Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 208 (2007) 497–530
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Interpolation categories for homology theories

Georg Biedermann

Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario N6A 5B7, Canada

Received 28 August 2005; received in revised form 13 January 2006
Available online 20 March 2006
Communicated by E.M. Friedlander

For Ralf Fütterer

Abstract

For a homological functor from a triangulated category to an abelian category satisfying some technical assumptions, we construct a tower of interpolation categories. These are categories over which the functor factorizes and which capture more and more information according to the injective dimension of the images of the functor. The categories are obtained by using truncated versions of resolution model structures. Examples of functors fitting in our framework are given by every generalized homology theory represented by a ring spectrum satisfying the Adams–Atiyah condition. The constructions are closely related to the modified Adams spectral sequence and give a very conceptual approach to the associated moduli problem and obstruction theory. As an application, we establish an isomorphism between certain $E(n)$-local Picard groups and some Ext-groups.

MSC: 55S35; 55S45; 55T15

1. Introduction

Algebraic topology, or more precisely homotopy theory, is the study of geometric objects up to weak homotopy equivalence by translating the geometrical or homotopical information into algebraic data. The mathematical devices to do this are functors from a topological category to an algebraic category. Computationally very useful are homological functors or homology theories, which are functors satisfying excision. Examples in classical algebraic topology are abundant: singular homology, K-theory and cobordism, and more. For the purpose of this article, we adopt the general definition that a homological functor $F\colon \mathcal{T} \to \mathcal{A}$ from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$ is an additive functor which maps distinguished triangles to long exact sequences. Here, $\mathcal{A}$ is graded in the following sense: there is a self equivalence [1] of $\mathcal{A}$ called shift, such that

$$F(\Sigma X) \cong (FX)[1]$$

via a natural isomorphism. The functors that we consider have to satisfy some standard technical conditions 3.10 met by most examples in topology.

E-mail address: gbiederm@uwo.ca.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.01.016

For such a functor we would like to study questions of the following kind: given an object A in the abelian target category $\mathcal{A}$, does there exist an object X in the triangulated source category $\mathcal{T}$ together with an isomorphism $FX \cong A$ and, if yes, how many different objects exist? We can ask the same question for morphisms. This is the realization or moduli problem for the homological functor F.

This article contributes a theory of interpolation categories for F and an obstruction calculus for lifting objects or morphisms. The categories are intended to give a very conceptual approach to realizations and moduli problems of this functor and to the associated obstruction theory. They interpolate in a precise sense between the topological source and the algebraic target category. The obstruction groups will lie in the E_2-term of the F-based Adams spectral sequence.

The way that we set up the obstruction calculus follows the philosophy of [5] and [17]. We use injective resolutions in $\mathcal{A}$ and try to realize them step by step as cosimplicial objects over $\mathcal{M}$, where $\mathcal{M}$ is a stable model category having $\mathcal{T}$ as its homotopy category. As we proceed by gluing certain layers 4.6 to our potential liftings to kill their higher cohomology, we run into difficulties giving us the obstruction groups; see Theorems 4.24 and 4.27. They can be described as certain mapping spaces 4.13 out of these layers.

In order to be able to identify cosimplicial objects stemming from different resolutions, we consider resolution model structures on the category $c\mathcal{M}$ of cosimplicial objects. These were invented in [14] and used for exactly this purpose. Recently, Bousfield in [7] gave a very general and elegant treatment of resolution model structures which exhibits all previous instances as special cases.

In [3], I truncate these resolution model structures, and this article is based on the results there. Weak equivalences are now given by maps which induce isomorphisms on cohomology just up to degree n 2.7. The successive stages in our realization process are now given as cofibrant objects in these truncated resolution model structures. For an object X in $\mathcal{M}$, they are given by the skeletons of a Reedy cofibrant replacement of the constant cosimplicial object over X and fit into a filtration called the Postnikov cotower 2.9. This organizes the obstruction calculus nicely and enables us to define interpolation categories 4.36. We carry out in full detail the obstruction calculus for realizing maps, Theorems 4.30 and 4.35, and prove in 5.1 that there is a full-fleshed theory of interpolation categories, as axiomatized in [1] which keeps track of how obstructions behave under composition. We also obtain a description 4.33 of the Adams differential $d_n\colon\ E_n^{0,0}(X,Y) \to E_n^{n,n-1}(X,Y)$ in terms of our co-k-invariants.

When we finally have realized an object as an ∞-stage, we apply $\mathrm{Tot}\colon\ c\mathcal{M} \to \mathcal{M}$. If the relevant spectral sequence converges, which happens in exactly those cases, when F-localization and F-completion coincide, then we obtain an object in $\mathcal{M}$ which is a realization of the object in $\mathcal{A}$ with which we started; see 4.40 and 5.11.

As in [5] and [17], we also study moduli spaces of objects and morphisms but, with the truncated model structures at hand, the proofs become a lot easier and shorter.

Finally, we apply our obstruction calculus to the problem of determining certain $E(n)$-local Picard groups. In Theorem 6.3, we establish an isomorphism between them and some Ext-groups for a certain range of n and p building on and extending results from [23]. However, this result just uses the obstruction calculus and not the full result on the existence of interpolation categories.

2. Resolution model structures

Let $\mathcal{M}$ be a model category. Let $c\mathcal{M}$ be the category of cosimplicial objects over $\mathcal{M}$. We refer to [18,19] or [21] for the necessary background, in particular for the internal simplicial structure, which is compatible with the Reedy structure, and for latching and matching objects. Beware of a degree shift between our matching objects and those in [18]. The theory of $\mathcal{G}$-structures does not require a simplicial structure on $\mathcal{M}$ but, since we want a good theory of $\mathrm{Tot}\colon\ c\mathcal{M} \to \mathcal{M}$ later on, we assume from the beginning that the model structure on $\mathcal{M}$ is simplicial. In Section 2.1, we review the relevant definitions from [7] on resolution model structures. In Section 2.2, we give a dualized account of the spiral exact sequence from [13]. In the last Section 2.3, we explain the truncated model structures from [3].

2.1. The $\mathcal{G}$-structure on $c\mathcal{M}$

The following definitions are taken from [7], who gave the definitive treatment on resolution model structures.

Definition 2.1. Let $\mathcal{M}$ be a left proper pointed model category. We call a class $\mathcal{G}$ of objects in $\mathcal{M}$ a class of **injective models** if the elements of $\mathcal{G}$ are fibrant and group objects in the homotopy category $Ho(\mathcal{M})$, and if $\mathcal{G}$ is closed under loops. We reserve the letter $\mathcal{G}$ for such a class.

Definition 2.2. A map $A \xrightarrow{i} B$ in $\mathcal{M}$ is called $\mathcal{G}$-**monic** when $[B, G] \xrightarrow{i^*} [A, G]$ is surjective for each $G \in \mathcal{G}$.

An object I is called $\mathcal{G}$**-injective** when $[B, I] \xrightarrow{i^*} [A, I]$ is surjective for each $\mathcal{G}$-monic map $A \xrightarrow{i} B$.

We call a fibration in $\mathcal{M}$ a $\mathcal{G}$**-injective fibration** if it has the right lifting property with respect to every $\mathcal{G}$-monic cofibration.

We say that $Ho(\mathcal{M})$ **has enough** $\mathcal{G}$**-injectives** if each object in $Ho(\mathcal{M})$ is the source of a $\mathcal{G}$-monic map to a $\mathcal{G}$-injective target. We say that $\mathcal{G}$ is **functorial** if these maps can be chosen functorially.

Definition 2.3. A map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is called:

(i) a $\mathcal{G}$**-equivalence** if the induced maps $[Y^\bullet, G] \to [X^\bullet, G]$ are weak equivalences of simplicial sets for each $G \in \mathcal{G}$;
(ii) a $\mathcal{G}$**-cofibration** if it is a Reedy cofibration and the induced maps $[Y^\bullet, G] \to [X^\bullet, G]$ are fibrations of simplicial sets for each $G \in \mathcal{G}$;
(iii) a $\mathcal{G}$**-fibration** if $f : X^n \to Y^n \times_{M^n Y^\bullet} M^n X^\bullet$ is a $\mathcal{G}$-injective fibration for $n \geq 0$.

Here, $M^n X^\bullet$ is the n-th matching object; see [19, 15.2.2.]. These three classes of $\mathcal{G}$-equivalences, $\mathcal{G}$-cofibrations and $\mathcal{G}$-fibrations will be called the $\mathcal{G}$**-structure on** $c\mathcal{M}$. We denote it by $c\mathcal{M}^{\mathcal{G}}$. If $\mathcal{G}$ is a class of injective models, then by [7, 3.3.] the $\mathcal{G}$-structure is a simplicial left proper model structure on $c\mathcal{M}$. The simplicial structure is the external one described in Appendix A.

2.2. Natural homotopy groups and the spiral exact sequence

It will be important for us to have a different view on the $\mathcal{G}$-structure. The reason is that there are no Postnikov-like truncations with respect to the groups $\pi_s[X^\bullet, G]$. We need a more (co-)homotopical description of the $\mathcal{G}$-equivalences. But first we rewrite the above approach and associate groups with a cosimplicial object, which the reader should consider as its cohomology. They are contravariant functors on $c\mathcal{M}$ and depend on two parameters. In the situation of Definition 2.1, let $X^\bullet$ be an object in $c\mathcal{M}$ and let ho $\mathcal{G}$ be the class $\mathcal{G}$ considered as a full subcategory of $Ho(\mathcal{M}) = \mathcal{T}$. Let $[\_, \_]$ denote the morphisms in $\mathcal{T}$. Note that $[X^\bullet, G]$ is a simplicial group. For every $s \geq 0$ we have a functor

$$\begin{aligned} \text{ho}\,\mathcal{G} &\to \text{groups} \\ G &\mapsto \pi_s[X^\bullet, G]. \end{aligned} \tag{2.1}$$

which takes values in abelian groups for $s > 0$. Obviously, $\mathcal{G}$-equivalences are characterized by these groups. On the other hand, we can consider the pointed simplicial set $\mathrm{Hom}_{\mathcal{M}}(X^\bullet, G)$, where the constant map $X^0 \to G$ of the pointed category $\mathcal{M}$ serves as basepoint. Note that, if $X^\bullet$ is Reedy cofibrant, then this simplicial set is fibrant and a homotopy group object, since G is so. It supplies a functor

$$\begin{aligned} \mathcal{G} &\to \text{fibrant homotopy group objects} \\ G &\mapsto \mathrm{Hom}_{\mathcal{M}}(X^\bullet, G), \end{aligned} \tag{2.2}$$

where $\mathcal{G}$ is considered as a full subcategory of $\mathcal{M}$. Its homotopy should be thought of as the (co-)homotopy of $X^\bullet$. Also observe the equality:

$$\mathrm{Hom}_{\mathcal{M}}(X^\bullet, G) = \mathrm{map}^{\mathrm{ext}}(X^\bullet, r^0 G), \tag{2.3}$$

where $r^0 G$ denotes the constant cosimplicial object over G. Here, $\mathrm{map}^{\mathrm{ext}}$ denotes the mapping space from the external simplicial structure on $c\mathcal{M}$ described in Appendix A. We will usually drop the superscript.

Definition 2.4. Following [17], we denote the homotopy groups of these H-spaces by

$$\pi_s^\natural(X^\bullet, G) := \pi_s \mathrm{Hom}_{\mathcal{M}}(X^\bullet, G) = \pi_s \mathrm{map}(X^\bullet, r^0 G)$$

for $s \geq 0$ and $G \in \mathcal{G}$ and call them the **natural homotopy groups of $X^\bullet$ with coefficients in $\mathcal{G}$**. Note that r^0G is Reedy fibrant so, again, these groups have homotopy meaning if $X^\bullet$ is Reedy cofibrant.

Remark 2.5. Obviously, the canonical functor $\mathcal{M} \to Ho(\mathcal{M})$ induces a map $\mathrm{Hom}_{\mathcal{M}}(X^\bullet, G) \to [X^\bullet, G]$, which in turn induces a natural transformation of functors

$$\pi_s^\natural(X^\bullet, G) \to \pi_s[X^\bullet, G].$$

This map is called the **Hurewicz map** and was constructed in [13, 7.1]. One of the main results is [13, 8.1] (also [17, 3.8]) that this Hurewicz homomorphism for each $G \in \mathcal{G}$ fits into a long exact sequence, the so-called **spiral exact sequence**,

$$\cdots \to \pi_{s-1}^\natural(X^\bullet, \Omega G) \to \pi_s^\natural(X^\bullet, G) \to \pi_s[X^\bullet, G] \to \pi_{s-2}^\natural(X^\bullet, \Omega G) \to \cdots$$
$$\cdots \to \pi_2[X^\bullet, G] \to \pi_0^\natural(X^\bullet, \Omega G) \to \pi_1^\natural(X^\bullet, G) \to \pi_1[X^\bullet, G] \to 0,$$

where Ω is the loop space functor on $\mathcal{M}$, plus an isomorphism

$$\pi_0^\natural(X^\bullet, G) \cong \pi_0[X^\bullet, G].$$

In the construction of the exact sequence, we rely on the external simplicial structure, but not on a simplicial structure on $\mathcal{M}$.

As explained in [13, 8.3.] or [17, (3.1)], these long exact sequences can be spliced together to give an exact couple and an associated spectral sequence

$$\pi_p[X^\bullet, \Omega^q G] \Longrightarrow \operatorname*{colim}_k \pi_k^\natural(X^\bullet, \Omega^{p+q-k} G). \tag{2.4}$$

Lemma 2.6. *A map $X^\bullet \to Y^\bullet$ is an $\mathcal{G}$-equivalence if and only if it induces isomorphisms*

$$\pi_s^\natural(\widetilde{Y}^\bullet, G) \to \pi_s^\natural(\widetilde{X}^\bullet, G)$$

for all $s \geq 0$ and all $G \in \mathcal{G}$ and some Reedy cofibrant approximation $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$.

Proof. This follows immediately from the spiral exact sequence by simultaneous induction over the whole class $\mathcal{G}$ and the five lemma. Remember that $\mathcal{G}$ is closed under loops by assumption. □

2.3. *Truncated $\mathcal{G}$-structures*

Now we will study the homotopy categories associated with the truncated structures. Observe that there is a natural isomorphism

$$\mathrm{map}(\mathrm{sk}_{n+1} X^\bullet, r^0 G) \cong \mathrm{cosk}_{n+1} \mathrm{map}(X^\bullet, r^0 G).$$

If $X^\bullet$ is Reedy cofibrant then $\mathrm{map}(X^\bullet, r^0G)$ is fibrant. Note also that, for a Kan-complex W, the space $\mathrm{cosk}_{n+1} W$ is a model for the n-th Postnikov section.

Definition 2.7. In [3, Theorem 3.5.], we prove the existence of a left proper simplicial model structure on $c\mathcal{M}$, called the ***n*-$\mathcal{G}$-structure**, whose equivalences are given by maps $X^\bullet \to Y^\bullet$ such that, for every $G \in \mathcal{G}$ and all $0 \leq s \leq n$, the induced maps

$$\pi_s^\natural(\widetilde{Y}^\bullet, G) \to \pi_s^\natural(\widetilde{X}^\bullet, G)$$

are isomorphisms, where $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ is a cofibrant approximation to $X^\bullet \to Y^\bullet$. These maps are called ***n*-$\mathcal{G}$-equivalences**. An ***n*-$\mathcal{G}$-cofibration** is a map $X^\bullet \to Y^\bullet$ which is a $\mathcal{G}$-cofibration such that, for every $G \in \mathcal{G}$ and all $s > n$, the induced maps

$$\pi_s^\natural(\widetilde{Y}^\bullet, G) \to \pi_s^\natural(\widetilde{X}^\bullet, G)$$

are isomorphisms. The fibration are the $\mathcal{G}$-fibrations.

Now we are going to determine the n-$\mathcal{G}$-cofibrant objects.

Remark 2.8. Remember that cofibrant objects in the $\mathcal{G}$-structure coincide with the Reedy cofibrant ones:

(i) an object $A^\bullet$ in $c\mathcal{M}$ is n-$\mathcal{G}$-cofibrant if and only if it is Reedy cofibrant and $\pi_s^\natural(A^\bullet, G) = 0$ for all $G \in \mathcal{G}$ and $s > n$;
(ii) an n-$\mathcal{G}$-cofibrant approximation functor is given by $Q = \mathrm{sk}_{n+1} \tilde{\_}$;
(iii) on n-$\mathcal{G}$-cofibrant objects, the n-$\mathcal{G}$-structure and the $\mathcal{G}$-structure coincide.

Definition 2.9. Let $X^\bullet$ be an object in $c\mathcal{M}$. The skeletal filtration of a Reedy cofibrant approximation to $X^\bullet$ consists of n-$\mathcal{G}$-cofibrant approximations $X_n^\bullet$ to $X^\bullet$ for the various n, and these assemble into a sequence

$$X_0^\bullet \to X_1^\bullet \to X_2^\bullet \to \cdots \to X^\bullet$$

which captures higher and higher natural homotopy groups. So this can be viewed as a **Postnikov-cotower** for $X^\bullet$.

Definition 2.10. As explained in [3, 3.13.], the functor $\mathrm{id}\colon\ c\mathcal{M}^{\mathcal{G}} \to c\mathcal{M}^{n-\mathcal{G}}$ is a right Quillen functor, whose left adjoint is given by $Q_n = \mathrm{sk}_{n+1} \tilde{\_}$. We have an induced pair of adjoint derived functors:

$$LQ_n \cong L(\mathrm{id})\colon\ Ho(c\mathcal{M}^{n-\mathcal{G}}) \leftrightarrows Ho(c\mathcal{M}^{\mathcal{G}}) : R(\mathrm{id}),$$

where $LQ_n \cong L(\mathrm{id})$ is an embedding of a full subcategory. In the same way, we can view $\mathrm{id}\colon\ c\mathcal{M}^{(n+1)-\mathcal{G}} \to c\mathcal{M}^{n-\mathcal{G}}$ as a right Quillen functor and $L(\mathrm{id})\colon\ Ho(c\mathcal{M}^{n-\mathcal{G}}) \to Ho(c\mathcal{M}^{(n+1)-\mathcal{G}})$ is again an embedding of a full subcategory. The tower of categories

$$\cdots \to Ho(c\mathcal{M}^{(n+1)-\mathcal{G}}) \stackrel{\sigma_n}{\to} Ho(c\mathcal{M}^{n-\mathcal{G}}) \to \cdots \to Ho(c\mathcal{M}^{1-\mathcal{G}}) \stackrel{\sigma_0}{\to} Ho(c\mathcal{M}^{0-\mathcal{G}})$$

can be identified as a tower of full subcategories of $Ho(c\mathcal{M}^{\mathcal{G}})$ given by coreflections.

