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Support theorems on $\mathbb{R}^n$ and non-compact symmetric spaces ☆

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Abstract

We consider convolution equations of the type $f * T = g$, where $f, g \in L^p(\mathbb{R}^n)$ and T is a compactly supported distribution. Under natural assumptions on the zero set of the Fourier transform of T, we show that f is compactly supported, provided g is. Similar results are proved for non-compact symmetric spaces as well.

Keywords: Support theorems; Paley–Wiener theorems; Analytic sets; Symmetric spaces; Spherical Fourier transform

1. Introduction

Support theorems have attracted a lot of attention in the past. We recall two such results. First, the famous result due to Helgason [8] (see page 107). This result states the following: If a measurable function f on $\mathbb{R}^n$ satisfies $(1+|x|)^N f \in L^1(\mathbb{R}^n)$, for each integer $N > 0$ and f integrates to zero over all spheres enclosing a fixed ball of radius $R > 0$, then f is supported in B_R, where B_R is the ball of radius R centered at the origin. An analogue holds also for rank one symmetric spaces of non-compact type [4]. The second is a result by A. Sitaram. In [17], he proved the following support theorem: If $f \in L^1(\mathbb{R}^n)$ is such that $f * \chi_{B_r} = g$, where χ_{B_r} is the indicator function of B_r and g is supported in B_R, then $\operatorname{supp} f \subseteq B_{R+r}$.

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In this paper we are interested in the second result. We consider convolution equations of the form $f * T = g$, where T is a compactly supported distribution on $\mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$. The question we are interested in is: can we conclude that f is compactly supported, if g is compactly supported? Combining methods from several complex variables and harmonic analysis we prove general support theorems under natural assumptions on the zero set of the entire function $\hat{T}$ (Fourier transform of T). When $T = \chi_{B_r}$ or μ_r (the normalized surface measure on the sphere of radius r on $\mathbb{R}^n$), this problem was studied by Sitaram [17], Volchkov [19], etc. When T is a distribution supported at the origin, this becomes a problem in PDE. In [18], Trèves proved that, if $P(D)u = v$ and v is compactly supported, then u is also compactly supported, provided $u \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) and the variety of zeros of each irreducible factor of P in $\mathbb{C}^n$ intersects $\mathbb{R}^n$. These questions were later taken up by Littman in [12] and [13]. Considering the principal value integral

$$\int_{\mathbb{R}^n} \frac{\hat{v}(y)}{P(y)} e^{ix\cdot y}\, dy$$

he was able to show that u is compactly supported with the assumption that $\{x \in \mathbb{R}^n \colon P(x) = 0\}$ has dimension $(n-1)$. Hörmander strengthened these results in [11]. Our results may be viewed as generalizations of these results. We end this section with the following theorems which will be needed later.

Theorem 1.1. *If $f \in L^p(\mathbb{R}^n)$ and* supp *$\hat{f}$ is carried by a C^1 manifold of dimension $d < n$ then $f = 0$ provided $1 \leqslant p \leqslant \frac{2n}{d}$ and $d > 0$. If $d = 0$ then $f = 0$ for $1 \leqslant p < \infty$.*

(See [2].)

Theorem 1.2. *Let f and g be entire functions of exponential type defined on $\mathbb{C}^n$ such that $h = f/g$ is entire, then h is of exponential type.*

This result is due to Malgrange. See [14].

2. Support theorems on $\mathbb{R}^n$

In this section we prove support theorems on $\mathbb{R}^n$ under natural assumptions on the zero set of the Fourier transform of the distribution T. Before we state our results we recall some notation from several complex variables which will be used throughout.

Let F be an entire function on $\mathbb{C}^n$. Then Z_F will denote the zero set of F, i.e. $Z_F = \{z \in \mathbb{C}^n \colon F(z) = 0\}$. The set Z_F is a complex analytic set and it can be written as a union of irreducible complex analytic sets, where, by an irreducible complex analytic set we mean a complex analytic set which cannot be written as a union of two non-empty complex analytic sets. For more details on complex analytic sets we refer to [5]. Let $\operatorname{Reg}(Z_F)$ denote the regular points of Z_F. If $z \in Z_F$ then $\operatorname{Ord}_z F$ will denote the order of F at z (see [5, page 16]). We also recall that the order is a constant on each connected component of $\operatorname{Reg}(Z_F)$. If A is a complex analytic set, $\operatorname{Sing} A$ will denote the singular points. That is, $\operatorname{Sing} A = A - \operatorname{Reg} A$.

We start with the following general result.

Theorem 2.1. *Let T be a compactly supported distribution on $\mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$ for some p with $1 \leqslant p \leqslant \frac{2n}{n-1}$. Assume the following*:

- **(a)** *If V is any irreducible component of $Z_{\hat{T}}$, then* $\dim_{\mathbb{R}}(V \cap \mathbb{R}^n) = n - 1$.
- **(b)** $\operatorname{grad} \hat{T} \neq 0$ *on* $\operatorname{Reg}(Z_{\hat{T}}) \cap \mathbb{R}^n$.

Suppose $f * T = g$, where g is compactly supported, then f is also compactly supported.

We need several lemmas for the proof of this theorem.

Lemma 2.2. *If $f \in L^p(\mathbb{R}^n)$, $p = \frac{2n}{n-1,}$ then $\exists r_k \to \infty$ such that, for any fixed constants $s_1, s_2 > 0$ we have*

$$\int\limits_{r_k - s_1 \leqslant |x| \leqslant r_k + s_2} |f(x)|^2 \, dx \to 0$$

as $k \to \infty$.

Proof. By contrary, assume that $\exists a > 0$ and $R > 0$ such that

$$\int\limits_{r - s_1 \leqslant |x| \leqslant r + s_2} |f(x)|^2 \, dx \geqslant a, \quad \forall r \geqslant R. \tag{2.1}$$

By Hölder's inequality we have

$$\int\limits_{r - s_1 \leqslant |x| \leqslant r + s_2} |f(x)|^2 \, dx \leqslant \Bigg(\int\limits_{r - s_1 \leqslant |x| \leqslant r + s_2} |f(x)|^{\frac{2n}{n-1}} \, dx \Bigg)^{\frac{n-1}{n}} \Bigg(\int\limits_{r - s_1 \leqslant |x| \leqslant r + s_2} dx \Bigg)^{\frac{1}{n}}.$$

From (2.1) and the above it follows that for some constant $c > 0$

$$\int\limits_{r - s_1 \leqslant |x| \leqslant r + s_2} |f(x)|^{\frac{2n}{n-1}} \, dx \geqslant \frac{c}{r}, \quad \forall r > R.$$

In particular, for each integer $k > R$, the above inequality is true for $r = s_1 + k(s_1 + s_2)$. Now, summing all these inequalities we get a contradiction to the fact that $f \in L^p(\mathbb{R}^n)$, $p = \frac{2n}{n-1}$. Hence the lemma is proved. □

Lemma 2.3. *Let F and G be two entire functions on $\mathbb{C}^n$ such that*:

- **(a)** *The intersection with $\mathbb{R}^n$ of each connected component of* $\operatorname{Reg}(Z_F)$ *has real dimension* $(n - 1)$.
- **(b)** $(\operatorname{Reg} Z_F) \cap \mathbb{R}^n \subseteq Z_G \cap \mathbb{R}^n$.
- **(c)** $\operatorname{Ord}_x F \leqslant \operatorname{Ord}_x G \ \forall x \in \mathbb{R}^n \cap \operatorname{Reg} Z_F$.

Then $\frac{G}{F}$ is an entire function.

Proof. Let $\operatorname{Reg} Z_F = \bigcup_{j\in J} S_j$ be the decomposition of $\operatorname{Reg} Z_F$ into connected components. Then $Z_F = \bigcup_{j\in J} A_j$ where $A_j = \bar{S_j}$ gives the decomposition of Z_F into irreducible components. If the complex dimension $\dim_{\mathbb{C}}(A_j \cap Z_G) \leqslant (n-2)$, then $\dim_{\mathbb{R}}(A_j \cap Z_G \cap \mathbb{R}^n) \leqslant (n-2)$ which contradicts **(a)** due to **(b)** in the assumptions. It follows that $\dim_{\mathbb{C}}(A_j \cap Z_G) = (n-1)$. Since A_j is an irreducible analytic set in $\mathbb{C}^n$, this will force A_j to be an irreducible component of Z_G (see [5]). It follows that $\operatorname{Reg}(Z_F) \subseteq \operatorname{Reg}(Z_G)$. Since the order is a constant on the regular part of an analytic set we also have $\operatorname{Ord}_z F \leqslant \operatorname{Ord}_z G\ \forall z \in \operatorname{Reg} Z_F$. Consequently $\frac{G}{F}$ is holomorphic in $\mathbb{C}^n - \operatorname{Sing}(Z_F)$. However, the $(2n-2)$ Hausdorff measure of $(\operatorname{Sing} Z_F)$ is zero (see [5, page 22]) and so by Proposition 2, page 298, in [5], $\frac{G}{F}$ extends to an entire function. □

Lemma 2.4. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \frac{2n}{n-1}$. Let T be a compactly supported distribution on $\mathbb{R}^n$ and $f * T = g$, where g is compactly supported. If $\hat{T}$ is zero on a smooth $(n-1)$-dimensional manifold $M \subseteq \mathbb{R}^n$, then $\hat{g}(x) = 0\ \forall x \in M$.*

Proof. By convolving with radial approximate identities we may assume that $f \in L^{p_0}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ where $p_0 = \frac{2n}{n-1}$ and $T \in L^1(\mathbb{R}^n)$. Let $\operatorname{supp} T \subseteq B_{R_1}$ and $\operatorname{supp} g \subseteq B_{R_2}$. For $r > 0$ define $f_r(x) = \chi_{|x|\leqslant r}(x) f(x)$ and write

$$f_r * T = g + g_r. \tag{2.2}$$

If r is very large, then $\operatorname{supp} g_r \subseteq \{x\colon r - R_1 \leqslant |x| \leqslant r + R_1\}$ and for $r - R_1 \leqslant |x| \leqslant r + R_1$ we have

$$g_r(x) = T * f_{r-2R_1,r}(x) \tag{2.3}$$

where

$$f_{r-2R_1,r}(x) = \chi_{r-2R_1 \leqslant |x| \leqslant r}(x) f(x).$$

Next, let $\phi \in C_c^\infty(\mathbb{R}^n)$ and consider the measure μ defined by

$$d\mu = \phi(x)\, dx_M$$

where dx_M is the surface measure on M. Then μ is a compactly supported measure on M. Since $\hat{T}$ is zero on M, it is easy to see by taking the Fourier transform that $T * f_r * \hat{\mu}$ vanishes identically.

