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Notes on the Wave Equation on Asymptotically Euclidean Manifolds

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wave equation on an asymptotically Euclidean manifold, when the initial data have compact support. By going over to an appropriate conformal metric, it is shown that (just as for the ordinary wave equation) such a solution has a forward ("future") and a backward ("past") radiation field. The same method is then used to define "end points" of bicharacteristics that begin and end above the boundary (the analogue of the sphere at infinity in the Euclidean case), and to derive a relation between the wave front sets of the two radiation fields. The extension of these results to solutions with finite energy is also briefly discussed.

Let Z be an *asymptotically Euclidean* manifold, that is to say a compact C^∞ manifold of dimension $n>1$, with non-empty boundary ∂Z, such that $Z^0 = Z \backslash \partial Z$ is Riemannian and has the following property: there is a positive real number a and a collar neighborhood Z_a of ∂Z given by $\{0 \leqslant x < a\}$, where x is a defining function for ∂Z, and the Riemann metric on $Z_a^0 = Z_a \backslash \partial Z$ is of the form

$$g = x^{-4}\, dx^2 + x^{-2} h(x, y, dy). \tag{1}$$

Here $(x, y) \in [0, a) \times \mathbf{R}^{n-1}$ are local coordinates on (coordinate patches of) Z_a, and h is a symmetric positive definite covariant 2-tensor, C^∞ on Z_a, so that, in particular, it defines a Riemann metric on ∂Z. Such a metric is a *scattering metric* as defined in [M]. Moreover, it is shown in [JS] that any manifold equipped with a scattering metric has this property.

If one puts

$$r = 1/x, \qquad 0 < x < a, \tag{2}$$

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then (1) becomes

$$g = dr^2 + r^2 h(1/r, y, dy), \qquad 1/a < r < \infty. \tag{1'}$$

The Euclidean metric $|dz|^2$ on $\mathbf{R}^n$ is of this form on $\mathbf{R}^n \backslash \{0\}$ when it is expressed in polar coordinates $r = |z|$, $\omega = z/|z| \in S^{n-1}$, with $h = |d\omega|^2$. (The y coordinates are then local coordinates on the unit sphere, and Z is $\mathbf{R}^n$ compactified by the addition of the sphere at infinity.)

It is proposed to study the wave equation on $Z^0 \times \mathbf{R}$,

$$\Box u = (\partial_t^2 - \Delta)\, u = 0, \tag{3}$$

where Δ is the Laplacian on Z^0. Occasionally, the inhomogeneous equation, with a forcing term—a second member which is a function or a distribution on $Z^0 \times \mathbf{R}$—will be considered instead.

In the main, these notes follow [F1]. But for the proof of the principal result, the existence of radiation fields of solutions of the wave equation with initial data that have compact support (Proposition 2), the energy estimates used in [F1] are replaced by a simpler version of the Penrose conformal method. The method is also utilised to discuss the asymptotics of singularities. But no attempt has been made to relate this to [MZ], which deals with the elliptic wave equation, other than to point out an analogy (probably an equivalence) between properties of bicharacteristics of the wave equation near $\partial T^*(Z \times \mathbf{R})$ and the "geodesics at infinity" of [MZ].

We first note an important property of the d'Alembertian on $Z \times \mathbf{R}$:

PROPOSITION 1. *For any $z' \in Z^0$, the wave equation has a unique fundamental solution $E(z, t, z')$ on $Z^0 \times \mathbf{R}$ such that*

$$\Box_{(z,t)} E = \delta_Z(z, z')\, \delta(t), \qquad \operatorname{supp} E \subset \{t \geqslant d(z, z')\}, \tag{4}$$

where δ_Z is the Dirac kernel on (Z^0, g) and $d(z, z')$ the distance determined by the metric. Furthermore, for $(z', t') \in Z^0 \times \mathbf{R}$, $E(z, t - t', z')$ and $E(z, t' - t, z')$ are, respectively, the unique (globally defined) forward and backward fundamental solutions of the wave equation with "pole" (z', t').

Note. It is well known that E lifts to a distribution on $Z^0 \times \mathbf{R} \times Z^0$ and is symmetric in z and z'.