We can characterize the objects in $Ho(c\mathcal{M}^{n-\mathcal{G}})$ viewed as a subcategory of $Ho(c\mathcal{M}^{\mathcal{G}})$ by their natural homotopy groups. An object $X^\bullet$ is in the image of $Ho(c\mathcal{M}^{n-\mathcal{G}})$ if and only if it is $\mathcal{G}$-equivalent to its n-$\mathcal{G}$-cofibrant replacement, i.e. if we have:

$$\pi_s^\natural(\widetilde{X}^\bullet, G) = 0 \qquad \text{for } s > n$$

for a Reedy cofibrant replacement $\widetilde{X}^\bullet \to X^\bullet$. We have to relate all this to $Ho(\mathcal{M})$ by the following statement, whose analogue for the Reedy structure is well known. The lemma is cited from [7, Proposition 8.1.].

Lemma 2.11. *The functors*

$$\mathcal{M} \underset{\mathrm{Tot}}{\overset{\_\otimes^{\mathrm{pro}}\Delta^\bullet}{\rightleftarrows}} c\mathcal{M}^{\mathcal{G}}$$

form a Quillen pair.

Remark 2.12. The natural transformation $\_ \otimes^{\mathrm{pro}} \Delta^\bullet \to r^0$ gives a Reedy cofibrant replacement by [19, 16.1.4.] and hence a $\mathcal{G}$-cofibrant replacement. It follows that both induce the same left derived functor:

$$Ho(\mathcal{M}) \underset{R\,\mathrm{Tot}}{\overset{Lr^0 = \_\otimes^{\mathrm{pro}}\Delta^\bullet}{\rightleftarrows}} Ho(c\mathcal{M}^{\mathcal{G}}).$$

We can look at the composition

$$\mathcal{M} \stackrel{r^0}{\to} c\mathcal{M}^{\mathcal{G}} \stackrel{\mathrm{id}}{\to} c\mathcal{M}^{n-\mathcal{G}}$$

and the composition of induced derived functors:

$$Ho(\mathcal{M}) \xrightarrow{Lr^0} Ho(c\mathcal{M}^{\mathcal{G}}) \xrightarrow{R(\mathrm{id})} Ho(c\mathcal{M}^{n-\mathcal{G}}). \tag{2.5}$$

Definition 2.13. We will denote the composition (2.5) of functors by $\boldsymbol{\theta_n}$. We arrive at the following diagram:

$$\begin{array}{ccccccc} & & & \mathcal{T} = Ho(\mathcal{M}) & & & \\ & & \swarrow^{\theta_{n+1}} & \downarrow^{\theta_n} & & \searrow^{\theta_0} & \\ \cdots \longrightarrow & Ho(c\mathcal{M}^{(n+1)-\mathcal{G}}) & \xrightarrow[\sigma_n]{} & Ho(c\mathcal{M}^{n-\mathcal{G}}) & \longrightarrow \cdots & \xrightarrow[\sigma_0]{} & Ho(c\mathcal{M}^{0-\mathcal{G}}) \end{array}$$

This diagram is a 2-commuting diagram of functors. For details on 2-commutativity, we refer to [21]. 2-commutativity is provided by the relation $QQ \simeq Q$. We call this **the tower of truncated homotopy categories** associated with $\mathcal{M}$ and $\mathcal{G}$.

3. Homological functors

This section begins in 3.1 with the introduction of the technical conditions which the homological functors that we consider have to satisfy. In particular, the general assumptions on F are summarized in 3.10. In Section 3.2, we derive the resolution model structures, which are relevant for the obstruction calculus, by applying the general machinery from Section 2. We study the associated homotopy category in Section 3.3.

3.1. Homological functors with enough injectives

Definition 3.1. From now on, let $\boldsymbol{\mathcal{T}}$ always denote a triangulated category. The set of morphisms for X and Y in $\mathcal{T}$ will be denoted by $[\boldsymbol{X}, \boldsymbol{Y}]$. The shift functor or **suspension** of $\mathcal{T}$ will be denoted by $\boldsymbol{\Sigma}$. It is, of course, an equivalence of categories.

Let $\boldsymbol{\mathcal{A}}$ always be a **graded abelian category**, which means that we require that $\mathcal{A}$ possesses a **shift functor** denoted by [1] which is an equivalence of categories. Let $[n]$ denote the n-fold iteration of [1].

Definition 3.2. Let $F_* : \mathcal{T} \to \mathcal{A}$ be a covariant functor, where the star stands for the grading of $\mathcal{A}$. We say that F_* is **homological** if it satisfies the following conditions:

(i) F_* is a graded functor, in other words, it commutes with suspensions, so there are natural equivalences

$$F_* \Sigma X \cong (F_* X)[1] =: F_{*-1} X,$$

which are part of the structure;
(ii) F_* is additive, saying that it commutes with arbitrary coproducts;
(iii) F_* converts distinguished triangles into long exact sequences.

Remark 3.3. Later we will assume that $\mathcal{T}$ has an underlying model category $\mathcal{M}$. The suspension functor Σ here is internal to the model structure on $\mathcal{M}$. The reader should be aware that this has nothing to do with the external construction Σ_{ext} from A.3, which is derived from the external simplicial structure on the cosimplicial objects $c\mathcal{M}$ over $\mathcal{M}$.

Definition 3.4. We say that $\boldsymbol{F} : \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{A}}$ **detects isomorphisms** or equivalently that $\boldsymbol{\mathcal{T}}$ **is $\boldsymbol{F}$-local** if a map $X \to Y$ in $\mathcal{T}$ is an isomorphism if and only if the induced map $F_* X \to F_* Y$ in $\mathcal{A}$ is an isomorphism.

Definition 3.5. Let $F_*: \mathcal{T} \to \mathcal{A}$ be a homological functor from a triangulated category to a graded abelian category and let I be an injective object in $\mathcal{A}$. Consider the following functor:

$$X \mapsto \mathrm{Hom}_{\mathcal{A}}(F_* X, I)$$

We require that this functor is representable by an object $E(I)$ of $\mathcal{T}$. If the canonical morphism $F_* E(I) \to I$ induced by

$$\mathrm{id}_{E(I)} \in [E(I), E(I)] \cong \mathrm{Hom}_{\mathcal{A}}(F_* E(I), I)$$

is an isomorphism, then we call $E(I)$ an $(\boldsymbol{F}, \boldsymbol{I})$**-Eilenberg–MacLane object**. Usually, we will just say that $E(I)$ is $\boldsymbol{F}$**-injective**.

Definition 3.6. We will say that the functor F_* **possesses enough injectives** if every object in $\mathcal{T}$ admits a morphism to an F-injective object that induces a monomorphism in $\mathcal{A}$.

Remark 3.7. There are a lot of examples of functors with enough injectives that detect isomorphisms. Every topologically flat ring spectrum E, where E_*E is commutative, induces a homological functor

$$E_* : Ho(\text{Spectra}) \to E_*E\text{-comod}$$

from the stable homotopy category of spectra to the category of E_*E-comodules. The phrase ring spectrum is to be interpreted here in the most naive sense: a monoid object in the homotopy category. The notion of topological flatness was earlier called the Adams(-Atiyah) condition. It ensures, see [11, Theorem 1.5.], that $E_*E(I) \cong I$ as E_*E-comodules. Hence, E_* possesses enough injectives, since F-injective objects exist by Brown representability. See also [22]. If we perform Bousfield localization and consider it to be a functor

$$E_* : Ho(\text{Spectra})_E \to E_*E\text{-comod}$$

from the E-local category, it also detects isomorphism. From this data we can construct a spectral sequence, which is known as the E-based modified Adams-spectral sequence; see 4.43.

Remark 3.8. There are two convenient facts about finding F-injectives that are derived from [16, 2.1.1.] or [11, Theorem 1.5.]:

(i) if F detects isomorphisms, then every representing object X in $\mathcal{T}$, whose image $F_*X \cong I$ is injective in $\mathcal{A}$, is an (F, I)-Eilenberg–MacLane object.
(ii) retracts of F-injective objects are again F-injective.

Remark 3.9. If F_* is a homological functor that possesses enough injectives and detects isomorphism, then there is the following observation taken from [16, 2.1. Lemma 1]: given another representing object $\widetilde{E}(I)$, there is a unique morphism

$$\widetilde{E}(I) \to E(I)$$

in $\mathcal{T}$ lifting the identity of I, and this is an isomorphism. This can be reformulated in the following way. Let $\mathcal{T}_{F\text{-inj}}$ denote the full subcategory of $\mathcal{T}$ consisting of the F-injective objects. Let $\mathcal{A}_{\text{inj}}$ denote the full subcategory of $\mathcal{A}$ consisting of the injective objects. Then the functor F induces an equivalence $\mathcal{T}_{F\text{-inj}} \to \mathcal{A}_{\text{inj}}$.

The following assumptions will be valid for the rest of the article.

Assumptions 3.10. From now on, let $\mathcal{T}$ be the homotopy category of a simplicial left proper stable model category $\mathcal{M}$ and let $\mathcal{A}$ be an abelian category with enough injectives. Let $F_* : \mathcal{T} \to \mathcal{A}$ be a homological functor with enough F-injectives, as explained in 3.2 and 3.6, which detects isomorphisms 3.4. We will call the composition $\mathcal{M} \to \mathcal{T} \to \mathcal{A}$ of F_* with the canonical functor from $\mathcal{M}$ to its homotopy category also F_*. By applying it levelwise, we can prolong it to a functor $c\mathcal{M} \to c\mathcal{A}$ which we will again call F_*.

3.2. The F-injective structure and its truncations

Definition 3.11. We take as our class of injective models $\mathcal{G}$ the class of all F-injective objects in $\mathcal{M}$, which were defined in 3.5. This class $\mathcal{G}$ will be fixed for the rest of this work. We denote our special choice by

$$\{F\text{-injectives}\} =: \{\boldsymbol{F}\textbf{-Inj}\}.$$

We will call the involved classes of maps ***F*-injective equivalences**, ***F*-injective fibrations** and **cofibrations**. Sometimes we will abbreviate even this, and simply say F-equivalent or F-fibrant and so on. We will call this model structure on $c\mathcal{M}$ the ***F*-injective model structure**. The truncated model structure from 2.7 will be called ***n*-*F*-injective structure** or just n-F-structure. We will denote them by $c\mathcal{M}^F$ and $c\mathcal{M}^{n-F}$.

To conclude that the choice of $\mathcal{G} = \{F\text{-injectives}\}$ is really admissible, we observe first of all that $\mathcal{M}$ is stable, so all objects are homotopy group objects. Next, it is easy to check from the definitions that the F-injectives are closed under equivalences and (de-)suspensions. Finally, it will follow that there are enough $\mathcal{G}$-injectives, from the assumption that F has enough injectives 3.6 and the following two consistency checks, which are easy to prove. A map will be called ***F*-monic** if it is $\{F\text{-Inj}\}$-monic.

Lemma 3.12. *A map is F-monic if and only if its image is a monomorphism.*

Lemma 3.13. *The two classes* $\{F\text{-Inj}\}$ *and* $\{\{F\text{-Inj}\}\text{-injectives}\}$ *coincide.*

As a consequence of [3] (see also 2.7), we have the following model structures at hand, where right properness is proved in 3.23.

Theorem 3.14. *Let $\mathcal{M}$ be a pointed simplicial left proper stable model category und set $Ho(\mathcal{M}) =: \mathcal{T}$ and let $\mathcal{A}$ be an abelian category. Let $F : \mathcal{T} \to \mathcal{A}$ be a homological functor that possesses enough injectives and that detects isomorphisms. On $c\mathcal{M}$ there is a pointed simplicial proper model structure given by the $(n\text{-})F$-injective equivalences, the $(n\text{-})F$-injective cofibrations and the $(n\text{-})F$-injective fibrations. The simplicial structure is always the external one.*

In fact, $Ho(c\mathcal{M})^F$ behaves like the category of non-negative cochain complexes inside the full derived category of an abelian category with enough injectives; see the discussion in Section 3.3.

Remark 3.15. If we view $\mathcal{A}$ as a discrete model category, we can equip the category $c\mathcal{A}$ of cosimplicial objects over $\mathcal{A}$ with the **$\mathcal{I}$-structure** where the class $\mathcal{I}$ of injective objects in $\mathcal{A}$ serves as a class of injective models. It follows from [7, 4.4] that this model structure corresponds to the classical model structure from [24] for the nonnegative cochain complexes $\mathrm{CoCh}^{\geq 0}(\mathcal{A})$ via the Dold–Kan-correspondence. So, in $\mathrm{CoCh}^{\geq 0}(\mathcal{A})$ we have: the $\mathcal{I}$-equivalences are the cohomology equivalences, the $\mathcal{I}$-cofibrations are the maps that are monomorphisms in positive degrees, and the $\mathcal{I}$-fibrations are those that are (split) surjective with injective kernel in all degrees. The fibrant objects are the degreewise injective objects, while all objects are cofibrant.

We will now list characterizations of F-injective equivalences, F-injective cofibrations and F-injective fibrations and their truncated analogues. In the next statements, let $\pi^s A^\bullet \cong H^s N A^\bullet$ be the usual thing with many names, e.g. the cohomology of the normalized cochain complex $NA^\bullet$.

Corollary 3.16. *A map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is an F-injective equivalence if and only if the induced maps*

$$H^s N F_* X^\bullet \to H^s N F_* Y^\bullet$$

are isomorphisms for all $s \geq 0$, i.e. it induces a quasi-isomorphism $NF_*X^\bullet \to NF_*Y^\bullet$.

Proof. We have, for F-injective G, the isomorphisms

$$\pi_s[X^\bullet, G] \cong H^s N\mathrm{Hom}_{\mathcal{A}}(F_*X^\bullet, F_*G) \cong \mathrm{Hom}_{\mathcal{A}}(H^s N F_* X^\bullet, F_*G).$$

Then the lemma follows from the fact mentioned in 3.9 that, if G runs through all F-injectives, then F_*G ranges over all injectives in $\mathcal{A}$. □

Remark 3.17. There is no obvious way to characterize n-F-equivalences in terms of $\pi^s F_*(\_)$ like the F-equivalences. The induction used to prove 2.6 crawling up the spiral exact sequence does not yield anything useful if it stops at some finite stage. So we do not offer another description of them as the one given in 2.7. Of course we have that on n-F-cofibrant objects n-F-equivalence and F-equivalence agree.

Lemma 3.18. *A map $i : X^\bullet \to Y^\bullet$ is an F-injective cofibration if and only if it is a Reedy cofibration that induces monomorphisms*

$$N^k F X^\bullet \to N^k F Y^\bullet$$

for all $k \geq 1$.

Proof. The map i is an F-cofibration if and only if it is a Reedy cofibration and the induced map

$$[Y^\bullet, G] \to [X^\bullet, G]$$

is a fibration of simplicial sets for all $G \in \mathcal{G}$. The result now follows from the fact that a map of simplicial abelian groups is a fibration if and only if it induces a surjection of the normalizations in positive degrees. □

Lemma 3.19. *Let $X^\bullet \to Y^\bullet$ be an F-cofibration with cofiber $C^\bullet$ that induces a monomorphism $N^0 F_* X^\bullet \to N^0 F_* Y^\bullet$. Then there is a long exact sequence*

$$0 \to H^0 N F_* X^\bullet \to H^0 N F_* Y^\bullet \to H^0 N F_* C^\bullet \to H^1 N F_* X^\bullet \to \cdots$$
$$\cdots \to H^s N F_* X^\bullet \to H^s N F_* Y^\bullet \to H^s N F_* C^\bullet \to H^{s+1} N F_* X^\bullet \to \cdots .$$

Proof. This can be proved by 3.18. □

We need a little bit more care to describe F-injective fibrations. First of all, we remind the reader that F-injective fibrations and n-F-injective fibrations coincide. By definition, a map $X^\bullet \to Y^\bullet$ is an F-injective fibration if and only if all the maps $X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$ for $s \geq 0$ are $\mathcal{G}$-injective fibrations in $\mathcal{M}$ in the sense of Definition 2.2 with $\mathcal{G} = \{F\text{-Inj}\}$. Thus, we describe $\{F\text{-Inj}\}$-injective fibrations in $\mathcal{M}$.

Lemma 3.20. *A map in $\mathcal{M}$ is an $\{F\text{-Inj}\}$-injective fibration if and only if it is a fibration with F-injective fiber and it induces an epimorphism under F.*

Proof. By [7, 3.10.], a map $X \to Y$ in $\mathcal{M}$ is a $\mathcal{G}$-injective fibration if and only if it is a retract of a $\mathcal{G}$-cofree map $X' \to Y'$. A $\mathcal{G}$-cofree map is a map that can be expressed as a composition $X' \to Y' \times E \to Y'$, where $X' \to Y' \times E$ is a trivial fibration in $\mathcal{M}$, $Y' \times E \to Y'$ is the projection onto Y' and E is $\mathcal{G}$-injective.

The assertion is true for $\{F\text{-Inj}\}$-cofree maps. Here, we use the fact that $\mathcal{T}$ is F-local (see 3.4), so weak equivalences in $\mathcal{M}$ induce isomorphisms under F. But the claim is also true for retracts. This is obvious for surjectivity. The fiber condition follows from 3.8, since the fiber of $X \to Y$ is a retract of the fiber of $X' \to Y'$ which is weakly equivalent to E and therefore itself F-injective.

