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Stanley–Reisner rings with large multiplicities are Cohen–Macaulay

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Abstract

We prove that Stanley–Reisner rings having sufficiently large multiplicities are Cohen–Macaulay using Alexander duality.

Keywords: Stanley–Reisner ring; Cohen–Macaulay; Multiplicity; Alexander duality; Linear resolution; Initial degree; Relation type

1. Introduction

Throughout this paper, let $S = k[X_1, \dots, X_n]$ be a homogeneous polynomial ring over a field k with $\deg X_i = 1$. For a simplicial complex Δ on vertex set $[n] = \{1, \dots, n\}$ (note that $\{i\} \in \Delta$ for all i), $k[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of Δ, where I_Δ is an ideal generated by all square-free monomials $X_{i_1} \cdots X_{i_p}$ such that $\{i_1, \dots, i_p\} \notin \Delta$. The ring $A = k[\Delta]$ is a homogeneous reduced ring with the unique homogeneous maximal ideal $\mathfrak{m} = (X_1, \dots, X_n)k[\Delta]$ and the Krull dimension $d = \dim \Delta + 1$. Let $e(A)$ denote the *multiplicity* $e_0(\mathfrak{m}A_\mathfrak{m}, A_\mathfrak{m})$ of A, which is equal to the number of facets (i.e., maximal

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faces) F of Δ with $\dim F = d-1$. Also, we frequently call it the multiplicity of Δ. If all facets of Δ have the same dimension, Δ is called *pure*. See [1,11] for more details.

Take a graded minimal free resolution of a homogeneous k-algebra $A = S/I$ over S:

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p,j}(A)} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{1,j}(A)} \xrightarrow{\varphi_1} S \to A \to 0.$$

Then the *initial degree* $\operatorname{indeg} A$ (respectively *the relation type* $\operatorname{rt}(A)$) of A is defined by $\operatorname{indeg} A = \min\{j \in \mathbb{Z}\colon\ \beta_{1,j}(A) \neq 0\}$ (respectively $\operatorname{rt}(A) = \max\{j \in \mathbb{Z}\colon\ \beta_{1,j}(A) \neq 0\}$). Let $\mu(I)$ denote the minimal number of generators of I, that is, $\mu(I) = \sum \beta_{1,j}(A)$. Note that the initial degree and relation type are simply the smallest and biggest degrees of a minimal set of generators of I, respectively. Also, $\operatorname{reg} A = \max\{j - i \in \mathbb{Z}\colon\ \beta_{i,j}(A) \neq 0\}$ is called the *Castelnuovo–Mumford regularity* of A. It is easy to see that $\operatorname{reg} A \geqslant \operatorname{indeg} A - 1$, and A has (q-)*linear resolution* if equality holds (and $q = \operatorname{indeg} A$).

The main purpose of this paper is to prove the following theorems:

Theorem 2.1. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Put* $\operatorname{codim} A = c$. *If $e(A) \geqslant \binom{n}{c} - c$, then A is Cohen–Macaulay.*

Theorem 3.1. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Put* $\operatorname{codim} A = c$. *Suppose that Δ is pure* (*i.e., A is equidimensional*). *If $e(A) \geqslant \binom{n}{c} - 2c + 1$, then A is Cohen–Macaulay.*

It is easy to prove the above theorems in the case of $d = 2$. In fact, when $d = 2$, A is Cohen–Macaulay if and only if Δ is connected. A disconnected graph has at most $\binom{n-1}{2}$ $(= \binom{n}{2} - (n-2) - 1)$ edges. This shows that Theorem 2.1 is true in this case. Similarly, a disconnected graph without an isolated point has at most $\binom{n-2}{2} + 1 (= \binom{n}{2} - 2(n-2))$ edges. Indeed, such a graph is contained in the disjoint union of an $(n-i)$-complete graph and an i-complete graph for some $2 \leqslant i \leqslant n-2$. When $i = 2$, the number of edges of the above union is maximal and just $\binom{n-2}{2} + 1$. Thus we also get Theorem 3.1 in this case.

For Theorem 2.1, we give several proofs by different methods in Section 2. One of powerful tools is the Alexander dual. As for Theorem 3.1, since it seems to be difficult to prove it directly, we give a proof using the Alexander dual complex in Section 3. The following Alexander dual versions of Theorems 2.1 and 3.1 have some interest in their own right.

Theorem 2.8. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that* $\operatorname{indeg} A = d$. *If $e(A) \leqslant d$, then A has d-linear resolution. In particular,* $\operatorname{rt}(A) = d$.

Theorem 3.3. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that* $\operatorname{indeg} A = \operatorname{rt}(A) = d$. *If $e(A) \leqslant 2d - 1$, then A has d-linear resolution. In particular, $a(A) < 0$.*

For a Stanley–Reisner ring A with $\operatorname{indeg} A = \dim A = d$, it has d-linear resolution if and only if $a(R) < 0$. Thus the assertion of Theorem 3.3 could be seen as an analogy of the following: *Let R be a homogeneous integral domain over an algebraically closed field of characteristic* 0. *If $e(R) \leqslant 2\dim R - 1$ and* $\operatorname{codim} R \geqslant 2$, *then $a(R) < 0$.*

In the last section, we will provide several examples related to the above results.

2. Complexes $\boldsymbol{\Delta}$ with $e(k[\Delta]) \geqslant \binom{n}{c} - c$

In this section, we use the following notation. Let Δ be a simplicial complex on $V = [n]$, and let $A = k[\Delta] = S/I_\Delta$ be the Stanley–Reisner ring of Δ. Put $d = \dim A$, and $c = \operatorname{codim} A = n - d$. Let $\binom{[n]}{d}$ denote the family of all d-subsets of $[n]$. For a subset W of V and a face G in Δ, we put

$$\begin{aligned} \Delta_W &= \{F \in \Delta\colon\ F \subseteq W\},\\ \operatorname{star}_\Delta G &= \{F \in \Delta\colon\ F \cup G \in \Delta\},\\ \operatorname{link}_\Delta G &= \{F \in \Delta\colon\ F \cup G \in \Delta,\ F \cap G = \emptyset\}. \end{aligned}$$

The main purpose of this section is to prove the following theorem.

Theorem 2.1. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. If $e(A) \geqslant \binom{n}{c} - c$, then A is Cohen–Macaulay.*

Let us begin the proof of this theorem with the following lemmas.