Proof. E is of course in any case defined locally and has the properties asserted in the Proposition and the Note. So it only remains to prove that it has a unique extension to all of $Z^0 \times \mathbf{R} \times Z^0$. Let

$$\Delta^+(z', t') = \{t \geqslant t' + d(z, z')\}, \qquad \Delta^-(z', t') = \{t \leqslant t' - d(z, z')\} \tag{5}$$

denote the forward and backward dependence domains of a point (z', t'), respectively. Now $Z^0 \times \mathbf{R}$ is *globally hyperbolic* in the sense of Leray [L] if

$$\Delta^+(z^{(1)}, t_1) \cap \Delta^-(z^{(2)}, t_2)$$

is compact or empty for any pair of points in $Z^0 \times \mathbf{R}$, that is to say if

$$\{(z, t): t_1 \leqslant t \leqslant t_2, d(z^{(1)}, z) + d(z, z^{(2)}) \leqslant t_2 - t_1\}$$

is compact or empty. As $d(z, z') \to \infty$ when $z \to \partial Z$, it is evident that this is the case. The proposition therefore follows from a theorem of Leray.

Both r and x will be used, interchangeably; it is sometimes easier to compute (and to visualize) in the coordinates (r, y). In $Z_a \times \mathbf{R}$, the manifolds $t \pm r = \text{constant}$ are characteristics of $\Box$. So, if one thinks of Z^0 as "space" and t as "time," the hypersurfaces $\{(r, y): t \pm r = \text{constant}, \ r > 1/a\}$ are "wave fronts" in $Z_a^0 \times \mathbf{R}$. This observation is at the back of

DEFINITION 1. A solution of (3) will be called an *expanding wave* if there are real numbers t_0, c and $b \in (0, a]$ such that

$$\Box u = 0 \qquad \text{on} \quad \{(x, y, t): 0 < x < b, t > t_0\} \tag{6a}$$

and

$$u = 0 \qquad \text{if} \quad 0 < x < b, \quad t_0 < t < r + c. \tag{6b}$$

When u is a distribution, one can restrict it to the manifolds $\{x = \text{constant}\}$ in $\{(x, y, t): 0 < x < b, t > t_0\}$, because they are not characteristics. This gives a function

$$u_x \in C^\infty((0, a); \mathscr{D}'(\partial Z \times \mathbf{R})).$$

For a C^∞ function, u_x is just u regarded as a function $(0, b) \to C^\infty(\partial Z \times \mathbf{R})$. With this in mind, we state

PROPOSITION 2. *Let u be an expanding wave. Put*

$$m = (n-1)/2$$

and let $s \in \mathbf{R}$. Then: (i) *if $u \in C^\infty(Z_a^0 \times \mathbf{R})$, there is a $w \in C^\infty(\partial Z \times \mathbf{R})$ such that*

$$x^{-m} u(x, y, s + 1/x) \to w(y, x) \tag{8}$$

as $x \to 0$, *in the topology of* $C^\infty(\partial Z \times \mathbf{R})$; (ii) *if* $u \in \mathscr{D}'(Z_a^0 \times \mathbf{R})$, *then*

$$x^{-m} u_x(y, s+1/x) \to w(y, s) \in \mathscr{D}'(\partial Z \times \mathbf{R}) \tag{8'}$$

in $\mathscr{D}'(\partial Z \times \mathbf{R})$. *Furthermore, there is a complete asymptotic expansion*

$$u(x, y, t) \sim \sum x^{m+k} w_k(y, t-r), \qquad w_0 = w \tag{9}$$

where $t - r = s$, *in the appropriate sense in each case; w will be called the radiation field of u.*

Note. In terms of (r, y, t), (8) asserts that

$$r^m u(1/r, y, r+s) \to w(y, s) \tag{8''}$$

as $r \to \infty$, and there is a similar version of (8′). So the limit is taken along bicharacteristic curves that generate the characteristics $t - r =$ constant. It can be thought of as the signal recorded by a distant observer arising from a wave generated at a finite time in $(Z \backslash Z_a) \times \mathbf{R}$.

We need a technical lemma.

