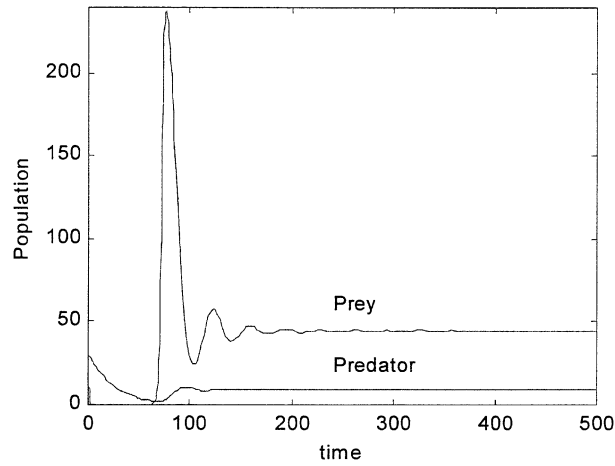


Fig. 1. Both the prey and predator population converge to their equilibrium values.

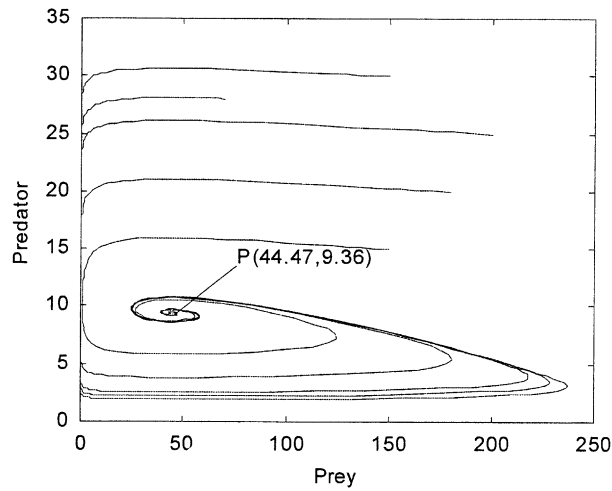


Fig. 2. Phase space trajectories corresponding to different initial levels, which shows that $P_2(44.47, 9.36)$ is a global attractor.

Now we consider the following theorem (see [15]) regarding uniqueness of limit cycles of the above system.

Theorem 2. Suppose for system (4.6)

$$\frac{d}{dx} \left(\frac{xg'(x) + g(x) - xg(x)(p'(x)/p(x))}{-\gamma - q_2E_2 + q(x)} \right) \leq 0$$

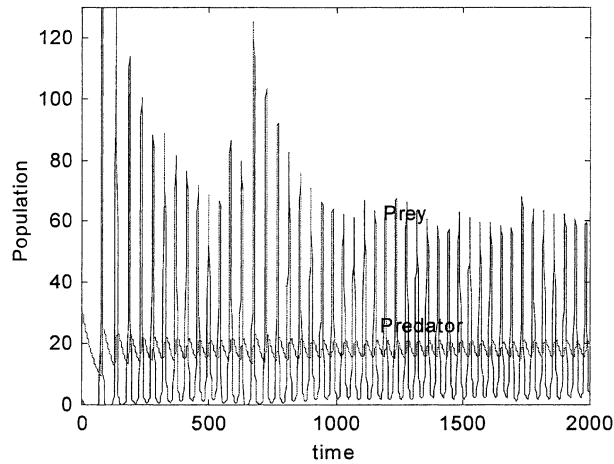


Fig. 3. There exist Hopf-bifurcating small amplitude periodic solutions.