Lemma 2.2. *If $e(A) \geqslant \binom{n}{c} - c$, then* $\operatorname{indeg} A \geqslant d$ *and Δ is pure.*

Proof. Let Γ be the $(d-1)$-skeleton of 2^V. Every $(d-2)$-face of Γ is contained at least $c+1$ facets of Γ. Consider Δ as a subcomplex of Γ. Then Δ is obtained from Γ by removing at most c facets. Hence Δ contains all $(d-2)$-faces of Γ, that is, $\operatorname{indeg} A \geqslant d$ and Δ is pure, as required. □

By the following two lemmas we may assume that $\operatorname{indeg} A = d$ and $c = \operatorname{codim} A \geqslant 2$ to prove Theorem 2.1.

Lemma 2.3. *Under the above notation, the following conditions are equivalent*:

(1) $\operatorname{indeg} A = d + 1$.
(2) $e(A) = \binom{n}{d}$.
(3) $I_\Delta = (X_{i_1} \cdots X_{i_{d+1}}\colon\ 1 \leqslant i_1 < \cdots < i_{d+1} \leqslant n)$.
(4) *A has $(d+1)$-linear resolution.*

When this is the case, A is Cohen–Macaulay with $\operatorname{rt}(A) = d + 1$.

Proof. See, e.g., [12, Proposition 1.2]. □

Lemma 2.4. *Suppose $n = d + 1$. If $e(A) \geqslant d$, then A is a hypersurface.*

Proof. Suppose that A is not a hypersurface. Then we can write

$$I_\Delta = X_{i_1} \cdots X_{i_p} J$$

for some monomial ideal $J (\neq S)$ with height $J \geqslant 2$ since height $I_\Delta = 1$. In particular, A is not Cohen–Macaulay. Thus $\operatorname{indeg} A \leqslant d$ by Lemma 2.3. Then $e(A) = p \leqslant d - 1$. This contradicts the assumption. □

In what follows, we put $\Gamma_i = \operatorname{link}_\Delta\{i\}$ for each $i \in V$. Also, using the following lemma one can show that every Γ_i satisfies the assumption of the theorem if so does Δ with $\dim k[\Delta] \geqslant 3$.

Lemma 2.5. *Suppose that* $\operatorname{indeg} A = d$ *and Δ is pure. Then*

$$e(A) - \binom{n}{d} \leqslant e\bigl(k[\Gamma_i]\bigr) - \binom{n-1}{d-1}.$$

Also, equality holds if and only if $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$.

Proof. If we put $W_i = \{F \in \binom{[n]}{d}\colon\ i \in F \notin \Delta\}$, then $\bigcup_{j=1}^n W_j = \binom{[n]}{d} \setminus \Delta$ and $W_i = \{\{i\} \cup G\colon\ G \in \binom{[n]-\{i\}}{d-1} \setminus \Gamma_i\}$. Thus

$$\binom{n-1}{d-1} - e\bigl(k[\Gamma_i]\bigr) \leqslant \binom{n}{d} - e(A),$$

and equality holds if and only if $W_i = \binom{[n]}{d} \setminus \Delta$, that is, $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$. □

Proof of Theorem 2.1. We use induction on $n \geqslant d$ and $d \geqslant 2$. By the observation in the introduction, the case of $d = 2$ is true. Also, when $n = d + 1$, A is Cohen–Macaulay by Lemma 2.4. We may assume that $d \geqslant 3$ and $n \geqslant d + 2$. By Lemmas 2.2, 2.3 we may assume that $\operatorname{indeg} A = d$ and Δ is pure. In particular, $\dim k[\Gamma_i] = d - 1$ for every $i \in V$. Thus Γ_i is Cohen–Macaulay by Lemma 2.5 and the induction hypothesis on d.

Now suppose that $e(k[\Gamma_i]) \geqslant \binom{n-1}{d-1}$ for all $i \in V$. Then

$$e(A) = \frac{1}{d} \sum_{i=1}^n e\bigl(k[\Gamma_i]\bigr) \geqslant \frac{n}{d}\binom{n-1}{d-1} = \binom{n}{d}.$$

This implies that Δ is the $(d-1)$-skeleton of 2^V; hence it is Cohen–Macaulay. Thus we may assume that $e(k[\Gamma_i]) \leqslant \binom{n-1}{d-1} - 1$ for some $i \in V$. Fix such $i \in V$. Then there exists a $(d-1)$-facet F of Δ such that $i \notin F$. Hence $\dim k[\Delta_{V\setminus\{i\}}] = d$ and

$$\begin{aligned} e\bigl(k[\Delta_{V\setminus\{i\}}]\bigr) &= e\bigl(k[\Delta]\bigr) - e\bigl(k\bigl[\mathrm{star}_{\Delta}\{i\}\bigr]\bigr) \\ &= e\bigl(k[\Delta]\bigr) - e\bigl(k\bigl[\mathrm{link}_{\Delta}\{i\}\bigr]\bigr) \\ &= \binom{n-1}{d} - (c-1). \end{aligned}$$

By the induction hypothesis on n, $\Delta_{V\setminus\{i\}}$ is Cohen–Macaulay. Take the Mayer–Vietoris sequence with respect to $\Delta = \Delta_{V\setminus\{i\}} \cup \mathrm{star}_{\Delta}\{n\}$ as follows:

$$\tilde{H}_{d-2}(\Delta_{V\setminus\{i\}}) \oplus \tilde{H}_{d-2}\bigl(\mathrm{star}_{\Delta}\{i\}\bigr) \to \tilde{H}_{d-2}(\Delta) \to \tilde{H}_{d-3}(\Gamma_i).$$

Note that both sides are zero since $\Delta_{V\setminus\{i\}}$, Γ_i are Cohen–Macaulay and since $\tilde{H}_j(\mathrm{star}_{\Delta}\{i\}) = 0$ for all j. Hence $\tilde{H}_{d-2}(\Delta) = 0$ and Δ is Cohen–Macaulay by Hochster's formula on Betti numbers. □

In the rest of this section we give another proof of Theorem 2.1 using Alexander dual complex. Assume that $c = \operatorname{codim} A \geqslant 2$. Let Δ^* be the *Alexander dual* of Δ:

$$\Delta^* = \bigl\{F \in 2^V \colon\ V \setminus F \notin \Delta\bigr\}.$$

Then Δ^* is a simplicial complex on the same vertex set V of Δ for which the following properties are satisfied:

