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Global Solution to the Compressible Isothermal Multipolar Fluid

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The global existence of weak solutions of the initial boundary value problem in bounded domains to the system of partial differential equations for viscous compressible isothermal bipolar and multipolar fluids is proved. Some other properties as cavitation, regularity up to the strong solution and uniqueness are discussed.

I. INTRODUCTION

The paper of M. Padula [10] concerns the same topics of the classical ideal gas and two-dimensional space domains. Padula's paper is pioneer, especially in its use of Orlicz spaces characterizing the finite entropy and theorems of the compensated compactness type. Nevertheless, the main existence theorem is false, cf. [11].

In this paper we follow the ideas presented in the work by M. Feistauer, J. Nečas, and V. Šverák [1] inspired by [10]. The main topic is the study of multipolar fluids. The physical background is studied in the paper by J. Nečas and M. Šilhavý [8], where higher stress tensors to constitution laws are introduced. There it is proved that it is possible (as we corroborate) to satisfy all thermodynamical laws. Higher stress tensors

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imply the use of higher derivatives of the velocity field. This point of view expresses some space nonlocality and also seems to better describe the turbulence phenomena. It is interesting that the proof of the existence of the central manifold to the incompressible fluid requires in fact higher stress tensors; see C. Foias, G. R. Sell, and R. Temam [2].

We prove a global existence of Hopf solutions to the bipolar fluid under general initial data and volume forces in the time cylinder $(0, T) \times \Omega$ with $T > 0$ and $\Omega \subset R^N$, $N = 2$ or 3 provided the temperature $\theta = \theta_0 = \text{const.} > 0$. In the same spirit, the general multipolar gas can be treated. We are also concerned with the problem of cavitation, regularity up to the strong solution, and uniqueness. In all these studies we look for the lowest multipolarity.

In the present case only one new stress tensor is needed such that the momentum equations are of the fourth order. So we handle a bipolar fluid. The corresponding stress–strain relations are supposed to be linear.

II. Formulation of the Problem

We suppose the classical state equation

$$p = R\rho\theta, \tag{2.1}$$

where p, ρ, θ are pressure, density, and temperature, respectively, and R is the universal gas constant. The isothermal process implies

$$p = \beta\rho, \qquad \beta = \text{const.} > 0. \tag{2.2}$$

We denote, as usual, the velocity vector by v; hence the continuity equation has its standard form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0. \tag{2.3}$$

In (2.3) as well as throughout the paper we use summation convention.

A standard symmetric stress tensor τ_{ij} is considered such that

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^d; \tag{2.4}$$

its power on an elementary surface dS with outer normal ν is

$$\tau_{ij} v_i \nu_j \, dS. \tag{2.5}$$

We consider a further 3-order stress tensor τ_{ijk}^d, whose power on an elementary surface dS with outer normal ν is

$$\tau_{ijk}^d \frac{\partial v_i}{\partial x_j} \nu_k \, dS. \tag{2.6}$$

The general linear form for τ_{ij}^d (with coefficients depending on the temperature θ only and therefore constant in our case), provided τ_{ij}^d is symmetric, reads

$$\tau_{ij}^d(v) = \gamma \frac{\partial v_l}{\partial x_l} \delta_{ij} + 2\mu e_{ij} - \gamma_1 \Delta \frac{\partial v_l}{\partial x_l} \delta_{ij} - 2\mu_1 \Delta e_{ij} + \gamma_2 \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial v_l}{\partial x_l} \right); \tag{2.7}$$

see [8] (Δ denotes as usual the Laplace operator). We suppose $\gamma \geq (-\frac{2}{3})\mu$, $\mu > 0$, $\gamma_1 > (-\frac{2}{3})\mu_1$, $\mu_1 > 0$, $\gamma_2 = 0$, $2e_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$.

For the stress tensor τ_{ijk}^d we require the symmetry in i, j; then the general form, according to [8], is

$$\begin{aligned}\tau_{ijk}^d(v) &= 2\mu_1 \frac{\partial}{\partial x_k} e_{ij} + \gamma_1 \delta_{ij} \frac{\partial}{\partial x_k} e_{ll} + \gamma_3 \delta_{ij} \Delta v_k \\ &+ \gamma_4 \delta_{ik} \Delta v_j + \gamma_4 \delta_{jk} \Delta v_i + \gamma_5 \delta_{ik} \frac{\partial}{\partial x_j} e_{ll} \\ &+ \gamma_5 \delta_{jk} \frac{\partial}{\partial x_i} e_{ll} + \gamma_6 \frac{\partial^2}{\partial x_i \partial x_j} v_k + \gamma_7 \frac{\partial^2}{\partial x_j \partial x_k} v_i \\ &+ \gamma_7 \frac{\partial^2}{\partial x_i \partial x_k} v_j.\end{aligned} \tag{2.8}$$

We restrict ourselves to the case $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 0$. The Clasius–Duhem inequality implies (see [8])

$$\tau_{ij}^d(v)\, e_{ij} + \tau_{ijk}^d(v) \frac{\partial^2 v_i}{\partial x_j \partial x_k} + \frac{\partial}{\partial x_k} (\tau_{ijk}^d(v)) \frac{\partial v_i}{\partial x_j} \geq 0; \tag{2.9}$$

this is satisfied in our example, which work with and we also show that the corresponding Korn's inequality will be satisfied.

Let $\Omega \subset R^N$, $N = 2$ or 3 be a bounded domain with a boundary smooth enough and let $I = (0, T)$, $Q_T = I \times \Omega$ be the time cylinder. Let F be the density of volume forces. The momentum equation combined with (2.3) give

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j + \beta \rho \delta_{ij} - \tau_{ij}^d(v)) = \rho F_i. \tag{2.10}$$

In addition to initial conditions for v and ρ we suppose $v = 0$ on $(0, T) \times \Omega$. The further boundary condition is in the general case

$$\tau_{ijk}^d(v)\, v_j v_k = 0 \qquad \text{on} \quad (0, T) \times \Omega. \tag{2.11}$$

First let us suppose that we consider a solution smooth enough and $\rho > 0$ in $\bar{Q}_T$. Then we get

2.12 THEOREM.

$$\int_{\Omega_t} \rho\, dx - \int_{\Omega_0} \rho\, dx = 0,$$

$$\frac{1}{2}\int_{\Omega_t} \rho\, |v|^2\, dx - \frac{1}{2}\int_{\Omega_0} \rho\, |v|^2\, dx + \int_{Q_t} (\gamma(e_{ll})^2 + 2\mu e_{ij} e_{ij}$$

$$+ \gamma_1 \frac{\partial}{\partial x_k}(e_{ll}) \frac{\partial}{\partial x_k}(e_{ll}) + 2\mu_1 \frac{\partial}{\partial x_k}(e_{ij}) \frac{\partial}{\partial x_k}(e_{ij}))\, d\tau\, dx$$

$$+ \beta \int_{\Omega_t} \rho \ln \rho\, dx - \beta \int_{\Omega_0} \rho \ln \rho\, dx$$

$$= \int_{Q_t} \rho F_i v_i\, dx\, d\tau, \qquad t \in (0, T\rangle, Q_t = (0, t)\, x\Omega.$$