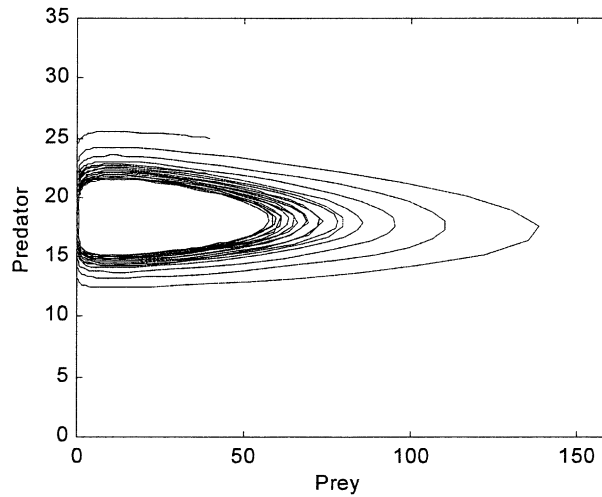


Fig. 4. Phase diagram of the limit cycle surrounding $P_2(10.79, 18.2)$.

in $0 \leq x < x^*$ and $x^* < x \leq k$. Then the above system has exactly one limit cycle which is globally asymptotically stable with respect to the set

$$\{(x, y) \mid x > 0, y > 0\} \setminus \{P_2(x^*, y^*)\}.$$

Following Theorem 2, we may state that when

$$E_1 \in \left[0, \frac{\alpha}{q} \left(1 - \frac{c\beta + a\gamma}{k(c\beta - a\gamma)} \right) \right] \quad \text{and} \quad E_2 < \frac{c\beta}{q_2 a} \left[\frac{ka(1-m)(\alpha - q_1 E_1) - \alpha}{ka(1-m)(\alpha - q_1 E_1) + \alpha} \right] - \frac{\gamma}{q_2},$$

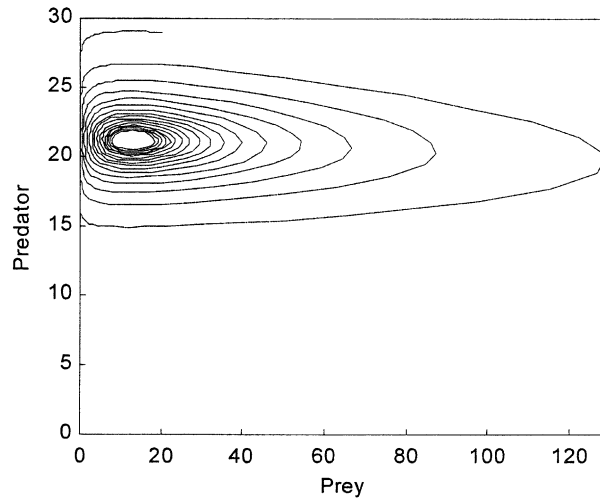


Fig. 5. Phase diagram of the limit cycle for $m = 0.15$ ($E_1 = 1.0$ and $E_2 = 0.5$).

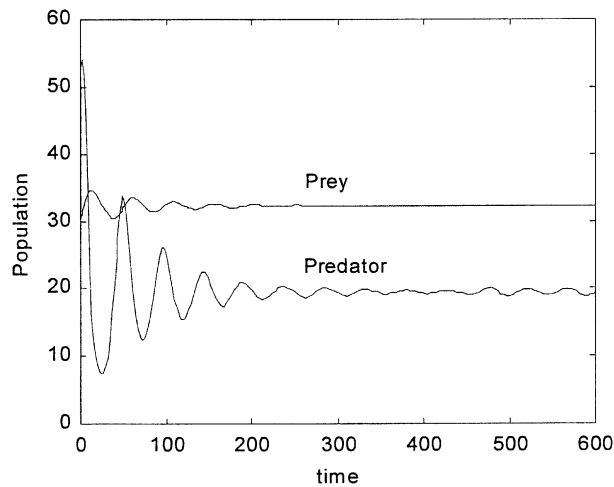


Fig. 6. Both the population converge to their equilibrium values for $m = 0.45$ ($E_1 = 1.0$ and $E_2 = 0.5$).

then system (1.2) has unique globally stable limit cycle. Thus we see that when the system is unstable, there exist unique globally stable limit cycle.

4.2. Simulation

Let $\alpha = 2.0$, $k = 600$, $\beta = 0.1$, $m = 0.01$, $a = 0.002$, $\gamma = 0.00046$, $c = 0.01$, $q_1 = 0.2$, $q_2 = 0.02$ in appropriate units.