Proposition 2.6. *Under the above notation, we have*

(1) $\operatorname{indeg} k[\Delta^*] + \dim k[\Delta] = n$.
(2) $\mathrm{rt}(k[\Delta^*]) = \operatorname{bight} I_{\Delta}$, *where*

$$\operatorname{bight} I_{\Delta} = \max\{\operatorname{height}\mathfrak{p}\colon\ \mathfrak{p} \text{ is a minimal prime divisor of } I_{\Delta}\}.$$

In particular, Δ is pure if and only if $\mathrm{rt}(k[\Delta^*]) = \operatorname{indeg} k[\Delta^*]$.
(3) $\beta_{0,q^*}(I_{\Delta^*}) = e(k[\Delta])$, *where* $q^* = \operatorname{indeg} k[\Delta^*]$.
(4) $(\Delta^*)^* = \Delta$.

Also, the following theorem is fundamental. See [3] for more details.

Theorem 2.7. (Eagon–Reiner [3]) $k[\Delta]$ *is Cohen–Macaulay if and only if* $k[\Delta^*]$ *has linear resolution.*

We want to reduce Theorem 2.1 to its Alexander dual version. Let Δ^* be the Alexander dual of Δ. Then $\operatorname{indeg} k[\Delta^*] = n - \dim k[\Delta] = c$ and $\dim k[\Delta^*] = n - \operatorname{indeg} k[\Delta] = n - d = c$. Also, since $\operatorname{indeg} k[\Delta^*] = \dim k[\Delta^*] = c$, we have

$$e\bigl(k[\Delta^*]\bigr) = \binom{n}{c} - \beta_{0,c}(I_{\Delta^*}) = \binom{n}{c} - e(A) \leqslant c.$$

Also, $\tilde{H}_{c-1}(\Delta^*) = 0$ if and only if $\tilde{H}_{d-2}(\Delta) = 0$. Note that $\tilde{H}_{c-1}(\Delta^*)$ is the top reduced homology of Δ^*, while $\tilde{H}_{d-2}(\Delta)$ is an intermediate reduced homology of Δ. The top homology is easier to treat than the intermediate one in most case. For example, the top homology of Δ^* vanishes if Δ^* is homotopy equivalent to a lower-dimensional simplicial complex. This is the reason why we consider Alexander dual version.

Theorem 2.8 *(Alexander dual version of Theorem 2.1). Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that* $\operatorname{indeg} A = d$. *If $e(A) \leqslant d$, then A has d-linear resolution. In particular,* $\operatorname{rt}(A) = d$.

Proof. It is enough to show that $\tilde{H}_{d-1}(\Delta) = 0$ whenever $e(k[\Delta]) \leqslant d$. Assume that there exists a complex Δ such that $e(k[\Delta]) \leqslant d$, $\tilde{H}_{d-1}(\Delta) \neq 0$ and $\dim \Delta = d - 1$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Choose any $(d-1)$-facet F of Δ. Then F contains just d subfacets of Δ; say $G_1, \ldots, G_d$. Then G_i is not a *free face* (see [9]). That is, G_i is contained in at least two facets of Δ. Indeed, if $G = G_i$ is a free face of Δ, then the simplicial complex $\Delta' := \Delta \setminus \{F, G\}$ is homotopy equivalent to Δ and thus $\tilde{H}_{d-1}(\Delta') \cong \tilde{H}_{d-1}(\Delta) \neq 0$. This contradicts the minimality of $e(k[\Delta])$ since $e(k[\Delta']) < e(k[\Delta])$.

Thus for each $i \in V$ there exists a $(d-1)$-facet F_i of Δ such that $G_i \subseteq F_i \neq F$. In particular, $F_1, \ldots, F_d, F$ are $(d+1)$ distinct facets of Δ. This is a contradiction. $\square$

Now we give a slight generalization of the theorem. A ring homomorphism $A \to B$ is called *pure* if for every A-module M, $M \to M \otimes_A B (m \mapsto m \otimes 1)$ is injective. Let $A = S/I$ be an arbitrary homogeneous reduced k-algebra over a field k of characteristic $p > 0$. The ring A is called *F-pure* if the Frobenius map $F : A \to A (a \mapsto a^p)$ is pure in the above sense.

Proposition 2.9. *Let $A = S/I$ be a homogeneous F-pure k-algebra. Put* $\dim A = \operatorname{indeg} A = d \geqslant 2$. *If $e(A) \leqslant d$, then A has d-linear resolution. In particular,* $\operatorname{rt}(A) = d$ *and* $a(A) < 0$.

Proof. Put $a(A) = \sup\{j \in \mathbb{Z}\colon [H^d_{\mathfrak{m}}(A)]_j \neq 0\}$, the *$a$-invariant* of A. From the assumption we obtain that

$$a(A) + d \leqslant e(A) - 1 \leqslant d - 1,$$

where the first inequality follows from e.g. [7, Lemma 3.1]. Hence $a(A) < 0$. On the other hand, $[H^i_{\mathfrak{m}}(A)]_j = 0$ for all i and $j \geqslant 1$ since A is F-pure. Then by [4], we have

$$\operatorname{reg} A = \inf\bigl\{m \in \mathbb{Z}\colon \bigl[H^i_{\mathfrak{m}}(A)\bigr]_j = 0 \text{ for all } i + j > m\bigr\} \leqslant d - 1 = \operatorname{indeg} A - 1.$$

This means that A has d-linear resolution, as required. $\square$

It is known that a Stanley–Reisner ring is F-pure over a field of characteristic $p > 0$. If Δ has linear resolution over a positive characteristic field then so does it over a field of characteristic zero. Thus the above proof gives a ring-theoretic proof of Theorem 2.8.

Remark 2.10. After submitting the paper, Ngô Viêt Trung informed us about another proof of Theorem 2.8: In fact, in order to prove the theorem, it is enough to show that $\operatorname{reg} k[\Delta] \leqslant e(k[\Delta]) - 1$ in the case where Δ is pure. But it follows from [6, Theorem 3.1] or [8, Theorem 1.1].

3. Complexes Δ with $e(k[\Delta]) \geqslant \binom{n}{c} - 2c + 1$

We use the same notation as in the previous section unless otherwise specified. The main purpose of this section is to prove the following theorem.