From (2.2) it follows that

$$g * \hat{\mu} + g_r * \hat{\mu} \equiv 0.$$

We will show that $g_r * \hat{\mu}(x)$ goes to zero $\forall x \in \mathbb{R}^n$ as $r \to \infty$, which implies that $g * \hat{\mu}$ vanishes identically. Taking the Fourier transform again we obtain that $\hat{g}$ vanishes on $(\operatorname{supp} \phi) \cap M$. Since ϕ was arbitrary this proves the lemma.

Fix $x_0 \in \mathbb{R}^n$ and consider $g_r * \hat{\mu}(x_0)$. We have, by (2.3)

$$\left|g_r * \hat{\mu}(x_0)\right| \leqslant \int\limits_{r-R_1 \leqslant |y| \leqslant r+R_1} \left|T * f_{r-2R_1,r}(y)\right| \left|\hat{\mu}(x_0 - y)\right| dy. \tag{2.4}$$

Now if ν is a compactly supported smooth measure on M then $\exists c > 0$ such that

$$\left(\int\limits_{S^{n-1}} \left|\hat{\nu}(s\omega)\right|^2 d\omega \right)^{\frac{1}{2}} \leqslant \frac{c}{s^{\frac{n-1}{2}}}, \quad s > 0.$$

(See [3, Proposition 1, page 2563].) Apply the above to the measure $e^{ix_0 \cdot y}\phi(y)\,dy_M$ on M to obtain

$$\left(\int\limits_{S^{n-1}} \left|\hat{\mu}(x_0 - s\omega)\right|^2 d\omega \right)^{\frac{1}{2}} \leqslant \frac{c(x_0)}{s^{\frac{n-1}{2}}},$$

implying

$$\left(\int\limits_{r-R_1 \leqslant |y| \leqslant r+R_1} \left|\hat{\mu}(x_0 - y)\right|^2 dy \right)^{\frac{1}{2}} \leqslant C(x_0),$$

where $c(x_0)$ and $C(x_0)$ are some constants depending on x_0. A simple application of the Cauchy–Schwarz inequality to (2.4) along with the above estimates gives us

$$\left|g_r * \hat{\mu}(x_0)\right| \leqslant C(x_0) \left\| T * f_{r-2R_1,r} \right\|_2.$$

Therefore, by Young's inequality we get

$$\left|g_r * \hat{\mu}(x_0)\right| \leqslant C(x_0) \|T\|_1 \left(\int\limits_{r-2R_1 \leqslant |y| \leqslant r} \left|f(y)\right|^2 dy \right)^{\frac{1}{2}}.$$

Choosing $\{r_k\}$ as in Lemma 2.2 we finish the proof. □

Proof of Theorem 2.1. Without loss of generality we may assume that $f \in L^{p_0}(\mathbb{R}^n)$, $p_0 = \frac{2n}{n-1}$. Since $f * T = g$ and $(\operatorname{Reg} Z_{\hat{T}}) \cap \mathbb{R}^n$ is a smooth $(n-1)$-dimensional manifold, Lemma 2.4 implies that $\hat{g}(x) = 0$ if $\hat{T}(x) = 0$. Since $\operatorname{grad} \hat{T}$ is non-zero on $\operatorname{Reg} Z_{\hat{T}}$ we have $\operatorname{Ord}_x \hat{T} = 1$ if $x \in \operatorname{Reg} Z_{\hat{T}}$. Since $\hat{g}(x) = 0\ \forall x \in (\operatorname{Reg} Z_{\hat{T}}) \cap \mathbb{R}^n$ it follows that $\operatorname{Ord}_x \hat{g} \geqslant \operatorname{Ord}_x \hat{T}\ \forall x \in \operatorname{Reg} Z_{\hat{T}} \cap \mathbb{R}^n$. By Lemma 2.3 we have that $\frac{\hat{g}}{\hat{T}}$ is an entire function. Hence we have

$$\hat{f} = \frac{\hat{g}}{\hat{T}} + \delta, \tag{2.5}$$

where δ is a distribution supported on $Z_{\hat{T}} \cap \mathbb{R}^n$. We will show that $\delta \equiv 0$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Multiplying (2.5) with ϕ and taking the inverse Fourier transform we obtain

$$(\phi\delta)\check{} = \check{\phi} * f - h$$

where $h \in \mathcal{S}(\mathbb{R}^n)$. Notice that $\check{\phi} * f \in L^{p_0}(\mathbb{R}^n)$, $p_0 = \frac{2n}{n-1}$. From Theorem 1.1 it follows that $\phi\delta = 0$. Since ϕ was arbitrary it follows that $\hat{f} = \frac{\hat{g}}{\hat{T}}$. By Malgrange's theorem $\hat{f}$ is an entire function of exponential type. If $\hat{T}$ is slowly decreasing this readily implies that f is compactly supported. However, this extra assumption is not needed as can be seen below. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\hat{\psi}$ is compactly supported. Then

$$\begin{aligned}(\hat{\psi f})(x) &= \hat{\psi} * \hat{f}(x)\\ &= \int\limits_{\mathbb{R}^n} \hat{\psi}(t)\hat{f}(x-t)\,dt\end{aligned}$$

clearly extends to an entire function of exponential type. Since $\psi f \in L^1(\mathbb{R}^n)$, $(\hat{\psi f})$ is bounded on R^n. By the Paley–Wiener theorem we obtain that ψf is compactly supported which finishes the proof. □

Remark 2.5. It is possible to weaken the condition $\operatorname{grad}\hat{T} \neq 0$ on $\operatorname{Reg} Z_{\hat{T}} \cap \mathbb{R}^n$ as follows. Let V be any global irreducible component of $Z_{\hat{T}}$. Then there exists an entire function f_V whose zero locus is exactly V and there exists a positive integer k such that $\frac{\hat{T}}{f_V^k}$ is non-zero on V. This is an application of the Cousin **II** problem on $\mathbb{C}^n$. See [7]. This function f_V is unique up to multiplication by units. A close examination of the proof shows that it suffices to assume that $\operatorname{grad} f_V \neq 0$ on $V \cap \mathbb{R}^n$ for all V. In particular when $\hat{T} = f_1^{m_1} f_2^{m_2} \cdots f_k^{m_k}$ where $f_1, f_2, \ldots, f_k$ are irreducible entire functions then it suffices to assume that $\operatorname{grad} f_j \neq 0$ on $Z_{f_j} \cap \mathbb{R}^n$. Also see Hörmander [11, Theorem 3.1].

Next we show that if $1 \leqslant p \leqslant 2$ or T is a radial distribution then the condition on $\operatorname{grad}\hat{T}$ is not needed in Theorem 2.1.

Theorem 2.6. *Let T be a compactly supported distribution on $\mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$ for some p with $1 \leqslant p \leqslant 2$. If $f * T$ is compactly supported and condition* **(a)** *of the previous theorem is satisfied then f is compactly supported.*

Proof. Let $f * T = g$. Convolving with compactly supported approximate identities we may assume that $f \in L^2(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n)$. Since $\hat{T}\hat{f} = \hat{g}$ and $f \in L^2(\mathbb{R}^n)$ we have $\int_{\mathbb{R}^n} |\frac{\hat{g}}{\hat{T}}|^2 < \infty$. We will show that, if $x_0 \in \operatorname{Reg}(Z_{\hat{T}}) \cap \mathbb{R}^n$ then $\operatorname{Ord}_{x_0}(\hat{T}) \leqslant \operatorname{Ord}_{x_0}(\hat{g})$. Then we may argue as in Theorem 2.1 to conclude that $\frac{\hat{g}}{\hat{T}}$ is entire which will prove the theorem. As in the proof of Theorem 2.1 we have $Z_{\hat{T}} \subset Z_{\hat{g}}$. Without loss of generality we can assume $x_0 = 0$. If $\operatorname{Ord}_{x_0}(\hat{T}) = m_1$ and $\operatorname{Ord}_{x_0}(\hat{g}) = m_2$ then there exist holomorphic functions φ, ψ_1 and ψ_2 such that

$$\hat{T}(z) = \big(z_n - \varphi(z')\big)^{m_1}\psi_1(z)$$

and

$$\hat{g}(z) = \big(z_n - \varphi(z')\big)^{m_2}\psi_2(z)$$

in a neighborhood V (in $\mathbb{C}^n$) of the origin, where ψ_1 and ψ_2 are zero free in V. Here $z' = (z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$.