For the above values of parameters, if $E_1 \in [0, 9.98]$ and $E_2 > 0.813$ then $P_2(x^*, y^*)$ exists and stable and if $E_1 > 9.98$ then $P_2(x^*, y^*)$ exists and stable for all $E_2 \geq 0$. On the other hand if $E_1 \in [0, 9.98]$ and $E_2 < 0.813$, then there exists unique globally stable limit cycle.

Let us take $E_1 = 5.0$ and $E_2 = 2.0$, then the corresponding equilibrium point $P_2(44.47, 9.36)$ is asymptotically stable (see Figs. 1 and 2).

Next let us take $E_1 = 1.0$ and $E_2 = 0.813$ then oscillation occurs as shown in Fig. 3.

Again taking $E_1 = 1.0$ and $E_2 = 0.5$, we see that the corresponding equilibrium point $P_2(10.79, 18.2)$ is unstable. The phase diagram, as shown in Fig. 4 is a limit cycle.

Thus taking E_1 and E_2 as control it is possible to drive the prey–predator system to required equilibrium point and to prevent the cyclic behaviour of the system.

From Figs. 5 and 6 we see that refuge parameter may also be used as control for system (1.2)

5. Bionomic equilibrium

Let

c_1 = fishing cost per unit effort for prey species,

c_2 = fishing cost per unit effort for predator species,

p_1 = price per unit biomass of the prey,

p_2 = price per unit biomass of the predator.

Therefore, the economic rent (net revenue) at any time is given by

$$\begin{aligned}\Pi &= (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 \\ &= \Pi_1 + \Pi_2 \text{ (say),}\end{aligned}$$

where $\Pi_1 = (p_1q_1x - c_1)E_1$, $\Pi_2 = (p_2q_2y - c_2)E_2$ i.e., Π_1 and Π_2 represent the net revenues for the prey and predator species, respectively.

The bionomic equilibrium $[x_\infty, y_\infty, E_{1\infty}, E_{2\infty}]$ is given by the following simultaneous equations:

$$\alpha \left(1 - \frac{x}{k}\right) - \frac{\beta(1-m)y}{1+a(1-m)x} - q_1E_1 = 0, \quad (5.1)$$

$$-\gamma + \frac{c\beta(1-m)x}{1+a(1-m)x} - q_2E_2 = 0, \quad (5.2)$$

$$\Pi = (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 = 0. \quad (5.3)$$

In order to determine the bionomic equilibrium, we now consider the following cases:

Case I: If $c_2 > p_2q_2y$, i.e. the cost is greater than the revenue for the predator, then the predator fishing will be stopped ($E_2 = 0$). Only the prey fishery remains operational (i.e. $c_1 < p_1q_1x$).

We then have $x_\infty = c_1/p_1q_1$, since $c_1 < p_1q_1x < p_1q_1k$. Hence $1 - c_1/kp_1q_1 > 0$ and $(y_\infty, E_{1\infty})$ will be any point on the line

$$\alpha \left(1 - \frac{c_1}{p_1q_1k}\right) = \frac{\beta(1-m)y}{1-a(1-m)(c_1/p_1q_1)} + q_1E_1$$

in the first quadrant of the yE_1 -plane.

Case II: If $c_1 > p_1q_1x$ i.e. the cost is greater than the revenue in the prey fishery, then prey fishery will be closed ($E_1 = 0$). Only predator fishery remains operational (i.e. $c_2 < p_2q_2y$).

We then have $y_\infty = c_2/p_2q_2$. Now substituting y_∞ into (5.1) we get $Ax^2 + Bx + C = 0$ where $A = (a\alpha/k)(1-m) > 0$, $B = \alpha/k - a\alpha(1-m)$, $C = -[\alpha - (\beta(1-m)c_2)/p_2q_2]$.