Theorem 3.1. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Put $c = \operatorname{codim} A$. Suppose that Δ is pure. If $e(A) \geqslant \binom{n}{c} - 2c + 1$, then A is Cohen–Macaulay.*

Now suppose that $c = 1$ (respectively $\operatorname{indeg} A \geqslant d + 1$). Then the assertion of Theorem 3.1 follows from Lemma 2.4 (respectively Lemma 2.3). Thus we may assume that $c \geqslant 2$ and $q = \operatorname{indeg} A \leqslant d$. The following lemma corresponds to Lemma 2.2.

Lemma 3.2. *If $e(k[\Delta]) \geqslant \binom{n}{c} - 2c + 1$, then $\operatorname{indeg} k[\Delta] \geqslant d - 1$, i.e.,*

(1) $\operatorname{indeg} k[\Delta] = d$ *or*
(2) $\operatorname{indeg} k[\Delta] = d - 1$.

Proof. Suppose that $\operatorname{indeg} k[\Delta] < d - 1$. Take a squarefree monomial $M \in I_\Delta$ with $\deg M = d - 2$. Then there are $\binom{n-d+2}{2}$ squarefree monomials in degree d in I_Δ. Note $\binom{n-d+2}{2} = \binom{c+2}{2} \geqslant 2c$. Hence

$$e(k[\Delta]) \leqslant \binom{n}{c} - 2c.$$

This contradicts the assumption. □

By the same reason in the previous section, we take the Alexander dual. First, we consider the Alexander dual version of Theorem 3.1 in the case of $\operatorname{indeg} k[\Delta] = d$. Namely, we will prove the following theorem.

Theorem 3.3 *(Alexander dual version of Theorem 3.1, Case (1)). Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that $\operatorname{indeg} A = \operatorname{rt}(A) = d$. If $e(A) \leqslant 2d - 1$, then A has d-linear resolution. In particular, $a(A) < 0$.*

The proof of the above theorem can be reduced to that of the following theorem, which is a key result in this paper.

Theorem 3.4. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that $\operatorname{rt}(A) \leqslant d$. If $e(A) \leqslant 2d - 1$, then $\operatorname{reg} A \leqslant d - 1$, equivalently, $\tilde{H}_{d-1}(\Delta) = 0$.*

Proof. Put $e = e(A)$. Let Δ' be the subcomplex that is spanned by all facets of dimension $d-1$. Replacing Δ with Δ', we may assume that Δ is pure.

We use induction on $d = \dim A \geqslant 2$. First suppose $d = 2$. The assumption shows that Δ does not contain the boundary complex of a triangle. Hence $\tilde{H}_1(\Delta) = 0$ since $e(A) \leqslant 3$.

Next suppose that $d \geqslant 3$, and that the assertion holds for any complex the dimension of which is less than $d-1$. Assume that Δ is a $(d-1)$-dimensional pure complex with $\mathrm{rt}(k[\Delta]) \leqslant d$, $e(k[\Delta]) \leqslant 2d-1$ and $\tilde{H}_{d-1}(\Delta) \neq 0$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Then Δ does not contain any free face by a similar argument as in the proof of Theorem 2.8.

First consider the case of $\mathrm{rt}(A) = d$. Take a generator $X_{i_1} \cdots X_{i_d}$ of I_Δ. For every $j = 1, \ldots, d$, each $G_j = \{i_1, \ldots, \hat{i}_j, \ldots, i_d\}$ is contained in at least two facets as mentioned above. Then $e(A) \geqslant 2d$ since those facets are different from each other. This is a contradiction.

Next we consider the case of $\mathrm{rt}(A) < d$. Let $V = [n]$ be the vertex set of Δ. Take the Mayer–Vietoris sequence with respect to $\Delta = \Delta_{V \setminus \{n\}} \cup \mathrm{star}_\Delta\{n\}$ as follows:

$$\tilde{H}_{d-1}(\Delta_{V\setminus\{n\}}) \oplus \tilde{H}_{d-1}\big(\mathrm{star}_\Delta\{n\}\big) \to \tilde{H}_{d-1}(\Delta) \to \tilde{H}_{d-2}\big(\mathrm{link}_\Delta\{n\}\big).$$

The minimality of $e(k[\Delta])$ yields that $\tilde{H}_{d-1}(\Delta_{V\setminus\{n\}}) = 0$ since $e(k[\Delta_{V\setminus\{n\}}]) < e(k[\Delta])$. Hence $\tilde{H}_{d-1}(\Delta) \hookrightarrow \tilde{H}_{d-2}(\mathrm{link}_\Delta\{n\})$. In particular, $\tilde{H}_{d-2}(\mathrm{link}_\Delta\{n\}) \neq 0$.

Set $\Delta' = \mathrm{link}_\Delta\{n\}$. Then Δ' is a complex on $V \setminus \{n\}$ such that $\dim k[\Delta'] = d-1$ and $\mathrm{rt}(k[\Delta']) \leqslant \mathrm{rt}(k[\Delta]) \leqslant d-1$. In order to apply the induction hypothesis to Δ', we want to see that $e(k[\Delta']) \leqslant 2d-3$. In order to do that, we consider $e(k[\Delta_{V\setminus\{n\}}])$. As $\Delta \neq \mathrm{star}_\Delta\{n\}$, one can take $F = \{i_1, \ldots, i_m, n\} \notin \Delta$ for some $m \leqslant d-2$ such that $X_{i_1} \cdots X_{i_m} X_n$ is a generator of I_Δ. Then $G := \{i_1, \ldots, i_m\} \in \Delta$, but it is not a facet of Δ. Thus it is contained in at least two facets of Δ, each of which does not contain n. Hence $e(k[\Delta_{V\setminus\{n\}}]) \geqslant 2$. Thus we get

$$e\big(k[\Delta']\big) = e\big(k\big[\mathrm{star}_\Delta\{n\}\big]\big) = e\big(k[\Delta]\big) - e\big(k[\Delta_{V\setminus\{n\}}]\big) \leqslant 2d-3.$$

By induction hypothesis, we have $\tilde{H}_{d-2}(\mathrm{link}_\Delta\{n\}) = 0$. This is a contradiction. □

Remark 3.5. Theorem 2.8 also follows from Theorem 3.4. In fact, one can easily see that $e(k[\Delta]) \geqslant d+1$ whenever $\mathrm{rt}(k[\Delta]) \geqslant d+1$.