Conversely, let $X \to Y$ be a fibration that has an F-injective fiber E and that induces a surjection under F. $\mathcal{M}$ is stable, hence we get a long exact F-sequence for $X \to Y$ and it follows $X \simeq E \times Y$. We deduce that $X \to Y$ has the right lifting property with respect to every $\{F\text{-Inj}\}$-monic cofibration. So, it is an $\{F\text{-Inj}\}$-injective fibration. □

Corollary 3.21. *A Reedy fibration $X^\bullet \to Y^\bullet$ between F-fibrant objects is an F-fibration if and only if, for $s \geq 0$, the induced maps $N^s F_* X^\bullet \to N^s F_* Y^\bullet$ are surjective with injective kernel. In other words, $X^\bullet \to Y^\bullet$ is an F-fibration if and only if $F_* X^\bullet \to F_* Y^\bullet$ is an $\mathcal{I}$-fibration.*

Proof. This follows from 3.20, since for an F-fibrant $Y^\bullet$ and all $s \geq 0$ we have an isomorphism

$$F_*(Y^s \times_{M^s Y^\bullet} M^s X^\bullet) \cong F_* Y^s \times_{F_* M^s Y^\bullet} F_* M^s X^\bullet. \quad \square$$

Lemma 3.22. *The functor $F_* : c\mathcal{M} \to c\mathcal{A}$ maps F-homotopy pullbacks to $\mathcal{I}$-homotopy pullbacks.*

Proof. It is sufficient to prove that F_* preserves pullbacks after F-fibrant replacement. Let $X^\bullet \to Z^\bullet \leftarrow Y^\bullet$ be F-fibrations between F-fibrant objects. It follows by [7, 5.3.] that all maps $X^s \to Z^s \leftarrow Y^s$ are F-injective fibrations in $\mathcal{M}$, in particular that they are fibrations and induce surjections under F_*. The pullback square

$$\begin{array}{ccc} X^s \times_{Z^s} Y^s & \longrightarrow & X^s \\ \downarrow & & \downarrow \\ Y^s & \longrightarrow & Z^s \end{array}$$

is also a homotopy pullback square in $\mathcal{M}$, and hence a homotopy pushout. The long exact F-sequence resulting from this collapses to the short exact sequences

$$0 \to F_*(X^s \times_{Z^s} Y^s) \to F_* X^s \oplus F_* Y^s \to F_* Z^s \to 0,$$

which proves the lemma. □

Corollary 3.23. *The* (n-)F*-structure for* $0 \leq n \leq \infty$ *is proper.*

Proof. We only need to prove right properness. For $n = \infty$ this follows from the characterization of F-equivalences and F-fibrations in 3.16 and 3.21 and from 3.22. This passes down to smaller n by theorem [3, 3.5.]. □

3.3. The F-injective homotopy category

The category $c\mathcal{M}$ equipped with the F-structure behaves very much like the full subcategory $\mathrm{CoCh}^{\geq 0}(\mathcal{A})$ of nonnegative cochain complexes within the derived category $D(\mathcal{A})$. This is displayed by the statements 3.29 and 3.30. We are going to need a dual version of the functor $\overline{W} : s\mathrm{Ab} \to s\mathrm{Ab}$ which is sometimes called the Eilenberg–MacLane functor or the Kan suspension.

Definition 3.24. Let $\mathcal{N}$ be a pointed model category. We define a functor $W : c\mathcal{N} \to c\mathcal{N}$. Let $X^\bullet$ be a cosimplicial object. Let $\boldsymbol{WX^\bullet}$ be defined by the following equations:

$$(WX^\bullet)^s := \prod_{i=0}^{s} X^i.$$

The structural maps of a cosimplicial object are constructed by the process dual to the one described in [18, p. 192]. There is a map $WX^\bullet \to X^\bullet$ given by projection

$$\prod_{i=0}^{s} X^i \to X^s.$$

Let $\boldsymbol{\overline{W}X^\bullet}$ be the fiber of $WX^\bullet \to X^\bullet$.

Remark 3.25. Let $X^\bullet$ be in $c\mathcal{M}$. The map $WX^\bullet \to X^\bullet$ is a Reedy fibration if and only if every $X^\bullet$ is Reedy fibrant. It is a $\mathcal{G}$-fibration for some general $\mathcal{G}$ if and only if, in addition, all X^s are $\mathcal{G}$-injective. In both cases, $\overline{W}X^\bullet$ has homotopy meaning; see 3.27.

Lemma 3.26. *If we take* $\mathcal{G} = \{F\text{-Inj}\}$*, then* $WX^\bullet$ *is* F*-equivalent to* $*$*.*

Proof. Since $FWX^\bullet \cong WFX^\bullet$, it suffices by 3.16 to show that $WA^\bullet$ is $\mathcal{I}$-equivalent to $*$ for arbitrary $A^\bullet$ in $c\mathcal{A}$. This follows by dualizing [18, III.5.]. □

Remark 3.27. Hence $\overline{W}X^\bullet$ is another different model for the loop object $\Omega_{\mathrm{ext}}X^\bullet$. If $A^\bullet$ is in $c\mathcal{A}$, this object can also be obtained in the following way:

$$\begin{array}{ccc} c\mathcal{A} & \xrightarrow{\overline{W}} & c\mathcal{A} \\ {\scriptstyle N}\downarrow & & \uparrow{\scriptstyle \Gamma} \\ \mathrm{CoCh}^{\geq 0}(\mathcal{A}) & \xrightarrow[{[1]_{\mathrm{ext}}}]{} & \mathrm{CoCh}^{\geq 0}(\mathcal{A}) \end{array}$$

where $(A^*[1]_{\mathrm{ext}})^s = A^{s+1}$ is the external shift functor of cochain complexes (which should not be confused with the internal shift [1] from 3.1), N is normalization, and Γ is the Dold–Kan functor. In particular, if $A^\bullet$ is in $c\mathcal{A}$ we have:

$$H^s N\overline{W}A^\bullet = \begin{cases} 0, & \text{for } s = 0 \\ H^{s-1}NA^\bullet, & \text{for } s \geq 1. \end{cases} \tag{3.1}$$

For every F-fibrant $X^\bullet$ we get a map

$$\Sigma_{\mathrm{ext}}\overline{W}X^\bullet \to X^\bullet \tag{3.2}$$

in $c\mathcal{M}$ which descends to a natural transformation $\Sigma_{\mathrm{ext}}\Omega_{\mathrm{ext}} \to \mathrm{Id}$ of endofunctors of $Ho(c\mathcal{M}^{\mathcal{G}})$.

Lemma 3.28. *For every F-fibrant object $X^\bullet$ in $c\mathcal{M}$, the map $\Sigma_{\text{ext}}\overline{W}X^\bullet \to X^\bullet$ is an F-equivalence.*

Proof. We note that $F\Sigma_{\text{ext}}\overline{W}X^\bullet = \Sigma_{\text{ext}}\overline{W}FX^\bullet$, because F is applied levelwise and commutes with finite products. Now, the fact follows from 3.16 and (3.1). □

Corollary 3.29. *The map (3.2) induces a natural equivalence $\Sigma_{\text{ext}}\Omega_{\text{ext}} \cong \text{Id}$ of endofunctors of $Ho(c\mathcal{M}^F)$.*

Proof. This follows from 3.25, 3.27 and 3.28. □

Furthermore we have an isomorphism $\Omega_{\text{ext}}\Sigma_{\text{ext}}X^\bullet \cong X^\bullet$ in $Ho(c\mathcal{M}^{\mathcal{G}})$ as long as the objects in question are "connected".

Lemma 3.30. *Let $X^\bullet$ be an F-fibrant object such that $\pi^0 F_* X^\bullet = 0$. Then the canonical map $X^\bullet \to \overline{W}\Sigma_{\text{ext}}X^\bullet$ is an F-equivalence.*

Proof. The condition $\pi^0 F_* X^\bullet = 0$ is equivalent to $0 = \pi_0[X^\bullet, G] \cong \pi_0^\natural(X^\bullet, G)$. Hence the map $X^\bullet \to \overline{W}\Sigma_{\text{ext}}X^\bullet$ induces isomorphisms on $H^s N F_*(\_)$ for all $s \geq 0$, so it is an F-equivalence. □

Remark 3.31. In a stable model category $\mathcal{M}$ finite products and finite coproducts are weakly equivalent. It follows that, for the Reedy structure and in particular for every $\mathcal{G}$-structure on $c\mathcal{M}$ and their truncated versions, finite products and coproducts are weakly equivalent.

Corollary 3.32. *For every $0 \leq n \leq \infty$, the category $Ho(c\mathcal{M}^{n-\mathcal{G}})$ is additive and the functors $\sigma_n : Ho(c\mathcal{M}^{(n+1)-F}) \to Ho(c\mathcal{M}^{n-F})$ and $\theta_n : \mathcal{T} \to Ho(c\mathcal{M}^{n-F})$ are additive.*

Proof. By 3.29, every object in $Ho(c\mathcal{M}^{n-F})$ for $0 \leq n \leq \infty$ is isomorphic to a double suspension, hence every object is an abelian cogroup object in the homotopy category. Both functors σ_n and θ_n commute with Σ_{ext}. □

Definition 3.33. We will denote the biproduct of a pair of objects $X^\bullet$ and $Y^\bullet$ in $Ho(c\mathcal{M}^{n-F})$ for $0 \leq n \leq \infty$ by $\boldsymbol{X^\bullet \oplus Y^\bullet}$.

4. The realization problem

In Section 4.1 we do the hard work and construct the obstruction calculus. The main theorems are 4.24, 4.27, 4.30, 4.32 and 4.35. In subsection 4.2 we define our interpolation categories for a homological functor F and in subsection 4.3 we describe the spectral sequences that play a role in the realization problem.

4.1. Realizations and obstruction calculus

In this subsection we develop an obstruction calculus for realizing objects and morphism along a homological functor F_* with enough injectives. In general, we follow [5] and [17]; see also [1]. Since we are in a completely linear or stable situation, the theory required to set up the obstruction calculus simplifies compared to the other settings. Nevertheless, the simplifications in paragraph 5.2 compared to [5] result from the use of truncated resolution model structures. An obstruction calculus for realizing objects using only the triangulated structure is described, among other things, in [2]. We apologize in advance for using so much notation, but it seems unavoidable.

Our task was to look out for realizations in $Ho(\mathcal{M}) = \mathcal{T}$ of objects in $\mathcal{A}$. To motivate our next definition, let X be an object in $\mathcal{M}$ and let $X^\bullet \to r^0 X$ be an n-$\mathcal{G}$-cofibrant approximation. We know that:

$$\pi_s[r^0 X, G] = \begin{cases} [X, G], & \text{if } s = 0 \\ 0, & \text{else.} \end{cases}$$

With the spiral exact sequence, we can calculate:

$$\pi_s^\natural(X^\bullet, G) = \begin{cases} [X, \Omega^s G], & \text{if } 0 \leq s \leq n \\ 0, & \text{for } s > n. \end{cases}$$

Also, respectively:

$$\pi_s[X^\bullet, G] = \begin{cases} [X, G], & \text{if } s = 0 \\ [X, \Omega^{n+1} G], & \text{if } s = n+2 \\ 0, & \text{else.} \end{cases}$$

All these sets of isomorphisms determine each other. Of course, this is not the way we will encounter such spaces, since we are seeking realization and not starting with them. Instead, we will take these equations as the defining conditions of our successive realizations.

Definition 4.1. Let A be an object in the abelian target category $\mathcal{A}$. We will call a Reedy cofibrant object $X^\bullet$ in $c\mathcal{M}$ a **potential *n*-stage for *A*** following [5] and [17], if there are natural isomorphisms

$$\pi_s^\natural(X^\bullet, G) \cong \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A, F_{*+s}G), & \text{if } 0 \le s \le n \\ 0, & \text{for } s > n \end{cases},$$

where we consider both sides as functors on $\mathcal{G}$ as a subcategory of $\mathcal{T}$. Note that this also makes sense for $n = \infty$. In this case, an object satisfying these equations is simply called an **∞-stage**. The reason is that, by 5.12, it is not "potential" any more.

Remark 4.2. If $X^\bullet$ is a potential n-stage for an object A in $\mathcal{A}$, then $\mathrm{sk}_n X^\bullet$ is a potential $(n-1)$-stage for A.

Remark 4.3. Since $\mathcal{G}$ was the class of F-injectives, the class $\{F_* G \mid G \in \mathcal{G}\}$ is cogenerating the category $\mathcal{A}$, and we derive, for a potential n-stage $X^\bullet$ from the previous properties and the spiral exact sequence, the following equations:

$$\pi^s F_* X^\bullet \cong H^s N F_* X^\bullet = \begin{cases} A, & \text{if } s = 0 \\ A[n+1], & \text{if } s = n+2 \\ 0, & \text{else.} \end{cases}$$

The shift functor $[\_]$ is the internal shift from 3.1.

Following the philosophy outlined, we start the process of realizing an object A in $\mathcal{A}$ with a potential 0-stage. Then we proceed by gluing on special objects to get to higher n-stages. We define these layers in 4.6. They have a certain representation property (see 4.10) and 4.13 will provide the obstruction groups that we are looking for. To prove this property we have to consider algebraic analogues of these layers defined in 4.4, and they should not be confused with each other. We also describe the moduli space of these different sorts of objects. In particular, we will see that they are connected, which ensures that the layers all look alike.

Definition 4.4. Let N be an object of $\mathcal{A}$ and let $n \ge 0$. We call an object $I^\bullet$ in $c\mathcal{A}$ an **object of type *K*(*N*, *n*)** if the following conditions are satisfied:

$$\pi^s I^\bullet \cong \begin{cases} N, & \text{if } s = n \\ 0, & \text{else.} \end{cases}$$

We denote $I^\bullet$ by $K(N, n)$. These objects are essentially unique according to the following remark. For a quick introduction to moduli spaces, we refer to Appendix B.

Remark 4.5. If $I^\bullet$ is an object of type $K(A, 0)$, then there is a weak equivalence $r^0 A = r^0(\pi^0 I^\bullet) \to I^\bullet$. It follows that the moduli space is weakly equivalent to $B\mathrm{Aut}(A)$.

Objects of type $K(N, n)$ exist, for example $\Omega^n_{\mathrm{ext}} r^0 N$ or equivalently $\overline{W}^n r^0 N$ is such an object. The moduli space is given by $B\mathrm{Aut}(N)$, since the functor Ω_{ext} induces an obvious equivalence $\mathcal{M}(K(N, n)) \to \mathcal{M}(K(N, n+1))$ for $n \ge 0$. In particular, this space is connected.

Definition 4.6. Let N be an object in $\mathcal{A}$ and $n \ge 0$. We call an object $Y^\bullet$ in $c\mathcal{M}$ an **object of type *L*(*N*, *n*)** if the following conditions are satisfied:

$$\pi_s^\natural(Y^\bullet, G) = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(N, F_* G), & \text{if } s = n \\ 0, & \text{else.} \end{cases}$$

Their existence and moduli space is described in 4.7. We denote a generic object by $L(N, n)$. Do not confuse these objects with objects of type $K(N, n)$ in $c\mathcal{A}$, see 4.4, the end of the Remarks 4.8 and 4.9.

Remark 4.7. Objects of type $L(A, 0)$ exist: We choose an exact sequence

$$0 \to A \to I^0 \xrightarrow{d} I^1$$

with I^0 and I^1 injective. The map d is induced by a map $E(I^0) \to E(I^1)$ in $\mathcal{T} = Ho(\mathcal{M})$ between F-injective objects by 3.5 that we will also call d. This d is again represented by a map d in $\mathcal{M}$ if we choose the models for $E(I^0)$ and $E(I^1)$ to be fibrant and cofibrant. Now define a 1-truncated cosimplicial object

$$E(I^0) \rightrightarrows E(I^0) \times E(I^1)$$

with

$$d^0 = \begin{pmatrix} 1 \\ d \end{pmatrix}, \quad d^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad s^0 = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

By applying a Reedy cofibrant approximation and left Kan extension, we get an entire cosimplicial object, which is of type $L(A, 0)$.

Objects of type $L(N, n)$ exist, since they can be given by setting:

$$L(N, n) := \Omega^n_{\text{ext}} L(N, 0) \quad \text{or} \quad L(N, n) := \overline{W}^n L(N, 0).$$

We will compute the moduli space of objects of type $L(A, 0)$ in 4.12; it is given by BAut(A). Then the moduli space of objects of type $L(N, n)$ is given by $B\text{Aut}(N)$. This is proved by observing that it follows from 3.28 and 3.30 that Σ_{ext} and $\overline{W}$ induce mutually inverse homotopy equivalences of $\mathcal{M}(L(N, n))$ and $\mathcal{M}(L(N, n+1))$ for $n \geq 0$. From Lemma 3.30, we also get that $\Sigma_{\text{ext}} L(N, n+1) \cong L(N, n)$.

Remark 4.8. By the spiral exact sequence, we compute, from 4.6:

$$\pi_s[L(N, n), G] = \begin{cases} \text{Hom}_{\mathcal{A}}(N, F_* G), & \text{if } s = n \\ \text{Hom}_{\mathcal{A}}(N, F_{*+1} G), & \text{if } s = n+2 \\ 0, & \text{else.} \end{cases}$$

By the defining property of the F-injective objects in $\mathcal{G}$, we get:

$$\pi^s F_* L(N, n) = \begin{cases} N, & \text{if } s = n \\ N[1], & \text{if } s = n+2 \\ 0, & \text{else.} \end{cases}$$

Both sets of data are equivalent to the defining equations of an object of type $L(N, n)$ in Definition 4.6. In particular, it follows that $F_* L(N, n)$ is not an object of type $K(N, n)$. We point out that, for an object of type $L(A, 0)$, the image $F_* L(A, 0)$ is 1-$\mathcal{I}$-equivalent to $r^0 A$, as we can see with these equations.

Remark 4.9. Despite the fact that $F_* L(N, n)$ is not of type $K(N, n)$ in $c\mathcal{A}$, there is a close connection explained in the following lemma. First, we have to prepare ourselves. Let $n \geq 1$ and let N be an object in $\mathcal{A}$. The isomorphism of N and $\pi^n F_* L(N, n)$ defines a map $K(N, n) \to F_* L(N, n)$. So, by first applying F_* and then pulling back along this arrow, we obtain a map

$$\phi_n(Y^\bullet) : \text{map}(L(N, n), Y^\bullet) \to \text{map}(K(N, n), F_* Y^\bullet). \tag{4.1}$$

Here we assume that $L(N, n)$ and $K(N, n)$ are Reedy cofibrant.