For usual notation for Sobolev, Orlicz, and Bochner spaces see, for example, A. Kufner, O. John, S. Fučík [3]. The equalities in Theorem 2.12 show the function spaces that we work in:

$$v \in L^2(I, W^{2,2}(\Omega, R^N)), \tag{2.13}$$

$$\rho \in L_\infty(I, L_\psi(\Omega)), \qquad \rho \geqslant 0. \tag{2.14}$$

We denote by $\psi(t) = (1+t)\ln(1+t) - t$, $\Phi(t) = e^t - t - 1$ resp. $\psi_{1/2}(t)$, $\Phi_2(t) = e^{t^2} - 1$ the pairs of Young complementary functions and by $L_\psi(\Omega), L_\Phi(\Omega)$ resp. $L_{\psi_{1/2}}(\Omega), L_{\Phi_2}(\Omega)$ the Orlicz spaces of functions for which the Luxemburg norm

$$\|u\|_{L_f}(\Omega) = \inf_h \left\{ h > 0; \int_\Omega f\left(\frac{|u(x)|}{h}\right) dx \leqslant 1 \right\} < +\infty, \tag{2.15}$$

where f stands for ψ, Φ, $\psi_{1/2}$, Φ_2.

Let $\mathscr{B}(\Omega)$ be the set of all bounded measurable functions defined on Ω. Let us denote further by $C_f(\Omega)$ the closure of $\mathscr{B}(\Omega)$ in $L_f(\Omega)$. We have (see [3])

$$\begin{aligned} (C_\Phi(\Omega))^* &= L_\psi(\Omega), & (C_{\Phi_2}(\Omega))^* &= L_{\psi_{1/2}}(\Omega), \\ (C_\psi(\Omega))^* &= L_\Phi(\Omega), & (C_{\psi_{1/2}}(\Omega))^* &= L_{\Phi_2}(\Omega). \end{aligned} \tag{2.16}$$

Of course, $C_f(\Omega)$ are separable Banach spaces and $\mathscr{C}_0^\infty(\Omega)$ is dense in every $C_f(\Omega)$. We realize that $\psi, \psi_{1/2}$ satisfy the Δ_2 condition; hence

$$C_\psi(\Omega) = L_\psi(\Omega), C_{\psi_{1/2}}(\Omega) = L_{\psi_{1/2}}(\Omega). \tag{2.17}$$

The weak formulation to Eq. (2.10) is also useful. It reads

$$-\int_{Q_T} v_i \frac{\partial w_i}{\partial t}\, dx\, dt - \int_\Omega \rho_0 v_{oi} w_i(0)\, dx$$
$$+\int_0^T ((v, w))\, dt - \int_{Q_T} (\rho v_i v_j + \beta\rho\delta_{ij}) \frac{\partial w_i}{\partial x_j}\, dx\, dt$$
$$= \int_{Q_T} \rho F_i w_i\, dx\, dt \tag{2.18}$$

for every $w \in \mathscr{C}^\infty(\bar{Q}_T, R^N)$, $w(t) \in W_0^{1,2}(\Omega, R^N)$, $t \in I$, $w(T) = 0$; v_0 is the initial condition for v. In our situation, for $v, w \in V = W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$, $((v, w))$ is defined by (3.1) in the following section.

III. A Modified Galerkin Method

Let us denote $V = W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$ and for $v, w \in V$ let

$$((v, w)) = \int_\Omega \bigg(\gamma e_{ll}(v)\, e_{kk}(w) + 2\mu e_{ij}(v)\, e_{ij}(w)$$
$$+ \gamma_1 \frac{\partial}{\partial x_k} e_{ll}(v) \frac{\partial}{\partial x_k} e_{ss}(w)$$
$$+ 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(v) \frac{\partial}{\partial x_k} e_{ij}(w) \bigg)\, dx. \tag{3.1}$$

From the coerciveness of deformations (see, e.g., J. Nečas, I. Hlaváček [7]) and from the very strong ellipticity of the bilinear form $((v, w))$ it follows for $v \in V$

$$((v, v)) \geq k_2 \int_\Omega (|v|^2 + |\nabla v|^2 + |\nabla_2 v|^2)\, dx, \qquad k_2 > 0 \tag{3.2}$$

(as usual ∇ resp. ∇_2 denotes first resp. second gradient). Let $\{w^k\}_{k=1}^{+\infty}$ be a complete orthonormal system of eigenfunctions in V, solutions to the following eigenproblem in V

$$((v, w)) = \lambda(v, w) \qquad \text{for every} \quad v \in V, \qquad w \in V, \tag{3.3}$$

where $(v, w) = \int_\Omega v_i w_i \, dx$. We have

$$((v, w^k)) = \lambda_k (v, w^k); \qquad k = 1, 2, ...; \qquad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \qquad ((w^k, w^l)) = \delta_{kl}. \tag{3.4}$$

From the regularity of solutions to the linear elliptic problem (6.1) (see Appendix) we get

$$w^k \in C^\infty(\bar{\Omega}, R^N). \tag{3.5}$$

Let us put for $w \in L^2(\Omega, R^N)$

$$P_m w = \sum_{k=1}^{m} \lambda_k (w^k, w)\, w^k. \tag{3.6}$$

If $L_m^2 = \operatorname{span}\{w^1, ..., w^m\}$ in $L^2(\Omega, R^N)$, $V_m = \operatorname{span}\{w^1, ..., w^m\}$ in V, then P_m is an orthogonal projector of L^2 onto L_m^2 and of V onto V_m.

By (6.1) one defines the operator $\mathscr{A}$: $((v, w)) = (\mathscr{A}v, w)$ for every $w \in V$. Its definition domain is denoted by $D(\mathscr{A})$ and of course $W_0^{4,2}(\Omega, R^N) \subset D(\mathscr{A})$. It is the consequence of Theorem 6.1 (see Appendix) that

$$\|v\|_{W^{4,2}(\Omega, R^N)} \leqslant k_3 \|\mathscr{A}v\|_{L^2(\Omega, R^N)} \qquad (k_3 > 0) \tag{3.7}$$

for every $v \in D(\mathscr{A})$; hence

$$\begin{aligned} \|P_m v\|_{W^{4,2}(\Omega, R^N)} &\leqslant k_3 \|\mathscr{A} P_m v\|_{L^2(\Omega, R^N)} \\ &\leqslant k_3 \|P_m \mathscr{A} v\|_{L^2(\Omega, R^N)} \leqslant k_4 \|v\|_{W^{4,2}(\Omega, R^N)} \\ &\qquad (k_4 > 0) \text{ for every } v \in W_0^{4,2}(\Omega, R^N). \end{aligned}$$

Thus, due to the interpolation theorem it holds for every $v \in W_0^{3,2}(\Omega, R^N)$

$$\|P_m v\|_{W^{3,2}(\Omega, R^N)} \leqslant k_5 \|v\|_{W^{3,2}(\Omega, R^N)} \qquad (k_5 > 0). \tag{3.8}$$

Let $c_i \in \mathscr{C}^1(\bar{I})$ and let us put $v^m(t, x) = \sum_{i=1}^m c_i(t)\, w^i(x)$. Let us look for $\rho_m \in \mathscr{C}^1(\bar{Q}_T)$ such that