Next, we prove the following proposition as the Alexander dual version of Theorem 3.1 in the case of $\operatorname{indeg} k[\Delta] = d-1$. Note that $f_i(\Delta)$ denotes the number of i-faces of Δ.

Proposition 3.6 *(Alexander dual version of Theorem 3.1, Case (2)). Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 2$. Suppose that* $\operatorname{indeg} A = \mathrm{rt}(A) = d-1$. *If $\mu(I_\Delta) \geqslant \binom{n}{d-1} - 2d + 2$, then A has $(d-1)$-linear resolution with $e(A) = 1$.*

Proof. First we show that $e(A)=1$. Now suppose that $e(A)\geqslant 2$. Then there exist at least two facets F_1 and F_2 with $|F_1|=|F_2|=d$. This implies that $f_{d-2}(\Delta)\geqslant 2d-1$. However, by the assumption, we have

$$f_{d-2}(\Delta)=\binom{n}{d-1}-\beta_{0,d-1}(I_\Delta)=\binom{n}{d-1}-\mu(I_\Delta)\leqslant 2d-2.$$

This is a contradiction. Hence we get $e(A)=1$.

In order to prove that A has $(d-1)$-linear resolution, it is enough to show that $\beta_{i,j}(A)=0$ for all $i\geqslant c$ and $j\geqslant i+d-1$ by [10, Theorem 5.2]. Also, it suffices to show that $\tilde{H}_{d-1}(\Delta)=\tilde{H}_{d-2}(\Delta)=\tilde{H}_{d-2}(\Delta_W)=0$ for all subsets $W\subset V$ with $|W|=n-1$ by virtue of Hochster's formula on the Betti numbers:

$$\beta_{i,j}(A)=\sum_{\substack{W\subseteq V\\ |W|=j}}\dim_k \tilde{H}_{j-i-1}(\Delta_W;k).$$

Claim 1. $\tilde{H}_{d-1}(\Delta)=\tilde{H}_{d-2}(\Delta)=0$.

It is easy to check $\tilde{H}_{d-1}(\Delta)=0$ using simplicial argument; see also Theorem 3.4. Now let $F=\{1,2,\ldots,d\}$ be the unique facet with $|F|=d$. Consider a simplicial subcomplex $\Delta':=\Delta\setminus\{F,G\}$ where $G=\{1,2,\ldots,d-1\}$. Then $\dim k[\Delta']=d-1$ and $e(k[\Delta'])\leqslant 2d-3=2(d-1)-1$. Also, since $\operatorname{rt}(k[\Delta'])\leqslant\operatorname{rt}(k[\Delta])\leqslant d-1$, applying Theorem 3.4 to Δ', we obtain that $\tilde{H}_{d-2}(\Delta)\cong\tilde{H}_{d-2}(\Delta')=0$, as required.

Claim 2. $\tilde{H}_{d-2}(\Delta_W)=0$ *for all subsets* $W\subset V$ *with* $|W|=n-1$.

Let W be a subset of V such that $|W|=n-1$. Put $V\setminus W=\{v\}$. If v is not contained in F, then $\tilde{H}_{d-2}(\Delta_W)=0$ by a similar argument as in the proof of the previous claim. So we may assume that $v\in F$. Then $\dim k[\Delta_W]=d-1$ and $e(k[\Delta_W])\leqslant(d-3)+1=d-2\leqslant 2(d-1)-1$. Also, since $\operatorname{rt}(k[\Delta_W])\leqslant d-1$, we have $\tilde{H}_{d-2}(\Delta_W)=0$ by Theorem 3.4 again.

Hence $k[\Delta]$ has $(d-1)$-linear resolution, as required. □

Remark 3.7. The above proposition gives a little bit stronger result than the desired one. Indeed, the Alexander dual version of the above proposition yields that if Δ is pure, $\operatorname{indeg} k[\Delta]=d-1$, $e(k[\Delta])\geqslant\binom{n}{c}-2c$ then $k[\Delta]$ is Cohen–Macaulay.

Example 3.8. Let ρ, d be an integers with $0\leqslant\rho\leqslant d-3$. Let Δ be a simplicial complex on $V=[n]$ spanned by $F=\{1,2,\ldots,d\}$, any distinct ρ elements from $\binom{[n]}{d-1}\setminus\binom{[d]}{d-1}$ and all elements of $\binom{[n]}{d-2}$. Then $\dim k[\Delta]=d$, $\operatorname{indeg} k[\Delta]=\operatorname{rt}(k[\Delta])=d-1$. Also, we have

$$\mu(I_\Delta)=\beta_{0,d-1}(I_\Delta)=\binom{n}{d-1}-\rho-d\geqslant\binom{n}{d-1}-2d+3.$$

Hence Δ satisfies the assumption of the above proposition.

On the other hand, we have no results for F-pure k-algebras corresponding to Theorem 3.3. But we remark the following.

Remark 3.9. As mentioned in the introduction, if A is a homogeneous integral domain over an algebraically closed field k of $\operatorname{char} k = 0$ with $\operatorname{codim} A \geqslant 2$ and $e(A) \leqslant 2d - 1$ then one has $a(A) < 0$. In fact, it is known that an inequality

$$a(A) + d \leqslant \left\lceil \frac{e(A) - 1}{\operatorname{codim} A} \right\rceil$$

holds; see, e.g., the remark after Theorem 3.2 in [7]. Moreover, Chikashi Miyazaki told us that this inequality is also true in positive characteristic.

Question 3.10. *Let $A = k[A_1]$ be a homogeneous F-pure, equidimensional k-algebra. Put* $\dim A = \operatorname{indeg} A = d \geqslant 2$. *If $e(A) \leqslant 2d - 1$, then does $a(A) < 0$ hold?*

4. Buchsbaumness

A Stanley–Reisner ring $A = k[\Delta]$ is *Buchsbaum* if and only if Δ is pure and $k[\operatorname{link}_\Delta\{i\}]$ is Cohen–Macaulay for every $i \in [n]$. As an application of Theorem 3.1, we can provide sufficient conditions for $k[\Delta]$ to be Buchsbaum.