Since $\hat{g}/\hat{T} \in L^2$, the above implies that,

$$\int\limits_{[-a,a]^n} \frac{1}{|x_n - \varphi(x')|^{2(m_1-m_2)}}\, dx < \infty,$$

for some $a > 0$. By a change of variables we get

$$\int\limits_{[-a,a]^{n-1}} \left(\int\limits_{-a-\varphi(x')}^{a-\varphi(x')} \frac{1}{r^{2(m_1-m_2)}}\, dr \right) dx' < \infty.$$

Now, since $\varphi(0) = 0$, if we choose $0 < \varepsilon < a$, then there exists $0 < \delta < a$ such that $|\varphi(x')| < \varepsilon$ $\forall x' \in [-\delta, \delta]^{n-1}$. Therefore,

$$\int\limits_{[-\delta,\delta]^{n-1}} \left(\int\limits_{-a+\varepsilon}^{a-\varepsilon} \frac{1}{r^{2(m_1-m_2)}}\, dr \right) dx' < \infty$$

implying that

$$\int\limits_{-a+\varepsilon}^{a-\varepsilon} \frac{1}{r^{2(m_1-m_2)}}\, dr < \infty.$$

Hence $m_2 \geqslant m_1$, which finishes the proof. □

Next, suppose that T is a radial distribution on $\mathbb{R}^n$. Then $\hat{T}$ is a function of $(z_1^2+z_2^2+\dots+z_n^2)^{\frac{1}{2}}$ and the assignment

$$\hat{T}(z_1, z_2, \dots, z_n) = G_T(s),$$

where $s^2 = z_1^2 + z_2^2 + \dots + z_n^2$, defines an even entire function G_T on the complex plane $\mathbb{C}$ of exponential type and at most polynomial growth on $\mathbb{R}$. The converse also holds. If the entire function G_T has only real zeros then $Z_{\hat{T}}$ (in $\mathbb{C}^n$) is a disjoint union of sets of the form $\{(z_1, z_2, \dots, z_n)\colon z_1^2 + z_2^2 + \dots + z_n^2 = a\}$ for $a > 0$. It is easy to see that such T satisfies the condition **(a)** of Theorem 2.1. Our next theorem shows that condition **(b)** of Theorem 2.1 is not necessary if we are dealing with radial distributions of the above kind.

Theorem 2.7. *Let T be a compactly supported radial distribution on $\mathbb{R}^n$ such that the zeros of the entire function $G_T(s)$ are contained in $\mathbb{R} - \{0\}$. If $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \frac{2n}{n-1}$ and $f * T$ is compactly supported then f is compactly supported.*

Proof. Let $f * T = g$ and let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the positive zeros of $G_T(s)$ with multiplicities $m_1, m_2, \ldots$. We have $\hat{T}\hat{f} = \hat{g}$. As in the previous case we will show that $\frac{\hat{g}}{\hat{T}}$ is entire. It clearly suffices to show that $(z_1^2 + z_2^2 + \cdots + z_n^2 - \lambda_k^2)^{m_k}$ divides $\hat{g}$. Now $\frac{G_T(s)}{s^2-\lambda_k^2}$ is an even entire function of exponential type on $\mathbb{C}$ and is of at most polynomial growth on $\mathbb{R}$. It follows that there exists a compactly supported radial distribution V on $\mathbb{R}^n$ such that

$$G_V(s) = \frac{G_T(s)}{s^2 - \lambda_k^2}.$$

Now,

$$\left(z_1^2 + z_2^2 + \cdots + z_n^2 - \lambda_k^2\right) \frac{\hat{T}}{(z_1^2 + z_2^2 + \cdots + z_n^2 - \lambda_k^2)} \hat{f} = \hat{g}$$

implies that

$$\left(-\Delta - \lambda_k^2\right)(V * f) = g. \tag{2.6}$$

Convolving f with a radial C_c^∞ function we may assume that $V * f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \frac{2n}{n-1}$. Note that $-\Delta - \lambda_k^2$ is a distribution supported at the origin and satisfies the conditions in Theorem 2.1. It follows that $V * f$ is compactly supported. Taking Fourier transform in (2.6) we obtain that $(z_1^2 + z_2^2 + \cdots + z_n^2 - \lambda_k^2)$ divides $\hat{g}$. This surely can be repeated to prove that $\frac{\hat{g}}{\hat{T}}$ is entire. The proof now can be completed as in the previous case. □

In our next result we show that assuming T is a compactly supported positive distribution (i.e. $T(\phi) \geqslant 0$ if $\phi \geqslant 0$) gives us precise information about the support of the function f. Recall that a positive distribution is a positive measure.

Theorem 2.8. *Let T be a compactly supported radial positive measure with* $\operatorname{supp} T = \bar{B}_{R_1}$. *Assume that the entire function $G_T(s)$ has only real zeros. If $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \frac{2n}{n-1}$ and $f * T = g$ with* $\operatorname{supp} g \subseteq B_{R_2}$ *then f is compactly supported and* $\operatorname{supp} f \subseteq B_{R_2 - R_1}$.

We start with the following lemma which is a simple application of the Phragmén–Lindelöf theorem.

Lemma 2.9. *Let $A(s)$ be an entire function of exponential type on $\mathbb{C}$ and $0 < R_1 < R_2 < \infty$. Suppose that $|A(s)| \leqslant e^{R_2|s|}$ $\forall s \in \mathbb{C}$ and*

(a) $|A(is)| \leqslant e^{(R_2 - R_1)|s|}$ $\forall s \in \mathbb{R}$.
(b) $|A(s)| \leqslant e^{(R_2 - R_1)|s|}$ $\forall s \in \mathbb{R}$.

Then $|A(s)| \leqslant e^{(R_2 - R_1)|s|}$ $\forall s \in \mathbb{C}$.

Proof. Define

$$H(s) = \frac{A(s)}{e^{(R_2 - R_1)(s)}}, \quad s \in \mathbb{C}.$$

By the given condition H is an entire function of exponential type on $\mathbb{C}$. Also H is bounded on real and imaginary axis. Now consider the region $\Omega = \{s\colon \operatorname{Im} s > 0 \text{ and } \operatorname{Re} s > 0\}$ which is a sector of angle $\frac{\pi}{2}$. Then H is bounded on $\partial\Omega$ and we can find $P > 0$ and $b < 2$ such that $H(s) \leqslant Pe^{|s|^b}$ $\forall z \in \Omega$. By the Phragmén–Lindelöf theorem H is bounded on Ω. We can repeat the argument in other quadrants. Hence the lemma follows. □

Proof of Theorem 2.8. Let μ be the compactly supported radial positive measure which defines the distribution T. Then $f * \mu = g$. By Theorem 2.7 we already know that f is compactly supported. In particular $f \in L^1(\mathbb{R}^n)$. Also $\hat{f} = \frac{\hat{g}}{\hat{\mu}}$ is an entire function of exponential type (by Malgrange's theorem). Proof will be completed by Lemma 2.9 and the Paley–Wiener theorem once we prove that for each $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\left|\hat{\mu}(iy)\right| \geqslant c_\epsilon e^{(R_1-\epsilon)|y|} \quad \forall y \in \mathbb{R}^n.$$

Now,

$$\hat{\mu}(iy) = \int\limits_{|x|\leqslant R_1} e^{x\cdot y}\, d\mu(x).$$

Given $\epsilon > 0$, it is possible to choose a fixed radius $\delta > 0$ such that

$$x \cdot y \geqslant (R_1 - \epsilon)|y|$$

for all x in a δ-neighborhood B_δ of $R_1 \frac{y}{|y|}$. Hence

$$\begin{aligned}\hat{\mu}(iy) &\geqslant \int\limits_{x\in B_\delta} e^{x\cdot y}\, d\mu(x)\\ &\geqslant c_\epsilon e^{(R_1-\epsilon)|y|},\end{aligned}$$

for some constant c_ϵ. Notice that we need $\operatorname{supp}\mu = \bar{B}_{R_1}$ here. This finishes the proof. □

Remark 2.10. When $T = \chi_{B_r}$ or μ_r this improves the result of Sitaram in [17]. Theorem 2.8 is also proved by Volchkov in [19] in a different way. See also [1].

The following theorem shows that the class of distributions which satisfies the conditions in Theorem 2.7 is large. Notice that if G is an even entire function of exponential type on $\mathbb{C}$ whose zeros are all non-zero reals and T is a radial, compactly supported distribution on $\mathbb{R}^n$ defined by

$$\hat{T}(z_1, z_2, \ldots, z_n) = G\bigl(\bigl(z_1^2 + z_2^2 + \cdots + z_n^2\bigr)^{\frac{1}{2}}\bigr)$$

then T satisfies the conditions of Theorem 2.7.

Theorem 2.11. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a positive even C^2 function. Assume that ϕ is increasing on* $[0, 1]$. *Then the entire function* (*on* $\mathbb{C}$)

$$G(z) := \int_{-1}^{1} \phi(t) e^{-itz}\, dt$$

has only real zeros.

Proof of the above requires several lemmas.