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x_i} (\rho_m v_i^m) = 0. \tag{3.9}$$

We suppose throughout the paper

$$\rho_m(0, x) = \rho_0(x) \in \mathscr{C}^1(\bar{\Omega}), \qquad \rho_0(x) > 0 \text{ in } \bar{\Omega}. \tag{3.10}$$

Let

$$\dot{x}^m(t) = v^m(t, x^m(t)), \qquad x^m(0) = y, \qquad y \in \bar{\Omega}. \tag{3.11}$$

For every $t \in \bar{I}$, $y \to x^m(t)$ is a diffeomorphism of $\bar{\Omega}$ onto $\bar{\Omega}$. For $\sigma_m = \ln \rho_m$ we have

$$\frac{d}{dt}(\sigma_m(t, x^m(t))) = -\frac{\partial}{\partial x_i} v_i^m(t, x^m). \tag{3.12}$$

Hence

$$\rho_m(t, x) = \rho_0(y) \exp\left(-\int_0^t \frac{\partial}{\partial x_i} v_i^m(\tau, x^m(\tau))\, d\tau\right), \tag{3.13}$$

where $x = x^m(t)$, $y = x^m(0)$. Let us look now for $\bar{v}^m$ such that for every $t \in I$

$$\int_\Omega \left(\rho_m \frac{\partial \bar{v}_i^m}{\partial t} + \rho_m v_j^m \frac{\partial \bar{v}_i^m}{\partial x_j} + \beta \frac{\partial \rho_m}{\partial x_i} - \rho_m F_i\right) w_i^k\, dx$$
$$= -((\bar{v}^m, w^k)), \qquad k = 1, 2, ..., m. \tag{3.14}$$

Let the initial conditions be given by

$$\int_\Omega \bar{c}_j(0)\, w_k^j w_k^i\, dx = \int_\Omega \bar{v}_k(0, x)\, w_k^i(x)\, dx. \tag{3.15}$$

We suppose in the sequel

$$v_0(x) = \bar{v}(0, x) \in L^2(\Omega, R^N), \qquad F_i \in L^2(I, L^\infty(\Omega)). \tag{3.16}$$

Because $\det(\int_\Omega \rho_m w_i^k w_i^r\, dx) \neq 0$, we can solve (3.14), (3.15) uniquely in I. We get the standard estimate

$$\frac{1}{2}\int_{\Omega_t} \rho_m |\bar{v}^m|^2\, dx - \frac{1}{2}\int_\Omega \rho_0 |\bar{v}_0^m|^2\, dx - \beta \int_{Q_t} \rho_m \frac{\partial}{\partial x_i} \bar{v}_i^m\, dx\, d\tau$$
$$-\int_{Q_t} \rho F_i \bar{v}_i^m\, d\tau\, dx + \int_0^t ((\bar{v}^m, \bar{v}^m))\, d\tau = 0, \qquad \bar{v}_0^m = \bar{v}^m(0). \tag{3.17}$$

If we start with $c_i(t)$ in the ball

$$\max_{[0,\alpha]} |c_i(t) - c_i(0)| \leqslant 1; \qquad i = 1, 2, ..., m, \tag{3.18}$$

we get

$$\max_{[0,\alpha]} |\bar{c}_i(t) - c_i(0)| \leqslant 1, \qquad i = 1, ..., m; \tag{3.19}$$

$$\max_{[0,\alpha]} |\dot{\bar{c}}_i| \leqslant k_6(\alpha), \qquad k_6 > 0 \tag{3.20}$$

provided α is small enough. So applying the Schauder fixed point theorem, we get on $[0, \alpha]$, $\bar{c}_i = c_i$. But for this solution we get

$$\int_{\Omega_t} \rho_m \, dx - \int_{\Omega} \rho_0 \, dx = 0,$$

$$\frac{1}{2}\int_{\Omega_t} \rho_m |\bar{v}^m|^2 \, dx - \frac{1}{2}\int_{\Omega} \rho_0 |\bar{v}_0^m|^2 \, dx + \int_0^t ((\bar{v}^m, \bar{v}^m)) \, d\tau$$

$$+ \beta \int_{\Omega_t} (1+\rho_m) \ln(1+\rho_m) \, dx - \beta \int_{\Omega} (1+\rho_0) \ln(1+\rho_0) \, dx$$

$$= \int_{Q_t} \rho_m F_i \bar{v}_i^m \, dx \, d\tau + \beta \int_{Q_t} \ln(1+\rho_m) \frac{\partial \bar{v}_j^m}{\partial x_j} \, dx \, d\tau. \tag{3.21}$$

We have

$$\int_{Q_t} \rho_m F_i \bar{v}_i^m \, dx \, d\tau \leqslant k_7 \, \|F_i\|_{L^2(I_t, L^\infty(\Omega))} \, \|\bar{v}^m\|_{L^2(I_t, V)} \, \|\rho^m\|_{L^\infty(I_t, L^1(\Omega))},$$

$$\int_{Q_t} \ln(1+\rho_m) \frac{\partial \bar{v}_i^m}{\partial x_j} \, dx \, d\tau \leqslant k_7 \left(\int_{Q_t} (1+\rho_m) \ln(1+\rho_m) \right)^{1/2} \|\bar{v}^m\|_{L^2(I_t, V)}, k_7 > 0,$$

$$\| \ \|_V = ((\ ,\ ))^{1/2}, \qquad I_t = (0, t), \qquad t \in (0, \alpha].$$

Using the Young inequality at the r.h.s. of (3.21) and applying the Gronwall lemma, we get

3.22 LEMMA.

$$\int_{\Omega_t} \left(\frac{1}{2} \rho_m |\bar{v}^m|^2 + (1+\rho_m) \ln(1+\rho_m) \right) dx$$

$$- \int_{\Omega} \left(\frac{1}{2} \rho_0 |v_0|^2 + (1+\rho_0) \ln(1+\rho_0) \right) dx$$

$$+ \int_0^t ((\bar{v}^m, \bar{v}^m)) \, d\tau \leqslant k_8, \qquad k_8 > 0.$$

This implies that we can continue with α to T. For more detailed proof see for example [9].

IV. THE LIMIT PROCESS

4.1 LEMMA. *Let B be Banach space, $B_i (i = 0, 1)$ separable reflexive Banach spaces. Let $B_0 \subset\subset B \subset B_1$ ($\subset\subset$ denotes compact imbedding), $1 < p_i < \infty$. Let $W = \{v, v \in L^{p_0}(I, B_0), \partial v / \partial t \in L^{p_1}(I, B_1)\}$. Then $W \subset\subset L^{p_0}(I, B)$.*

For proof we refer to Lions [5].