Proposition 4.1. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geqslant 3$. Put $c = \operatorname{codim} A$. Suppose that Δ is pure,* $\operatorname{indeg} A = d$ *and $e(A) \geqslant \binom{n}{c} - 2c$. Then*

(1) $e(k[\operatorname{link}_\Delta\{i\}]) \geqslant \binom{n-1}{c} - 2c$ *for all $i \in [n]$.*
(2) *If* $\operatorname{height}[I_\Delta]_d S \geqslant 2$, *then A is Buchsbaum.*
(3) *If* $\operatorname{rt}(A) = d$, *then A is Buchsbaum.*

Proof. (1) follows from Lemma 2.5. In order to prove (2) and (3), we may assume that $c \geqslant 2$ and $e(A) = \binom{n}{c} - 2c$ by virtue of Theorem 3.1.

(2) Suppose that $\operatorname{height}[I_\Delta]_d S \geqslant 2$. Then there is no element $i \in [n]$ for which $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$. Thus the latter assertion of Lemma 2.5 yields that

$$\binom{n}{d} - 2c = e(A) \leqslant \binom{n}{d} - \left[\binom{n-1}{d-1} - e\big(k[\Gamma_i]\big)\right] - 1,$$

that is, $e(k[\Gamma_i]) \geqslant \binom{n-1}{d-1} - 2c + 1$ for every $i \in [n]$. Also, we note that Γ_i is pure and $\operatorname{indeg} k[\Gamma_i] \geqslant \dim k[\Gamma_i] = d - 1$. Applying Theorem 3.1 to $k[\Gamma_i]$, we obtain that $k[\Gamma_i]$ is Cohen–Macaulay. Therefore A is Buchsbaum since Δ is pure.

(3) Now suppose that A is *not* Buchsbaum. Then since $\operatorname{height}[I_\Delta]_d S = 1$, one can take $i \in [n]$ for which $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$. We may assume $i = n$. Then $\{1, \ldots, \hat{j}, \ldots, d+1\} \in \Delta$ for all $j \in [d+1]$ because $n - 1 \geqslant d + 1$. This means that $X_1 \cdots X_{d+1}$ is a generator of I_Δ; thus $\operatorname{rt}(A) = d + 1$. $\square$

The next example shows that the assumptions "height$[I_\Delta]_d S \geqslant 2$" or "$\mathrm{rt}(k[\Delta]) = d$" are *not* superfluous when $e(k[\Delta]) = \binom{n}{c} - 2c$.

Example 4.2. Let Δ be a simplicial complex on $V = [5]$ which is spanned by

$$\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\} \quad \text{and} \quad \{3, 4, 5\}.$$

Then Δ is pure, $\operatorname{indeg} k[\Delta] = \dim k[\Delta] = 3$, $\mathrm{rt}(k[\Delta]) = 4$ and $e(k[\Delta]) = 6 = \binom{n}{c} - 2c$. Also, height$[I_\Delta]_3 S = 1$ since $[I_\Delta]_3 \subseteq X_1 S$. However $k[\Delta]$ is *not* Buchsbaum since $\mathrm{link}_\Delta\{1\}$ is spanned by two edges $\{2, 3\}$, $\{4, 5\}$ and thus is not connected.

5. Examples

Throughout this section, let c, d be given integers with $c, d \geqslant 2$. Set $n = c + d$.

Example 5.1. Put $F_{i,j} = \{1, 2, \ldots, \hat{i}, \ldots, d, j\}$ for each $i = 1, \ldots, d$; $j = d+1, \ldots, n$. For a given integer e with $1 \leqslant e \leqslant cd$, we choose e faces (say, $F_1, \ldots, F_e$) from $\{F_{i,j}\colon 1 \leqslant i \leqslant d,\ d + 1 \leqslant j \leqslant n\}$, which is a set of the facets of the simplicial join of $2^{[d]} \setminus \{[d]\}$ and c points.

Let Δ be a simplicial complex spanned by $F_1, \ldots, F_e$ and all elements of $\binom{[n]}{d-1}$. Then $k[\Delta]$ is a d-dimensional Stanley–Reisner ring with $\operatorname{indeg} k[\Delta] = \mathrm{rt}(k[\Delta]) = d$ and $e(k[\Delta]) = e$.

In particular, when $e \leqslant 2d - 1$, $k[\Delta]$ has d-linear resolution by Theorem 3.3. Thus their Alexander dual complexes provide examples satisfying the hypothesis of Theorem 3.1.

The following example shows that the assumption "$e(A) \leqslant 2d - 1$" is optimal in Theorem 3.3.

Example 5.2. There exists a complex Δ on $V = [n]$ for which $k[\Delta]$ does *not* have d-linear resolution with $\dim k[\Delta] = \operatorname{indeg} k[\Delta] = \mathrm{rt}(k[\Delta]) = d$ and $e(k[\Delta]) = 2d$.

In fact, let Δ_0 be a complex on $V_0 = [d + 2]$ such that $k[\Delta_0]$ is a complete intersection defined by $(X_1 \cdots X_d, X_{d+1}X_{d+2})$. Also, let Δ be a complex on V such that

$$\begin{aligned} I_\Delta = {} & (X_1 \cdots X_d)S + (X_{i_1} \cdots X_{i_{d-2}} X_{d+1} X_{d+2}\colon 1 \leqslant i_1 < \cdots < i_{d-2} \leqslant d)S \\ & + (X_{j_1} \cdots X_{j_d}\colon 1 \leqslant j_1 < \cdots < j_d \leqslant n,\ j_d \geqslant d + 3)S. \end{aligned}$$

Then $\tilde{H}_{d-1}(\Delta) \cong \tilde{H}_{d-1}(\Delta_0) \neq 0$ since $a(k[\Delta_0]) = 0$. Hence $k[\Delta]$ does not have d-linear resolution.

Remark 5.3. The case $n = d + 2$ in the above example is also obtained by considering the case $c = 2$, $e = 2d$ in Example 5.1.

The next example shows that the assumption "$\mathrm{rt}(A) = d$" is *not* superfluous in Theorem 3.3.

Example 5.4. Suppose that $d+1 \leqslant e \leqslant \binom{n}{d}-1$. Then there exists a simplicial complex Δ on $V=[n]$ such that $\dim k[\Delta]=\operatorname{indeg} k[\Delta]=d$, $\operatorname{rt}(k[\Delta])=d+1$ and $e(k[\Delta])=e$. In particular, $k[\Delta]$ does *not* have d-linear resolution.