The next lemma is one of the central ingredients in the obstruction calculus as well as for the proof of 5.3.

Lemma 4.10. *For F-fibrant $Y^\bullet$ in $c\mathcal{M}$ and Reedy cofibrant objects of type $L(N, n)$ and $K(N, n)$, the map $\phi_n(Y^\bullet)$ from (4.1) is a natural weak equivalence.*

Proof. The proof is exactly parallel to the proof of [5, Proposition 8.7.], although in our case linearity assures the result also for $n = 0, 1$. □

As is to be expected, 0-stages and 0-layers will coincide.

Corollary 4.11. *An object $X^\bullet$ is of type $L(A,0)$ if and only if it satisfies the following conditions:*

(i) *there is an isomorphism $\pi^0 F_* X^\bullet \cong A$ in $\mathcal{A}$.*
(ii) *for every $Y^\bullet$ in $c\mathcal{M}$, the natural map*

$$[X^\bullet, Y^\bullet] \to \mathrm{Hom}_{\mathcal{A}}(A, \pi^0 F_* Y^\bullet)$$

is an isomorphism.

Proof. Objects that satisfy (i) and (ii) are of type $L(A,0)$ because we can calculate:

$$\pi_s^\natural(X^\bullet, G) \cong [X^\bullet, \Omega^s_{\mathrm{ext}} r^0 G]_F \cong \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A, F_*G), & \text{for } s = 0 \\ 0, & \text{else.} \end{cases}$$

The other direction follows from 4.10. □

Finally, we can determine the moduli space of all objects of type $L(A,0)$.

Corollary 4.12. *The moduli space of all objects of type $L(A,0)$ is connected and we have the following weak equivalence:*

$$\mathcal{M}_F(L(A,0)) \simeq B\mathrm{Aut}(A).$$

Proof. The moduli space is connected. Let $L(A,0)$ be some reference object and let $X^\bullet$ be another object of type $L(A,0)$. By pulling back id_A along the isomorphism of 4.11(ii), we obtain a map $X^\bullet \to L(A,0)$ which induces an isomorphism on $\pi_0^\natural(\_, G)$ for every $G \in \{F\text{-Inj}\}$. Both are potential 0-stages, so this is the only group to check.

Now we will prove that the moduli space is weakly equivalent to the moduli space of objects of type $K(A,0)$. Then the result will follow from 4.5. By B.4, there are canonical weak equivalences

$$\mathcal{M}_F(L(A,0)) \simeq B\mathrm{haut}_F(L(A,0)) \quad \text{and} \quad \mathcal{M}_{\mathcal{I}}(K(A,0)) \simeq B\mathrm{haut}_{\mathcal{I}}(K(A,0)).$$

It suffices to prove that $\mathrm{haut}_F(L(A,0)) \simeq \mathrm{haut}(K(A,0))$, because both objects are fibrant grouplike simplicial monoids and B preserves weak equivalences between fibrant simplicial sets. By 4.10, we have the following weak equivalences:

$$\mathrm{map}(L(A,0), L(A,0)) \simeq \mathrm{map}(K(A,0), F_*L(A,0)) \simeq \ell_0 \mathrm{Hom}_{\mathcal{A}}(A, A).$$

Passing to appropriate components, we see that $\mathrm{haut}_F(L(A,0)) \simeq \ell_0 \mathrm{End}_{\mathcal{A}}(A)$, which finishes the proof. Here, $\ell_0(\_)$ denotes the constant simplicial object. □

Definition 4.13. Consider $K(N,n)$ in $c\mathcal{A}$ for $n \geq 0$. We assume that $K(N,n)$ is Reedy cofibrant. Let $\Lambda^\bullet$ be an object in $c\mathcal{A}$. Then we define

$$\mathrm{map}(K(N,n), \widetilde{\Lambda}^\bullet) =: \mathcal{H}^n(\Lambda^\bullet, N),$$

where $\Lambda^\bullet \to \widetilde{\Lambda}^\bullet$ is a fibrant approximation, to be the ***n*-th cohomology space** of $\Lambda^\bullet$ with coefficients in N. We define the ***n*-th cohomology** of $\Lambda^\bullet$ by

$$\pi_0 \mathcal{H}^n(\Lambda^\bullet, N) =: H^n(\Lambda^\bullet, N).$$

In the next lemma we will give an interpretation of these cohomology groups.

Remark 4.14. It follows for any $\Lambda^\bullet$ in $c\mathcal{A}$ that:

$$\Omega \mathcal{H}^n(\Lambda^\bullet, N) \simeq \mathcal{H}^{n-1}(\Lambda^\bullet, N).$$

Lemma 4.15. *Let $\Lambda^\bullet$ in $c\mathcal{A}$ be $\mathcal{I}$-fibrant and n-$\mathcal{I}$-equivalent to $r^0\pi^0\Lambda^\bullet$. Then there is a natural isomorphism*

$$H^n(\Lambda^\bullet, N[k]) \cong \mathrm{Ext}^{n,k}(N, \pi^0 \Lambda^\bullet)$$

of abelian groups.

Proof. The canonical map $r^0\pi^0\Lambda^\bullet \to \Lambda^\bullet$ obtained by adjunction factors as the composition $r^0\pi^0\Lambda^\bullet \to \mathrm{sk}_{n+1}\Lambda^\bullet \to \Lambda^\bullet$ of n-$\mathcal{I}$-equivalences, and we can approximate $\mathrm{sk}_{n+1}\Lambda^\bullet$ $\mathcal{I}$-fibrantly by $I^\bullet$, which yields an injective resolution of $\pi^0\Lambda^\bullet$ after normalization. Now, $K(N[k], n)$ is an $(n+1)$-skeleton and we compute:

$$H^n(\Lambda^\bullet, N[k]) = \pi_0\mathrm{map}(K(N[k], n), \Lambda^\bullet) \cong \pi_0\mathrm{map}(K(N[k], n), I^\bullet) \cong \mathrm{Ext}^{n,k}(N, \pi^0\Lambda^\bullet). \quad \square$$

Remark 4.16. Let $Y^\bullet$ be F-fibrant, such that $F_*Y^\bullet$ is n-$\mathcal{I}$-equivalent to $r^0\pi^0F_*Y^\bullet$. Altogether, Lemma 4.10 and Lemma 4.15 yield the following isomorphism of abelian groups:

$$\pi_0\mathrm{map}(L(A[k], n), Y^\bullet) \cong \mathrm{Ext}^{n,k}_{\mathcal{A}}(A, \pi^0F_*Y^\bullet).$$

Here we assume $L(A[k], n)$ and $Y^\bullet$ to be both Reedy cofibrant. This is functorial in $Y^\bullet$. It is not quite functorial in A but, for a morphism $A \to B$ after having chosen two objects $L(A[k], n)$ and $L(B[k], n)$, there is a uniquely determined homotopy class $L(A[k], n) \to L(B[k], n)$ inducing $A \to B$. The result tells us that an object $L(N, n)$ represents the cohomology functor $H^n(F_*(\_), N)$ in the homotopy category $Ho(c\mathcal{M}^F)$. Note that the isomorphism is in particular valid if $Y^\bullet$ is an F-fibrant n-stage.

We want to construct an obstruction calculus for lifting things from an interpolation category to the next one. In order to carry this out, we study the difference between potential $(n-1)$-stages and potential n-stages. In 4.18 and 4.19, we will prove the existence of certain homotopy pushout diagrams, where the difference between two stages is recognized as objects of type $L(N, n)$ for suitable N and n. This construction can be viewed as a (potential) Postnikov cotower; compare 4.20.

The following two lemmas are an example of the simplifications that we get for the stable case. The next lemma is the collapsed version of the so-called difference construction in [5, 8.4.].

Lemma 4.17. *Let $n \geq 1$ and provide $c\mathcal{M}$ with the F-structure. Let $f : X^\bullet \to Y^\bullet$ be a map in $c\mathcal{M}$, which induces an isomorphism on π^0F_* and whose homotopy cofiber $C^\bullet$ has the property that $\pi^sF_*C^\bullet = 0$ for $0 \leq s \leq n-1$. Let $P^\bullet$ be the homotopy fiber of f. Then there are isomorphisms $P^\bullet \cong \Omega_{\mathrm{ext}}C^\bullet$ and $\Sigma_{\mathrm{ext}}P^\bullet \cong C^\bullet$ in $Ho(c\mathcal{M}^F)$, and $\mathrm{sk}_{n+2}P^\bullet$ is an object of type $L(\pi^nF_*C^\bullet, n+1)$.*

Proof. This follows directly from 3.29 and 3.30 and the long exact $\pi^\natural_*$-sequence. $\square$

Lemma 4.18. *Let $X^\bullet_n$ be a potential n-stage for A. Then $\mathrm{sk}_nX^\bullet_n =: X^\bullet_{n-1}$ is a potential $(n-1)$-stage for A, and there is a homotopy cofiber sequence in $c\mathcal{M}^F$:*

$$L(A[n], n+1) \to X^\bullet_{n-1} \to X^\bullet_n.$$

This sequence is also a homotopy fiber sequence in $c\mathcal{M}^F$.

Proof. Call $C_n = \mathrm{hocofib}(X^\bullet_{n-1} \to X^\bullet_n)$. We know that $\pi^\natural_s$ of C_n vanishes except in dimension n. Hence $\mathrm{sk}_{n+2}C_n$ is F-equivalent to C_n. From 4.17, we see that $\Sigma_{\mathrm{ext}}\Omega_{\mathrm{ext}}C_n \simeq C_n \simeq \Omega_{\mathrm{ext}}\Sigma_{\mathrm{ext}}C_n$ and that $\Omega_{\mathrm{ext}}C_n$ is an object of type $L(A[n], n+1)$. We also see that the sequence is a homotopy cofiber sequence as well as a homotopy fiber sequence. $\square$

Lemma 4.19. *Let there be given a homotopy cofiber sequence in $c\mathcal{M}^F$:*

$$L(A[n], n+1) \xrightarrow{w_n} X^\bullet_{n-1} \longrightarrow X^\bullet_n .$$

Let $X^\bullet_{n-1}$ be a potential $(n-1)$-stage for A. A Reedy cofibrant approximation to $X^\bullet_n$ is a potential n-stage for A if and only if the map w_n induces an isomorphism $A[n] \cong \pi^{n+1}F_*X^\bullet_{n-1}$.

Proof. It follows from 3.19 that there is an exact sequence

$$0 \to \pi^nF_*X^\bullet_n \to \pi^{n+1}F_*L(A[n], n+1) \xrightarrow{\cong} \pi^{n+1}F_*X^\bullet_{n-1} \to \pi^{n+1}F_*X^\bullet_n \to 0$$

and an isomorphism $\pi^{n+2}F_*X^\bullet_n \cong \pi^{n+3}F_*L(A[n], n+1) \cong A[n+1]$. All other groups of the form $\pi^sF_*X^\bullet_n$ for $s > 0$ vanish, hence $X^\bullet_n$ has the right homotopy groups for a potential n-stage for A; we just need to approximate it Reedy cofibrantly. $\square$

Definition 4.20. We will call a map w_n as in 4.19 an ***n*-th attaching map**, i.e. if it is an F-cofibration of the form $L(A[n], n+1) \to X^\bullet_{n-1}$ between Reedy cofibrant objects, whose target is a potential n-stage. The induced homotopy class will be called ***n*-th co-k-invariant**. The concept is dual to that of k-invariants of a Postnikov cotower.

If we need to refer to an attaching map stemming from a specified potential n-stage $X^\bullet_n$ as in 4.18, we will write $w^{X^\bullet_n}$ instead of w_n. To an ∞-stage, we can associate attaching maps for each of its n-stages. By abuse of notation, we will denote them by $w^{X^\bullet_n}$ without specifying an actual n-stage.

Now we start to describe the obstruction against the existence of realizations of objects in $IP_{n-1}(F)$.

Definition 4.21. Let $X^\bullet_{n-1}$ be a potential $(n-1)$-stage for an object A. We call an object $X^\bullet$ a **potential *n*-stage over $X^\bullet_{n-1}$** if $X^\bullet$ is a potential n-stage and $\mathrm{sk}_n X^\bullet$ is F-equivalent to $X^\bullet_{n-1}$. This is equivalent to $X^\bullet_{n-1}$ being $(n-1)$-F-equivalent to $\mathrm{sk}_n X^\bullet$.

The obstruction against the existence of an n-stage over a given $(n-1)$-stage is the existence of an attaching map w_n, like in 4.19. We are now going to reformulate this in algebraic terms. We already know from Remark 4.3 that, for an $(n-1)$-stage $X^\bullet_{n-1}$, its image $F_* X^\bullet_{n-1}$ has the same cohomology groups as an object of type $K(A,0) \oplus K(A[n], n+1)$. Without loss of generality, we assume $X^\bullet_{n-1}$ to be F-fibrant. Hence we know that such an attaching map w_n exists if and only if we are able to construct a map

$$\omega_n : K(A[n], n+1) \to F_* X^\bullet_{n-1}$$

inducing an isomorphism on $\pi^{n+1} F_*(\_)$ because, by the representing property 4.10, it follows that we were then able to choose $w^{X^\bullet_n}$ such that

$$\pi_0[\phi(X^\bullet_{n-1})(w_n)] = \pi_0[\omega_n].$$

From 4.3, we have the homotopy cofiber sequence

$$K(A[n], n+2) \to \mathrm{sk}_{n+1} F_* X^\bullet_{n-1} \to F_* X^\bullet_{n-1}$$

and we can consider the following diagram:

$$\begin{array}{ccccc}
 & & \mathrm{sk}_1 F_* X^\bullet_{n-1} & \xrightarrow{\simeq} & K(A,0) \simeq r^0 A \\
 & & \downarrow \cong & & \\
K(A[n], n+2) & \xrightarrow{\beta_n} & \mathrm{sk}_{n+1} F_* X^\bullet_{n-1} & & \\
\downarrow & \text{ho-}\ulcorner & \downarrow & & \\
* & \longrightarrow & \mathrm{sk}_{n+2} F_* X^\bullet_{n-1} & \xrightarrow{\cong} & F_* X^\bullet_{n-1} \\
\downarrow & & & \nearrow_{\omega_n} & \\
K(A[n], n+1) & \cdots & \cdots & &
\end{array} \tag{4.2}$$

Observe also that we have isomorphisms

$$H^{n+2}(\mathrm{sk}_{n+1} F_* X^\bullet_{n-1}, A[n]) \xrightarrow{\cong} H^{n+2}(r^0 A, A[n]) = \mathrm{Ext}^{n+2,n}_{\mathcal{A}}(A, A)$$

of abelian groups by Lemma 4.15 or Remark 4.16.

Definition 4.22. The homotopy class b_n of the map β_n in

$$\pi_0 \mathrm{map}(K(A[n], n+2), r^0 A) = H^{n+2}(r^0 A, A[n]) \cong \mathrm{Ext}^{n+2,n}_{\mathcal{A}}(A, A)$$

will be called the **obstruction class** of the potential $(n-1)$-stage $X^\bullet_{n-1}$.

Lemma 4.23. *In (4.2) the map ω_n inducing an isomorphism on $\pi^{n+1}F_*(\_)$ exists if and only if β_n is nullhomotopic.*

Proof. Obvious. □

Theorem 4.24. *Let $n \geq 1$ and A be an object of $\mathcal{A}$. Let $X^\bullet_{n-1}$ be a potential $(n-1)$-stage of A. There exists a potential n-stage $X^\bullet_n$ over $X^\bullet_{n-1}$ if and only if the co-k-invariant b_n from Definition 4.22 in* $\mathrm{Ext}^{n+2,n}(A, A)$ *vanishes.*

Proof. From 4.23, we know that an attaching map for $X^\bullet_n$ exists if and only if $b_n = 0$. □

Now we are concerned with telling apart different realizations.

Definition 4.25. Let $X^\bullet_n$ and $Y^\bullet_n$ be potential n-stages for an object A with $\mathrm{sk}_n X^\bullet_n \simeq X^\bullet_{n-1} \simeq \mathrm{sk}_n Y^\bullet_n$. The homotopy fiber of the canonical maps from $X^\bullet_{n-1}$ to $X^\bullet_n$ and $Y^\bullet_n$ is $L(A[n], n+1)$ by 4.18. We obtain two attaching maps

$$w^{X^\bullet_n} \text{ and } w^{Y^\bullet_n} : L(A[n], n+1) \to X^\bullet_{n-1}.$$

The **difference class** of the objects $X^\bullet_n$ and $Y^\bullet_n$ is defined to be the class

$$\delta(X^\bullet_n, Y^\bullet_n) := \pi_0(w^{X^\bullet_n}) - \pi_0(w^{Y^\bullet_n}) \in \pi_0\mathcal{H}^{n+1}(F_* X^\bullet_{n-1}, A[n]) \cong \mathrm{Ext}^{n+1,n}(A, A).$$

Remark 4.26. The proof of the next theorem shows that this defines an action of $\mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A, A)$ on the class of F-equivalence classes of potential n-stages over a given potential $(n-1)$-stage. It is obviously transitive. This proves, first of all, that there is just a set of such equivalence classes or, what is the same, of realizations in $IP_n(F)$ of a given object in $IP_{n-1}(F)$.