MAIN THEOREM. *Let* (2.2), (2.7), (3.10), (3.16) *be satisfied. Then there exists*

$$\rho \in L^{\infty}(I, L_{\psi}(\Omega)), \qquad \rho > 0 \text{ a.e. in } Q_T, \tag{4.2}$$

$$v \in L^2(I, W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)), \tag{4.3}$$

$$\frac{\partial \rho}{\partial t} \in L^2(I, W^{-3,2}(\Omega)), \tag{4.4}$$

$$\frac{\partial}{\partial t}(\rho v) \in L^2(I, W^{-3,2}(\Omega, R^N)) \tag{4.5}$$

satisfying (2.3), (2.10) *in the sense of distributions and such that* (2.18) *holds. In addition*

$$\|\rho\|_{L^{\infty}(I, L_1(\Omega))} \leqslant \int_{\Omega_0} \rho_0 \, dx, \tag{4.6}$$

$$\frac{1}{2} \|\rho \, |v|^2\|_{L^{\infty}(I, L^1(\Omega))} + \|v\|^2_{L^2(I, W^{2,2}(\Omega, R^N))}$$
$$+ \beta \operatorname{supess}_I \int_{\Omega_t} (1+\rho) \ln(1+\rho) \, dx$$
$$\leqslant k_9 \left(1 + \sum_{i=1}^{N} \|F_i\|_{L^2(I, L_{\infty}(\Omega))}\right)^2, \qquad k_9 > 0. \tag{4.7}$$

Proof. Let $0 \leqslant k \leqslant 2$ and let $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$ be the usual Sobolev spaces with fractional derivatives; see, e.g., [3]. Let $V^k = \bar{V}$, where the closure is taken in $W^{k,2}(\Omega, R^N)$; of course the traces are zero only for $k > \frac{1}{2}$. Because

$$W_0^2(\Omega) \subset\subset W_0^{k_1,2}(\Omega) \subset\subset W_0^{k_2,2}(\Omega) \subset C_{\Phi}(\Omega) \subset C_{\Phi_2}(\Omega),$$
$$\text{for} \quad N/2 < k_2 < k_1 < 2, \tag{4.8}$$

we have

$$L_{\psi_{1/2}}(\Omega) \subset L_{\psi}(\Omega) \subset W^{-k_2,2}(\Omega) \subset\subset W^{-k_1,2}(\Omega) \subset\subset W^{-2,2}(\Omega). \tag{4.9}$$

Of course $W^{k,2}(\Omega) \subset C_{\psi}(\Omega)$; hence $L_{\psi}(\Omega) \subset (W^{k,2}(\Omega))^*$ $((N/2) < k)$. It follows from the interpolation theorem that

$$\sup_{\substack{v \in V^k \\ \|v\|_{W^{k,2}(\Omega, R^N)} \leqslant 1}} \|P_m v\|_{W^{k,2}(\Omega, R^N)} \leqslant k_{10}, \qquad k_{10} > 0 \qquad (0 \leqslant k \leqslant 2) \tag{4.10}$$

and

$$\sup_{\substack{v \in (V^k)^* \\ \|v\|_{(V^k)^*} \leqslant 1}} \|P_m^* v\|_{(V^k)^*} \leqslant k_{11}, \qquad k_{11} > 0 \qquad (0 \leqslant k \leqslant 2) \tag{4.11}$$

(P_m^* is the dual operator to P_m).

4.12. LEMMA. *Let $0 \leqslant k \leqslant 2$. Then for every $\varepsilon > 0$, l_0 exists such that for $l \geqslant l_0$ $\|u - P_l u\|_{W^{k,2}(\Omega, R^N)} < \varepsilon$ provided $\|u\|_{W^{2,2}(\Omega, R^N)} \leqslant 1$.*

Proof. Let us suppose the contrary. Then there exist P_{l_j}, $l_j \to \infty$ and $\|u_{l_j}\|_{W^{2,2}(\Omega, R^N)} \leqslant 1$ such that $\|u_{l_j} - P_{l_j} u_{l_j}\|_{W^{k,2}(\Omega, R^N)} \geqslant \varepsilon_0 \geqslant \varepsilon$. Because $W^{2,2}(\Omega, R^N) \subset\subset W^{k,2}(\Omega, R^N)$, we can suppose $u_{l_j} \to u$ strongly in $W^{k,2}(\Omega, R^N)$; hence $P_{l_j} u_{l_j} \to u$ strongly in $W^{k,2}(\Omega, R^N)$, which is contradictory.

Let ρ_m, v^m be an approximative solution from Section III. There exist subsequences (denoted $\{\rho_m\}_{m=1}^{+\infty}$, $\{v^m\}_{m=1}^{+\infty}$ again) such that $\rho_m \to \rho$ $*$-weakly in $L^2(I, L_\psi(\Omega)) = L^2(I, (C_\Phi(\Omega))^*)$;

$$v^m \to v, \frac{\partial}{\partial x_i} v^m \to \frac{\partial}{\partial x_i} v, \frac{\partial^2 v^m}{\partial x_i \, \partial x_j} \to \frac{\partial^2 v}{\partial x_j \, \partial x_i} \qquad (i, j = 1, ..., N)$$

weakly in $L^2(Q_T)$, $v^m \to v$ $*$-weakly in $L^2(I, L_\Phi(\Omega))$, $L^2(I, L_{\Phi_2}(\Omega))$.

Due to Lemma 3.22 and Eq. (4.9)

$$\|\rho_m\|_{L^\infty(I, W^{-k,2}(\Omega))} \leqslant k_{12}, \qquad k_{12} > 0, \qquad (N/2) < k \leqslant 2. \tag{4.13}$$

For $N = 2, 3$

$$\|v^m\|_{\mathscr{C}^0(\bar{\Omega}, R^N)} \leqslant k_{13} \|v^m\|_{W^{2,2}(\Omega, R^N)}, \qquad k_{13} > 0; \tag{4.14}$$

hence

$$\|\rho_m v^m\|_{L^2(I, L_\psi(\Omega, R^N))} \leqslant k_{14}, \qquad k_{14} > 0. \tag{4.15}$$

It follows from (3.9)

$$\left\| \frac{\partial \rho_m}{\partial t} \right\|_{L^2(I, W^{-3,2}(\Omega, R^N))} \leqslant k_{15}, \qquad k_{15} > 0. \tag{4.16}$$

Due to Lemma 4.1 $\rho_m \to \rho$ strongly in $L^2(I, W^{-2,2}(\Omega))$; hence due to (4.15) $\rho_m v^m \to \rho v$ $*$-weakly in $L^2(I, L_\psi(\Omega))$.

By (4.9), (4.15) we get

$$\|\rho_m v^m\|_{L^2(I, W^{-k,2}(\Omega, R^N))} \leqslant k_{16}, \qquad k_{16} > 0 \qquad ((N/2) < k \leqslant 2); \tag{4.17}$$

hence by (4.11)

$$\|P_m^*(\rho_m v^m)\|_{L^2(I, W^{-k,2}(\Omega, R^N))} \leqslant k_{17}, \qquad k_{17} > 0. \tag{4.18}$$

According to Lemma 3.22 $\rho_m |v^m|^2$ is bounded in $L^\infty(I, L^1(\Omega))$; therefore

$$\|\rho_m |v^m|^2\|_{L^2(I, W^{-2,2}(\Omega, R^N))} \leqslant k_{18}, \qquad k_{18} > 0. \tag{4.19}$$

By (3.14), (3.8), Lemma 3.22, (4.13), and (4.19),

$$\left\| \frac{\partial}{\partial t} (P_m^* \rho_m v^m) \right\|_{L^2(I, W^{-3,2}(\Omega, R^N))} \leqslant k_{19}, \qquad k_{19} > 0 \tag{4.20}$$

holds. So by Lemma 4.1 $P_m^*(\rho_m v^m) \to a$ strongly in $L^2(I, W^{-2,2}(\Omega, R^N))$ (eventually for a subsequence).