LEMMA 1. *Put*

$$s = t - r = t - 1/x, \qquad v(x, y, s) = x^{-m} u(x, y, t). \tag{10}$$

Then

$$x^{-m} \Box u = x^2 P v = x^2(\Box' v + m(m - 1 - x\,\partial_x \log(|h|^{1/2})\, v)), \tag{11}$$

where $|h| = \det(h_{ij})$ *and* $\Box'$ *is the d'Alembertian of the metric*

$$G' = x^2\, ds^2 - 2\, dx\, ds - h(x, y, dy). \tag{12}$$

Proof. The operator $\Box$ is the d'Alembertian of the Lorentz metric

$$G = dt^2 - x^{-4}\, dx^2 - x^{-2} h(x, y, dy). \tag{13}$$

In terms of (x, y, s) this is

$$G = ds^2 - (2/x^2)\, dx\, ds - (1/x^2)\, h = G'/x^2.$$

It follows from the well known relation between the d'Alembertians of conformal metrics that

$$x^{-m} \Box u = x^{m+2}(x^{-m} \Box' v - v \Box'(x^{-m}))$$

and a simple calculation gives (11).

Proof of Proposition 2. We first note that $\square$ and P have the same characteristics and bicharacteristics. The conformal metric G' is C^∞ up to the boundary $\partial Z \times \mathbf{R}$. (This observation corresponds to the Penrose compactification of a space-time in General Relativity; $\partial Z \times \mathbf{R}$ is then the *horizon.*) Now one can extend h as a Riemann metric $\bar{h}$ beyond $\{x=0\}$, say to

$$\bar{Z}_a = \{-c < x < a\} \times \partial Z, \qquad c > 0,$$

with coefficients in $C^\infty(\bar{Z}_a)$. This gives an extension of the metric G',

$$x^2\, ds^2 - 2dx\, ds - \bar{h}(x, y, dy), \qquad (x, y, s) \in \bar{Z}_a \times R \tag{14}$$

and of P to a strictly hyperbolic differential operator $\bar{P}$ defined on $\bar{Z}_A \times \mathbf{R}$, with C^∞ coefficients.

Let u be an expanding wave, and v as in (10). By choosing the origin of t, and hence of s, suitably, one can arrange for v to vanish when $s<0$, so that

$$Pv = 0 \quad \text{on } Z_a \times \mathbf{R}, \qquad v = 0 \quad \text{if } (x, y) \in Z_a, \quad s < 0. \tag{15}$$

With b_1 and b_2 such that $0 < b_1 < b_2 < a$, let $\chi \in C^\infty(\mathbf{R})$ be such that

$$\chi(x) = 1, \quad 0 < x < b_1, \qquad \chi(x) = 0, \quad x > b_2,$$

and put

$$V = \chi(x)\, v, \qquad q = P(\chi v). \tag{16}$$

Then

$$V = v, \quad 0 < x < b_1, \qquad \operatorname{supp} q \subset \{(x, y, s)\colon b_1 \leqslant x \leqslant b_2, s \geqslant 0\}. \tag{17}$$

By (14), the time-like or null tangent vectors at a point $(x, y, s) \in \bar{Z}_a \times \mathbf{R}$ are determined by

$$x^2\, ds\, d(s + 2/x) - \bar{h}(x, y, dy) \geqslant 0.$$

For $dy = 0$, the null directions are tangent to the bicharacteristic curves which generate the characteristics $s = \text{constant}$ and $s + 2/x = \text{constant}$ through the point. (They correspond to the relevant characteristics $t \pm r =$ constant, of course.) Now the past dependence domain $\bar{\Delta}(x, y, s)$ is the union of past-oriented time-like or null curves from (x, y, s), so it follows that

$$\bar{\Delta}(x, y, s) \cap (\bar{Z}_a \times \mathbf{R}) \subset \{(x', y', s') \colon x' \geqslant \max(-c, 2x/(2+(s-s')x)), s' \leqslant s\} \tag{18}$$

so that

$$\bar{\Delta}(x, y, s) \cap (\bar{Z}_a \times [0, \infty)) \subset \bar{Z}_a \times [0, \infty) \quad \text{if} \quad x > -2c/(2+2cs).$$

In view of (17), this implies that $\bar{\Delta}(x, y, s) \cap (\bar{Z}_a \times [0, \infty)) \cap \operatorname{supp} q$ is either compact or empty.