Theorem 4.27. *Let $n \geq 1$. There is an action of* $\mathrm{Ext}^{n+1,n}(A, A)$ *on the set of F-equivalence classes of potential n-stages of A over a given potential $(n-1)$-stages, if it is non-empty, which is transitive and free.*

Proof. Let $X^\bullet_n$ be a potential n-stage with $X^\bullet_{n-1} := \mathrm{sk}_n X^\bullet_n$ and take a class $\kappa \in \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A, A)$. We want to construct a potential n-stage $Y^\bullet_n$ over $X^\bullet_{n-1}$, such that $\kappa = \delta(X^\bullet_n, Y^\bullet_n)$. Consider the map γ given by the following composition

$$K(A[n], n+1) \xrightarrow{\kappa} K(A, 0) \xrightarrow{\text{incl.}} K(A, 0) \oplus K(A[n], n+1) \cong F_* X^\bullet_{n-1},$$

and let $c\colon\ L(A[n], n+1) \to X^\bullet_{n-1}$ be a realization of γ existing by 4.10. Take a map

$$\omega_n\colon\ K(A[n], n+1) \to F_* X^\bullet_{n-1}$$

from (4.2) representing the homotopy class of

$$w^{X^\bullet_n}\colon\ L(A[n], n+1) \to X^\bullet_{n-1}$$

associated to $X^\bullet_n$ by 4.18 and add it to c. The resulting map $\omega_n + \gamma$ will still induce an isomorphism on π^{n+1}, since γ itself induces the zero map on π^{n+1}. Thus the cofiber $Y^\bullet_n$ of the corresponding map $w^{X^\bullet_n} + c\colon\ L(A[n], n+1) \to X^\bullet_{n-1}$ is a potential n-stage over $X^\bullet_{n-1}$ by 4.19, which realizes the given difference class, hence $\kappa = \delta(X^\bullet_n, Y^\bullet_n)$. This process is obviously additive in $[\kappa]$, therefore we have a group action. It is also clear that $X^\bullet_n \cong Y^\bullet_n$ in $IP_n(F)$ if and only if $\kappa = 0$. □

Now we are going to describe the obstruction for lifting maps.

Definition 4.28. Let $n \geq 1$. Let $X^\bullet$ and $Y^\bullet$ be objects in $IP_n(F)$ and let $\varphi : \sigma_n X^\bullet \to \sigma_n Y^\bullet$ be a map in $IP_{n-1}(F)$. We say that $\boldsymbol{\varphi}$ **lifts** if there is a map $\Phi : X^\bullet \to Y^\bullet$ such that $\sigma_n \Phi = \varphi$. In this case, we call Φ a **lifting** of φ.

Remark 4.29. By definition, every object $W^\bullet$ in $IP_n(F)$ can be approximated by a potential n-stage, which corresponds to n-F-cofibrant approximation. On the other side it can be approximated F-fibrantly such that $F_* W^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $r^0\pi^0 F_* W^\bullet$.

Between a potential n-stage $X^\bullet$ and an F-fibrant $Y^\bullet$, where $F_* Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $\pi^0 F_* Y^\bullet$, every morphism in $IP_n(F)$ can be represented by a map $f : X^\bullet \to Y^\bullet$ in $c\mathcal{M}$.

Assume that we are given a morphism from $X^\bullet$ to $Y^\bullet$ in $IP_{n-1}(F)$, then this can be represented by a map $f : \mathrm{sk}_n X^\bullet \to Y^\bullet$. Now f lifts if and only if there is a map $\widetilde{f} : X^\bullet \to Y^\bullet$ such that

$$\mathrm{sk}_n X^\bullet \longrightarrow X^\bullet \longrightarrow Y^\bullet$$

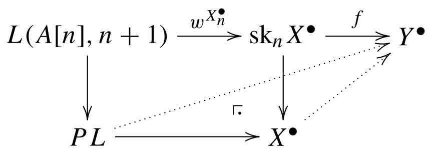
is homotopic to f in $c\mathcal{M}^F$.

Theorem 4.30. *A morphism $\sigma_n X^\bullet \to \sigma_n Y^\bullet$ in $IP_{n-1}(F)$ lifts to a morphism $X^\bullet \to Y^\bullet$ in $IP_n(F)$ if and only if* $\mathrm{ob}_n(f) \in \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(\pi^0 F_* X^\bullet, \pi^0 F_* Y^\bullet)$ *defined in (4.4) vanishes.*

Proof. We assume, without loss of generality, that $X^\bullet$ is a potential n-stage for an object A and that $Y^\bullet$ is F-fibrant such that $F_* Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $r^0 B$ in $c\mathcal{A}$. We can achieve this by approximations in the n-F-structure. Also without loss of generality, we can replace the homotopy cofiber sequence

$$L(A[n], n+1) \xrightarrow{w^{X^\bullet_n}} X^\bullet_{n-1} \longrightarrow X^\bullet_n$$

in $c\mathcal{M}^F$ of 4.19 by an actual cofiber sequence using factorizations in the F-structure. This means that we have constructed the following solid arrow diagram


(4.3)

where $PL \overset{F}{\simeq} *$ is a path object in the F-structure for $L(A[n], n+1)$. We conclude that the existence of the dotted liftings in diagram (4.3) are equivalent to each other. By 4.10, we deduce that an extension of f to $X^\bullet$ exists if and only if the map

$$K(A[n], n+1) \xrightarrow{\phi(f w^{X^\bullet_n})} F_* Y^\bullet$$

is null homotopic, where ϕ is the map from (4.1). $\phi(f w^{X^\bullet_n})$ defines an obstruction element

$$\mathrm{ob}_n(f) := [\phi(f w^{X^\bullet_n})] \in H^{n+1}(F_* Y^\bullet, A[n]) = \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(\pi^0 F_* X^\bullet, \pi^0 F_* Y^\bullet) \tag{4.4}$$

by 4.15. Recall that $F_* Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $r^0 \pi^0 F_* Y^\bullet = r^0 B$. So Remark 4.16 applies. □

Before we proceed to the next theorem, we have to reformulate the obstruction defined in (4.4). Here, we use the "almost stability" of $Ho(c\mathcal{M}^F)$ that is displayed in 3.29, 3.30 and 4.18. In the situation of (4.3), we want to achieve that f factors over a potential $(n-1)$-stage of $Y^\bullet$. Let

$$\mathrm{sk}_n Y^\bullet \xrightarrow{\upsilon} \widetilde{Y}^\bullet_{n-1} \xrightarrow{\widetilde{\upsilon}} Y^\bullet$$

be a factorization of the canonical inclusion map into an $(n-1)$-F-cofibration υ followed by an $(n-1)$-F-trivial fibration $\widetilde{\upsilon}$. The map υ will necessarily be an F-trivial cofibration. Obviously, $\widetilde{Y}^\bullet_{n-1}$ is a potential $(n-1)$-stage and the co-k-invariant of $Y^\bullet$ is represented by $\widetilde{\upsilon} \circ w^{\widetilde{Y}_n}$. Then the map $f : \mathrm{sk}_n X^\bullet \to Y^\bullet$ factors over a map $\widetilde{f} : \mathrm{sk}_n X^\bullet \to \widetilde{Y}^\bullet_{n-1}$ whose homotopy class is uniquely determined. The attaching map $w^{Y^\bullet_n}$ prolongs to an attaching map

$$L(B[n], n+1) \xrightarrow[w^{Y^\bullet_n}]{} \mathrm{sk}_n Y^\bullet \xrightarrow{\upsilon} \widetilde{Y}^\bullet_{n-1},$$

which we will denote by $w^{\widetilde{Y}^\bullet_n}$. Since υ is an F-equivalence, it induces the same class in $\mathrm{Ext}^{n+1,n}(B, B)$ as $w^{Y^\bullet_n}$. Consider the following diagram:

$$\begin{array}{ccc} L(A[n], n+1) & \xrightarrow{w^{X^\bullet_n}} & \mathrm{sk}_n X^\bullet \\ \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle \widetilde{f}} \\ L(B[n], n+1) & \xrightarrow[w^{\widetilde{Y}^\bullet_n}]{} & \widetilde{Y}^\bullet_{n-1} \end{array} \tag{4.5}$$

Here, $\overline{f}$ is induced by f in the following sense: it represents the uniquely determined homotopy class that induces the map $\pi^0 F_*(f) : A \to B$; compare 4.16.

We observe that, if we are given a diagram like (4.3), we get a diagram (4.5) and we have the following equation

$$\mathrm{ob}_n(f) = [\phi(f w^{X^\bullet_n})] = (\widetilde{\upsilon})_* \left(\left[w^{\widetilde{Y}^\bullet_n} \overline{f}\right] - \left[\widetilde{f} w^{X^\bullet_n}\right]\right) \in \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A, B), \tag{4.6}$$

since $L(B[n], n+1) \xrightarrow{w^{\widetilde{Y}^\bullet_n}} \widetilde{Y}^\bullet_{n-1} \xrightarrow{\widetilde{\upsilon}} Y^\bullet$ is a homotopy cofiber sequence.

Lemma 4.31. *The obstruction* $\mathrm{ob}_n(f)$ *from* (4.4) *vanishes if and only if the diagram* (4.5) *commutes in* $Ho(c\mathcal{M}^F)$.

Proof. If the square commutes up to homotopy, we can strictify it by changing $\overline{f}$ and $\widetilde{f}$ within their homotopy class. Then we can apply the pushout functor and obtain a map $X^\bullet \to Y^\bullet$. We can turn this process around if we remember that the homotopy cofiber sequence in 4.18 is also a homotopy fiber sequence. So, if a lifting exists, which is equivalent to $\mathrm{ob}_n(f) = 0$, then this diagram commutes. □

Theorem 4.32. *Let* $X^\bullet$ *and* $Y^\bullet$ *be objects in* $IP_n(F)$ *such that* $\pi^0 F_* X^\bullet = A$ *and* $\pi^0 F_* Y^\bullet = B$. *Then the map*

$$\mathrm{ob}_n(\_)\colon \mathrm{Hom}_{IP_{n-1}(F)}(\sigma_n X^\bullet, \sigma_n Y^\bullet) \to \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A, B)$$

is a homomorphism satisfying property (iii) *of* C.4.

Proof. There is a map of sets $\mathrm{Hom}_{IP_{n-1}(F)}(\sigma_n X^\bullet, \sigma_n Y^\bullet) \to \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A, B)$, where we map an f as in (4.3) to $\mathrm{ob}_n(f)$ as defined in (4.4). This is well defined by 4.10. We easily see that it is a homomorphism of abelian groups when we put the fold map $Y^\bullet \oplus Y^\bullet \to Y^\bullet$ in the place of $Y^\bullet$ in diagram (4.3).

Property (iii) of C.4 follows immediately by 4.31 by considering two squares, like (4.5), for $[f] : \sigma_n X^\bullet \to \sigma_n Y^\bullet$ and $[g] : \sigma_n Y^\bullet \to \sigma_n Z^\bullet$, respectively.

$$\begin{array}{ccc} L(A[n], n+1) & \xrightarrow{w^{X^\bullet_n}} & \mathrm{sk}_n X^\bullet \\ \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle \widetilde{f}} \\ L(B[n], n+1) & \xrightarrow[w^{Y^\bullet_n}]{} & \widetilde{Y}^\bullet_{n-1} \\ \Big\downarrow{\scriptstyle \overline{g}} & & \Big\downarrow{\scriptstyle \widetilde{g}} \\ L(C[n], n+1) & \xrightarrow[w^{Z^\bullet_n}]{} & \widetilde{Z}^\bullet_{n-1} \end{array}$$

Now the statement follows directly from 4.6 and 4.5. The homotopy classes involving the term $w^{Y^\bullet_n}$ cancel out. □

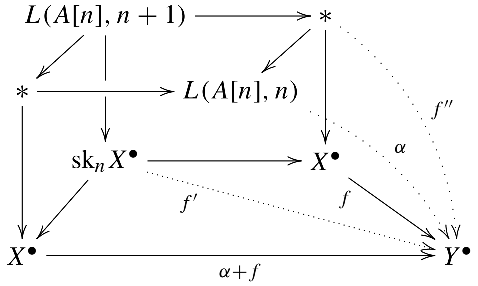
Remark 4.33. The homomorphism ob_n is identified by 4.43 and 5.9 with the differential $d_n\colon E^{0,0}_n(X, Y) \to \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(FX, FY)$ in the Adams spectral sequence for F. Here, $E^{0,0}_n(X, Y)$ is the intersection of the kernels of the previous differentials. In particular, we get from (4.6) the following formula

$$d_n f = w^Y_{n-1} f - f w^X_{n-1}$$

for $f \in \mathrm{Hom}_{\mathcal{A}}(FX, FY)$, which is reminiscent of [8, Proposition 8.10.]. Here, w^X_{n-1} and w^Y_{n-1} are the co-k-invariants of X and Y.

From diagram (4.3), we are now going to derive the obstruction against the uniqueness of the realization of f.

Definition 4.34. Let $f : X^\bullet \to Y^\bullet$ be a map of potential n-stages for A and B, respectively, and for $n \geq 1$. Let $\alpha \in \mathrm{Ext}^{n,n}_{\mathcal{A}}(A, B)$. We define a new map $\alpha + f : X^\bullet \to Y^\bullet$ by the following diagram:

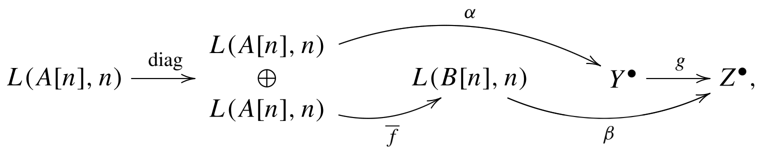


(4.7)

This diagram commutes up to homotopy; it can be strictified by choosing appropriate replacements for $*$ by external cone objects. The top square, the square on the left and the square in the back part of (4.7) are homotopy pushout squares. The datum of a map f is equivalent to giving maps f' and f'' making the obvious square (homotopy) commutative. Prescribing the homotopy class of α is equivalent to the existence of a map $\alpha + f$ whose homotopy class is, like that of f, a lifting of the homotopy class of f' in the sense of 4.28. Of course, the homotopy class of $\alpha + f$ is uniquely determined by the homotopy class of α (and of f). This defines a group action of $\mathrm{Ext}^{n,n}_{\mathcal{A}}(A, B)$ on the liftings of a map $\mathrm{sk}_n X^\bullet \to Y^\bullet$ in $IP_n(F)$, if they exist. We will denote this action by γ_n, so that $\alpha + f = \gamma_n(\alpha, f)$.

Theorem 4.35. *The map γ_n is an action of $\mathrm{Ext}^{n,n}_{\mathcal{A}}(A, B)$ on $\mathrm{Hom}_{IP_n(F)}(X^\bullet, Y^\bullet)$. Two morphisms agree on the n-skeleton if and only if they are in the same orbit. The restriction of the action to the set of realizations in $IP_n(F)$ of a given morphism in $IP_{n-1}(F)$ is free and transitive. The restricted action satisfies the linear distributivity law from C.4.*

Proof. All facts are straightforward. To prove that the linear distributivity law holds, we observe that the data to construct $(\beta + g)(\alpha + f)$ is contained in the map



where the homotopy class of $\overline{f}$ is induced by $f : X^\bullet \to Y^\bullet$. The induced map $X^\bullet \to Z^\bullet$ is $g_*\alpha + f^*\beta + gf$. □

4.2. The tower of interpolation categories

We will now plug the n-F-structures from Theorem 3.14 into the tower of truncated homotopy categories in 2.10. Suitable subcategories then supply a tower of interpolation categories for F. There are two ways to describe this tower. The first one is closer to the philosophy in [5].

Definition 4.36. Let $n \geq 0$. Let $IM_n(F)$ be the full subcategory of $c\mathcal{M}$ that consists of those objects $X^\bullet$, which are F-equivalent to a potential n-stage for some A in $\mathcal{A}$ defined in 4.1. We call this category the ***n*-th interpolation model of *F***.

Let $IP_n(F)$ be the image of $IM_n(F)$ in $Ho(c\mathcal{M}^F)$. We call this category the ***n*-th interpolation category of the functor *F***.

Remark 4.37. The second one is closer to our idea of truncated objects, where only the front end of the objects count and where we do not care for the higher degrees. Consider the full subcategory of $c\mathcal{M}$ of objects, which are n-F-equivalent to some potential n-stage. Then $IP_n(F)$ is equivalent to the image of this subcategory in $Ho(c\mathcal{M}^{n-F})$. The notion of isomorphism in $Ho(c\mathcal{M}^{0-F})$ is rather coarse, hence a lot of objects become identified. As n grows, fewer and fewer objects qualify for $IP_n(F)$, while the equivalences get finer and finer.

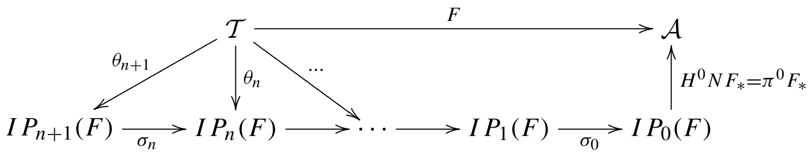
Note that whether $X^\bullet$ belongs to some interpolation category or not is detected by $FX^\bullet$; see Remark 4.3.

Remark 4.38. The functors σ_n from our tower of truncated homotopy categories restrict to our interpolation categories $IP_n(F)$. Also, $\theta_n X$ for some X in $\mathcal{T}$ lands in $IP_n(F)$. We will continue to denote these functors by θ_n and σ_n.

There is also an additional functor $\pi^0 F_* \cong H^0 NF_* : IP_n(F) \to \mathcal{A}$, which is derived from the functor $c\mathcal{M} \to \mathcal{A}$ given by

$$X^\bullet \mapsto \pi^0 F_* X^\bullet.$$

We arrive at the following tower of interpolation categories:



(4.8)

Again, this diagram 2-commutes in the 2-category of categories. It is worth emphasizing that the restricted functors σ_n and θ_n here in general do not possess left adjoints contrary to those in 2.10. The left adjoints there do not take values in some interpolation category.

4.3. Spectral sequences

Before we start the discussion about spectral sequences, let us explain what we want to find out. We would like to state that, if we have found an object $X^\bullet$ that looks like a fibrant replacement of a constant object and thus $\pi^s FX^\bullet = 0$ for all $s > 0$, then $\mathrm{Tot}X^\bullet$ is an actual realization of $\pi^0 FX^\bullet$. We will use this in 5.11 and 5.12. Unfortunately, there are problems with the spectral sequences in the cosimplicial case that do not occur in a simplicial setting.

Definition 4.39. For an object $Y^\bullet$ in $c\mathcal{M}$, let $\mathbf{Fib}_s\boldsymbol{Y}^\bullet$ denote the fiber of $\mathrm{Tot}_s Y^\bullet \to \mathrm{Tot}_{s-1} Y^\bullet$.