Let $w \in L^2(I, W_0^{2,2}(\Omega, R^N))$. Because of Lemma 4.12 for m large enough

$$\int_0^t \|P_m w - w\|^2_{W^{k,2}(\Omega, R^N)} \, d\tau \leqslant \varepsilon^2 \int_0^t \|w\|^2_{W^{2,2}(\Omega, R^N)} \, d\tau; \tag{4.21}$$

hence for $\int_0^t \|w\|^2_{W^{2,2}(\Omega, R^N)} \, d\tau \leqslant 1$, it follows uniformly with respect to w

$$\lim_{m \to \infty} \int_0^T \int_\Omega (P_m^*(\rho_m v_i^m) - \rho_m v_i^m) \, w_i \, dx \, d\tau = 0. \tag{4.22}$$

Therefore $\rho_m v^m$ is a Cauchy sequence in $L^2(I, W^{-2,2}(\Omega, R^N))$ and $\rho_m v^m \to a$ strongly in $L^2(I, W^{-2,2}(\Omega, R^N))$. But $\rho_m v^m \to \rho v$ in $\mathscr{D}'(Q_T)$ in the sense of distributions; hence $a = \rho v$. Therefore, due to (4.19)

$$\rho_m v_i^m v_j^m \to \rho v_i v_j \text{ weakly in } L^2(I, W^{-2,2}(\Omega, R^N)). \tag{4.23}$$

Due to estimate $\int_\Omega |\rho_m v^m| \, u \, dx \leqslant (\int_\Omega \rho_m |v^m|^2 \, dx)^{1/2} (\int_\Omega \rho_m u^2 \, dx)^{1/2}$ which holds for every $u \in L_{\Phi_2}(\Omega)$ and due to (4.14), we get

$$\|\rho_m v_i^m v_j^m\|_{L^2(I, L_{\psi_{1/2}}(\Omega))} \leqslant k_{20}, \qquad k_{20} > 0. \tag{4.24}$$

Therefore $\rho_m v_i^m v_j^m \to \rho v_i v_j$ $*$-weakly in $L^2(I, L_{\psi_{1/2}}(\Omega))$.

It follows from (3.14) that for every $\varphi \in \mathscr{C}^\infty(\bar{Q}_T, R^N)$ satisfying $\varphi(t) \in V_m$ for every $t \in [0, T]$ and $\varphi(T) = 0$,

$$\int_{Q_T} \rho v_i \frac{\partial \varphi_i}{\partial t} \, dx \, dt + \int_{Q_T} \rho v_i v_j \frac{\partial \varphi_j}{\partial x_j} \, dx \, dt + \beta \int_{Q_T} \rho \frac{\partial \varphi_j}{\partial x_j}$$

$$= \int_0^T ((v, \varphi)) \, dt - \int_{Q_T} \rho F_i \varphi_i \, dx \, dt - \int_\Omega \rho_0 v_{0i} \varphi_i \, dx$$

holds. Due to the density arguments (2.18) holds and (2.10) is satisfied in the sense of distributions. The continuity equation is obviously satisfied in the sense of distributions.

V. Higher Polarity and Qualitative Properties of the Solutions

In the case of a bipolar gas, we found for the limit density $\rho \geqslant 0$ only; i.e., in general, there can be a set of positive measures in Q_T such that $\rho = 0$ here. This means that there is a strong cavitation. For this reason and also because of uniqueness of the solution, it is worth considering k-polar gas ($k = 3, 4, ...$). We refer the reader to [8].

In our situation we consider on $V = W^{k,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$ a symmetric V-coercive bilinear form

$$((v, w)) = \int_\Omega \sum_{l=1}^{k} A^l_{iji_1 \cdots i_l j_1 \cdots j_l} \frac{\partial^l v_i}{\partial x_{i_1} \cdots \partial x_{i_l}} \frac{\partial^l w_j}{\partial x_{j_1} \cdots \partial x_{j_l}} \, dx, \tag{5.1}$$

where $A^l_{iji_1 \cdots i_l j_1 \cdots j_l}$ are constants. For $l = 1$ there are only combinations of $e_{ij}(v)$, $e_{ij}(w)$; we suppose that $A^l_{iji_1 \cdots i_l j_1 \cdots j_l}$ are symmetric under the permutation of indexes $(i_1 \cdots i_l)$, $(j_1 \cdots j_l)$. Of course, we suppose for $v \in V$

$$((v, v)) \geqslant \alpha_1 \, \|v\|^2_{W^{k,2}(\Omega, R^N)}, \qquad \alpha_1 > 0. \tag{5.2}$$

This follows for example from the conditions

$$A^1_{iji_1 j_1} \frac{\partial v_i}{\partial x_{i_1}} \frac{\partial v_j}{\partial x_{j_1}} \geqslant \alpha_2 e_{ij}(v) \, e_{ij}(v), \qquad \alpha_2 > 0, \tag{5.3}$$

$$\begin{aligned} &\sum_{l=2}^{k} A^l_{iji_1 \cdots i_l j_1 \cdots j_l} J^i_{i_1 \cdots i_l} J^i_{j_1 \cdots j_l} \\ &\qquad \geqslant \alpha_2 \sum_{l=2}^{k} J^i_{i_1 \cdots i_l} J^i_{i_1 \cdots i_l} \qquad \text{for every real vector } (J^i_{i_1 \cdots i_l}) \\ &\qquad i, i_1, ..., i_l = 1, ..., N. \end{aligned} \tag{5.4}$$

In our situation

$$(\tau^d_{ij})_{,j} = \sum_{l=1}^{k} (-1)^{l+1} A^l_{iji_1 \cdots i_l j_1 \cdots j_l} \frac{\partial^{2l} v_j}{\partial x_{i_1} \cdots \partial x_{i_l} \partial x_{j_1} \cdots \partial x_{j_l}}.$$

We consider $v=0$ on $(0, T)\, x\, \partial\Omega$ and unstable boundary conditions given by

$$\sum_{l=1}^{k}\sum_{s=0}^{l-1}\int_{\partial\Omega}(-1)^{s+1}A^{l}_{ij i_1\cdots i_l j_1\cdots j_l}\frac{\partial^{l+s}v_j}{\partial x_{i_1}\cdots\partial x_{i_l}\partial x_{j_1}\cdots\partial x_{j_s}}\nu_{j_{s+1}}$$
$$\times\frac{\partial^{l-s-1}w_j}{\partial x_{j_{s+2}}\cdots\partial x_{j_l}}\,dS=0$$

for every $w\in\mathscr{C}^{\infty}(\bar\Omega, R^N)\cap W^{1,2}_0(\Omega, R^N)$.