Therefore,

$$x^\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Now if

- (i) $C < 0$, then x^+ is the only positive solution.
(ii) $B < 0$, $C > 0$, $B^2 > 4AC$, then there are two solutions x^\pm . Substituting x^\pm into (5.2) we get

$$E_{2\infty}^\pm = \frac{1}{q_2} \left[\frac{c\beta(1-m)x_\infty^\pm}{1 + a(1-m)x_\infty^\pm} - \gamma \right].$$

$E_{2\infty} > 0$, provided $x_\infty^\pm > \gamma/(1-m)(c\beta - a\gamma)$ and $c\beta > a\gamma$.

Case III: If $c_1 > p_1q_1x$, $c_2 > p_2q_2y$, then the cost is greater than revenues for both the species and the whole fishery will be closed.

Case IV: If $c_1 < p_1q_1x$ and $c_2 < p_2q_2y$, the revenues for both the species being positive, then the whole fishery will be in operation.

In this case, $x_\infty = c_1/p_1q_1$ and $y_\infty = c_2/p_2q_2$.

Now substituting x_∞ and y_∞ into (5.1) and (5.2) we get

$$E_{1\infty} = \frac{\alpha}{q_1} \left(1 - \frac{c_1}{kp_1q_1} \right) - \frac{\beta(1-m)c_2p_1}{p_2q_2[p_1q_1 + a(1-m)c_1]},$$

and

$$E_{2\infty} = -\frac{\gamma}{q_2} + \frac{c\beta(1-m)c_1}{q_2[p_1q_1 + a(1-m)c_1]}.$$

Now,

$$E_{1\infty} > 0 \quad \text{if} \quad \frac{\alpha}{q_1} \left(1 - \frac{c_1}{kp_1q_1} \right) > \frac{\beta(1-m)c_2p_1}{p_2q_2[p_1q_1 + a(1-m)c_1]}, \quad (5.4)$$

$$E_{2\infty} > 0 \quad \text{if} \quad \gamma < \frac{c\beta(1-m)c_1}{[p_1q_1 + a(1-m)c_1]}. \quad (5.5)$$

Thus the nontrivial bionomic equilibrium point $[x_\infty, y_\infty, E_{1\infty}, E_{2\infty}]$ exists if and only if conditions (5.4) and (5.5) hold together.

6. Optimal harvesting policy

The fundamental problem in commercial exploitation of renewable resources is to determine the optimal trade-off between current and future harvests. This problem is too formidable to solve if the political, social and philosophical dimensions associated with it need to be examined. However, if we look at the

problem from the economic view point only, we have to use the standard technique of time discounting [7] to address questions of inter-temporal economic benefits. Time discounting of economic benefits (or costs) is a normal practice in business management. It has been shown by Clark [7] that the concept of maximizing sustained economic rent is unrealistic because it amounts to setting the discount rate equal to zero.

Our objective is to maximize the present value J of a continuous time-stream of revenues given by

$$J = \int_0^\infty e^{-\delta t} \{ (p_1 q_1 x - c_1) E_1(t) + (p_2 q_2 y - c_2) E_2(t) \} dt, \tag{6.1}$$

where δ denotes the instantaneous annual rate of discount. We intend to maximize (6.1) subject to the state equations (1.2) by invoking Pontryagin’s maximal principle [18]. The control variable $E_i(t)$ ($i = 1, 2$) are subjected to the constraints

$$0 \leq E_i(t) \leq (E_i)_{\max}.$$

The Hamiltonian for the problem is given by

$$H = e^{-\delta t} \{ (p_1 q_1 x - c_1) E_1 + (p_2 q_2 y - c_2) E_2 \} + \lambda_1 (F_1 - q_1 x E_1) + \lambda_2 (F_2 - q_2 y E_2), \tag{6.2}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint variables and

$$F_1 = \alpha x(1 - x/k) - \frac{\beta(1 - m)xy}{1 + a(1 - m)x},$$

$$F_2 = -\gamma y + \frac{c\beta(1 - m)xy}{1 + a(1 - m)x}.$$

The control variables E_1 and E_2 appear linearly in the Hamiltonian function H .