Remark 4.40. We already mentioned that the spiral exact sequence can be spliced together to an exact couple giving the spectral sequence (2.4):

$$E_2^{p,q} = \pi_p[X^\bullet, \Omega^q G] \Longrightarrow \operatorname*{colim}_k \pi_k^\natural(X^\bullet, \Omega^{p+q-k} G).$$

As in [17, 3.9], it follows that this spectral sequence is isomorphic from the E_2-term onward to a more familiar one, namely the G-cohomology spectral sequence of the total tower $\{\mathrm{Tot}_s X^\bullet\}$ for every $G \in \mathcal{G}$. Its E_1-term consists of

$$E_1^s = G^* \mathrm{Fib}_s X^\bullet = [\mathrm{Fib}_s X^\bullet, G]_*.$$

Since $X^\bullet$ is Reedy-fibrant, there is an isomorphism $\mathrm{Fib}_s X^\bullet \cong \Omega^s N^s X^\bullet$ by [18, p. 391], where $N^s X^\bullet := \mathrm{fiber}(X^s \to M^s X^\bullet)$ is the geometric normalization of $X^\bullet$. Moreover, it is true that there is an isomorphism

$$G^*(\mathrm{Fib}_s X^\bullet) = G^*(\Omega^s N^s X^\bullet) \cong N_s(G^{*+s} X^\bullet), \tag{4.9}$$

where, on the right hand side, N_s denotes the normalization of complexes. Also, the spectral sequence differential $d_1 : G^*(\mathrm{Fib}_{s+1} X^\bullet) \to G^*(\mathrm{Fib}_s X^\bullet)$ coincides up to sign with the boundary of the normalized cochain complex $N_*(G^* X^\bullet)$. Hence:

$$E_2^s = \pi_s[X^\bullet, G] \Longrightarrow \operatorname*{colim}_k [\mathrm{Tot}_k X^\bullet, G] \tag{4.10}$$

Theorem 6.1(a) from [6] states that the convergence of this spectral sequences is strong if and only if $\lim^1_r E^*_r = 0$. Problems arise now when we try to relate the target with the term $\mathrm{Hom}_{\mathcal{A}}(\pi^0 F X^\bullet, F_* G)$. Let us formulate all this still in another way. Consider the homology spectral sequence of a cosimplicial space

$$E_2^{s,t} = \pi^s F_t X^\bullet \Longrightarrow \lim_k F_{t-s} \mathrm{Tot}_k X^\bullet$$

with differentials

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

Again, a necessary and sufficient criterion for strong convergence is the vanishing of $\lim^1_r E^*_r$. Now the question arises, what has $\lim_s F\,\mathrm{Tot}_s X^\bullet$ to do with $F\,\mathrm{Tot} X^\bullet$. If our functor F commutes with countable products, we could set up a Milnor-type sequence that answers this question. But, in the cases of interest, F will not commute with infinite products, and the question has to remain open in general. The point is that Tot of a fibrant approximation computes a completion of the initial object, which may not coincide with the (localization of the) object. In the case of $F = E_*$ given by a suitable ring spectrum E, this will be the E-completion, which is proved in [4]. The answer we can offer is that, whenever the convergence results from [9] apply or the completion is known to be isomorphic to the localization, we can derive the existence of X in $\mathcal{T}$. Later, when it comes to realization questions, we will simply assume that the injective dimension of each object in $\mathcal{A}$ is finite, which will render all convergence problems trivial.

Lemma 4.41. *Let $X^\bullet$ be an F-fibrant object with the property that, for $s > 0$,*

$$\pi_s[X^\bullet, G] = 0.$$

(i) *Then there is a natural isomorphism*

$$\lim_s F_* \mathrm{Tot}_s X^\bullet \cong \pi^0 F_* X^\bullet.$$

For every $G \in \{F - \mathrm{inj}\}$, there are natural isomorphisms

$$\operatorname{colim}_k \pi_k^\natural (X^\bullet, \Omega^{p+q-k} G) \cong \operatorname{colim}_s [\mathrm{Tot}_s X^\bullet, G] \cong \mathrm{Hom}_{\mathcal{A}}(\pi^0 F X^\bullet, FG).$$

(ii) *If $\pi^0 F X^\bullet$ has finite injective dimension, then there are isomorphisms*

$$F\mathrm{Tot}\ X^\bullet \cong \pi^0 F X^\bullet$$

and, for every $G \in \{F - \mathrm{inj}\}$,

$$[\mathrm{Tot}\ X^\bullet, G] \cong \operatorname{colim}_s [\mathrm{Tot}_s X^\bullet, G] \cong \mathrm{Hom}_{\mathcal{A}}(\pi^0 F X^\bullet, FG).$$

Proof. The isomorphisms of part (i) follow from the fact that all the aforementioned spectral sequences collapse at the E_2-level with non-vanishing terms just in filtration 0 and converge strongly. Part (ii) follows from $\mathrm{Fib}_s \Omega^s N^s X^\bullet$ and (4.9), which imply that the constant tower $\{F_* \mathrm{Tot}\ X^\bullet\}$ and the tower $\{F_* \mathrm{Tot}_k X^\bullet\}$ are pro-equivalent. □

Corollary 4.42. *For every Y in $\mathcal{M}$, let $r^0 Y \to Y^\bullet$ be an F-fibrant approximation. Then the canonical map $Y \to \mathrm{Tot}\, Y^\bullet$ is an isomorphism in $\mathcal{T}$, if F detects isomorphisms and if FY has finite injective dimension.*

Proof. We immediately derive this result from 4.41. □

Remark 4.43. Finally, we remark that we get back to the modified Adams spectral sequence if we apply the functor $[X, \_]$ to the total tower of an F-fibrant approximation $Y^\bullet$ of an object Y from $\mathcal{T}$. The modified Adams spectral sequence is constructed in the same way as the original Adams spectral sequence, but it uses absolute injective resolutions instead of relative ones. It was introduced in [10]. Other accounts are given in [8,11] and [16]. It can be considered as the Bousfield–Kan spectral sequence of the simplicial space $\mathrm{map}^{\mathrm{pro}}(X, Y^\bullet)$. The E_1-term is given by

$$E_1^{s,t} = \pi_0 \mathrm{map}(\Sigma^t X, N^s Y^\bullet) \cong [X, N^s Y^\bullet]_t. \tag{4.11}$$

Since $Y^\bullet$ is an F-fibrant approximation to $r^0 Y$, it follows that the E_2-term takes the following form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(F_* X, F_* Y),$$

which is independent of the choice of the F-fibrant approximation and functorial in X and Y. The differentials are maps

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}. \tag{4.12}$$

We have another construction of the modified Adams spectral sequence obtained by applying the functor $[\_, Y^\bullet]_F$ to the Postnikov cotower from 2.9. We get an exact couple

$$\begin{array}{ccccccc} \cdots \longleftarrow & [\mathrm{sk}_s(X \otimes \Delta^\bullet), Y^\bullet]_F & \longleftarrow & [\mathrm{sk}_{s+1}(X \otimes \Delta^\bullet), Y^\bullet]_F & \longleftarrow & \cdots \\ & \downarrow & \nearrow^{+} & \downarrow & \nearrow^{+} & \\ & \mathrm{Ext}_{\mathcal{A}}^{s+1,s}(A, B) & & \mathrm{Ext}_{\mathcal{A}}^{s+2,s+1}(A, B) & & \end{array} \tag{4.13}$$

where the lower terms are identified by 4.10 and 4.18. The $+$ indicates that the map raises the external degree by one, so the differentials have the same form as in (4.12). It follows from the dual version of [17, Lemma 3.9.] that the exact couple from (4.13) is isomorphic to the derived couple of the E_1-exact couple in (4.11). Hence the spectral sequences coincide from the E_2-term onwards.

For convergence results, we have to take the usual precautions; see [11]. It is shown in [4] that it converges strongly to $[X, F^\wedge Y]$ if and only if $\lim_r^1 E_r^{*,*} = 0$. Here, F is a topologically flat ring spectrum with F_*F commutative (see [22]) and $F^\wedge Y$ is the F-completion of Y.

5. Properties of interpolation categories

In Section 5.1, we describe the properties that our tower of interpolation categories enjoys. In Section 5.2, we express these facts in terms of moduli spaces. The analogous results there have been obtained in [5], but our proofs are much shorter due to our truncated model structures.

5.1. Properties

The axioms that hold for our interpolation categories are taken from [1, VI.5] and briefly explained in Appendix C.

Theorem 5.1. *Let F be a homological functor as in 3.10 and $n \geq 1$. Then the following diagram*

$$\mathrm{Ext}_{\mathcal{A}}^{n,n}(\pi^0 F_*(\_), \pi^0 F_*(\_)) \to IP_n(F) \to IP_{n-1}(F) \\ \to \mathrm{Ext}_{\mathcal{A}}^{n+1,n}(\pi^0 F_*(\_), \pi^0 F_*(\_))$$

is an exact sequence of categories in the sense of C.4.

Proof. We have to check the various points in Definition C.4. That the Ext-terms here define natural systems of abelian groups is clear. Property (i) of C.4 is proved in 4.35; (ii) follows from 4.30; (iii) is shown in 4.32; and (iv) follows from the proof of Theorem 4.27. □

Theorem 5.2. *For each $n \geq 0$, the functor $\sigma_n\colon IP_{n+1}(F) \to IP_n(F)$ detects isomorphisms.*

Proof. This follows from 5.1 and C.5. □

Theorem 5.3. *The functor $\pi^0 F_*\colon IP_0(F) \to \mathcal{A}$ is an equivalence of categories.*

Proof. We will prove that $\pi^0 F$ is essentially surjective and induces a bijection

$$\mathrm{Hom}_{IP_0(F)}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{\mathcal{A}}(\pi^0 F_* X^\bullet, \pi^0 F_* Y^\bullet) \tag{5.1}$$

for arbitrary objects $X^\bullet$ and $Y^\bullet$ in $IP_0(F)$.

Let A be in $\mathcal{A}$. Choose an injective resolution $A \to I^\bullet$. Using Remark 3.9, we can realize $I^\bullet$ as a diagram in $\mathcal{T} = Ho(\mathcal{M})$. It was shown in 4.7 that we can realize the beginning part of this resolution by a 1-truncated cosimplicial object $E^\bullet$ in $c_1\mathcal{M}$. Let $X^\bullet$ in $c\mathcal{M}$ be $l^1E^\bullet$, where l^1 is the left Kan extension to $c\mathcal{M}$. Such an object is in $IM_0(F)$, because it is an object of the form $L(A,0)$. By construction, we have

$$\pi^0 F_* X^\bullet \cong \ker[\, F_* E^0 \xrightarrow{d^1-d^0} F_* E^1\,] \cong \ker[I^0 \to I^1] \cong A,$$

which proves that $\pi^0 F_*$ is essentially surjective. Now let $X^\bullet$ and $Y^\bullet$ be objects in $IP_0(F)$. Suppose that we are given a map $A \to B$ in $\mathcal{A}$ with $A = \pi^0 F_* X^\bullet$ and $B = \pi^0 F_* B^\bullet$. We can assume that $X^\bullet$ is 0-F-cofibrant and $Y^\bullet$ is F-fibrant. Then $X^\bullet$ is of type $L(A,0)$. A map from A to B can be extended to a map

$$K(A,0) \to F_* Y^\bullet,$$

since r^0 is left adjoint to taking the maximal augmentation $\pi^0(\_)$. Now 4.10 delivers us a map $L(A,0) = X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ inducing $A \to B$. Hence the functor is full.

Finally, let $X^\bullet \to Y^\bullet$ be a morphism in $IP_0(F)$ that is in the kernel of the map (5.1). Again, we assume that $X^\bullet$ is 0-F-cofibrant and $Y^\bullet$ is F-fibrant. This implies that the morphism is represented by a map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$, but also that $X^\bullet$ is of type $L(A,0)$. The induced map

$$K(A,0) \to F_* X^\bullet \to F_* Y^\bullet$$

is null homotopic by assumption. But then the fact that $L(A,0) = X^\bullet \to Y^\bullet$ is null homotopic follows again from 4.10. This proves that $\pi^0 F_*(\_)$ is faithful. □

Theorem 5.4. *Let A be an object in $\mathcal{A}$ of injective dimension $\le n+2$ for $n \ge 0$. The object A is realizable in $\mathcal{T}$ if and only if there exists an object $X^\bullet$ in $IP_n(F)$ with $\pi^0 F_* X^\bullet \cong A$ or equivalently a potential n-stage for A.*

Proof. The necessity of the existence of a potential n-stage is obvious, but also the sufficiency follows easily, since the obstructions against the existence of a realization as an ∞-stage $X^\bullet$ lie in $\mathrm{Ext}_{\mathcal{A}}^{n+3+s,n+1+s}(A,A)$ for $s \ge 0$ and these groups vanish by assumption. Now, for a fibrant approximation $X^\bullet \to \widetilde{X}^\bullet$, the total space $\mathrm{Tot}\,\widetilde{X}^\bullet$ is a realization of A by 4.41. □

Theorem 5.5. *Let A be an object in $\mathcal{A}$ with $\dim A \le n+1$ for $n \ge 0$. Let $X^\bullet$ in $IP_n(F)$ be an object with $\pi^0 F_* X^\bullet \cong A$. Then there exists an object X in $\mathcal{T}$ which is a lifting of $X^\bullet$ (and of A), and its isomorphism class in $\mathcal{T}$ is uniquely determined.*

Proof. Analogously to the previous proof, now all obstruction against uniqueness given by Theorem 4.27 also vanish. □

Definition 5.6. Recall that $\mathcal{A}$ has enough injectives by assumption. We consider the full subcategory $\boldsymbol{\mathcal{T}_n}$ of $\mathcal{T}$ consisting of those objects X such that F_*X has injective dimension $\le n$. This defines an increasing filtration of $\mathcal{T}$ with $\mathcal{T}_0$ equal to the full subcategory of F-injective objects $\mathcal{T}_{\mathrm{inj}}$. The inclusion functors $\mathcal{T}_n \hookrightarrow \mathcal{T}$ will be called $\boldsymbol{i_n}$.

Theorem 5.7. *The functors $\theta_k i_n : \mathcal{T}_n \to IP_k(F)$ are full for $k \ge n-1$. The functors $\theta_k i_n : \mathcal{T}_n \to IP_k(F)$ are faithful for $k \ge n$.*

Proof. If X is an object of $\mathcal{T}$, then its image $\theta_k X$ in $IP_k(F)$ is the k-F-equivalence class of r^0X. Let X and Y be in $\mathcal{T}_n$, where we assume from the beginning onwards that both are fibrant and cofibrant. We have to show that the map

$$\mathrm{Hom}_{\mathcal{T}_n}(X,Y) \to \mathrm{Hom}_{IP_k(F)}(\theta_k X, \theta_k Y) \tag{5.2}$$

is a bijection for $k \ge n-1$. To prove surjectivity, we take F-fibrant replacements $\widetilde{X}^\bullet$ and $Y^\bullet$ of $\theta_k X = r^0 X$ and $\theta_k Y = r^0 Y$, respectively, and then we replace $\widetilde{X}^\bullet$ Reedy cofibrantly by $X^\bullet$. Now each morphism $[f]$ in $\mathrm{Hom}_{IP_k(F)}(r^0X, r^0Y) \cong \mathrm{Hom}_{IP_k(F)}(X^\bullet, Y^\bullet)$ is represented by a map

$$f : \mathrm{sk}_{k+1} X^\bullet \to Y^\bullet$$

in $c\mathcal{M}$. The obstructions against extending this map to higher skeleta of $X^\bullet$ lie in $\mathrm{Ext}_{\mathcal{A}}^{k+2+s,k+1+s}(F_*X, F_*Y)$ for $s \ge 0$ by 4.30. All these groups vanish for $k \ge n-1$, because the injective dimension is smaller than or equal to

$n < k+2$. We end up with a map $f_\infty : X^\bullet \to Y^\bullet$. Now we get a morphism

$$\widetilde{f} : X \cong \operatorname{Tot} X^\bullet \xrightarrow{\operatorname{Tot} f_\infty} \operatorname{Tot} Y^\bullet \cong Y$$

in $\mathcal{T}$, where the isomorphisms are the canonical maps from 4.42, and they are isomorphisms, since F detects them. By Lemma 2.11 or Remark 2.12, $R\mathrm{Tot}$ and Lr^0 are a Quillen pair, and so $\sigma_n \widetilde{f}$ corresponds to $[f]$ via the isomorphism

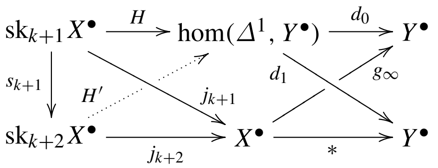
$$\operatorname{Hom}_{IP_k(F)}(r^0X, r^0Y) \cong \operatorname{Hom}_{IP_k(F)}(X^\bullet, Y^\bullet)$$

induced by the various replacements. So we have shown that θ_n is full.