Hence

$$\bar{P}\bar{V} = q \qquad \text{on} \quad \bar{Z}_a \times (0, \infty) \tag{19}$$

has a unique solution $\bar{V}$ in $\{x > -2c/(2+2cs)\}$ which, by (16), must be equal to V when $x > 0$. Hence $\bar{v} = (1-\chi)\, v + \bar{V}$ is an extension of v to this domain, and we extend it to $(-c, a) \times \mathbf{R}^-$ by setting it equal to 0 for $s < 0$. In the C^∞ case, $\bar{v} \in C^\infty(\bar{Z}_a \times \mathbf{R})$ so that the proposition follows, with

$$w(y, s) = \bar{v}(0, y, s), \quad w_k(y, s) = (\partial_x)^k\, v(0, y, s)/k!, \quad k = 1, 2, \dots. \tag{19}$$

When u is a distribution, one has to show that $\bar{v}$ can be restricted to the characteristic $\{x = 0\}$. To see that this is the case, we first observe that $WF(\bar{v}) \subset \operatorname{char} P$ since $\bar{P}\bar{v} = 0$. The bicharacteristic flow of $\bar{P}$ is generated by the Hamilton field of the principal symbol of $\bar{P}$. In the obvious notation, this is

$$-2\sigma\xi - x^2\xi^2 - \Sigma(\bar{h})^{ij}\, \eta_i \eta_j.$$

A simple computation shows that the only bicharacteristics that meet the conormal bundle of $\{x = 0\}$ are its bicharacteristic generators

$$\{(x, y, s, \xi, \eta, \sigma) \colon x = 0,\ y = \text{constant},\ \xi = \text{constant},\ \sigma = 0,\ \eta = 0\}.$$

But as $\bar{v} = 0$ also for $s < 0$, such bicharacteristics cannot be in its wave front set. So $\bar{v}$ can be restricted to manifolds of constant x in $\{-a' < x < a\}$ for some $a' \in (-c, a)$, yielding a C^∞ function $(-a' < x < a) \to \mathscr{D}'(\partial Z \times \mathbf{R})$, and we are done.

For fixed $z' \in Z$, the distribution $E(\cdot, z')$ is an expanding wave, and so has a radiation field

$$E_\infty(y, s, z') = \lim_{x \to 0} (x^{-m} E(x, y, s + 1/x, z')). \tag{20}$$

It is a distribution-valued function of z' that lifts to a member of $\mathscr{D}'(\partial Z \times \mathbf{R} \times Z)$. Let $q \in C_c^\infty(Z^0 \times \mathbf{R})$, so that the solution of the forcing problem

$$\Box u = q, \qquad u = 0 \qquad \text{for} \quad t \ll 0 \tag{21}$$

is an expanding wave. Hence u has a radiation field w. Now

$$u(s, t) = \int E(z, t - t', z')\, q(z', t')\, |d\mu(z')\, dt'|, \tag{22}$$

where the integral sign indicates the action of $E(z, t - t', z')$ as a Schwartz kernel. (Distributions are scalars; test functions are multiplied by the appropriate invariant density; $d\mu(z')$ is the invariant measure on Z, so that $|d\mu_z(z)\, dt|$ is the invariant density on $Z \times \mathbf{R}$, equipped with the metric (13). So (20) formally gives

$$w(y, s) = \int E_\infty(y, s - t', z')\, q(z', t')\, |d\mu(z')\, dt'|, \tag{23}$$

and it is not difficult to deduce from Proposition 2 that this is valid. In particular, for

$$q = \varphi_0(z)\, \delta'(t) + \varphi_1(z)\, \delta(t), \qquad \varphi_0, \varphi_1 \in C_c^\infty(Z^0)$$

(22) gives the solution of the initial value problem

$$\Box u = 0, \qquad u|_{t=0} = \varphi_0, \qquad \partial_t u|_{t=0} = \varphi_1 \tag{24}$$

in $\{t > 0\}$ as

$$u(z, t) = \int E(z, t, z')\, \varphi_1(z')\, |d\mu(z')| + \int \partial_t E(z, t, z')\, \varphi_0(z')\, |d\mu(z')|. \tag{24'}$$

So the (forward) radiation field of the solution of (24) is

$$w(y, s) = \int E_\infty(y, s, z')\, \varphi_1(z')\, |d\mu(z')| + \int \partial_s E_\infty(y, s, z')\, \varphi_0(z')\, |d\mu(z')|. \tag{25}$$