In fact, put $\mathcal{F}=\binom{[n]}{d} \setminus \binom{[d+1]}{d}$. Let Δ_0 be a simplicial complex on V such that

$$I_{\Delta_0}=(X_1\cdots X_d X_{d+1})S+\bigl(X_{i_1}\cdots X_{i_d}\colon \{i_1,\dots,i_d\}\in\mathcal{F}\bigr)S.$$

Then $\dim k[\Delta_0]=\operatorname{indeg} k[\Delta_0]=d$, $\operatorname{rt}(k[\Delta_0])=d+1$, and $e(k[\Delta_0])=d+1$.

For a given integer e which satisfies the above condition, one obtains the required simplicial complex by adding to Δ_0 any $(e-d-1)$ distinct d-subsets of $[n]$ which are not contained in $\binom{[d+1]}{d}$.

Remark 5.5. Now let Δ be a simplicial complex on $V=[n]$. Set $A=k[\Delta]$. Suppose that $\dim A=\operatorname{indeg} A=d\geqslant 2$. Then one can easily see that $d\leqslant \operatorname{rt}(A)\leqslant d+1$; $\operatorname{rt}(A)=d$ (respectively $d+1$) if $1\leqslant e(A)\leqslant d$ (respectively $e(A)=\binom{n}{d}$). So we put

$$f(n,d)=\min\left\{ m\in\mathbb{Z}\colon \begin{array}{l} \operatorname{rt} k[\Delta]=d+1 \text{ for all } (d-1)\text{-dimensional} \\ \text{complexes } \Delta \text{ on } V \text{ with } \operatorname{indeg} k[\Delta]=d \\ \text{and } e(k[\Delta])\geqslant m \end{array} \right\}.$$

Then $f(n,d)\geqslant cd+1$ by Example 5.1. From the definition of $f(n,d)$, one can easily see that there exists a simplicial complex Δ on V which satisfies $\operatorname{rt}(k[\Delta])=d$ and $e(k[\Delta])=e$ for each e with $d+1\leqslant e\leqslant f(n,d)-1$. On the other hand, according to Example 5.4, one can also find a simplicial complex Δ which satisfies $\operatorname{rt}(k[\Delta])=d+1$ and $e(k[\Delta])=e$ for each e with $d+1\leqslant e\leqslant \binom{n}{d}-1$.

It seems to be difficult to determine $f(n,d)$ in general. Let $T(n,p,k)$ be the so-called *Turan number*; see, e.g., [5]. Then we have

$$f(n,d)=\binom{n}{d}-T(n,d+1,d).$$

In particular, we get

$$f(n,2)=\begin{cases} \frac{n^2}{4}+1, & \text{if } n \text{ is even and } n\geqslant 4,\\ \frac{n^2-1}{4}+1, & \text{if } n \text{ is odd and } n\geqslant 3, \end{cases} \tag{5.1}$$

by Turan's theorem (e.g., [2, Theorem 7.1.1]). However, no formula is known for $T(n,4,3)$; see [5, pp. 1320].

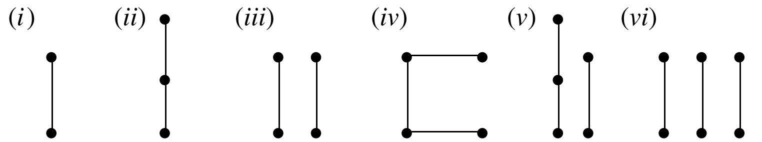
In the rest of this section, we show that the purity of Δ is a very strong condition in Theorem 3.3.

Proposition 5.6. *For integers $d\geqslant 3$, $c=n-d\geqslant 2$, the following conditions are equivalent*:

(1) *There exists a* $(d-1)$*-dimensional pure simplicial complex* $\varDelta$ *on* $V=[n]$ *such that* $\operatorname{indeg} k[\varDelta]=d$ *and* $e(k[\varDelta])=e \leqslant 2d-1$.
(2) $n=d+2$ *and* (d,e) *is one of the following pairs*:

$$(3,4),\ (3,5),\ (4,6),\ (4,7),\ (5,9).$$

Remark 5.7. Note that any 1-dimensional pure simplicial complex $\varDelta$ with $e(k[\varDelta]) \leqslant 3$, $\operatorname{indeg} k[\varDelta]=2$ is isomorphic to one of the following complexes:



In particular, when $d=2$, there exists a $(d-1)$-dimensional simplicial complex $\varDelta$ in which $c \geqslant 2$ and $\operatorname{indeg} k[\varDelta]=d$; see (v) or (vi).

To prove the proposition, we need the following lemma.

Lemma 5.8. *Let* $A=k[\varDelta]$ *be a* d*-dimensional Stanley–Reisner ring which is not a hypersurface. Suppose that* $\varDelta$ *is pure and* $\operatorname{indeg} A=d \geqslant 3$. *Then there exists a vertex* $i \in [n]$ *such that* $e(k[\varDelta_{V\setminus\{i\}}]) \geqslant 2$.

Proof. Note that $n \geqslant d+2$ by the assumption. Put $e=e(A)$. Suppose that $e(k[\varDelta_{V\setminus\{i\}}])=1$ for all i. Then since there exist $(e-1)$ facets containing i for each $i \in [n]$, we have $(d+2)(e-1) \leqslant n(e-1) \leqslant de$; hence $e \leqslant \frac{d+2}{2}$.

On the other hand, by counting the number of subfacets (i.e., the maximal faces among all faces except facets) of $\varDelta$ we get

$$de \geqslant \binom{n}{d-1}$$

since $\operatorname{indeg} A=d$ and $\varDelta$ is pure. It follows from these inequalities that

$$\frac{d(d+2)}{2} \geqslant de \geqslant \binom{n}{d-1} \geqslant \binom{d+2}{d-1}=\binom{d+2}{3}.$$

Hence $d \leqslant 2$. This is a contradiction. $\square$

Proof of Proposition 5.6. We first show (1) $\Rightarrow$ (2). Let $A=k[\varDelta]$ be a d-dimensional Stanley–Reisner ring for which $\varDelta$ is pure, $\operatorname{indeg} A=d$, and $e=e(A) \leqslant 2d-1$. We may assume that $e(k[\varDelta_{V\setminus\{7\}}]) \geqslant 2$ by Lemma 5.8. Since $\varDelta$ is pure, any subfacet is contained in some d-subset of $\varDelta$. By counting the number of subfacets that contain n, we obtain that

$$\binom{n-1}{d-2} \leqslant \bigl(e-e\bigl(k[\varDelta_{V\setminus\{n\}}]\bigr)\bigr)(d-1) \leqslant (e-2)(d-1).$$

Now let us see that $n = d + 2$. Suppose that $n \geqslant d + 3$. Then we get

$$\binom{d+2}{4} \leqslant \binom{n-1}{d-2} \leqslant (e-2)(d-1) \leqslant (2d-3)(d-1)$$

by the assumption. This implies that $d \leqslant 4$.