Lemma 2.12.

(I) *Let g be a positive C^1 function on $[0, a]$ such that both g and g' are strictly increasing on $[0, a]$. Then*

$$I = \int_0^a g(t) \cos t\, dt$$

is non-zero if $a = 2n\pi + \theta$ or $2n\pi + \pi + \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

(II) *Let g be as above with $g(0) = 0$. Then*

$$J = \int_0^a g(t) \sin t\, dt$$

is non-zero if $a = 2n\pi + \frac{\pi}{2} + \theta$ or $2n\pi + \frac{3\pi}{2} + \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Proof. **(I)** Case 1: Let $a = 2n\pi + \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Then

$$I \geqslant \int_0^{2n\pi} g(t) \cos t\, dt = \sum_{k=0}^{n-1} I_k$$

where

$$I_k = \int_{2k\pi}^{2k\pi+2\pi} g(t) \cos t\, dt = \int_0^{2\pi} g(2k\pi + t) \cos t\, dt.$$

First, consider I_0.

$$I_0 = \int_0^{\frac{\pi}{2}} G_0(t) \cos t\, dt$$

where

$$G_0(t) = g(2\pi - t) - g(\pi + t) - g(\pi - t) + g(t).$$

Now, $G_0(\frac{\pi}{2}) = 0$ and

$$G_0'(t) = -g'(2\pi - t) - g'(\pi + t) + g'(\pi - t) + g'(t)$$

is negative by the assumption on g. It follows that $G_0(t) > 0$ for $t \in [0, \frac{\pi}{2})$. Hence $I_0 > 0$. Notice that each I_k is given by an integral $\int_0^{2\pi} G_k(t)\,dt$ where G_k is just G_0 translated by a multiple of π. Hence each $I_k > 0$ which implies that I is non-zero.

Case 2: Let $a = 2n\pi + \pi + \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Then

$$-I \geqslant -\int_0^{2n\pi+\pi} g(t)\cos t\,dt = \bar{I} + \sum_{k=0}^{n-1} \bar{I}_k$$

where $\bar{I} = -\int_0^{\pi} g(t)\cos t\,dt$ and

$$\bar{I}_k = -\int_{(2k+1)\pi}^{(2k+1)\pi+2\pi} g(t)\cos t\,dt = \int_0^{2\pi} g\big((2k+1)\pi + t\big)\cos t\,dt.$$

Now $\bar{I} = \int_0^{\frac{\pi}{2}} [g(\pi - t) - g(t)]\cos t\,dt > 0$. Also as in the previous case $\bar{I}_k > 0$. Therefore I is non-zero.

(II) Case 1: Let $a = 2n\pi + \frac{\pi}{2}\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Then

$$J \geqslant \int_0^{2n\pi+\frac{\pi}{2}} g(t)\sin t\,dt = \sum_{k=0}^{n-1} J_k$$

where

$$J_k = \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{\pi}{2}+2\pi} g(t)\sin t\,dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2\pi} g(2k\pi + t)\sin t\,dt.$$

First consider J_0.

$$J_0 = \int_0^{\frac{\pi}{2}} E_0(t)\sin t\,dt$$

where

$$E_0(t) = g(2\pi + t) - g(2\pi - t) - g(\pi + t) + g(\pi - t).$$

Now, $E_0(0) = 0$ and

$$E_0'(t) = g'(2\pi + t) + g'(2\pi - t) - g'(\pi + t) - g'(\pi - t)$$

is positive by assumption on g. It follows that $E_0(t) > 0$ for $t \in (0, \frac{\pi}{2}]$. Hence $J_0 > 0$. Similarly each $J_k > 0$ which implies that J is non-zero.

Case 2: Let $a = 2n\pi + \frac{3\pi}{2} + \theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Then

$$-J \geqslant -\int_0^{2n\pi + \frac{3\pi}{2}} g(t)\sin t\,dt = \bar{J} + \sum_{k=0}^{n-1} \bar{J}_k$$

where $\bar{J} = -\int_0^{\frac{3\pi}{2}} g(t)\sin t\,dt$ and

$$\bar{J}_k = -\int_{2k\pi + \frac{3\pi}{2}}^{2k\pi + \frac{3\pi}{2} + 2\pi} g(t)\sin t\,dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + 2\pi} g\big((2k+1)\pi + t\big)\sin t\,dt.$$

$\bar{J} = \int_0^{\frac{\pi}{2}} E(t)\sin t\,dt$ where

$$E(t) = g(\pi + t) - g(\pi - t) - g(t).$$

Now $E(0) = 0$ and

$$E'(t) = g'(\pi + t) + g'(\pi - t) - g'(t)$$

is positive by assumptions on g. It follows that $E(t) > 0$ for $t \in (0, \frac{\pi}{2}]$. Hence $\bar{J} > 0$. Also as in the previous case $\bar{J}_k > 0$. Therefore J is non-zero. □

Lemma 2.13.

(I) *Let g be a non-negative continuous strictly increasing function on* $[0, a]$. *Then,*

$$I := \int_0^a g(t)\cos t\,dt$$

is non-zero if $a = \frac{\pi}{2} + k\pi$ *for some non-negative integer* k.

(II) *Let g be as above. Then,*

$$J := \int_0^a g(t)\sin t\,dt$$

is non-zero if $a = k\pi$ *for some positive integer* k.

Proof. Let $a = \frac{\pi}{2} + k\pi$ for some non-negative integer k. Then,

$$I = \int_0^{\frac{\pi}{2}} g(t)\cos t + \sum_{j=0}^{(k-1)} I_j$$

where

$$I_j = \int_{\frac{\pi}{2}+j\pi}^{\frac{\pi}{2}+(j+1)\pi} g(t)\cos t\,dt.$$

If k is even we can write

$$I = \int_0^{\frac{\pi}{2}} g(t)\cos t\,dt + \sum_{j=0}^{\frac{k-2}{2}} (I_{2j} + I_{2j+1}).$$

By a change of variable we get

$$I_0 + I_1 = \int_0^{\pi} \left[g\left(\pi + \frac{\pi}{2} + t\right) - g\left(\frac{\pi}{2} + t\right)\right] \sin t\,dt$$

which is positive since g is strictly increasing. Similarly each $I_{2j} + I_{2j+1}$ is positive. Hence I is positive. If k is odd then we can write

$$I = \int_0^{\frac{3\pi}{2}} g(t)\cos t\,dt + \sum_{j=1}^{\frac{k-1}{2}} (I_{2j-1} + I_{2j}).$$

Again using a change of variable we get

$$I_1 + I_2 = \int_0^{\pi} \left[g\left(\frac{\pi}{2} + \pi + t\right) - g\left(\frac{\pi}{2} + 2\pi + t\right)\right] \sin t\,dt$$

which is negative since g is strictly increasing. Similarly each $I_{2j-1} + I_{2j}$ is negative. Also

$$\begin{aligned}\int_0^{\frac{3\pi}{2}} g(t)\cos t\,dt &< \int_0^{\frac{\pi}{2}} g(t)\cos t\,dt + \int_{\pi}^{\pi+\frac{\pi}{2}} g(t)\cos t\,dt \\ &< \int_0^{\frac{\pi}{2}} \left[g(t) - g(\pi + t)\right] \cos t\,dt\end{aligned}$$

is negative. Therefore I is negative. Hence **(I)** is proved. **(II)** can be proved using a similar type of argument. □

Lemma 2.14.

(I) *Let g be a non-negative increasing C^2 function on* $[0,1]$ *such that for some* $M>1$, $Mg(t)+g''(t)\geqslant 0\ \forall t\in[0,1]$. *Then, for each fixed* $y>M$ *the function*

$$F_y(x) := \int_0^1 g(t)\bigl(e^{yt}+e^{-yt}\bigr)\cos(xt)\,dt$$

can vanish at most once in each of the intervals $[\frac{\pi}{2}+k\pi, \frac{\pi}{2}+(k+1)\pi]$, *where k is a non-negative integer.*

(II) *Let g be as above. Then, for each fixed* $y>M$ *the function*

$$G_y(x) = \int_0^1 g(t)\bigl(e^{yt}+e^{-yt}\bigr)\sin(xt)\,dt$$

can vanish at most once in each of the intervals $[k\pi, (k+1)\pi]$, *where k is a non-negative integer.*

Proof. To prove **(I)** first note that we can write $F_y(x)$ and $F'_y(x)$ in the following way:

$$F_y(x) = \frac{1}{x}\int_0^x g\left(\frac{t}{x}\right)\bigl(e^{t\frac{y}{x}}+e^{-t\frac{y}{x}}\bigr)\cos t\,dt$$

and

$$F'_y(x) = -\frac{1}{x}\int_0^x \frac{t}{x}g\left(\frac{t}{x}\right)\bigl(e^{t\frac{y}{x}}+e^{-t\frac{y}{x}}\bigr)\sin t\,dt.$$

Now, if possible assume that there exists $y_0>M$ and a non-negative integer k_0 such that the interval $[\frac{\pi}{2}+k_0\pi, \frac{\pi}{2}+(k_0+1)\pi]$ contains at least two zeros of the function $F_{y_0}(x)$. Because of the given conditions an easy calculation shows that the functions $g(\frac{t}{x})(e^{t\frac{y_0}{x}}+e^{t\frac{y_0}{x}})$ and $\frac{t}{x}g(\frac{t}{x})(e^{t\frac{y_0}{x}}+e^{t\frac{y_0}{x}})$ on the interval $[0,x]$ satisfy the conditions of **(I)** and **(II)** of Lemma 2.11 respectively. Hence, $F_{y_0}(x)$ and $F'_{y_0}(x)$ cannot vanish in the intervals $[\frac{\pi}{2}+k\pi+\frac{\pi}{2}, \frac{\pi}{2}+(k+1)\pi]$ and $[\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi+\frac{\pi}{2}]$ respectively. Therefore, $F_{y_0}(x)$ vanishes at least twice in the interval $[\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi+\frac{\pi}{2}]$ which implies, by Rolle's theorem, that $F'_{y_0}(x)$ has at least one zero in the same interval, which is a contradiction. This finishes the proof of **(I)**. Using a similar type of argument, we can also prove **(II)**. □

Lemma 2.15. *Let g be an even or odd continuous function on $[-1,1]$ such that on $[0,1]$ it is non-negative, increasing and C^2. Assume that for some $M>1$, $Mg(t)+g''(t)\geqslant 0\ \forall t\in[0,1]$. Let the entire function*

$$H_1(z):=\int_{-1}^{1} g(t)e^{-izt}\,dt$$

have a non-real zero. Then the entire function

$$H_2(z):=\int_{-1}^{1} tg(t)e^{-izt}\,dt$$

also has a non-real zero.