The solution of the initial value problem (24) is of course defined for all $t \in \mathbf{R}$. In terms of the *wave group* $U(t)$, which maps initial values of u and $\partial_t u$ to the corresponding pairs at time t, it is

$$U(t)\,\varphi = (u(\cdot, t), \partial_t u(\cdot, t)), \qquad t \in \mathbf{R}, \quad \varphi = (\varphi_0, \varphi_1). \tag{26}$$

The wave equation is invariant under $(z, t) \to (z, -t)$ ("reversal of the time orientation"). Let us note in passing that, for the wave group, one can use the following observation. The solution of the initial value problem (24) is an even function of t if $\varphi_1 = \partial_t u|_{t=0} = 0$, odd in t if $\varphi_0 = u|_{t=0} = 0$. Hence

$$U(-t)\,\varphi = U(t)\,\varphi^*, \qquad \varphi^* = (\varphi_0, -\varphi_1), \qquad t \in \mathbf{R}. \tag{27}$$

So one can derive results on "backward" (or "past") asymptotics by reflection in t. To preserve the time orientation at ∂Z ("at infinity"), let

$$s' = t + 1/x = t + r. \tag{28}$$

If $u \in C^\infty(Z^0 \times \mathbf{R})$ or $u \in \mathscr{D}'(Z^0 \times \mathbf{R})$ and there are real numbers t_0, c and $b \in (0, a)$ such that

$$\Box u = 0 \quad \text{on } Z_a^0 \times (-\infty, t_0), \qquad u = 0 \quad \text{if} \quad 0 < x < b, \quad t + 1/x > c, \tag{29}$$

u will be called a *contracting wave*, by analogy with Definition 1. The time-reversed version of Proposition 2 asserts the existence of a *backward radiation field* w^-, that is to say that

$$v^-(x, y, s') = x^{-m} u(x, y, s' - 1/x) \to w^-(y, s') \qquad \text{as} \quad x \to 0, \tag{30}$$

in $C^\infty(\partial Z \times \mathbf{R})$ or in $\mathscr{D}'(\partial Z \times \mathbf{R})$, as the case may be, and of a complete asymptotic expansion at $x = 0$.

Consider the initial value (24), with initial data $\varphi = (\varphi_0, \varphi_1) \in \mathscr{D}'(Z^0) \times \mathscr{D}'(Z^0)$. If the initial data have compact support, there is a $b \in (0, a)$ such that

$$\operatorname{supp} \varphi \subset (Z \backslash Z_a) \cup \{(x, y) \in Z_a : x \geqslant b\}$$

so that, by the compact dependence domain property,

$$\operatorname{supp} u(\cdot, t) \subset (Z \backslash Z_a) \cup \{(x, y) \in Z_a : x \geqslant b/(b + |t|)\}, \qquad t \in \mathbf{R}. \tag{31}$$

Hence $u \mid \{t > 0\}$ is an expanding wave and $u \mid \{t < 0\}$ is a contracting wave; let us call u an *hourglass wave*. It will have both a forward and a backward radiation field. For consistency of notation, replace (8) by

$$v^+(x, y, s) = x^{-m} u(x, y, s + 1/x) \to w^+(y, s) \qquad \text{as} \quad x \to 0. \tag{32}$$

One can establish a relation between the wave front sets u, w^+ and w^-, because $WF(u)$ is a union of bicharacteristics.

We need some preliminary remarks. As (32) could be proved directly by making appropriate changes in the proof of Proposition 2, one can say that the existence of the radiation fields is proved by the "conformal method" (which goes back to Penrose's construction of "horizons" in general relatively). It can be summarized as follows. Denote $Z_a^0 \times \mathbf{R}$ the sub-manifold of $Z \times \mathbf{R}$, equipped with the Lorentz metric $G = dt^2 - g$ and the canonical coordinate system (x, y, t), so that g is given by (1). Let M be the same manifold equipped with the conformal metric $G' = x^2 G$, and let M^+, M^- be the coordinate neighbourhoods obtained by going over to the two sets of local coordinates (x, y, s_+), (x, y, s_-) in $Z_a^0 \times \mathbf{R}$, where

$$s_- = t + 1/x, \qquad s_+ = t - 1/x. \tag{33}$$

Then

$$G' = x^2\, ds_-^2 + 2dx\, ds\_h(x, y, dy) = x^2\, ds_+^2 - 2dx\, ds_+ - h(x, y, dy).$$