The second part of the theorem amounts to proving the injectivity of the map

$$\operatorname{Hom}_{\mathcal{T}_k}(X, Y) \to \operatorname{Hom}_{IP_k(F)}(\theta_k X, \theta_k Y) \tag{5.3}$$

for $k \geq n$. This map is a homomorphism of abelian groups, since θ_k is additive. Let $g : X \to Y$ represent a morphism that is mapped to zero. Again, we pick replacements $X^\bullet$ and $Y^\bullet$ of r^0X and r^0Y as above. We find a map $g_\infty : X^\bullet \to Y^\bullet$ whose homotopy class is uniquely determined by $r^0g : r^0X \to r^0Y$ and which is nullhomotopic in $c\mathcal{M}^F$ when we restrict it to the $(k+1)$-skeleton of $X^\bullet$. This is displayed in the following solid arrow diagram, which strictly commutes:



The evaluation maps d_0 and $d_1 : \hom(\Delta^1, Y^\bullet) \to Y^\bullet$ are F-equivalences, so for both objects their $\pi^0 F_*$-term is isomorphic to F_*Y in $\mathcal{T}_n$. In particular, it follows from the first part of the theorem that H' exists with $H's_{k+1} \simeq H$ in the F-structure. Actually, the proof of 4.30 shows that we can arrange this to be strictly equal. It tells us that d_0H' and $g_\infty j_{k+2}$ are both extensions of the map $g_\infty j_{k+1} = d_0 H$. The obstructions against uniqueness of liftings, which are the homotopy classes of these extensions, lie in $\operatorname{Ext}^{k+1,k+1}_{\mathcal{A}}(F_*X, F_*Y)$ and this group vanishes, since the injective dimension is smaller than or equal to $n < k+1$ by assumption. It follows that $g_\infty j_{k+2}$ is F-homotopic to d_0H'. The same argument works with the other evaluation map d_1 and shows that $g_\infty j_{k+2}$ is nullhomotopic. By induction, we can extend this over all skeleta, since all higher obstruction groups also vanish. The skeletal tower of $X^\bullet$ is a tower of F-cofibrations between Reedy cofibrant (aka. F-cofibrant) objects, since $X^\bullet$ is Reedy cofibrant, therefore we have

$$X^\bullet \cong \operatorname*{colim}_k \mathrm{sk}_k X^\bullet \cong \operatorname*{hocolim}_k \mathrm{sk}_k X^\bullet.$$

Hence, the successive extensions give us a map $g'_\infty : X^\bullet \to Y^\bullet$, which on one side is homotopic to g_∞ and on the other to $*$. Because the homotopy class of g_∞ or g'_∞ corresponds under the isomorphism

$$\pi_0\mathrm{map}(X^\bullet, Y^\bullet) \cong \pi_0\mathrm{map}(r^0X, r^0Y)$$

to r^0g, this shows that our original map r^0g is nullhomotopic in $c\mathcal{M}$. Finally, constant cosimplicial objects over fibrant objects are Reedy fibrant, so we can apply and conclude

$$[g] = 0 \in \pi_0\mathrm{map}_{\mathcal{M}}(X, Y),$$

since Tot maps external homotopies between Reedy fibrant objects to homotopies in $\mathcal{M}$. □

Actually, the previous statement can be strengthened since, for both assertions, only the fact that Y is in $\mathcal{T}_n$ was needed. A plausible extension of 5.7 is the statement that the functor $\mathcal{T} \to IP_n(F)$ given by

$$X \mapsto \operatorname{Hom}_{IP_n(F)}(\theta_n X, Y^\bullet)$$

for some $Y^\bullet$ in $IP_n(F)$ is representable if and only if $\dim \pi^0 FY^\bullet \leq n$, but we have not been able to prove that.

The following theorem relates the tower of interpolation categories to the Adams spectral sequence associated with F.

Definition 5.8. A map f in $\mathcal{T}$ is said to be of **Adams filtration *n*** if it admits a factorization $f = f_1 \dots f_n$, where the maps f_i induce the zero map via F in $\mathcal{A}$. Let $F^n[X, Y]$ be the set of all maps of Adams filtration n, where we set $F^0[X, Y] := [X, Y]$. We obtain a decreasing filtration of $[X, Y]$.

Theorem 5.9. *For $n \geq 0$, there is a natural isomorphism*

$$F^{n+1}[X, Y] \cong \ker[[X, Y] \to \mathrm{Hom}_{IP_n(F)}(\theta_n X, \theta_n Y)].$$

Proof. Let $r^0 Y \to Y^\bullet$ be an F-fibrant approximation and remember that $NF_*Y^\bullet$ is an injective resolution of F_*Y. For $n = 0$, the claim follows from the equivalence $IP_0(F) \cong \mathcal{A}$. For $n = 1$, a map $f \in F^2$ in particular induces the zero map in $\mathcal{A}$, thus it admits a factorization

$$X \to \widetilde{Y}^1 \to Y$$

where $\widetilde{Y}^1$ is the fiber of the map $Y \to Y^0$. It follows easily that such maps f are characterized by the fact that the map $X \to \widetilde{Y}^1$ induces the zero map in $\mathcal{A}$. On the other hand, a map $f : X \to Y$ is in the kernel above if and only if the map $\mathrm{sk}_2(X \otimes \Delta^\bullet) \to Y^\bullet$ representing $\theta_1 f : \theta_1 X \to \theta_1 Y$ admits a lifting:

$$\begin{array}{ccc} & & \mathrm{hom}(\Delta^1, Y^\bullet) \times_{Y^\bullet} * \\ & \nearrow & \downarrow \\ \mathrm{sk}_2(X \otimes \Delta^\bullet) & \longrightarrow & Y^\bullet \end{array}$$

Considering the non-degenerate 1-simplex in $X \otimes \Delta^1$ shows that the map $X \to Y^1$ is null homotopic. This map also factorizes over $X \to \widetilde{Y}^1$ and, since $F\widetilde{Y}^1 \to F_*Y^1$ is injective, it follows that $X \to \widetilde{Y}^1$ induces the zero map. We can work backwards and show that, if $X \to \widetilde{Y}^1$ induces the zero map, we can construct null homotopy on the 2-skeleton, which proves the isomorphism. For higher n, we proceed inductively. We show that the map $X \to \widetilde{Y}^n$, where $\widetilde{Y}^n$ is the fiber of the map $\widetilde{Y}^{n-1} \to Y^{n-1}$, induces the zero map if and only if there is a diagram:

$$\begin{array}{ccc} & & \mathrm{hom}(\Delta^1, Y^\bullet) \times_{Y^\bullet} * \\ & \nearrow & \downarrow \\ \mathrm{sk}_{n+1}(X \otimes \Delta^\bullet) & \longrightarrow & Y^\bullet \end{array}$$

□

5.2. *Moduli spaces of realizations*

Again, we let F be a homological functor as in 3.10; F **detects isomorphisms** in $\mathcal{T}$. Note that this assumption means that a map $X \to Y$ in $\mathcal{M}$ induces an isomorphism $F_*X \to F_*Y$ if and only if it was a weak equivalence. We summarize the necessary theory of moduli spaces in Appendix B.

Definition 5.10. Let A be an object in $\mathcal{A}$. We define the **space of realizations** of A to be the moduli space of all objects X in $\mathcal{M}$, such that their image F_*X is isomorphic to A (see Definition B.3). We will write $\mathbf{Real}(\boldsymbol{A})$.

We define the **space of *n*-th partial realizations** of A to be the moduli space of all objects $X^\bullet$ in $c\mathcal{M}$ that are potential n-stages for A (see Definition 4.1). We will write $\mathbf{Real}_{\boldsymbol{n}}(\boldsymbol{A})$. Everything also makes sense for $n = \infty$, and hence we define in the same way the **space of ∞-stages** of A and denote it by $\mathbf{Real}_\infty(\boldsymbol{A})$. Recall that ∞-$\mathcal{G}$-structure is just another name for the $\mathcal{G}$-structure.

The first theorem that we are heading for is 5.12, which tells us that ∞-stages are the same as actual realizations in $\mathcal{T} = Ho(\mathcal{M})$. The next step is Theorem 5.13, which relates the moduli space of ∞-stages to the spaces $\mathrm{Real}_n(A)$ of potential n-stages. Finally, we establish in 5.19 a fiber sequence involving $\mathrm{Real}_{n-1}(A)$ and $\mathrm{Real}_n(A)$.

Remark 5.11. To relate an ∞-stage of an object in $\mathcal{A}$ to an actual realization, we use the functor $\mathrm{Tot} : c\mathcal{M} \to \mathcal{M}$. By 4.40, there is a spectral sequence:

$$E_2^{s,t} = \pi_s F_t X^\bullet \Longrightarrow \lim_k F_{t-s} \mathrm{Tot}_k X^\bullet.$$

From Lemma 4.41, we can read off that, for an ∞-stage $X^\bullet$ of an object A with finite injective dimension, the spectral sequence collapses and its edge homomorphism gives an isomorphism

$$F_*\mathrm{Tot}\, X^\bullet \cong \lim_k F_*\mathrm{Tot}_k X^\bullet \cong A.$$

More generally, the spectral sequence gives such an isomorphism whenever the results in [9] say so. We see that, under these assumptions, the functor Tot induces a natural map

$$\mathrm{Real}_\infty(A) \to \mathrm{Real}(A). \tag{5.4}$$

Theorem 5.12. *The map (5.4) is a weak equivalence of spaces if A has finite injective dimension or if the convergence results from* [9] *apply.*

Proof. Let X be a realization of A in $\mathcal{M}$. Then the canonical map $X \to \mathrm{Tot}\, r^0 X = \mathrm{Tot}_0\, r^0 X = X$ is even an isomorphism in $\mathcal{M}$.

Let $X^\bullet$ be a vertex in $\mathrm{Real}_\infty(A)$, in other words an ∞-stage of A. Without loss of generality, we assume that $X^\bullet$ is F-fibrant and Reedy cofibrant, because these manipulations induce self equivalences of the moduli space $\mathrm{Real}_\infty(A)$. But now the map

$$r^0\mathrm{Tot}\, X^\bullet \to X^\bullet$$

is an F-equivalence by 4.41. Since F detects isomorphisms in $\mathcal{T}$, this shows that the maps induced by Tot and r^0 are mutually inverse homotopy equivalences. □

The rest of this subsection is true without any restriction on A.

Theorem 5.13. *The canonical map*

$$\mathrm{Real}_\infty(A) \to \operatorname*{holim}_n \mathrm{Real}_n(A)$$

is a weak equivalence.

We prove this theorem after having established two lemmas.

Definition 5.14. Let $\mathbf{weak}_S(\boldsymbol{A}^\bullet, \boldsymbol{B}^\bullet)$ denote the simplicial set given by

$$\mathrm{weak}_S(A^\bullet, B^\bullet)_n := \mathrm{Hom}_{\mathcal{W}_S}(A^\bullet \otimes \Delta^n, B^\bullet),$$

where $\mathcal{W}_S$ is the subcategory of weak equivalences in some simplicial model structure S on $c\mathcal{M}$. If $A^\bullet$ is fibrant and cofibrant in S, then

$$\mathrm{weak}_S(A^\bullet, A^\bullet) = \mathrm{haut}_S(A^\bullet)$$

by Definition B.1. Analogously to Remark B.2, we observe that $\mathrm{weak}_S(A^\bullet, B^\bullet)$ is a union of connected components of $\mathrm{map}_S(A^\bullet, B^\bullet)$.

Lemma 5.15. *Let $\mathcal{G}$ be a class of injective models for $\mathcal{M}$. Let $X^\bullet$ be a Reedy cofibrant object and $Y^\bullet$ be a $\mathcal{G}$-fibrant object in $c\mathcal{M}$. Then there is a canonical map*

$$\begin{aligned}\operatorname*{holim}_n \mathrm{weak}_{n-\mathcal{G}}(\mathrm{sk}_{n+1}X^\bullet, Y^\bullet) &\xrightarrow{\simeq} \lim_n \mathrm{weak}_{n-\mathcal{G}}(\mathrm{sk}_{n+1}X^\bullet, Y^\bullet)\\ &\cong \quad \mathrm{weak}_{\mathcal{G}}(X^\bullet, Y^\bullet)\end{aligned}$$

where the first one is a weak equivalence and the second one is an isomorphism which are natural in both variables for $\mathcal{G}$-equivalences.

Proof. First we observe that the corresponding statement for the functor $\mathrm{map}(\_, \_)$ is true. Here, $\mathrm{map}(\_, \_)$ that is the external mapping space from A.2 always has homotopy meaning, since $\mathrm{sk}_{n+1}X^\bullet \to X^\bullet$ is an n-$\mathcal{G}$-cofibrant approximation. Also, the tower maps are fibrations by (SM7'), because they are induced by the $\mathcal{G}$-cofibration $\mathrm{sk}_n X^\bullet \to \mathrm{sk}_{n+1}X^\bullet$. Finally, $\mathrm{map}(\_, \_)$ turns colimits in the first variable into limits and $\mathrm{colim}_n\ \mathrm{sk}_{n+1}X^\bullet \cong X^\bullet$.

The proof is finished by the above remark that $\text{weak}_{\mathcal{S}}(X^\bullet, Y^\bullet)$ is a union of components in $\text{map}_{\mathcal{S}}(X^\bullet, Y^\bullet)$ and that these components form a tower because n-$\mathcal{G}$-equivalences are mapped to $(n-1)$-equivalences by the restriction of the upper maps. □

Lemma 5.16. *Let $X^\bullet$ be F-fibrant and Reedy cofibrant, then the canonical map*

$$\text{haut}_F(X^\bullet) \to \operatorname{holim}_n \text{haut}_{n-F}(\text{sk}_{n+1} X^\bullet)$$

is a weak equivalence.

Remark 5.17. Note that the homotopy self-equivalences on the right hand side can also be taken in the F-structure, since n-F-equivalences and F-equivalences agree on n-F-cofibrant objects.

Proof of 5.16. The inclusions of the skeletons into $X^\bullet$ induce the following commutative diagram:

$$\begin{array}{ccc} \text{haut}_{n-F}(\text{sk}_{n+1} X^\bullet) & \longrightarrow & \text{haut}_{(n-1)-F}(\text{sk}_n X^\bullet) \\ \simeq \downarrow & & \downarrow \simeq \\ \text{weak}_{n-F}(\text{sk}_{n+1} X^\bullet, X^\bullet) & \xrightarrow{\omega_n} & \text{weak}_{(n-1)-F}(\text{sk}_n X^\bullet, X^\bullet) \end{array}$$

Both horizontal maps fit into a tower of maps when we vary n. We want to compute the homotopy limit of the upper tower. To do this, we have to replace this tower by an objectwise weakly equivalent one in which the tower maps are fibrations. This is provided by the lower tower, as we proved in 5.15. The vertical maps are homotopy equivalences, because $\text{sk}_{n+1} X^\bullet \to X^\bullet$ is a cofibrant approximation in the n-F-structure. The result now follows from 5.15. □

Proof of 5.13. By Theorem B.4, we have the following weak equivalences:

$$\text{Real}_\infty(A) \simeq \coprod_{\langle X^\bullet \rangle_F} B\text{haut}_F(X^\bullet)$$

where the coproduct is taken over all F-equivalence classes $\langle X^\bullet \rangle$ of ∞-stages $X^\bullet$ of A. By the same theorem, we obtain the first of the next two weak equivalences:

$$\text{Real}_n(A) \simeq \coprod_{\langle X_n^\bullet \rangle_F} B\text{haut}_F(X_n^\bullet) \simeq \coprod_{\langle X_n^\bullet \rangle_{n-F}} B\text{haut}_{n-F}(X_n^\bullet),$$

where the coproduct is taken over all F-equivalence classes $\langle X_n^\bullet \rangle$ of potential n-stages $X_n^\bullet$ of A. Because the F- and the n-F-equivalences agree on n-F-cofibrant objects, there is a one-to-one correspondence between equivalence classes of potential n-stages in the F-structure and in the n-F-structure and there is a weak equivalence $\text{haut}_F(X^\bullet) \simeq \text{haut}_{n-F}(X^\bullet)$. Hence we get the second weak equivalence, where the coproduct is taken over weak equivalence classes in the n-F-structure. The theorem now follows from Lemma 5.16 and the fact that the classifying space functor B from simplicial monoids to $\mathcal{S}$ preserves weak equivalences, fibrations and limits. □

Theorem 5.18. *Let A be an object in $\mathcal{A}$. Then we have a weak equivalence*

$$\text{Real}_0(A) \simeq B\text{Aut}(A).$$

Proof. This is just a restatement of the equivalence in 4.12. □

Theorem 5.19. *Let $X_{n-1}^\bullet$ be a potential $(n-1)$-stage for an object A in $\mathcal{A}$. Then there is a fiber sequence*

$$\mathcal{H}^{n+1}(r^0 A, A[n]) \to \text{Real}_n(A)_{X_{n-1}^\bullet} \to \mathcal{M}(X_{n-1}^\bullet),$$

where $\text{Real}_n(A)_{X_{n-1}^\bullet}$ are those components of $\text{Real}_n(A)$ that correspond to objects $X^\bullet$ with $\text{sk}_n X^\bullet \simeq X_{n-1}^\bullet$.

Proof. By 4.18, there is a cofiber sequence

$$X_{n-1}^\bullet \to X_n^\bullet \to L(A[n], n)$$

in $c\mathcal{M}^F$ inducing the following fiber sequence in $\mathcal{S}$:

$$\operatorname{map}(L(A[n], n), X_n^\bullet) \to \operatorname{map}(X_n^\bullet, X_n^\bullet) \to \operatorname{map}(X_{n-1}^\bullet, X_n^\bullet).$$

Passing to appropriate components gives a fiber sequence

$$\operatorname{map}(L(A[n], n), X_n^\bullet) \to \operatorname{weak}_{n-F}(X_n^\bullet, X_n^\bullet) \to \operatorname{weak}_{(n-1)-F}(X_{n-1}^\bullet, X_n^\bullet)$$

of grouplike simplicial monoids. Applying the classifying space functor B to this sequence yields a fiber sequence

$$B\operatorname{map}(L(A[n], n), X_n^\bullet) \to \mathcal{M}(X_n^\bullet)_{X_{n-1}^\bullet} \to \mathcal{M}(X_{n-1}^\bullet).$$

Let $\Gamma : \mathrm{CoCh}^{\geq 0}(\mathcal{A}) \to c\mathcal{A}$ be the Dold–Kan functor. We finally compute, using 4.10:

$$\begin{aligned} B\operatorname{map}(L(A[n], n), X_n^\bullet) &\simeq B\operatorname{map}(K(A[n], n), \mathrm{sk}_{n+1} F_* X_n^\bullet) \\ &\simeq B\operatorname{map}(K(A[n], n), r^0 A) \\ &\simeq B\Gamma(\mathrm{Hom}_{\mathcal{A}}(A, A)[n]_{\mathrm{ext}}) \\ &\simeq \Gamma(\mathrm{Hom}_{\mathcal{A}}(A, A)[n+1]_{\mathrm{ext}}) \\ &\simeq \mathcal{H}^{n+1}(A, A[n]), \end{aligned}$$

where $\mathrm{Hom}_{\mathcal{A}}(A, A)[n]_{\mathrm{ext}}$ is viewed as a cochain complex concentrated in degree n. Here, $[1]_{\mathrm{ext}}$ is the external shift from 3.27. □

Theorem 5.20. *Let f be a morphism in $\mathcal{A}$. Then we have a weak equivalence*

$$\mathrm{Real}_0(f) \simeq B\mathrm{Aut}(f).$$

Proof. This follows readily from the equivalence $\mathcal{A} \cong IP_0(F)$ of categories of 5.3. □

Theorem 5.21. *Let $f : X_n^\bullet \to Y_n^\bullet$ be a map of potential n-stages for objects A and B, respectively, in $\mathcal{A}$. Then there is a fiber sequence*

$$\mathcal{H}^n(A[n], B) \to \mathcal{M}(f)_{\mathrm{sk}_n f} \to \mathcal{M}(\mathrm{sk}_n f).$$

Proof. We can assume, without loss of generality, that $X_n^\bullet$ and $Y_n^\bullet$ are Reedy cofibrant and F-fibrant. As in the proof of 5.19, we obtain a fiber sequence

$$\operatorname{map}(L(A[n], n), Y_n^\bullet) \to \operatorname{map}(X_n^\bullet, Y_n^\bullet) \to \operatorname{map}(\mathrm{sk}_n X_n^\bullet, Y_n^\bullet).$$

Proceeding as in the previous proof, we arrive at the conclusion. □

6. Examples and applications

All the applications given here will involve just the obstruction calculus. We do not have yet applications of the interpolation categories themselves.

