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Fuzzy real valued lacunary I-convergent sequences

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ABSTRACT

In this article, we introduce the concept of lacunary I-convergent sequence of fuzzy real numbers and study some basic properties.

1. Introduction

The works on I-convergence of real valued sequences was initially studied by Kostyrko et al. [1]. Later on, it was further studied by Šalàt et al. [2,3], Tripathy et al. [4] and many others.

Let S be a non-empty set. A non-empty family of sets $I \subseteq P(S)$ (power set of S) is called an *ideal* on S if (i) for each A, $B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$. Let S be a non-empty set. A family $F \subseteq P(S)$ (power set of S) is called a *filter* on S if (i) $\phi \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; (iii) for each $A \in F$ and $B \supseteq A$, we have $B \in F$. An ideal I is called *non-trivial* if $I \neq \phi$ and $S \notin I$. It is clear that $I \subseteq P(S)$ is a non-trivial ideal if and only if the class $F = F(I) = \{S - A : A \in I\}$ is a filter on S. The filter $F(I)$ is called the filter associated with the ideal I. A non-trivial ideal $I \subseteq P(S)$ is called an *admissible ideal* on S if and only if it contains all singletons, *i.e.*, if it contains $\{\{x\} : x \in S\}$. A non-trivial ideal I is *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset (for details see [1]).

The concept of fuzzy sets was initially introduced by Zadeh [5]. Later on, sequences of fuzzy real numbers have been discussed by Nanda [6], Nuray and Savas [7] and many others.

A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals are determined by θ and it will be defined by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r.

Freedman et al. [8] defined the space N_θ in the following way. For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_θ is a BK space with the norm

$$\| (x_k) \|_\theta = \sup_r h_r^{-1} \sum_{k \in I_r} |x_k|.$$

N_θ^0 denotes the subset of these sequences in N_θ for which $\theta = 0$, $(N_\theta^0, \|\cdot\|_\theta)$ is also a BK space.

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Lemma 1. *Every solid space is monotone.*

Lemma 2 (*Kostyrko et al. [1, Lemma 5.1]*)**.** *If $I \subset 2^N$ is a maximal ideal, then for each $A \subset N$ we have either $A \in I$ or $N - A \in I$.*

2. Definitions and notations

A fuzzy real number X is a fuzzy set on R *i.e.* a mapping $X: R \to J\ (= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

The α-*level* set of a fuzzy real number X, $0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in R : X(t) \geq \alpha\}$.

A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon])$, for all $a \in J$ is open in the usual topology of R. The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $R(J)$.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line R. For $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ in D, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Define a metric d on D by

$$d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It is known that (D, d) is a complete metric space and "$\leq$" is a partial order on D.

The absolute value $|X|$ of $X \in R(J)$ is defined as

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : R(J) \times R(J) \to R$ be defined by

$$\bar{d}\,(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $R(J)$.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. The additive identity and multiplicative identity in $R(J)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to a fuzzy real number X_0 if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq n_0$.

A sequence (X_k) of fuzzy real numbers is said to be *I-convergent* if there exists a fuzzy real number X_0 such that for each $\varepsilon > 0$, the set

$$\{k \in N : \bar{d}(X_k, X_0) \geq \varepsilon\} \in I.$$

We write I-$\lim X_k = X_0$.

If $I = I_f$ (class of all finite subsets of N), then I_f- convergence coincides with the usual convergence.

Let E_F denote the sequence space of fuzzy numbers. Then E_F is said to be *solid* (or *normal*) if $(Y_k) \in E_F$, whenever $(X_k) \in E_F$ and $|Y_k| \leq |X_k|$, for all $k \in N$.

A sequence space E_F is said to be *symmetric* if $(X_k) \in E$ implies $(X_{\pi(k)}) \in E$, where π is a permutation of N.

A sequence space E_F is said to be *monotone*if it contains the canonical preimages of its step space.

Throughout the article, we assume that I is an admissible ideal of N.

3. Lacunary *I*-convergent sequence of fuzzy real numbers

Definition 1. Let $\theta = (k_r)$ be a lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I-convergent* if for every $\varepsilon > 0$ such that

$$\left\{r \in N : h_r^{-1} \sum_{k \in I_r} \bar{d}(X_k, X) \geq \varepsilon\right\} \in I.$$

We write $I_\theta - \lim X_k = X$.

Definition 2. Let $\theta = (k_r)$ be a lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I-null* if for every $\varepsilon > 0$ such that

$$\left\{r \in N : h_r^{-1} \sum_{k \in I_r} \bar{d}(X_k, \bar{0}) \geq \varepsilon\right\} \in I.$$

We write $I_\theta - \lim X_k = \bar{0}$.

Definition 3. Let $\theta = (k_r)$ be a lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I-Cauchy* if there exists a subsequence $(X_r^{/}(r))$ of (X_k) such that $k'(r) \in J_r$, for each r, $\lim_{r\to\infty} X_k^{/}(r) = X'$ and for every $\varepsilon > 0$ such that

$$\left\{r \in N : h_r^{-1} \sum_{k\in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \varepsilon\right\} \in I.$$

Definition 4. A lacunary sequence $\theta' = (k'(r))$ is said to be a *lacunary refinement* of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (k'(r))$.