So both M^+ and M^- can be partially compactified by adding the "horizons"

$$\partial^+ M = \partial M^+ \backslash \{x = a\}, \qquad \partial^- M = \partial M^- \backslash \{x = a\}. \tag{34}$$

Whereas M^+ and M^- cover the same underlying domain $Z_a^0 \times \mathbf{R}$, the horizons are disjoint. In terms of $r = 1/x$, one can visualize M^+ as the limit of the characteristics $r + t =$ constant as $t \to \infty$ for bounded $t - r$, and M^- as the limit of the characteristics $t - r =$ constant as $t \to -\infty$ for bounded $t + r$. Now M^+, M^- and G' can be extended to full neighbourhoods $\bar{M}^+$, $\bar{M}^-$ of $\partial^+ M$ and $\partial^- M$, respectively. One then finds that the differential operators P^+, P^- which are such that

$$x^{-m} \Box u = x^2 P^+ v^+ = x^2 P^- v^-$$

can be extended to differential operators $\bar{P}^+$, $\bar{P}^-$ which have C^∞ coefficients, and are strictly hyperbolic, in each extended domain. (P^+ is the old P, as in Lemma 1.) The crucial observation in the proofs of Proposition 2 and its backward counterpart is that v^+ and v^- can be extended to solutions $\bar{v}^+$ and $\bar{v}^-$ of $\bar{P}^+(\bar{v}^+) = 0$, $\bar{P}^-(\bar{v}^-) = 0$, respectively, and that w^+, w^- are then obtained by restriction to the relevant horizon.

The conformal method can be applied to bicharacteristics. As the symbol of the d'Alembertian does not depend on t, the dual fibre coordinate τ, which is necessarily nonzero on char, is constant on bicharacteristics: $\tau = \tau_0 \neq 0$. So the equation of a bicharacteristic γ can be put in the form

$$t \to \gamma(x, y, \xi/\tau_0, \eta/\tau_0).$$

DEFINITION 2. A bicharacteristic γ will be called a *scattering bicharacteristic* if $x \to 0$ on γ both as $t \to -\infty$ and as $t \to \infty$.

(It may be of interest to correlate this with the terminology in [W]). Two examples, where the bicharacteristics above $Z_a \times R$ can be obtained explicitly, are summarized in the Appendix. They strongly suggest that bicharacteristics close to $\partial T^*(Z \times R)$ are scattering bicharacteristics. But in one of these examples (h independent of y) it is easy to construct a.e. manifolds with an ample supply of geodesics that remain above a compact subset of Z_a^0.

We note first that the principal symbols of $\square$, P^+, and P^- are related by

$$\Sigma(\square) = x^2 \Sigma(P^+) = x^2 \Sigma(P^-), \tag{35}$$

where

$$\begin{aligned} \Sigma(P^+) &= -x^2\xi_+^2 - 2\xi_+\sigma_+ - h^c(x, y, \eta), \\ \Sigma(P^-) &= -x^2\xi_-^2 + 2\xi_-\sigma_- h^c(x, y, \eta) \end{aligned} \tag{35'}$$

with

$$h^c(x, y, \eta) = \sum h^{ij}(x, y)\, \eta_i \eta_j.$$

(Σ is used to denote symbols, to avoid confusion with the fibre coordinates σ, σ^+ and σ_-.) Each symbol vanishes on the respective characteristic variety. It follows that the restrictions of the Hamilton field of $\Sigma(\square)$ to char $\square$ is x^2 times the restrictions of the Hamilton fields of $\Sigma(P^\pm)$ to char $P^\pm$, respectively: they are related to that of $\Sigma(\square)$ by the relevant coordinate transformations lifted to the cotangent bundle, and division by x^2. So the bicharacteristics of P^+ and P^- are obtained from those of $\square$ by re-parametrization: if β and α are corresponding orientation preserving affine parameters on a bicharacteristic γ of $\square$ and its images γ^+, γ^- as bicharacteristics of P^+ or P^-, respectively, then $d\alpha = d\beta/x^2$. One can identify γ with $\gamma^\pm$ above $Z_a^0 \times \mathbf{R}$. Now these bicharacteristics can be extended to bicharacteristics $\bar{\gamma}^\pm$ of $\bar{P}^\pm$, respectively, above

$$\bar{M}^\pm = \{(x, y, s_\pm) : x \in (-c, a),\ y \in \partial Z,\ s_\pm \in \mathbf{R}\}.$$

We can now define the *end points* of γ as the intersections of the extended bicharacteristics with the relevant cotangent bundles above the respective horizons,

$$\partial^+\gamma = \gamma^+ \cap T^*(\bar{M}^+)|_{\{x=0\}}, \qquad \partial^-\gamma = \gamma_- \cap T^*(\bar{M}^-)|_{\{x=0\}}. \tag{36}$$

As these end points are limits when approached above M^- or M^+, respectively, the definition is independent of the choice of extension of the metric.