5.5 THEOREM. *Let* $k=3$ *and* (2.2), (3.10), (3.16), (5.1), *and* (5.2) *be satisfied. Then there exist* ρ, v

$$\rho\in L^{\infty}(Q_T),\qquad \rho\geqslant\gamma>0\qquad in\ Q_T, \tag{5.6}$$

$$v\in L^2(I, W^{3,2}(\Omega, R^N)\cap W^{1,2}_0(\Omega, R^N)), \tag{5.7}$$

$$\frac{\partial\rho}{\partial t}\in L^2(I, W^{-1,2}(\Omega)), \tag{5.8}$$

$$\frac{\partial}{\partial t}(\rho v)\in L^2(I, W^{-3,2}(\Omega, R^N)) \tag{5.9}$$

such that (2.3) *is satisfied in the sense of distributions and also in the sense of duality in* $L^2(I, W^{-1,2}(\Omega))$ *and* (2.10) *in the sense of distributions and also in the weak sense; i.e.,* (2.18) *is fulfilled. Besides* (4.6), (4.7) *hold and*

$$\min_{\Omega}\rho_0(x)\exp(-k_{20}t^{1/2})\leqslant\rho(t,x)$$
$$\leqslant\max_{\Omega}\rho_0(x)\exp(+k_{20}t^{1/2}),\qquad k_{20}>0,\qquad t\in I. \tag{5.10}$$

5.11 THEOREM. *Let* $k\geqslant 4$ *and* (2.2), (3.10), (3.17), (5.1), *and* (5.2) *be satisfied. Let* $\rho_0\in\mathscr{C}^{k-3}(\bar\Omega)$. *Then there exist* ρ, v

$$\frac{\partial^s\rho}{\partial x_1^{s^1}\cdots\partial x_N^{s^N}}\in L^{\infty}(Q_T)\qquad for\quad 0\leqslant s\leqslant k-3,\qquad s=s^1+\cdots s^N, \tag{5.12}$$

$$\frac{\partial^s}{\partial x_1^{s^1}\cdots\partial x_N^{s^N}}\frac{\partial\rho}{\partial t}\in L^2(I, L^{\infty}(\Omega))\qquad for\quad 0\leqslant s\leqslant k-4, \tag{5.13}$$

$$v\in L^2(I, W^{k,2}(\Omega, R^N)\cap W^{1,2}_0(\Omega, R^N)), \tag{5.14}$$

$$\frac{\partial}{\partial t}(\rho v)\in L^2(I, W^{-k,2}(\Omega, R^N)) \tag{5.15}$$

such that (2.3), (2.10) *are satisfied in the sense of distributions,* (2.3) *also almost everywhere, and* (2.10) *in the weak sense; i.e.,* (2.18) *holds.*

Proof of 5.5 *and* 5.11. We take w^k ($k = 1, 2, ...$) from (3.4). We define P_m by (3.6). We get ρ_m, v^m as before. Of course (3.21) and Lemma 3.22 hold. Due to

$$\|v^m(t)\|_{\mathscr{C}^{k-2}(\bar{\Omega}, R^N)} \leqslant k_{21} \|v^m(t)\|_{W^{k,2}(\Omega, R^N)}, \tag{5.16}$$

it follows from (3.13) that

$$(\min_{\bar{\Omega}} \rho_0(x)) \exp\left(-\int_0^t k_{21} \|v^m(\tau)\|_{W^{k,2}(\Omega, R^N)} d\tau\right) \leqslant \rho^m(t, x)$$
$$\leqslant (\max_{\bar{\Omega}} \rho_0(x)) \exp\left(+\int_0^t k_{21} \|v^m(\tau)\|_{W^{k,2}(\Omega, R^N)} d\tau\right). \tag{5.17}$$

Using (3.11), Lemma 3.22, (5.16), and the Gronwall lemma, we obtain

$$\max_{\bar{Q}_T} \left|\frac{\partial^s x_i}{\partial y_1^{s^1} \cdots \partial y_N^{s^N}} (t, y)\right| \leqslant k_{22}, \tag{5.18}$$

where $0 \leqslant s \leqslant k-3$, $k_{22} > 0$ and $(s^1, ..., s^N)$ is any multiindex such that $s = s^1 + \cdots s^N$. Due to

$$\rho_m(t, x(t, y)) \det\left(\frac{\partial x_i}{\partial y_j}\right) = \rho_0(y) \tag{5.19}$$

and also

$$\max_{\bar{Q}_T} \left|\frac{\partial^s y_i}{\partial x_1^{s^1} \cdots \partial x_N^{s^N}} (t, x)\right| \leqslant k_{23}, k_{23} > 0, \qquad s \leqslant k-3. \tag{5.20}$$

It follows from (5.20) and (3.13) that

$$\|\rho_m\|_{\mathscr{C}^0(I, \mathscr{C}^s(\bar{\Omega}))} \leqslant k_{24}, k_{24} > 0 \tag{5.21}$$

provided $s \leqslant k-3$. Now from (3.3) one gets

$$\left\|\frac{\partial \rho_m}{\partial t}\right\|_{L^2(I, \mathscr{C}^{s-1}(\bar{\Omega}))} \leqslant k_{25}, \qquad k_{25} > 0 \qquad (s \leqslant k-3, k \geqslant 4) \tag{5.22}$$

or

$$\left\|\frac{\partial \rho_m}{\partial t}\right\|_{L^2(I, W^{-1,2}(\Omega, R^N))} \leqslant k_{25} \qquad \text{for} \quad k = 3.$$

In any case, we can suppose $\rho_{m_l} \to \rho$ weakly in $L^2(Q_T)$ and $\rho_{m_l} \to \rho$ strongly in $L^2(I, W^{-1,2}(\Omega))$. We have

$$\|\rho_m v^m\|_{L^2(Q_T, R^N)} \leqslant k_{26}, \qquad k_{26} > 0; \tag{5.23}$$

therefore $\rho_m v^m \to \rho v$ weakly in $L^2(Q_T, R^N)$ at least for a chosen subsequence. It follows from (3.14) with $\bar{v}^m = v^m$

$$\left\| \frac{d}{dt} P_m^*(\rho_m v^m) \right\|_{L^2(I, V^*)} \leqslant k_{27}, \qquad k_{27} > 0, \tag{5.24}$$

hence for a subsequence if necessary $P_m^*(\rho_m v^m) \to a$ strongly in $L^2(I, W^{-1,2}(\Omega, R^N))$. But we have for every $\varepsilon > 0$ $\|P_m w - w\|_{L^2(\Omega, R^N)} \leqslant \varepsilon$ provided $w \in W_0^{1,2}(\Omega, R^N)$, $\|w\|_{W^{1,2}(\Omega, R^N)} \leqslant 1$, and m is large enough (see Lemma 4.12); so $\rho_m v^m \to a = \rho v$ strongly in $L^2(I, W^{-1,2}(\Omega, R^N))$. We have the estimate

$$\|\rho_m |v^m|^2\|_{L^2(Q_T)} \leqslant k_{28}, \qquad k_{28} > 0; \tag{5.25}$$

therefore $\rho_m v_i^m v_j^m \to \rho v_i v_j$ weakly in $L^2(Q_T)$ at least for a chosen subsequence. The other is obvious. The proof of Theorems 5.5 and 5.11 is finished.