Now consider the case of $d = 4$. Then $n = d + 3 = 7$, $e = 2d - 1 = 7$. Let $\{F_1, \ldots, F_7\}$ be the set of facets of Δ. Since $e(k[\Delta_{V\setminus\{7\}}]) = 2$, we may assume that $7 \in F$ if and only if $1 \leqslant i \leqslant 5$. Note that F_i contains only one subfacet that does not contain 7 for each $1 \leqslant i \leqslant 5$. On the other hand, one can find at most 4×2 subfacets as faces of F_6 or F_7. Therefore the total number of subfacets that do not contain 7 is at most 13. However the number of all subfacets which do not contain 7 is $\binom{7-1}{4-1} = 20$ since $\operatorname{indeg} A = 4$. This is a contradiction.

By a similar observation as in the case of $d = 4$, one can also prove that the case of $d = 3$ does not occur. Therefore we conclude that $n = d + 2$.

Under the assumption that $n = d + 2$, let us determine (d, e). Let Δ^* be the Alexander dual complex of Δ and put $R = k[\Delta^*]$. Then R is a 2-dimensional Stanley–Reisner ring with $\operatorname{indeg} R = 2$. Also, $\operatorname{rt}(R) = \operatorname{indeg} R = 2$ since Δ is pure. Thus by virtue of Turan's theorem (see Eq. (5.1)), we have

$$\binom{d+2}{2} - e = e(R) \leqslant f(d+2, 2) - 1 = \left\lfloor \frac{(d+2)^2}{4} \right\rfloor,$$

where $\lfloor a \rfloor$ denotes the maximum integer that does not exceed a. Namely, we have

$$2d - 1 \geqslant e \geqslant \left\lfloor \frac{(d+1)^2}{4} \right\rfloor.$$

It immediately follows from this that (d, e) is one of the pairs listed above.

Conversely, in order to prove (2) $\Rightarrow$ (1), it is enough to find (n, e')-graphs (i.e., 1-dimensional simplicial complexes Γ on $[n]$ with e' edges) which does not contain any triangle for each $(n, e') = (5, 6)$, $(5, 5)$, $(6, 9)$, $(6, 8)$, $(7, 12)$. One can easily construct those complexes using the following example. $\square$

Example 5.9. Let $d \geqslant 2$ be a given integer, and put $n = d + 2$. Let $T^2(n)$ be the so-called *Turan graph*, that is, it is the unique complete bipartite graph on $[n]$ whose partition sets differ in size by at most 1:

$$T^2(n) = \begin{cases} K_{m,m} & (n = 2m), \\ K_{m,m+1} & (n = 2m+1), \end{cases}$$

where $K_{r,s}$ denotes the complete bipartite graph which has two partition classes containing exactly r vertices, s vertices, respectively.

If we can regard $T^2(n)$ as a 1-dimensional simplicial complex Γ_n, then $e' := e(k[\Gamma_n]) = \lfloor \frac{n^2}{4} \rfloor (= f(n, 2) - 1)$ and $\operatorname{rt}(k[\Gamma_n]) = \operatorname{indeg} k[\Gamma_n] = 2$.

Let Δ_n be the Alexander dual complex of Γ_n. Then Δ_n is pure and $k[\Delta]$ is a d-dimensional Stanley–Reisner ring with $e(k[\Delta_n]) = \binom{n}{2} - e' = \lfloor \frac{(d+1)^2}{4} \rfloor$ and $\operatorname{indeg} k[\Delta_n] = d$.

Using Proposition 5.6, one can determine non-Cohen–Macaulay Buchsbaum complexes Δ with $\operatorname{indeg} k[\Delta] = \dim k[\Delta] = d$ and $e(k[\Delta]) \leqslant 2d - 1$.

Corollary 5.10. *Let $A = k[\Delta]$ be a d-dimensional Buchsbaum Stanley–Reisner ring which is not a hypersurface. Suppose that* $\operatorname{indeg} A = d \geqslant 3$ *and* $e(A) \leqslant 2d - 1$. *Then $d = 3$ and Δ is isomorphic to the simplicial complex spanned by* $\{1,2,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}$ *and* $\{2,4,5\}$.

Proof. Since A is Buchsbaum and $\operatorname{indeg} A = d$ we have

$$e = e(A) \geqslant \frac{c+d}{d}\binom{c+d-2}{d-2}$$

by [12, Proposition 2.1]. Also, $n = d + 2$ by Proposition 5.6 since Δ is pure. Thus

$$2d - 1 \geqslant e \geqslant \frac{d+2}{d}\binom{d}{d-2} = \frac{(d+2)(d-1)}{2}.$$

This implies that $d \leqslant 3$, and thus $d = 3$, $e = 5$ and $k[\Delta]$ is not Cohen–Macaulay.

Now let $\Gamma = \Delta^*$ be the Alexander dual complex of Δ. Then Γ is a 1-dimensional connected simplicial complex which contains a cycle which is not a triangle. Thus Γ is isomorphic to either one of the complexes Γ_1 which is spanned by

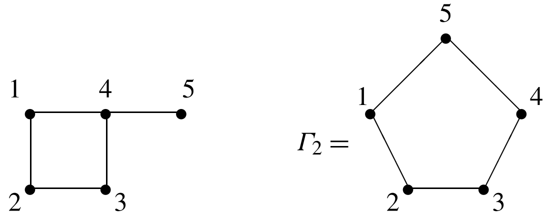
$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{4,5\}$$

or Γ_2 which is spanned by

$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\};$$

see below.

$\Gamma_1 =$ $\qquad$ $\Gamma_2 =$



Since $k[\Gamma_1^*]$ is not Buchsbaum, Δ is isomorphic to Γ_2^*, as required. $\square$

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