Assuming that the control constraints are not binding i.e., the optimal solution does not occur at $(E_i)_{\min}$ or $(E_i)_{\max}$, we have singular control [7] given by

$$\frac{\partial H}{\partial E_i} = 0, \quad i = 1, 2.$$

Now

$$\frac{\partial H}{\partial E_1} = 0 \Rightarrow \lambda_1 = e^{-\delta t} \left(p_1 - \frac{c_1}{q_1 x} \right), \tag{6.3}$$

$$\frac{\partial H}{\partial E_2} = 0 \Rightarrow \lambda_2 = e^{-\delta t} \left(p_2 - \frac{c_2}{q_2 y} \right). \tag{6.4}$$

Thus the shadow prices $e^{\delta t} \lambda_i(t)$, ($i = 1, 2$) do not vary with time in optimal equilibrium.

Hence they satisfy the transversality condition at ∞ , i.e. they remain bounded as $t \rightarrow \infty$.

Again $\partial H / \partial E_1 = 0 \Rightarrow \lambda_1 q_1 x = \partial \Pi_1 / \partial E_1$ and $\partial H / \partial E_2 = 0 \Rightarrow \lambda_2 q_2 y = \partial \Pi_2 / \partial E_2$.

This implies that, for each species, the user cost of harvest per unit effort must equal the discounted value of the future marginal profit of effort at the steady-state effort level.

We intend to derive here an optimal equilibrium solution of the problem. Since we are considering an equilibrium solution, x , y and E are to be treated as constants in the subsequent steps.

Now

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x} \\ &= -\left[e^{\delta t} (p_1 q_1 E_1) + \lambda_1 \left\{ \alpha \left(1 - \frac{2x}{k} \right) - \frac{\beta(1-m)y}{(1+a(1-m)x)^2} - q_1 E_1 \right\} \right. \\ &\quad \left. + \lambda_2 \left\{ \frac{c\beta(1-m)y}{(1+a(1-m)x)^2} \right\} \right]. \end{aligned} \tag{6.5}$$

Substituting λ_1 and λ_2 into (6.4) and after simplification we get

$$\begin{aligned} &\delta p_1 q_1 a^2 (1-m)^2 x^3 + \{2\delta p_1 q_1 a(1-m) - \delta c_1 a^2 (1-m)^2\} x^2 + \{\delta p_1 q_1 - 2\delta c_1 a(1-m)\} x - \delta c_1 \\ &= \left\{ p_1 q_1 \alpha - q_1 c \beta (1-m) \frac{c_2}{q_2} \right\} x - \frac{2p_1 q_1 \alpha x^2}{k} + [\{p_1 q_1 \beta a(1-m)^2 - p_1 q_1 \beta (1-m)\} x^2 \\ &\quad - \{c_1 a \beta (1-m)^2 + p_1 q_1 \beta a(1-m)^2\} x + (1-m) p_2 q_1 c \beta x] y. \end{aligned} \tag{6.6}$$

Again,

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\left[e^{-\delta t} p_2 q_2 E_2 + \lambda_1 \left\{ \frac{-\beta(1-m)x}{1+a(1-m)x} \right\} \right]. \tag{6.7}$$

Substituting λ_1 and λ_2 into (6.7) and after simplifying we get

$$y = \frac{\delta c_2 \{1+a(1-m)x\}}{p_2 q_2 (\delta + \gamma) - q_2 \beta (1-m) \frac{c_1}{q_1} + (1-m) \{p_2 q_2 a (\delta + \gamma) - q_2 \beta (c p_2 - p_1)\} x}. \tag{6.8}$$