5.26 Theorem. *Let $k \geqslant 4$. Let the conditions of Theorem 5.5 be satisfied and further let $v_0 \in V$. Then for the solution it holds*

$$\int_0^T \int_\Omega \left| \frac{\partial v}{\partial t} \right|^2 dx\, dt + \sup_I \|v(t)\|^2_{W^{k,2}(\Omega, R^N)} \leqslant k_{29}, \qquad k_{29} > 0, \tag{5.27}$$

$$v \in L^2(I, W^{2k,2}(\Omega, R^N)). \tag{5.28}$$

Equation (2.10) is fulfilled a.e. in Q_T.

Proof. For $\bar{v}^m = v^m$, we use in (3.14) $\partial v^m / \partial t$ for the test function. Because of Lemma 3.22 and (5.17), $\int_{\Omega_t} |v^m|^2\, dx \leqslant k_{30}$ ($k_{30} > 0$); hence using (5.12), Lemma 3.22, (5.16), and the Hölder and Young inequality, we get

$$\int_0^T \int_\Omega \left| \frac{\partial v^m}{\partial t} \right|^2 dx\, dt + \sup_I \|v^m(t)\|^2_{W^{k,2}(\Omega, R^N)} \leqslant k_{31}, \qquad (k_{31} > 0). \tag{5.29}$$

Now, (5.28) follows from the regularity to the elliptic systems $((v, w)) = \int_\Omega g_i w_i\, dx$, where $g \in L^2(\Omega, R^N)$ (see Eq. (6.1)).

5.30 Theorem. *Let the conditions of Theorems 5.11 and 5.26 be satisfied. Then in the set of solutions satisfying Theorems 5.11 and 5.23 there exists at most one solution to the problem (2.3), (2.10).*

Proof. Let (ρ, v), $(\bar{\rho}, \bar{v})$ be two solutions to the problem. Then for $(\xi, w) = (\rho - \bar{\rho}, v - \bar{v})$ it holds

$$\frac{\partial \xi}{\partial t} = -\xi \frac{\partial v_j}{\partial x_j} - \bar{\rho} \frac{\partial w_j}{\partial x_j} - \frac{\partial \xi}{\partial x_j} v_j - \frac{\partial \bar{\rho}}{\partial x_j} w_j \tag{5.31}$$

$$\int_{\Omega_t} \bar{\rho} \frac{\partial w_i}{\partial t} \varphi_i \, dx + ((w, \varphi))$$
$$= -\int_{\Omega_t} \left(\xi \frac{\partial v_i}{\partial t} + \xi v_j \frac{\partial v_i}{\partial x_j} + \bar{\rho} w_j \frac{\partial v_i}{\partial x_j} + \bar{\rho} \bar{v}_j \frac{\partial w_i}{\partial x_j} \right) \varphi_i \, dx$$
$$+ \int_{\Omega_t} \xi \frac{\partial \varphi_j}{\delta x_j} dx + \int_{\Omega_t} \xi F_i \varphi_i \, dx \tag{5.32}$$

for every $\varphi \in V$ a.e. in I.

We multiply (5.31) by ξ and integrate over Ω. After some computation we get the estimate

$$\frac{d}{dt} (\|\xi(t)\|_{L^2(\Omega)})^2 \leqslant a_1(t) \, \|\xi(t)\|^2_{L^2(\Omega)} + \frac{1}{2} ((w, w)), \tag{5.33}$$

where $a_1 \leqslant k_{32}\{((v, v)) + (\|\bar{\rho}\|_{L^\infty(\Omega)} + \|\bar{\rho}\|_{W^{1,2}(\Omega)})^2\}$, $k_{32} > 0$; hence $a_1 \in L^1(I)$.

We put $\varphi = w$ in (5.32). We get

$$\frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \bar{\rho} \, |w|^2 \, dx + ((w(t), w(t)))$$
$$= -\int_{\Omega_t} \left(\frac{\partial v_i}{\partial t} w_i \xi + \xi v_j w_i \frac{\partial v_i}{\partial x_j} + \bar{\rho} w_i w_j \frac{\partial v_i}{\partial x_j} + \bar{\rho} \bar{v}_j \frac{\partial w_i}{\partial x_j} w_i \right) dx$$
$$+ \int_{\Omega_t} \left(\xi \frac{\partial w_j}{\partial x_j} + \frac{1}{2} \frac{\partial \bar{\rho}}{\partial t} |w|^2 \right) dx \qquad \text{a.e. in } I. \tag{5.34}$$

Using Holder and Young inequalities, r.h.s. of (5.34) can be estimated by

$$a_2(t)(\|w(t)\|^2_{L^2(\Omega, R^N)} + \|\xi(t)\|^2_{L^2(\Omega)}) + \tfrac{1}{2}((w(t), w(t))),$$

where

$$a_2 \leqslant k_{33} \left\{ \left(1 + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega, R^N)} + \|v\|_{W^{k,2}(\Omega, R^N)} + \|\bar{v}\|_{W^{k,2}(\Omega, R^N)} \right)^2 \right.$$
$$\left. \times \left(\|\bar{\rho}\|_{L^\infty(\Omega)} + \left\| \frac{\partial \bar{\rho}}{\partial t} \right\|_{L^\infty(\Omega)} + \|\bar{\rho}\|_{W^{1,2}(\Omega)} + 1 \right)^2 \right\}, \qquad k_{33} > 0.$$

From (5.27), (5.12), (2.3) we have $\partial\bar{\rho}/\partial t \in L^\infty(Q_T)$; hence $a_2 \in L^1(I)$. Due to (5.6)

$$k_{34} \|w\|^2_{L^2(\Omega, R^N)} \leqslant \int_\Omega \bar{\rho}\, |w|^2\, dx \leqslant k_{35} \|w\|^2_{L^2(\Omega, R^N)} \qquad (0 < k_{34} < k_{35}).$$

Hence from (5.34)

$$\frac{d}{dt} \|w(t)\|^2_{L^2(\Omega, R^N)} \leqslant a_2(t)(\|w(t)\|^2_{L^2(\Omega, R^N)} + \|\xi(t)\|^2_{L^2(\Omega)}). \tag{5.35}$$

We add (5.33), (5.35), and apply the Gronwall lemma. Because of $\xi(0)=0$, $w(0)=0$, $\xi=0$, and $w=0$ a.e. in Q_T. The proof is finished.

VI. Appendix

For the construction of the basis for the Galerkin method we have used the following regularity property to the weak solution of the elliptic problem

$$u \in V, \qquad f \in L^2(\Omega, R^N), ((v, u)) = \int_\Omega v_i f_i\, dx \qquad \text{for every } v \in V. \tag{6.1}$$

6.2 Theorem. *Let $u \in V$ be a solution to* (6.1). *Then $u \in W^{2k,2}(\Omega, R^N)$ and*

$$\|u\|_{W^{2k,2}(\Omega, R^N)} \leqslant c\, \|f\|_{L^2(\Omega, R^N)}, \qquad c > 0. \tag{6.3}$$

The detailed proof can be constructed according to the procedure from J. Nečas [6]. We restrict ourselves to the case $\Omega = R^N_+ = \{x \in R^N, x_N > 0\}$ provided the solution $u \in W^{2k,2}(R^N_+, R^N)$. For exact proof one must use differences instead of derivatives. For general domain $\Omega \subset R^N$ one uses in the usual way the partition of unity to replace the original problem by the problem in R^N_+.