Proof. First assume that g is even. Since g is also real valued, there exist $x_0>0$ and $y_0>0$ such that H_1 is zero at $z_0=x_0+iy_0$. Now, if possible assume that H_2 has only real zeros, i.e. for any $z=x+iy$, $y\neq 0$,

$$\operatorname{Re} H_2(z)=\int_{0}^{1} tg(t)\bigl(e^{yt}-e^{-yt}\bigr)\cos(xt)\,dt,$$

and

$$\operatorname{Im} H_2(z)=-\int_{0}^{1} tg(t)\bigl(e^{yt}+e^{-yt}\bigr)\sin(xt)\,dt$$

cannot vanish simultaneously. But this implies that, if we define the smooth function $F:\mathbb{R}^2\to\mathbb{R}$ by

$$F(x,y)=\operatorname{Re} H_1(x+iy)=\int_{0}^{1} g(t)\bigl(e^{yt}+e^{-yt}\bigr)\cos(xt)\,dt$$

then the gradient vector

$$\nabla F(x,y)=\left(-\int_{0}^{1} tg(t)\bigl(e^{yt}+e^{-yt}\bigr)\sin(xt)\,dt,\ \int_{0}^{1} tg(t)\bigl(e^{yt}-e^{-yt}\bigr)\cos(xt)\,dt\right)\neq 0$$

whenever $z=x+iy$ is not real i.e. $y\neq 0$. Therefore, the zero set of F is closed and except on the real axis it defines a smooth one-dimensional manifold.

By **(I)** of Lemma 2.13, the connected component of the zero set through (x_0, y_0) (call it C) is contained in the region $R := \{(x, y)\colon \frac{\pi}{2} + k\pi < x < \frac{\pi}{2} + (k+1)\pi\}$ for some non-negative integer k. Since the curve C is closed and $\nabla F \neq 0$ on the non-real points of C, there are three possibilities:

(a) C intersects the real axis,
(b) C is a smooth closed loop in the region $\{(x, y) \in R\colon\ y > 0\}$,
(c) C is a smooth curve in the region $\{(x, y) \in R\colon\ y > 0\}$, with both ends going upwards to infinity along the direction of y-axis.

Notice that F is a harmonic function, hence (b) cannot occur. By Lemma 2.14, (c) is also ruled out. Consider the first case. Parametrize a portion of C by a continuous function $\gamma : [0, 1] \to C$ such that $\gamma(0) = (x_0, y_0)$, $\gamma(1) = (u_0, 0)$ $(\frac{\pi}{2} + k\pi < u_0 < \frac{\pi}{2} + (k+1)\pi)$, and for all $s \in (0, 1)$ γ is smooth, $\gamma(s) \notin \mathbb{R}$, $\gamma'(s) \neq 0$. Now, identifying $\mathbb{R}^2$ with $\mathbb{C}$, consider the function $H_1 \circ \gamma$. It is easy to see that, this is a purely imaginary-valued continuous function on $[0, 1]$, smooth on $(0, 1)$, which vanishes at 0 and 1. Since γ' is non-zero on $(0, 1)$, applying Rolle's theorem to the function $i(H_1 \circ \gamma)$ we get that $\int_{-1}^{1} tg(t)e^{-i\gamma(s_0)t}\,dt = 0$ for some $s_0 \in (0, 1)$, which is a contradiction, because $\gamma(s_0)$ is not real. This finishes the proof when g is even. When g is odd the proof is almost similar except the fact that instead of finding a path (C) on which H_1 is purely imaginary (0 included) we find a path on which H_1 is real. □

Proof of Theorem 2.11. If possible assume that G has a non-real zero. Now, from the given conditions it is easy to see that for some large $M > 0$ $M\phi(t) + \phi''(t) \geqslant 0$ and hence for any positive integer n, $M(t^n\phi(t)) + (t^n\phi(t))'' \geqslant 0$, for all $t \in [0, 1]$. By Lemma 2.15 and using induction we can say that for each positive integer n the entire function

$$G_n(s) := \int_{-1}^{1} \phi_n(t)e^{-its}\,dt$$

has a non-real zero, where

$$\phi_n(t) := t^n\phi(t) \quad \forall t \in \mathbb{R}.$$

Since

$$\phi_n'(t) = nt^{(n-1)}\phi(t) + t^n\phi'(t)$$

and

$$\begin{aligned}\phi_n''(t) &= n(n-1)t^{(n-2)}\phi(t) + 2nt^{n-1}\phi'(t) + t^n\phi''(t)\\ &= t^{(n-2)}\big[n(n-1)\phi(t) + t^2\phi''(t)\big] + 2nt^{(n-1)}\phi'(t),\end{aligned}$$

by the given conditions it follows that, for some large positive integer N (we can take N to be even) $\phi_N'(t) \geqslant 0$ and $\phi_N''(t) \geqslant 0$ for all $t \in [0, 1]$, i.e. ϕ_N and ϕ_N' both are increasing on $[0, 1]$.

Now, since ϕ_N is even and real valued, we will get a contradiction if we can prove that $G_N(s)$ has no zero in $\{s \in \mathbb{C}\colon\ s = x + iy,\ x > 0,\ y > 0\}$. Now,

$$\begin{aligned} G_N(s) &= 2\int_0^1 \phi_N(t)\bigl(e^{-its} + e^{its}\bigr)\,dt \\ &= \int_0^1 \phi_N(t)\bigl(e^{-itx}e^{ty} + e^{itx}e^{-ty}\bigr)\,dt \\ &= \frac{2}{x}\int_0^x \phi_N\left(\frac{t}{x}\right)\bigl(e^{-it}e^{t\frac{y}{x}} + e^{it}e^{-t\frac{y}{x}}\bigr)\,dt. \end{aligned}$$

Therefore,

$$\operatorname{Re} G_N(s) = \frac{2}{x}\int_0^x \phi_N\left(\frac{t}{x}\right)\bigl(e^{t\frac{y}{x}} + e^{-t\frac{y}{x}}\bigr)\cos t\,dt$$

and

$$-\operatorname{Im} G_N(s) = \frac{2}{x}\int_0^x \phi_N\left(\frac{t}{x}\right)\bigl(e^{t\frac{y}{x}} - e^{-t\frac{y}{x}}\bigr)\sin t\,dt.$$

Since ϕ_N and ϕ_N' both are increasing on $[0,1]$, it is easy to see that the functions $\phi_N(\frac{t}{x})(e^{t\frac{y}{x}} + e^{-t\frac{y}{x}})$ and $\phi_N(\frac{t}{x})(e^{t\frac{y}{x}} - e^{-t\frac{y}{x}})$ on the interval $[0,x]$ satisfy the assumptions in Lemma 2.12. Therefore, both $\operatorname{Re} G_N(s)$ and $\operatorname{Im} G_N(s)$ cannot be simultaneously zero in the first quadrant which finishes the proof. □

3. Support theorems on non-compact symmetric spaces

In this section we prove support theorems on non-compact symmetric spaces. Let G be a connected, non-compact semisimple Lie group with finite center. Let $K \subseteq G$ be a fixed maximal compact subgroup and $X = G/K$, the associated Riemannian space of non-compact type. Endow X with the G-invariant Riemannian structure induced from the Killing form. Let dx denote the Riemannian volume element on X. We study convolution equations of the form $f * T = g$, where $f \in C^\infty(X) \cap L^p(X)$, T is a K-biinvariant compactly supported distribution on X and $g \in C_c^\infty(X)$. We show that under natural assumptions on the zero set of the spherical Fourier transform of T, f turns out to be compactly supported. (The function f is assumed to be smooth only to make sure that the convolution $f * T$ is well defined.) Before we state our results we recall necessary details. We follow the notation in [8] and [9].