6.1. Very low dimensions

Example 6.1. If the injective dimension of the target category is 0, then the tower of interpolation categories simply collapses to the equivalences:

$$\begin{array}{ccc} IP_0(F) & \xrightarrow{\cong} & \mathcal{A} \\ \cong\uparrow & & \uparrow\cong \\ \mathcal{T} = \mathcal{T}_0 & \underset{\cong}{\xrightarrow{F}} & \mathcal{A}_{\mathrm{inj}} \end{array}$$

Here, the lower equivalence was already stated in 3.9.

Example 6.2. If the injective dimension of $\mathcal{A}$ is 1, then the tower of interpolation categories has one non-trivial step:

$$\begin{array}{ccccc} \mathcal{T} = \mathcal{T}_1 & \xrightarrow{\cong} & IP_1(F) & & \\ & & \downarrow & & \\ & & IP_0(F) & \xrightarrow{\cong} & \mathcal{A} \\ & & \uparrow & & \uparrow \\ & & \mathcal{T}_0 & \xrightarrow[\cong]{} & \mathcal{A}_{\text{inj}} \end{array}$$

We can express this using 4.35 or 5.1 by saying that

$$\operatorname{Ext}^{1,1}_{\mathcal{A}}(F_*(\_), F_*(\_)) \to \mathcal{T} \to \mathcal{A}$$

is a linear extension of categories which is defined in [1, VI.5].

6.2. Some $E(n)$-local Picard groups

Let $\mathcal{C}$ be a symmetric monoidal category whose pairing is called smash product and denoted by $\wedge$. An invertible object X is one such that there exists a Y in $\mathcal{C}$ with $X \wedge Y \cong S$, where S denotes the unit of the monoidal structure. The isomorphism classes of invertible objects inherit an abelian group structure that we will call the Picard group $\operatorname{Pic}(\mathcal{C})$. It is an abelian group, but sometimes in a higher universe. It was defined by Hopkins and we refer to [20], where it is proved that the Picard group of the whole stable homotopy category of spectra is $\mathbb{Z}$. There are also computations involving the Picard group of the $K(n)$-local category, where $K(n)$ denotes n-th Morava K-theory.

Fix a prime p. The problem of computing $\operatorname{Pic}(E(n))$ for the $E(n)$-local category of spectra was considered in [23]. Here, $E(n)$ denotes the n-th Johnson–Wilson spectrum. It is a Landweber exact theory with

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$$

where $|v_i| = 2(p^i - 1)$. $E(1)$ is a retract of $K_{(p)}$, which is sometimes called the Adams summand. Hovey and Sadofsky prove that there is a splitting

$$\operatorname{Pic}(E(n)_*) \cong \mathbb{Z} \oplus \operatorname{Pic}^0(E(n)_*),$$

where $\mathbb{Z}$ is generated by the unit of the smash product, the stable localized sphere $L_n S^0$, and they show that

$$\operatorname{Pic}^0(E(n)) = 0,$$

whenever $2p - 1 > n^2 + n$. Their argument is the following. First they prove in [23, 2.4.] that, for an element $X \in \operatorname{Pic}^0(E(n))$, there is an isomorphism

$$E(n)_* X \cong E(n)_* \tag{6.1}$$

as $E(n)_*E(n)$-comodules. This turns the question into a moduli problem adressed in this paper. Actually, the Picard group in the presence of the isomorphism (6.1) is nothing but $\pi_0 \operatorname{Real}(E(n)_*)$. Then they show in [23, 5.1.] that, for $p > n + 1$, the category of $E(n)_*E(n)$-comodules has injective dimension $\leq n^2 + n$. This is also proved in [16, Theorem 9]. Considering the $E(n)$-Adams spectral sequence, which they prove to converge nicely, they see that the first obstruction for realizing the isomorphism (6.1) as a map $X \to L_n S^0$ lies in $\operatorname{Ext}^{2p-1,2p-2}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$ by the usual sparseness in the chromatic setting. Now the statement is clear since, for $2p - 1 > n^2 + n$, this obstruction group is zero. Using their vanishing line, we can extend the range of calculations of Picard groups slightly.

Theorem 6.3. *Fix a prime p and a natural number n such that $p > n + 1$ and $4p - 3 > n^2 + n$. Then we have:*

$$\operatorname{Pic}^0(E(n)) \cong \operatorname{Ext}^{2p-1,2p-2}_{E(n)_*E(n)}(E(n)_*, E(n)_*).$$

Proof. According to (6.1), we are trying to realize the object $E(n)_*$. In the range under consideration, the only obstruction against existence lies in

$$\mathrm{Ext}^{2p,2p-2}_{E(n)_*E(n)}(E(n)_*, E(n)_*),$$

but the obstruction vanishes, because we know that there is a realization, L_nS^0. Now, theorem 4.27 tells us that the uniqueness obstruction group

$$\mathrm{Ext}^{2p-1,2p-2}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$$

acts freely and transitively on the realizations. All these elements give actual spectra, since all other obstruction groups in the asserted range vanish by [23, 5.1.]. □

For the range $2p-1 > n^2+n$, we get back the result of Hovey and Sadofsky.

Acknowledgements

This article is the second part of my thesis, carried out at the Universität Bonn. I would like to thank my advisor Jens Franke and my coadvisor Stefan Schwede for their interest and help over the years. I would also like to thank Paul Goerss for his encouragement. Finally, I am deeply indebted to Pete Bousfield for providing me with a copy of his then unpublished preprint [7] and for several extremely enlightening e-mails. I was supported by the Bonner Internationale Graduiertenschule (BIGS), which gave me the opportunity to visit several conferences on homotopy theory.

Appendix A. The external simplicial structure on $c\mathcal{M}$

The resolution model structures are not compatible with the internal simplicial structure. Here we describe the external simplicial structure, which will be compatible with the resolution structure and its truncated versions.

Remark A.1. For $X^\bullet$ in $c\mathcal{M}$ and L in $\mathcal{S}$, we can perform the following coend construction. Let $\coprod_{L_\ell} X^m$ be the coproduct in $\mathcal{M}$ of copies of X^m indexed by the set L_ℓ, and view this as a functor $\Delta^{\mathrm{op}} \times \Delta \to \mathcal{M}$. Then we can take the coend

$$X^\bullet \otimes_\Delta L := \int^\Delta \coprod_{L_\ell} X^m \in \mathcal{M}.$$

We are now ready to describe the functors that will enrich all our model structures to simplicial model categories.

Definition A.2. We define a simplicial structure on $c\mathcal{M}$. Let K be in $\mathcal{S}$ and $X^\bullet$ and $Y^\bullet$ in $c\mathcal{M}$, then set

$$(X^\bullet \otimes^{\mathrm{ext}} K)^n := X^\bullet \otimes_\Delta (K \times \Delta^n),$$

where $\times$ denotes the usual product of simplicial sets and Δ^n is the standard n-simplex,

$$\mathrm{hom}^{\mathrm{ext}}(K, X^\bullet)^n := \prod_{K_n} X^n,$$

where the product is taken over the set of n-simplices of K, and finally

$$\mathrm{map}^{\mathrm{ext}}(X^\bullet, Y^\bullet)_n := \mathrm{Hom}_{c\mathcal{M}}(X^\bullet \otimes^{\mathrm{ext}} \Delta^n, Y^\bullet).$$

We call this the **external (simplicial) structure** on $c\mathcal{M}$. Note that we do not refer to any simplicial structure of $\mathcal{M}$. We will usually drop the superscripts.

Definition A.3. For an object $X^\bullet$ in $c\mathcal{M}$, we define its ***s*-th external suspension** $\Sigma^s_{\text{ext}} X^\bullet$ by the following pushout diagram:

$$\begin{array}{ccc} X^\bullet = X^\bullet \otimes^{\text{ext}} * & \longrightarrow & X^\bullet \otimes^{\text{ext}} \Delta^s/\partial\Delta^s \\ \downarrow & & \downarrow \\ * & \longrightarrow & X^\bullet \wedge^{\text{ext}} \Delta^s/\partial\Delta^s =: \Sigma^s_{\text{ext}} X^\bullet \end{array}$$

There is a dual construction called $\Omega_{\text{ext}} X^\bullet$.

Appendix B. Moduli spaces in model categories

Definition B.1. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{W}$ be its subcategory of weak equivalences. For a cofibrant and fibrant object X, we define the **simplicial monoid of self equivalences** denoted by **haut** (X) by setting

$$\text{haut}(X)_n := \text{Hom}_{\mathcal{W}}(X \otimes \Delta^n, X).$$

If we need to specify a model structure on $\mathcal{M}$, because there are several possible choices, we write the name of the structure as an index, so e.g. $\text{haut}_{\text{Reedy}}(X^\bullet)$ denotes the simplicial monoid in the Reedy structure on $c\mathcal{M}$.

Remark B.2. It is an easy, but quite important, observation that the space $\text{haut}(X)$ consists of those connected components of the space $\text{map}(X, X)$ stemming from the simplicial structure of $\mathcal{M}$ that are given by the vertices corresponding to weak self-equivalences.

Definition B.3. Let $\mathcal{M}$ be a model category. We define the **moduli space of an object** X in $\mathcal{M}$ to be the nerve of the following category: objects are the objects of $\mathcal{M}$ that are weakly equivalent to X and morphisms are the weak equivalences. It is denoted by $\boldsymbol{\mathcal{M}}(X)$. Note that, for each X in $\mathcal{M}$, this moduli space is non-empty and connected. If S is a set of objects in $\mathcal{M}$, we define $\boldsymbol{\mathcal{M}}(S)$ to be the nerve of the full subcategory of $\mathcal{W}$, whose objects are weakly equivalent to an element of S.

We define the **moduli space of a morphism** in $\mathcal{M}$ in the same way as for objects. Let f be an object in the category $\text{Mor}(\mathcal{M})$. It can be given a model structure with objectwise weak equivalences, for which we refer to [15]. The moduli space of f is the space $\mathcal{M}(f)$ from Definition B.3 in the category $\text{Mor}(\mathcal{M})$.

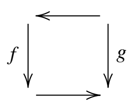
The important theorem about moduli spaces is the following one proved in [12, Proposition 2.3.].

Theorem B.4. *Let $\mathcal{M}$ be a simplicial model category and let X be an object of $\mathcal{M}$. Then the moduli space $\mathcal{M}(X)$ is weakly equivalent to the space $B\text{haut}(X)$.*

Appendix C. Extension of categories

The definitions in this paragraph are taken from [1, VI.5].

Definition C.1. Let $\mathcal{C}$ be a category. Let $\text{Fac}\,\mathcal{C}$ be the **category of factorizations** of $\mathcal{C}$. It is the Grothendieck construction on $\mathcal{C}^{\text{op}} \times \mathcal{C}$ with respect to the functor $\text{Hom}_{\mathcal{C}}(\_, \_)$. Explicitly, it has the morphisms of $\mathcal{C}$ as objects, and a morphism $f \to g$ is given by a commutative diagram of the following shape:



Definition C.2. A **natural system of abelian groups** on a category $\mathcal{C}$ is a functor from $\text{Fac}\,\mathcal{C}$ to the category Ab of abelian groups.

Remark C.3. There is a canonical functor $\text{Fac}\,\mathcal{C} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$, sending a morphism to its source and target. Hence each bifunctor $\Gamma\colon \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}$ induces a natural system on $\mathcal{C}$. In this case, we will also write $\Gamma(X, Y)$ for $\Gamma(f)$ if $f\colon X \to Y$ is a morphism in $\mathcal{C}$.

The following definition is taken from [1, VI(5.4)].

Definition C.4. Let $\sigma : \mathcal{C} \to \mathcal{D}$ be a functor, and let Γ and Ξ be natural systems on $\mathcal{D}$. We write symbolically

$$\Gamma \xrightarrow{\gamma} \mathcal{C} \xrightarrow{\sigma} \mathcal{D} \xrightarrow{\text{ob}} \Xi$$

and call this diagram an **exact sequence of categories**, if the following conditions are satisfied:

(i) For all objects A and B in $\mathcal{C}$ and for each morphism $f \in \text{Hom}_{\mathcal{D}}(\sigma(A), \sigma(B))$, there is a transitive action γ of the group $\Gamma(f)$ on the set $\sigma^{-1}(f) \subset \text{Hom}_{\mathcal{C}}(A, B)$. This action satisfies the **linear distributivity law**:

$$(\alpha + \widetilde{f})(\beta + \widetilde{g}) = f_*\alpha + g^*\beta + \widetilde{f}\widetilde{g}$$

for all $\widetilde{f} \in p^{-1}(f)$, $\widetilde{g} \in p^{-1}(g)$, $\alpha \in G(f)$ and $\beta \in G(g)$, and where we have abbreviated $\gamma(\alpha, f)$ by $\alpha + f$.

(ii) For all objects A and B in $\mathcal{C}$ and all morphisms $f : \sigma(A) \to \sigma(B)$ in $\mathcal{D}$, there is an obstruction element $\text{ob}(f) \in \Xi(f)$ given such that

$$\text{ob}(f) = 0$$

if and only if there exists a morphism $\widetilde{f} : A \to B$ with $\sigma(\widetilde{f}) = f$.

(iii) For all $f : \sigma(A) \to \sigma(B)$ and $g : \sigma(B) \to \sigma(C)$, we have the following equation:

$$\text{ob}(gf) = g_*\text{ob}(f) + f^*\text{ob}(g).$$

(iv) For all objects A in $\mathcal{C}$ and for all $\alpha \in \Xi(\text{id}_{\sigma(A)})$, there is an object B in $\mathcal{C}$ with the property that $\sigma(A) = \sigma(B)$ and $\text{ob}(\text{id}_{\sigma(A)}) = \alpha$.

The next lemma follows from the axioms and is taken from [1, V(5.7)].

Lemma C.5. *If the functor σ is part of an exact sequence of categories as in C.4, then it detects isomorphism.*

References

[1] H. Baues, Combinatorial Foundations of Homology and Homotopy, in: Monographs in Mathematics, Springer, 1999.

[2] D. Benson, H. Krause, S. Schwede, Realizability of modules over Tate cohomology, Trans. Amer. Math. Soc. 356 (9) (2004) 3621–3668 (electronic).

[3] G. Biedermann, Truncated resolution model structures. J. Pure Appl. Algebra. (in press).

[4] G. Biedermann, Injective completion with respect to homology, arxiv:math.AT/0412387, 2004.

[5] D. Blanc, W. Dwyer, P. Goerss, The realization space of a Π-algebra: a moduli problem in algebraic topology, http://www.nd.edu/≃wgd, 2001.

[6] J. Boardman, Conditionally convergent spectral sequences, in: J.-P. Meyer, J. Morava, W. Wilson (Eds.), Homotopy Invariant Algebraic Structures: A Conference in Honor of J. Michael Boardman, in: Contemporary Mathematics, vol. 239, AMS, 1999, pp. 49–84.

[7] A. Bousfield, Cosimplicial resolutions and homotopy spectral sequences in model categories, Geom. Topol. 7 (2003) 1001–1053.

[8] A.K. Bousfield, On the homotopy theory of K-local spectra at an odd prime, Amer. J. Math. 107 (4) (1985) 895–932.

[9] A.K. Bousfield, On the homology spectral sequence of a cosimplicial space, Amer. J. Math. 109 (2) (1987) 361–394.

[10] H.-B. Brinkmann, Relative homological algebra and the Adams spectral sequence, Arch. Math. (Basel) 19 (1968) 137–155.

[11] E.S. Devinatz, Morava modules and Brown-Comenetz duality, Amer. J. Math. 119 (4) (1997) 741–770.

[12] W. Dwyer, D. Kan, A classification for diagrams of simplicial sets, Topology 23 (2) (1984) 139–155.

[13] W. Dwyer, D. Stover, The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space, J. Pure Appl. Algebra 103 (1995) 167–188.

[14] W.G. Dwyer, D.M. Kan, C.R. Stover, An E^2 model category structure for pointed simplicial spaces, J. Pure Appl. Algebra 90 (2) (1993) 137–152.

[15] W.G. Dwyer, J. Spaliński, Homotopy theories and model categories, in: Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[16] J. Franke, Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence, in: K-Theory Archives, vol. 139, 1996, http://www.math.uiuc.edu/K-Theory.

[17] P.G. Goerss, M.J. Hopkins, Moduli spaces of commutative ring spectra, in: Structured Ring Spectra, in: London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.

[18] P.G. Goerss, J.F. Jardine, Simplicial homotopy theory, in: Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[19] P.S. Hirschhorn, Model Categories and their Localizations, in: Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

[20] M.J. Hopkins, M. Mahowald, H. Sadofsky, Constructions of elements in Picard groups, in: Topology and Representation Theory (Evanston, IL, 1992), in: Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 89–126.

[21] M. Hovey, Model Categories, in: Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[22] M. Hovey, Homotopy theory of comodules over a Hopf algebroid, in: Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, in: Contemporary Mathematics, vol. 346, American Mathematical Society, 2004, pp. 261–304.
[23] M. Hovey, H. Sadofsky, Invertible spectra in the $E(n)$-local stable homotopy category, J. London Math. Soc. (2) 60 (1) (1999) 284–302.
[24] D.G. Quillen, Homotopical Algebra, in: Lecture Notes in Mathematics, vol. 43, Springer-Verlag, Berlin, 1967.