Again from (6.6) and (6.8) we get

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \tag{6.9}$$

where

$$\begin{aligned} A &= \delta p_1 q_1 a^2 q_2 (1-m)^3 \{p_2 a (\delta + \gamma) - \beta (c p_2 - p_1)\}, \\ B &= \delta p_1 q_1 a^2 (1-m)^2 \left\{ p_2 q_2 (\delta + \gamma) - q_2 \beta (1-m) \frac{c_1}{q_1} \right\} + \left[\delta a (1-m) \{2q_1 p_1 - c_1 a (1-m)\} \right. \\ &\quad \left. + \frac{2p_1 q_1 \alpha}{k} \right] + [q_2 (1-m) \{p_2 a (\delta + \gamma) - \beta (c p_2 - p_1)\}] \\ &\quad - [p_1 q_1 \beta (1-m) \{a(1-m) - 1\} \delta c_2 a (1-m)], \\ C &= [q_2 \{p_2 (\delta + \gamma) - \beta (1-m)\}] \left[a \delta (1-m) \{2p_1 q_1 - c_1 a (1-m)\} + \frac{2q_1 p_1 \alpha}{k} \right] \\ &\quad + \left[p_1 q_1 (\delta - \alpha) + (1-m) \left\{ q_1 \beta \frac{c_2}{q_2} c - 2\delta c_1 a \right\} \right] [(1-m) q_2 \{p_2 a (\delta + \gamma) - \beta (c p_2 - p_1)\}] \\ &\quad - \delta c_2 \beta (1-m) [p_2 q_1 c - a(1-m)(c_1 + p_1 q_1)], \end{aligned}$$

$$D = q_2(1 - m)\{p_2a(\delta + \gamma) - \beta(cp_2 - p_1)\} \\ + q_2[p_2(\delta + \gamma) - \beta(1 - m)] \left[p_1q_1(\delta - \alpha) + (1 - m) \left\{ q_1\beta \frac{c_2}{q_2}c - 2\delta c_1a \right\} \right] - \delta c_2a(1 - m), \\ E = q_2 \left\{ p_2(\delta + \gamma) - \beta(1 - m) \frac{c_1}{q_1} \right\} - \delta c_1.$$

After finding possible values of $x = x^*$ from (6.9), we have $y = y^*$ from (6.8).

We then have

$$E_1^* = \frac{1}{q_1} \left\{ \alpha \left(1 - \frac{x^*}{k} \right) - \frac{\beta(1 - m)y^*}{1 + a(1 - m)x^*} \right\}$$

and

$$E_2^* = \frac{1}{q_2} \left\{ -\gamma + \frac{c\beta(1 - m)y^*}{1 + a(1 - m)x^*} \right\}.$$

Hence once the optimal equilibrium (x^*, y^*) is obtained, the optimal harvesting effort E_1^* and E_2^* can be determined.

Simulation: For simulation let us take $\alpha=2.0$, $k=600$, $\beta=0.1$, $m=0.01$, $a=0.002$, $\gamma=0.00046$, $c=0.01$, $q_1 = 0.2$, $q_2 = 0.02$, $\delta = 0.005$, $p_1 = 1.0$, $p_2 = 6.0$, $c_1 = 5.0$, $c_2 = 6.0$.

For the above values of parameters we get the optimal equilibrium (26.86, 19.37) and corresponding optimal harvesting efforts are $E_1^* = 0.45$, $E_2^* = 1.24$.

7. Concluding remarks

In this paper a harvested predator–prey system with Holling type II response function is considered. Incorporating a prey refuge into system (1.2) we have made the model more realistic since many prey mite populations do have some form of refuge available. This work presents analysis of the effect of harvesting efforts and prey refuge on the prey–predator system. We have proved that exactly one stable limit cycle occurs in the system when the positive equilibrium is unstable. This proof also enables us to conclude that local asymptotic stability of the positive equilibrium implies its global asymptotic stability. It was also found that it is possible to control the system in such a way that the system approaches a required state, using the efforts E_1 and E_2 as controls.

We then examine the possibilities of existence of bionomic (biological as well as economic) equilibria of the system. Next the optimal harvesting policy is discussed using Pontryagin's maximal principle.

Before ending this article, we would like to mention that there is still tremendous amount of work to do in this model. For example,

- (i) One can consider the effort levels E_i ($i = 1, 2$) to be dynamic (that is time dependent) variable.
- (ii) Optimal approach path consisting of a combination of bang-bang controls and nonequilibrium singular controls may be found though it is very difficult as pointed out by Clark [7].
- (iii) Gestation period for predator is also an important characteristic to be considered. We leave it for future considerations.

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