Let $G = KAN$ be an Iwasawa decomposition of G and $\mathbf{a}$ be the Lie algebra of A. Let $\mathbf{a}^*$ be the real dual of $\mathbf{a}$ and $\mathbf{a}^*_{\mathbb{C}}$ its complexification. Then for any $g \in G$, $g = k(g)\exp H(g) n(g)$ where

$k(g) \in K$, $H(g) \in \mathbf{a}$, $n(g) \in N$. Let M be the centralizer of A in K. For a suitable function f on X, the Helgason–Fourier transform is defined by

$$\tilde{f}(\lambda, k) = \int_G f(x) e^{(i\lambda - \rho)H(x^{-1}k)}\, dx,$$

where ρ is the half sum of positive roots and $\lambda \in \mathbf{a}^*$. We note that $\tilde{f}(\lambda, k) = \tilde{f}(\lambda, kM)$ and so sometimes we will write $\tilde{f}(\lambda, b)$ where $b = kM$.

For each $\lambda \in \mathbf{a}^*_{\mathbb{C}}$, let ϕ_λ be the elementary spherical function given by:

$$\phi_\lambda(x) = \int_K e^{(i\lambda - \rho)H(x^{-1}k)}\, dk.$$

They are the matrix elements of the spherical principal representations π_λ of G defined for $\lambda \in \mathbf{a}^*_{\mathbb{C}}$ on $L^2(K/M)$ by

$$\bigl(\pi_\lambda(x)v\bigr)(b) = e^{(i\lambda - \rho)H(x^{-1}b)} v\bigl(k\bigl(x^{-1}b\bigr)\bigr),$$

where $v \in L^2(K/M)$. The representations π_λ are unitary if and only if $\lambda \in \mathbf{a}^*$. They are also irreducible if $\lambda \in \mathbf{a}^*$. For $f \in L^1(X)$, the group Fourier transform $\pi_\lambda(f)$, defined by

$$\pi_\lambda(f) = \int_G f(xK)\pi_\lambda(x)\, dx$$

is a bounded linear operator on $L^2(K/M)$. Its action is given by

$$\bigl(\pi_\lambda(f)v\bigr)(b) = \biggl(\int_{K/M} v(k)\, dk\biggr) \tilde{f}(\lambda, b).$$

We also have the Plancherel formula which says that $f \to \tilde{f}(\lambda, b)$ is an isometry from $L^2(X)$ onto $L^2(\mathbf{a}^* \times K/M, |c(\lambda)|^{-2}\, d\lambda)$ where $c(\lambda)$ is the Harish-Chandra c-function. In particular,

$$\int_X \bigl|f(x)\bigr|^2\, dx = |W|^{-1} \int_{\mathbf{a}^*} \int_{K/M} \bigl|\tilde{f}(\lambda, w)\bigr|^2 \bigl|c(\lambda)\bigr|^{-2}\, d\lambda\, dk.$$

Next we comment on the pointwise existence of the Helgason–Fourier transform. For $1 \leqslant p \leqslant 2$, define $S_p = \mathbf{a}^* + iC^p_\rho$, where C^p_ρ is the convex hull of $\{s(\frac{2}{p} - 1)\rho\colon s \in W\}$, W being the Weyl group. Let S^0_p be the interior of S_p. The following result from [15] proves the existence of Helgason–Fourier transform pointwise.

Theorem 3.1. *Let $f \in L^p(X)$, $1 \leqslant p \leqslant 2$. Then $\exists$ a subset $B(f) \subseteq K$, of full measure such that $\tilde{f}(\lambda, b)$ exists $\forall b \in B$ and $\lambda \in S^0_p$. Moreover, for every $b \in B(f)$ fixed, $\lambda \to \tilde{f}(\lambda, b)$ is holomorphic on S^0_p and $\|\tilde{f}(\lambda, \cdot)\|_{L^1(K)} \to 0$ as $|\lambda| \to \infty$ in S^0_p.*

Remark 3.2. **(1)** When $p = 1$ we have $\|\tilde{f}(\lambda, \cdot)\|_{L^1(K)} \leqslant \|f\|_1$ $\forall \lambda \in S_1$.

(2) When $p = 2$, existence of $\tilde{f}(\lambda, b)$ is provided by the Plancherel theorem.

We also have the Paley–Wiener theorem for compactly supported functions and distributions.

Theorem 3.3. *The Fourier transform is a bijection from $C_c^\infty(X)$ to C^∞ functions ψ on $\mathbf{a}_{\mathbb{C}}^* \times K/M$ satisfying*

- **(a)** *$\psi(\lambda, b)$ is holomorphic as a function of λ.*
- **(b)** *There is a constant $R \geqslant 0$ such that $\forall N > 0$*

$$\sup_{\lambda \in \mathbf{a}_{\mathbb{C}}^*,\, b \in K/M} e^{-R|\mathrm{Im}\,\lambda|}\big(1 + |\lambda|\big)^N \big|\psi(\lambda, b)\big| < \infty.$$

- **(c)** *For any σ in the Weyl group and $g \in G$*

$$\int_{K/M} e^{-(i\sigma\lambda+\rho)H(g^{-1}k)}\psi(\sigma\lambda, kM)\,dk = \int_{K/M} e^{-(i\lambda+\rho)H(g^{-1}k)}\psi(\lambda, kM)\,dk.$$

See [9, page 270]. We restate the above as in [16]. Let v_j, $j = 0, 1, 2, \ldots$ be an orthonormal basis for $L^2(K/M)$ where each v_j transforms according to some irreducible unitary representation of K and v_0 is the constant function 1 on K/M. (Note that, for any $\lambda \in \mathbf{a}^*$ $\pi_\lambda(k)v_0 = v_0$ and v_0 is the essentially unique vector with this property.) Let $\hat{K_M}$ consist of all unitary irreducible representations of K which have an M fixed vector. For $\delta \in \hat{K_M}$ let χ_δ be its character and $d(\delta)$ its dimension. If $f \in C^\infty(X)$, then

$$f = \sum_{\delta \in \hat{K_M}} d(\delta)\chi_\delta * f,$$

where the convergence is absolute (see [8, page 532]). It follows that f is compactly supported if and only if $\chi_\delta * f$ is compactly supported for all δ. We now state the Paley–Wiener theorem in the following form:

Theorem 3.4. *Let $f \in L^p(X)$, $1 \leqslant p \leqslant 2$ and $f = \chi_\delta * f$ for some $\delta \in \hat{K_M}$. Then $\tilde{f}(\lambda, b) = a_1(\lambda)v_{i_1}(b) + a_2(\lambda)v_{i_2}(b) + \cdots + a_n(\lambda)v_{i_n}(b)$.*

- **(a)** *If* supp $f \subseteq B_R$*, then each $a_i(\lambda)$ extends to an entire function on $\mathbf{a}_{\mathbb{C}}^*$ of exponential type R.*
- **(b)** *Conversely, if each a_i extends to an entire function of exponential type R then* supp $f \subseteq B_R$.

Remark 3.5. In [16] the above theorem is stated only for $f \in L^1(X)$. But, this clearly extends to $f \in L^p(X)$, $1 \leqslant p \leqslant 2$.

We also recall that if f is K-biinvariant, then the Helgason–Fourier transform is independent of b, and it reduces to the spherical Fourier transform of f defined by

$$\tilde{f}(\lambda) = \int f(x)\phi_\lambda(x)\,dx.$$

If T is a K-biinvariant compactly supported distribution, then $\tilde{T}(\lambda)$ is defined by $\tilde{T}(\lambda) = T(\phi_\lambda)$. We also have a Paley–Wiener theorem for distributions. See [6].

Theorem 3.6. *The spherical Fourier transform is a bijection from the space of K-biinvariant compactly supported distributions on X onto the space of Weyl group invariant entire functions of exponential type on $\mathbf{a}_{\mathbb{C}}^*$ which are of at most polynomial growth on $\mathbf{a}^*$.*

We start with the following proposition.

Proposition 3.7. *Let $f \in L^p(X) \cap C^\infty(X)$, $1 \leqslant p \leqslant 2$ and T be a compactly supported K-biinvariant distribution such that $f * T$ is compactly supported. Then*

$$(f * T)\tilde{}(\lambda, b) = \tilde{f}(\lambda, b)\tilde{T}(\lambda).$$

Proof. Since $L^p \subseteq L^1 + L^2$, it suffices to prove this for L^1 and L^2. If $\phi \in C_c^\infty(K\backslash G/K)$ then $T * \phi = \phi * T \in C_c^\infty(K\backslash G/K)$ and

$$(T * \phi)\tilde{}(\lambda, b) = \tilde{T}(\lambda)\tilde{\phi}(\lambda).$$

Also if $f \in L^1$ or L^2 and $g \in C_c^\infty(K\backslash G/K)$ then

$$(f * g)\tilde{}(\lambda, b) = \tilde{f}(\lambda, b)\tilde{g}(\lambda).$$

Now, by assumption $f * T \in C_c^\infty(X)$. So

$$\big((f * T) * \phi\big)\tilde{}(\lambda, b) = (f * T)\tilde{}(\lambda, b)\tilde{\phi}(\lambda).$$

But $(f * T) * \phi = f * (T * \phi)$ and

$$\big(f * (T * \phi)\big)\tilde{}(\lambda, b) = \tilde{f}(\lambda, b)\tilde{T}(\lambda)\tilde{\phi}(\lambda)$$

which proves the proposition. □

Now we are in a position to state the analogue of Theorem 2.6 in the previous section. We first deal with the case $1 \leqslant p < 2$.