Our local coordinates are all well defined above $Z_a^0 \times \mathbf{R}$. Obviously, one can identify (x, y), (x_+, y_+) and (x_-, y_-)—this has already been done, tacitly—and hence one can also identify the dual coordinates η, η_+, η_-. The remaining fibre coordinates are related by

$$\xi\, dx + \tau\, dt = \xi_+\, dx + \sigma_+\, ds_+ = \xi_-\, dx + \sigma_-\, ds_-$$

whence $\tau = \sigma_+ = \sigma_-$. In particular, if γ is a bicharacteristic are above Z_a^0 and γ^+, γ^- are the corresponding bicharacteristics of P^+ and P^-, respectively, then

$$\sigma_+ = \sigma_- = \tau_0 \neq 0,$$

bearing in mind that τ is constant on a bicharacteristic and that this carries over to its end points.

Let $\omega_- = (y, s_-, \eta, \sigma_-) \in T^*(\partial^- M)$ be given, with $\sigma_- \neq 0$. Then there is a unique bicharacteristic γ of $\Box$ such that $\pi^-(\partial^-\gamma) = \omega_-$. The lift of ω_- to $T^*(\bar{M}^-)$ is

$$\{(0, y, s_-, \xi, \eta, \sigma_-) \colon \xi \in \mathbf{R}\}.$$

By (35′), this meets char P^- in just one point, when $\xi = h^c(0, y, \eta)/2s_-$. The bicharacteristic of P^- through this point determines a bicharacteristic γ of $\Box$ such that $\pi^-(\partial^-\gamma) = \omega_-$. There is a similar construction for a given $\omega_+ \in T^*(\partial^+ M)$, with $\sigma_+ \neq 0$, which gives a unique bicharacteristic γ such that $\pi^+(\partial^+\gamma) = \omega_+$. In either case, γ may not be a scattering bicharacteristic: one cannot exclude the possibility that it remains above a compact subset of Z^0 as $t \to \infty$ or $t \to -\infty$, respectively. So, let Π^+ and Π^- be the subsets of $T^*(\partial^+ M)$ and $T^*(\partial^- M)$ defined by

$$\Pi^\pm = \{\pi^\pm(\partial^\pm\gamma) \colon \gamma \text{ is a scattering bicharacteristic}\}, \tag{37}$$

where

$$\pi^\pm \colon T^*(\bar{M}^+)|_{\{x=0\}} \to T^*(\partial^\pm M)$$

are the natural projections. One then has a map $\lambda \colon \Pi^- \to \Pi^+$ which sends ω_- to $\omega_+ = \pi^+(\partial^+\gamma)$, where γ is the bicharacteristic such that $\pi^-(\partial^-\gamma) = \omega_-$. Similarly, one can define a map $\Pi^+ \to \Pi^-$. As the end points of a scattering bicharacteristic are interchanged when its orientation is reversed, this must be the inverse of λ. Summing up, we have

LEMMA 2. *If $\omega_- \in \Pi^-$, then there is one and only one bicharacteristic γ such that $\pi^-(\partial^-\gamma) = \omega_-$, and the map λ: $\omega_- \to \omega_+ = \pi^+(\partial^+\gamma)$ is a bijection $\Pi^- \to \Pi^+$.*

We can now return to the wave front sets of the radiation fields of an hourglass wave.