Throughout w^F, ℓ_∞^F, c_θ^{IF} and $(c_0^{IF})_\theta$ denotes *all*, *bounded*, *lacunary I-convergent*, and *lacunary I-null* class of sequences of fuzzy real numbers, respectively.

Theorem 1. *A sequence (X_k) of fuzzy real numbers is I_θ-convergent if and only if it is an I_θ-Cauchy sequence.*

Proof. Let (X_k) be a sequence of fuzzy real numbers with $I_\theta - \lim X_k = X$.

Write $H_{(i)} = \{r \in N : h_r^{-1} \sum_{k\in J_r} \bar{d}(X_k, X) < \frac{1}{i}\}$, for each $i \in N$.

Hence for each i, $H_{(i)} \supseteq H_{(i+1)}$ and $\{r \in N : h_r^{-1} \sum_{k\in J_r} \bar{d}(X_k, X) < \frac{1}{i}\} \notin I$.

We choose k_1 such that $r \geq k_1$, then $\{r \in N : h_r^{-1} \sum_{k_1\in J_r} \bar{d}(X_{k_1}, X) < 1\} \notin I$.

Next we choose $k_2 > k_1$ such that $r \geq k_2$, then $\{r \in N : h_r^{-1} \sum_{k_2\in J_r} \bar{d}(X_{k_2}, X) < \frac{1}{2}\} \notin I$.

For each r satisfying $k_1 \leq r < k_2$, choose $k'(r) \in J_r$ such that

$$\left\{r \in N : h_r^{-1} \sum_{k'(r)\in J_r} \bar{d}(X_{k'(r)}, X) < 1\right\} \notin I.$$

In general, we choose $k_{p+1} > k_p$, such that $r > k_{p+1}$ then

$$\left\{r \in N : h_r^{-1} \sum_{k_{p+1}\in J_r} \bar{d}(X_{k_{p+1}}, X) < \frac{1}{p}\right\} \notin I.$$

Then for all r satisfying $k_p \leq r < k_{p+1}$, such that

$$\left\{r \in N : h_r^{-1} \sum_{k'(r)\in J_r} \bar{d}(X_{k'(r)}, X) < \frac{1}{p}\right\} \notin I.$$

Thus we get $k'(r) \in J_r$, for each r and $\lim_{r\to\infty} X_{k'(r)} = X$.

Therefore, for every $\varepsilon > 0$, we have

$$\left\{r \in N : h_r^{-1} \sum_{k,k'\in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \varepsilon\right\} \subseteq \left\{r \in N : h_r^{-1} \sum_{k\in J_r} \bar{d}(X_k, X) \geq \frac{\varepsilon}{2}\right\} \cup \left\{r \in N : h_r^{-1} \sum_{k'(r)\in J_r} \bar{d}(X_{k'(r)}, X) \geq \frac{\varepsilon}{2}\right\}.$$

i.e. $\left\{r \in N : h_r^{-1} \sum_{k,k'\in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \varepsilon\right\} \in I$.

Then (X_k) is an I_θ-Cauchy sequence.

Conversely, suppose (X_k) is an I_θ-Cauchy sequence. Then for every $\varepsilon > 0$, we have

$$\left\{r \in N : h_r^{-1} \sum_{k\in J_r} \bar{d}(X_k, X) \geq \varepsilon\right\} \subseteq \left\{r \in N : h_r^{-1} \sum_{k,k'\in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \frac{\varepsilon}{2}\right\} \cup \left\{r \in N : h_r^{-1} \sum_{k'(r)\in J_r} \bar{d}(X_{k'(r)}, X) \geq \frac{\varepsilon}{2}\right\}.$$

It follows that (X_k) is a I_θ-convergent sequence. □

Theorem 2. *If θ' is a lacunary refinement of a lacunary sequence θ and $(X_k) \in c_{\theta'}^{IF}$, then $(X_k) \in c_\theta^{IF}$.*

Proof. Suppose that for each J_r of θ contains the points $(k^{/}_{r,t})_{t=1}^{\eta(r)}$ of θ' such that

$$k_{r-1} < k^{/}_{r,1} < k^{/}_{r,2} < \cdots < k^{/}_{r,\eta(r)} = k_r, \quad \text{where } J^{/}_{r,t} = (k^{/}_{r,t-1}, k^{/}_{r,t}].$$

Since $k_r \subseteq (k'(r))$, so $r, \eta(r) \geq 1$.

Let $(J_j^*)_{j=1}^{\infty}$ be the sequence of intervals $(J^{/}_{r,t})$ ordered by increasing right end points. Since $(X_k) \in c_{\theta'}^{IF}$, then for each $\varepsilon > 0$,

$$\left\{ j \in N : (h_j^*)^{-1} \sum_{J_j^* \subset J_r} \bar{d}(X_k, X) \geq \varepsilon \right\} \in I.$$

Also since $h_r = k_r - k_{r-1}$, so $h^{/}_{r,t} = k^{/}_{r,t} - k^{/}_{r,t-1}$.