Theorem 3.8. *Let $f \in L^p(X) \cap C^\infty(X)$, $1 \leqslant p < 2$ and T be a compactly supported K-biinvariant distribution. Assume that $f * T$ is compactly supported. If all irreducible components of $Z_{\tilde{T}}$ intersect S_p^0, then f is compactly supported.*

Proof. Let $f * T = g$, for $g \in C_c^\infty(X)$. We may assume that $f = \chi_\delta * f$ and so $g = \chi_\delta * g$ as T is K-biinvariant. We have

$$\tilde{g}(\lambda, b) = a_1(\lambda)v_{i_1} + a_2(\lambda)v_{i_2} + \cdots + a_n(\lambda)v_{i_n},$$

where each $a_i(\lambda)$ extends to an entire function on $\mathbf{a}_{\mathbb{C}}^*$ of exponential type R (for some $R > 0$), whose restriction to $\mathbf{a}^*$ is bounded. Next, by Proposition 3.7

$$(f * T)\tilde{}(\lambda, b) = \tilde{f}(\lambda, b)\tilde{T}(\lambda).$$

It follows that

$$\tilde{f}(\lambda, b) = b_1(\lambda)v_{i_1}(b) + b_2(\lambda)v_{i_2}(b) + \cdots + b_n(\lambda)v_{i_n}(b),$$

where

$$a_j(\lambda) = \tilde{T}(\lambda)b_j(\lambda).$$

Now, $b_j(\lambda) = a_j(\lambda)/\tilde{T}(\lambda)$ are holomorphic functions in the open set S_p^0 and all the irreducible components of $Z_{\tilde{T}}$ intersect S_p^0. Hence, in the open set S_p^0, all the irreducible components of $Z_{\tilde{T}}$ intersected with S_p^0 are contained in the zero set of $a_j(\lambda)$. By irreducibility, this will force all the components of $Z_{\tilde{T}}$ to be contained in the zero set of a_j. It immediately follows that $\frac{a_j}{\tilde{T}}$ is an entire function of exponential type. This finishes the proof. □

To prove the L^2 case we need to recall details about the δ-spherical transform and analyze the c-function in detail. If $f \in C^\infty(X)$ then we have

$$f = \sum_{\delta \in \hat{K_M}} d(\delta)\chi_\delta * f,$$

where $\hat{K_M}$ consists of all unitary irreducible representations of K which have M-fixed vector. We also have $L^2(K/M) = \bigoplus_{\delta \in \hat{K_M}} V_\delta$, where V_δ consists of the vectors in $L^2(K/M)$ that transform according to the representation δ under the K-action. Let $V_\delta^M = \{v \in V_\delta\colon\ \delta(m)v = v\ \forall m \in M\}$. For $\delta \in \hat{K_M}$ define spherical functions of type δ by

$$\Phi_{\lambda\delta}(x) = \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))}\delta(k)\,dk, \quad \lambda \in \mathbf{a}_{\mathbb{C}}^*,\ x \in X.$$

Then,

$$\Phi_{\lambda,\delta}(kx) = \delta(k)\Phi_{\lambda,\delta}(x),$$

and

$$\Phi_{\lambda,\delta}(x)\delta(m) = \Phi_{\lambda,\delta}(x), \quad m \in M.$$

If $f = d(\delta)\chi_\delta * f$, define its δ-spherical Fourier transform by

$$\tilde{f}(\lambda) = d(\delta)\int_X f(x)\Phi_{\bar{\lambda},\delta}^*(x)\,dx,$$

where $*$ denotes the adjoint. If δ is the trivial representation then $f \to \tilde{f}$ is the spherical Fourier transform. In general $\delta(m)\tilde{f}(\lambda) = \tilde{f}(\lambda)$ and so $\tilde{f}(\lambda) \in \mathrm{Hom}(V_\delta, V_\delta^M)$. If $\tilde{f}(\lambda, kM)$ is the Helgason–Fourier transform of f then we have

$$\tilde{f}(\lambda) = d(\delta) \int\limits_K \tilde{f}(\lambda, kM)\delta\bigl(k^{-1}\bigr)\, dk, \qquad \tilde{f}(\lambda, kM) = \mathrm{Trace}\bigl(\delta(k)\tilde{f}(\lambda)\bigr).$$

The δ-spherical Fourier transform is inverted by

$$f(x) = \frac{1}{|W|}\,\mathrm{Trace}\biggl(\int\limits_{\mathbf{a}^*} \Phi_{\lambda,\delta}(x)\tilde{f}(\lambda)\bigl|c(\lambda)\bigr|^{-2}\, d\lambda\biggr).$$

For each $\delta \in \hat{K_M}$, we also have the $Q_\delta(\lambda)$ matrices which are $l(\delta) \times l(\delta)$ matrices whose entries are polynomial factors in λ (see [9, page 238]). Here $l(\delta) = \dim V_\delta^M$. Let $\check{\delta}$ denote the contragredient representation of K on the dual space of V_δ. Then, the Paley–Wiener theorem for the δ-spherical transform (see [9, page 285]) says the following: Let $H^\delta(\mathbf{a}^*)$ stand for all the functions $F : \mathbf{a}^*_{\mathbb{C}} \to \mathrm{Hom}(V_\delta, V_\delta^M)$ such that

- **(i)** F is holomorphic and is of exponential type,
- **(ii)** $Q_{\check{\delta}}^{-1}F$ is holomorphic and Weyl group invariant.

Theorem 3.9. *The δ-spherical transform $f \to \tilde{f}$ is a bijection from $\{f \in C_c^\infty(X){:}\ f = d(\delta)\chi_\delta * f\}$ onto $H^\delta(\mathbf{a}^*)$.*

We are now in a position to state the L^2 version of Theorem 3.8. Also recall that if G is a real rank one group then $\mathbf{a}$ and $\mathbf{a}^*$ may be identified with $\mathbb{R}$ and $\mathbf{a}^*_{\mathbb{C}}$ with $\mathbb{C}$.

Theorem 3.10.

- **(1)** *Let G be a real rank one group and T be a compactly supported K-biinvariant distribution such that all the zeros of $\tilde{T}(\lambda)$ are real. If $f \in L^2 \cap C^\infty(G/K)$ and $f * T$ is compactly supported then f is compactly supported.*
- **(2)** *Let G have only one conjugacy class of Cartan subgroups. Let T be a K-biinvariant compactly supported distribution such that any irreducible component of $Z_{\tilde{T}}$ intersected with $\mathbf{a}^*$ has real dimension $(n-1)$. If $f \in L^2 \cap C^\infty(X)$ and $f * T$ is compactly supported then f is compactly supported.*

Proof. **(1)** In the rank one case it is known that $\lambda \to c(\lambda)$ is a meromorphic function on $\mathbb{C}$ with simple poles, all lying on the imaginary axis. In particular, $\lambda = 0$ is a simple pole. It follows that $|c(\lambda)|^{-2} = c(\lambda)c(-\lambda)$ is a holomorphic function in a small strip containing the real line and the only zero of $|c(\lambda)|^{-2}$ in that strip is $\lambda = 0$, of order 2. As in the previous theorem we assume that $f = d(\delta)\chi_\delta * f$ and so $g = d(\delta)\chi_\delta * g$. Applying the δ-spherical transform to $f * T = g$ we obtain

$$\tilde{T}(\lambda)\tilde{f}(\lambda) = \tilde{g}(\lambda).$$

Since $l(\delta) = \dim V_\delta^M = 1$, both $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$ are $1 \times d(\delta)$ vectors. So, to be consistent with the previous notation we write

$$\tilde{g}(\lambda) = \bigl(a_1(\lambda), a_2(\lambda), \ldots, a_{d(\delta)}(\lambda)\bigr),$$

and

$$\tilde{f}(\lambda) = \bigl(b_1(\lambda), b_2(\lambda), \ldots, b_{d(\delta)}(\lambda)\bigr),$$

where

$$b_j(\lambda) = \frac{a_j(\lambda)}{\tilde{T}(\lambda)}.$$

By the Paley–Wiener theorem (Theorem 3.4) $\lambda \to a_j(\lambda)$ is an entire function of exponential type and $\frac{a_j(\lambda)}{Q_{\check{\delta}}(\lambda)}$ is an even entire function on $\mathbb{C}$. We also have

$$\int_{\mathbf{a}^*} \left|\frac{a_j(\lambda)}{\tilde{T}(\lambda)}\right|^2 \bigl|c(\lambda)\bigr|^{-2}\, d\lambda < \infty. \tag{3.1}$$

Now if $0 \neq \lambda_0$ is a zero of $\tilde{T}(\lambda)$ of order k, since $|c(\lambda_0)|^{-2} \neq 0$ it readily follows from (3.1) that λ_0 is a zero of $a_j(\lambda)$ of order at least k. Next, suppose that $\lambda = 0$ is a zero $\tilde{T}(\lambda)$. Since $\tilde{T}(\lambda)$ is even it follows that $\exists$ a positive integer l such that $\tilde{T}(\lambda) \sim \lambda^{2l}$ in a neighborhood of $\lambda = 0$. Recall that $Q_{\check{\delta}}(\lambda) \neq 0$ on $\mathbf{a}^*$ and $h(\lambda) = \frac{a_j(\lambda)}{Q_{\check{\delta}}(\lambda)}$ is even, holomorphic. Now (3.1) implies that

$$\int_{|\lambda| \leqslant \varepsilon} \left|\frac{h(\lambda)}{\tilde{T}(\lambda)}\right|^2 \bigl|c(\lambda)\bigr|^{-2}\, d\lambda < \infty, \tag{3.2}$$

for some $\varepsilon > 0$. Since $|c(\lambda)|^{-2} \sim \lambda^2$ near zero (3.2) implies that $h(\lambda) = 0$ if $\lambda = 0$. Since $h(\lambda)$ is even $h(\lambda) \sim \lambda^{2m}$ in a neighborhood of $\lambda = 0$. Then (3.2) implies that $m \geqslant l$ which in turn implies that $\frac{a_j(\lambda)}{\tilde{T}(\lambda)}$ is entire which is of exponential type by Malgrange's theorem. This finishes the proof.