Idea of the proof. We proceed by induction. Let us suppose that for $2k-1 \geqslant l \geqslant k$ we have

$$\|u\|_{W^{l,2}(R^N_+, R^N)} \leqslant c\, \|f\|_{L^2(R^N_+, R^N)}. \tag{6.4}$$

First, let $v \in V \cap W^{k+1,2}(R^N_+, R^N)$. Then $\partial v/\partial x_j \in V$, $j = 1, 2, ..., N-1$. So from (6.1) it follows that

$$-\left(\left(v, \frac{\partial u}{\partial x_j}\right)\right) = \int_{R^N} \frac{\partial v_i}{\partial x_j} f_i\, dx. \tag{6.5}$$

Especially for $v = \partial u/\partial x_j$

$$\left(\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j}\right)\right) = \int_{R_+^N} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} f_i \, dx; \tag{6.6}$$

hence

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{W^{k,2}(R_+^N, R^N)} \leqslant c \, \|f\|_{L^2(R_+^N, R^N)}. \tag{6.7}$$

Let $\varphi \in \mathscr{D}(R_+^N, R^N)$. We have

$$-\left(\left(\varphi, \frac{\partial u}{\partial x_N}\right)\right) = \int_{R_+^N} \frac{\partial \varphi_i}{\partial x_N} f_i \, dx. \tag{6.8}$$

Put $(0, ..., 0, \psi, 0, ..., 0)$, where ψ is on the j-th position in the vector; of course $\psi \in \mathscr{D}(R_+^N)$. So by the theorem about negative norms [6], we obtain

$$\left\| A^k_{ij \underbrace{N \cdots N}_{k\text{-times}} \underbrace{N \cdots N}_{k\text{-times}}} \frac{\partial^{k+1} u_i}{\partial x_N^{k+1}} \right\|_{L^2(R_+^N)} \leqslant c \qquad (j = 1, 2, ..., N). \tag{6.9}$$

Therefore in virtue of (5.4),

$$\det(A^k_{ij \underbrace{N \cdots N}_{k\text{-times}} \underbrace{N \cdots N}_{k\text{-times}}}) \neq 0$$

and we have

$$\|u\|_{W^{k+1,2}(R_+^N, R^N)} \leqslant c \, \|f\|_{L^2(R_+^N, R^N)}. \tag{6.10}$$

Let $\partial^{l+1-k}/\partial x_{i_1} \cdots \partial x_{i_{l+1-k}}$ be some derivative for $i_1, ..., i_{l+1-k} \leqslant N-1$. Then we have as before

$$\left(\left(\frac{\partial^{l+1-k} u}{\partial x_{i_1} \cdots \partial x_{i_{l+1-k}}}, \frac{\partial^{l+1-k} u}{\partial x_{i_1} \cdots \partial x_{i_{l+1-k}}}\right)\right)$$
$$= (-1)^{l+1} \int_{R_+^N} \frac{\partial^{2(l+1-k)} u_i}{\partial x_{i_1}^2 \cdots \partial x^2_{\partial x_{i_{l+1-k}}}} f_i \, dx. \tag{6.11}$$

Hence

$$\left\| \frac{\partial^{l+1-k} u}{\partial x_{i_1} \cdots \partial x_{i_{l+1-k}}} \right\|_{W^{k,2}(R_+^N, R^N)} \leqslant c \, \|f\|_{L^2(R_+^N, R^N)}.$$

Let

$$\frac{\partial^{l+1-k}}{\partial x_N\, \partial x_{i_1} \cdots \partial x_{i_{l-k}}} \quad \text{with} \quad i_s \leqslant N-1 \ (s=1, ..., l-k).$$

Then we get from (6.1) for $\varphi \in \mathscr{D}(R_+^N, R^N)$

$$\left(\left(\frac{\partial^{l+1-k} u}{\partial x_N\, \partial x_{i_1} \cdots \partial x_{i_{l-k}}}, \varphi\right)\right) = (-1)^{l+1} \int_{R_+^N} \frac{\partial^{l+1-k} \varphi_i}{\partial x_N\, \partial x_{i_1} \cdots \partial x_{i_{l-k}}} f_i \, dx.$$

So we get as before

$$\Bigg\| A^k_{ij\, \underbrace{N \cdots N}_{k\text{-times}}\, \underbrace{N \cdots N}_{k\text{-times}}} \frac{\partial^{l+1} u_i}{\partial x_N^{k+1}\, \partial x_{i_1} \cdots \partial x_{i_{l-k}}} \Bigg\|_{L^2(R_+^N,\, R^N)} \leqslant c\, \|f\|_{L^2(R_+^N,\, R^N)};$$

hence

$$\left\| \frac{\partial^{l+1} u}{\partial x_{i_1} \cdots \partial x_{i_{l-k}}\, \partial x_N^{k+1}} \right\|_{L^2(R_+^N,\, R^N)} \leqslant c\, \|f\|_{L^2(R_+^N,\, R^N)}. \tag{6.12}$$

The next step is to consider $\partial^{l+1}/\partial x_{i_1} \cdots \partial x_{i_{l-k-1}}\, \partial x_N^{k+2}$. By the same reasoning we get

$$\left\| \frac{\partial^{l+1} u}{\partial x_{i_1} \cdots \partial x_{i_{l-k-1}}\, \partial x_N^{k+2}} \right\|_{L^2(R_+^N,\, R^N)} \leqslant c\, \|f\|_{L^2(R_+^N,\, R^N)}.$$

REFERENCES

1. M. FEISTAUER, J. NEČAS, AND V. ŠVERÁK, On the weak compactness of the equations of compressible flow, *Appl. Anal.* **34** (1989).
2. C. FOIAS, G. R. SELL, AND R. TEMAM, Inertial manifolds for nonlinear evolution equations, preprint series No. 234, Inst. Math. Appl., Univ. Minnesota, 1986.
3. A. KUFNER, O. JOHN, AND S. FUČÍK, "Functional Spaces," Academia, Prague, 1977.
4. J. L. LIONS AND E. MAGENES, "Problèmes aux limites non-homogènes et application," Dunod, Paris, 1968.
5. J. L. LIONS, "Quelques méthodes de résolution des problèmes aux limites non-linéaires," Dunod, Paris, 1969.
6. J. NEČAS, "Les méthodes directes en la théorie des équations elliptiques," Academia, Prague, 1967.
7. J. NEČAS AND I. HLAVÁČEK, "Introduction to the Theory of Elastic and Elastoplastic Bodies," SNTL, Prague, 1983. [in Czech]
8. J. NEČAS AND M. ŠILHAVÝ, Viscous multipolar fluids, to appear.
9. J. NEUSTUPA AND A. NOVOTNÝ, Global weak solvability to the regularized viscous heat conductive flow, to appear.
10. M. PADULA, Existence of global solutions for 2-dimensional viscous compressible flow, *J. Funct. Anal.* **69** (1986).
11. M. PADULA, Erratum, *J. Funct. Anal.* **76** (1988), and **77** (1988).