For each $\varepsilon > 0$, we have

$$\left\{ r \in N : h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) \geq \varepsilon \right\} \subseteq \left\{ r \in N : h_r^{-1} \sum_{k \in J_r} \left\{ j \in N : (h_j^*)^{-1} \sum_{\substack{J_j^* \subset J_r \\ k \in J_j^*}} \bar{d}(X_k, X) \right\} \geq \varepsilon \right\}.$$

Therefore $\{r \in N : h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) \geq \varepsilon\} \in I$.

Hence $(X_k) \in c_\theta^{IF}$. □

Theorem 3. *Let ψ be a set of all lacunary sequences.*

(a) *If ψ is closed under arbitrary union, then $c_\mu^{IF} = \bigcap_{\theta \in \psi} c_\theta^{IF}$, where $\mu = \bigcup_{\theta \in \psi} \theta$;*
(b) *If ψ is closed under arbitrary intersection, then $c_\upsilon^{IF} = \bigcup_{\theta \in \psi} c_\theta^{IF}$, where $\upsilon = \bigcap_{\theta \in \psi} \theta$;*
(c) *If ψ is closed under union and intersection, then $c_\mu^{IF} \subseteq c_\theta^{IF} \subseteq c_\upsilon^{IF}$.*

Proof. (a) By hypothesis, we have $\mu \in \psi$ which is a refinement of each $\theta \in \psi$. Then from Theorem 2, we have if $(X_k) \in c_\mu^{IF}$ implies that $(X_k) \in c_\theta^{IF}$.

Thus for each $\theta \in \psi$, we get $c_\mu^{IF} \subseteq c_\theta^{IF}$. The reverse inclusion is obvious.

Hence $c_\mu^{IF} = \bigcap_{\theta \in \psi} c_\theta^{IF}$.

(b) By part (a) and Theorem 2, we have $c_\upsilon^{IF} = \bigcup_{\theta \in \psi} c_\theta^{IF}$.

(c) By part (a) and (b), we get $c_\mu^{IF} \subseteq c_\theta^{IF} \subseteq c_\upsilon^{IF}$. □

Theorem 4. *$c_\theta^{IF} \cap \ell_\infty^F$ is a closed subset of ℓ_∞^F.*

Proof of the theorem is easy, so omitted.

Theorem 5. *Let $\theta = (k_r)$ be a lacunary sequence. Then the spaces c_θ^{IF} and $(c_0^{IF})_\theta$ are normal and monotone, in general.*

Proof. We shall give the proof of the theorem for $(c_0^{IF})_\theta$ only. Let $(X_k) \in (c_0^{IF})_\theta$ and (Y_k) be such that $\bar{d}(Y_k, \bar{0}) \leq \bar{d}(X_k, \bar{0})$, for all $k \in N$.

Then for a given $\varepsilon > 0$, we have

$$B = \left\{ r \in N : h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, \bar{0}) \geq \varepsilon \right\} \in I.$$

Again the set $D = \left\{ r \in N : h_r^{-1} \sum_{k \in J_r} \bar{d}(Y_k, \bar{0}) \geq \varepsilon \right\} \subseteq B$.

Hence $D \in I$ and so $(Y_k) \in (c_0^{IF})_\theta$. Thus the space $(c_0^{IF})_\theta$ is normal. Also from Lemma 1, it follows that $(c_0^{IF})_\theta$ is monotone. □

Theorem 6. *Let $\theta = (k_r)$ be a lacunary sequence. Then the spaces $(c_0^{IF})_\theta$ and c_θ^{IF} are symmetric, in general.*

Proof. We will give the proof for c_θ^{IF} only. Suppose I is not maximal and $I \neq I_f$. Let us consider a sequence $X = (X_k)$ of fuzzy real numbers defined by

$$X_k(t) = \begin{cases} 1 + t - 2k, & \text{if } t \in [2k-1, 2k]; \\ 1 - t + 2k, & \text{if } t \in [2k, 2k+1]; \\ 0, & \text{otherwise.} \end{cases}$$

for $k \in A \subset I$ an infinite set.

Then $(X_k) \in c_\theta^{IF}$. Let $K \subseteq N$ be such that $K \notin I$ and $N - K \notin I$ (the set K exists by Lemma 2, as I is not maximal). Consider a sequence $Y = (Y_k)$, a rearrangement of the sequence (X_k) defined as follows:

$$Y_k = \begin{cases} X_k, & \text{if } k \in K, \\ \bar{1}, & \text{otherwise.} \end{cases}$$

Then $(Y_k) \notin c_\theta^{IF}$. Therefore the space c_θ^{IF} is not symmetric. This completes the proof of the theorem. □

4. Conclusions

In this article, we have investigated the notion of lacunary convergence from I-convergence of sequences point of view. Still there are a lot to be investigated on sequence spaces applying the notion of I-convergence. The workers will apply the techniques used in this article for further investigations on I-convergence.

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