(2) If G has only one conjugacy class of Cartan subgroups then the Plancherel density $|c(\lambda)|^{-2}$ is given by a polynomial which we describe now. Let Σ_0^+ be the set of positive indivisible roots. If $\alpha \in \Sigma_0^+$ then the multiplicity m_α is even $\forall \alpha$ and $m_{2\alpha} = 0$. For $\alpha \in \Sigma_0^+$ define

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}, \quad \lambda \in \mathbf{a}_{\mathbb{C}}^*.$$

With the convention that the product over an empty set is 1 the explicit expression for $|c(\lambda)|^{-2}$ is given by

$$\bigl|c(\lambda)\bigr|^{-2} = c \prod_{\alpha \in \sum_0^+} \lambda_\alpha^2 \prod_{k=1}^{m_\alpha/2-1} \bigl(\lambda_\alpha^2 + k^2\bigr),$$

(see [10]) where c is a positive constant.

Proceeding as in the previous case we obtain that

$$\tilde{f}(\lambda) = \frac{\tilde{g}(\lambda)}{\tilde{T}(\lambda)}.$$

Notice that both $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$ belong to $\mathrm{Hom}(V_\delta, V_\delta^M)$. Write $\tilde{f}(\lambda) = (\tilde{f_{ij}}(\lambda))$ and $\tilde{g}(\lambda) = (\tilde{g_{ij}}(\lambda))$. By the Plancherel theorem we have

$$\int_{\mathbf{a}^*} \left| \frac{g_{ij}(\lambda)}{\tilde{T}(\lambda)} \right|^2 \left| c(\lambda) \right|^{-2} d\lambda < \infty.$$

From the above and the expression for $|c(\lambda)|^{-2}$ we also have

$$\int_{\mathbf{a}^*} \left| \frac{p(\lambda) g_{ij}(\lambda)}{\tilde{T}(\lambda)} \right|^2 d\lambda < \infty, \tag{3.3}$$

where $p(\lambda)$ is the polynomial given by

$$p(\lambda) = \prod_{\alpha \in \Sigma_0^+} \lambda_\alpha.$$

Let $\dim \mathbf{a}^* = l$. Since $\lambda \to p(\lambda) g_{ij}(\lambda)$ is an entire function of exponential type with rapid decay on $\mathbf{a}^*$, we have $H \in C_c^\infty(\mathbb{R}^l)$ such that the Euclidean Fourier transform of H, $\hat{H}(\lambda) = p(\lambda) g_{ij}(\lambda)$. Similarly, let S be the compactly supported distribution on $\mathbb{R}^l$ such that $\hat{S}(\lambda) = \tilde{T}(\lambda)$. From (3.3) it follows that there exists $F \in L^2(\mathbb{R}^l)$ such that $F *_{\mathbb{R}^l} S = H$. Since $\hat{S}(\lambda) = \tilde{T}(\lambda)$ satisfies the conditions in Theorem 2.6 we obtain that $F \in C_c^\infty(\mathbb{R}^l)$. It follows that $\frac{p(\lambda) g_{ij}(\lambda)}{\tilde{T}(\lambda)}$ is an entire function of exponential type with rapid decay on $\mathbf{a}^*$. However we need to show that $\frac{g_{ij}(\lambda)}{\tilde{T}(\lambda)}$ is entire. This follows from applying the following lemma to matrix entries of $\frac{Q_{\check{\delta}}(\lambda)^{-1} \tilde{g}(\lambda)}{\tilde{T}(\lambda)}$. □

Lemma 3.11. *Let $p(\lambda)$ be as above and $\psi(\lambda)$ be a holomorphic function defined on $\mathbf{a}_{\mathbb{C}}^* - \{\lambda\colon\ p(\lambda) = 0\}$ such that $p(\lambda)\psi(\lambda)$ has an entire extension. If $\psi(\lambda)$ is Weyl group invariant then $\psi(\lambda)$ is an entire function.*

Proof. Since $p(\lambda)$ is a product of irreducibles it suffices to show that $R(\lambda) = p(\lambda)\psi(\lambda)$ vanishes on $\{\lambda \in \mathbf{a}_{\mathbb{C}}^*\colon\ p(\lambda) = 0\}$. This will follow if we show that $R(\lambda)$ vanishes on $\{\lambda \in \mathbf{a}^*\colon\ p(\lambda) = 0\}$. Fix $\alpha \in \Sigma_0^+$ and let $0 \neq \lambda_0 \in \mathbf{a}^*$ be such that $\langle \alpha, \lambda_0 \rangle = 0$ and $\langle \beta, \lambda_0 \rangle \neq 0$ if $\beta \neq \alpha$. It is easy to see that, in a small enough neighborhood of λ_0, $\langle \alpha, \lambda \rangle$ takes both positive and negative values while $\mathrm{sgn}(\langle \beta, \lambda \rangle)$ is constant $\forall \beta \in \Sigma_0^+$, $\beta \neq \alpha$. Since $\psi(\lambda)$ is Weyl group invariant this will force $R(\lambda) = 0$ if $\lambda = \lambda_0$. This proves that $R(\lambda)$ is zero on (real) $(n-1)$-dimensional strata of the set $\{\lambda \in \mathbf{a}^*\colon\ p(\lambda) = 0\}$. This clearly implies that $R(\lambda) = 0$ whenever $p(\lambda) = 0$. This finishes the proof. □

Remark 3.12. The Paley–Wiener theorem (Theorem 3.6) and Theorem 2.11 provide a large class of compactly supported distributions which satisfy the assumptions in Theorem 3.10. Also, the main result in [16] may be improved as in Theorem 2.8.

Our proof works well for many other cases as well. To explain this, first we reproduce the computation of the c-function from [10]. The Plancherel density $|c(\lambda)|^{-2}$ is given by the product formula

$$\left|c(\lambda)\right|^{-2} = c \prod_{\alpha \in \Sigma_0^+} \left|c_\alpha(\lambda)\right|^{-2},$$

where

$$c_\alpha(\lambda) = \frac{2^{-i\lambda_\alpha}\Gamma(i\lambda_\alpha)}{\Gamma(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2})\Gamma(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2})}.$$

Recall that if both α and 2α are roots, then m_α is even and $m_{2\alpha}$ is odd. Consider the following cases:

(a) m_α even, $m_{2\alpha} = 0$,
(b) m_α odd, $m_{2\alpha} = 0$,
(c) $m_\alpha/2$ even, $m_{2\alpha}$ odd,
(d) $m_\alpha/2$ odd, $m_{2\alpha}$ odd.

If $\lambda_\alpha = \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$, with the convention that product over an empty set is 1, the explicit expression for $|c_\alpha(\lambda)|^{-2}$ is given (up to a constant) by $\lambda_\alpha p_\alpha(\lambda)q_\alpha(\lambda)$ where p_α and q_α are the following, in the four cases listed above, respectively:

(a) $p_\alpha(\lambda) = \prod_{k=1}^{\frac{m_\alpha}{2}-1}[\lambda_\alpha^2 + k^2]$,
$q_\alpha(\lambda) = 1$.
(b) $p_\alpha(\lambda) = \prod_{k=0}^{\frac{m_\alpha-3}{2}}[\lambda_\alpha^2 + (k+\frac{1}{2})^2]$,
$q_\alpha(\lambda) = \tanh \pi\lambda_\alpha$.
(c) $p_\alpha(\lambda) = \prod_{k=0}^{\frac{m_\alpha}{4}-1}[(\frac{\lambda_\alpha}{2})^2 + (k+\frac{1}{2})^2]\prod_{k=0}^{\frac{m_\alpha}{4}+\frac{m_\alpha-1}{2}-1}[(\frac{\lambda_\alpha}{2})^2 + (k+\frac{1}{2})^2]$,
$q_\alpha(\lambda) = \tanh \frac{\pi\lambda_\alpha}{2}$.
(d) $p_\alpha(\lambda) = \prod_{k=0}^{\frac{m_\alpha-2}{4}}[(\frac{\lambda_\alpha}{2})^2 + k^2]\prod_{k=1}^{\frac{m_\alpha+2m_{2\alpha}}{4}-1}[(\frac{\lambda_\alpha}{2})^2 + k^2]$,
$q_\alpha(\lambda) = \coth \frac{\pi\lambda_\alpha}{2}$.

The case **(a)** corresponds to the case dealt with in Theorem 3.9. It is clear from the above expression that if m_α is large enough $\forall\alpha \in \sum_0^+$ then

$$\lambda_\alpha p_\alpha(\lambda)q_\alpha(\lambda) \geqslant \lambda_\alpha^2, \quad \forall\alpha \in \Sigma_0^+$$

and consequently we obtain (3.3). Hence the theorem holds for all groups with this property. Simple Lie groups with this property can be read off from the list in [20] (see pages 30–32).

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