PROPOSITION 3. *Let $u \in \mathscr{D}'(Z^0 \times \mathbf{R})$ be a solution of the wave equation, such that $z \to u(z, t)$ has compact support for one, and hence for all $t \in \mathbf{R}$ (i.e., an hourglass wave). Assume that the bicharacteristics whose union is $WF(u)$ are either scattering bicharacteristics, or remain above a compact subset of Z^0 for all $t \in \mathbf{R}$. Let w^+ and w^- be its forward and backward radiation field, respectively. Then*

$$WF(w^+) = \lambda(WF(w^-)). \tag{38}$$

Proof. By hypothesis, $WF(w^+) \subset \Pi^+$ and $WF(w^-) \subset \Pi^-$. As w^- and w^+ are the restrictions to the respective horizons of the associated solutions of $\bar{P}^-(\bar{v}^-) = 0$, $\bar{P}^+(\bar{v}^+) = 0$, respectively, it is clear that

$$WF(w^-) \subset \pi^-\{\partial^-\gamma\colon \gamma \text{ is a bicharacteristic and } \gamma \in WF(u)\}$$

$$WF(w^+) \subset \pi^+\{\partial^+\gamma\colon \gamma \text{ is a bicharacteristic and } \gamma \in WF(u)\}.$$

If γ is a scattering bicharacteristic, it follows that $\lambda(\partial^-\gamma) \in WF(w^+)$, whence $\lambda(WF(w^-)) \subset WF(w^+)$. But λ is a bijection; so the proposition follows.

Remark. If $Wf(u)$ contains bicharacteristics which begin above $\partial^- M$ or end above $\partial^+ M$ but are not scattering bicharacteristics, one has to replace (38) by

$$WF(w^+) \cap \Pi^+ = \lambda(WF(w^-) \cap \Pi^-).$$

One can obviously also define the radiation fields as the limits of

$$t^m u(1/(t-s), y, t), \qquad t^m u(1/(s-t), y, t)$$

as $t \to \infty$ and $t \to -\infty$, respectively. (Up to the choice of sign of s, the "local time at the horizon," this is equivalent to Lax and Phillips' definition for the wave equation in an exterior domain [LP].) So, roughly speaking, the proposition extends the well known relation between $WF(u(\cdot, t_1))$ and $WF(u(\cdot, t_2))$, where $-\infty < t_1 < t_2 < \infty$, by restriction of the bicharacteristic flow to $\{t = t_1\}$ and $\{t = t_2\}$, and $t_1 \to -\infty$ and $t_2 \to \infty$.

Proposition 3, which relates the singularities of w^- to those of w^+, is restricted to the ranges of the maps $\varphi \to w^\pm$ when the data φ have compact support. (These ranges are related by (27).) But no intrinsic characterization of these ranges is known (except in the simple case of the wave equation on $\mathbf{R}^n \times \mathbf{R}$, n odd.) If Z is *essentially analytic*, in the sense that Z^0 is (real) analytic and has a (real) analytic extension $\bar{Z} \supset Z$, the radiation fields of expanding waves have *coherence properties*. For example, if the forward radiation field w^+ of an expanding wave u vanishes on $\Omega \times \mathbf{R}$, where Ω is an open subset of ∂Z, as well as for $s \ll 0$, then $u = 0$ on a neighborhood of $\partial^+ M$. This can be deduced from properties of analytic wave front sets by arguments similar to those used in [F2]. (This deals with the solution of the ordinary wave equation in the exterior of a characteristic double cone.) However, this does imply the injectivity of the map $\varphi \to w^+$ when the data have compact support.

On the face of it, the *inverse problem*, to find an expanding wave u of the wave equation with given (forward) radiation field w^+, is a characteristic initial value problem for $P^+ v^+ = 0$, with $v^+ = w^+$ when $x = 0$ and $v^+ = 0$ if $s \ll 0$. But the forward dependence domain of a point (x', y', s') in $\{0 < x < a\}$ is in $\{x \leqslant x',\ s \geqslant s'\}$, so does not meet any set $\{s \geqslant c\}$ in a relatively compact subset of $(0, a) \times \partial Z \times \mathbf{R}$. Hence this problem is not well posed. On the other hand, the characteristic initial value problem

$$P^+ v = 0, \qquad v|_{x=0} = w, \qquad w \in C^\infty(\partial Z \times \mathbf{R}), \qquad w = 0 \quad \text{if } s \gg 0. \tag{39}$$

is well posed in a certain subset of $(0, a) \times \partial Z \times \mathbf{R}$, and its solution can be used to construct what may be called a *receding wave*, that is to say a solution u of the wave equation such that $u = 0$ for $t - 1/x \gg 0$, that has the (forward) radiation field w. (In the terminology of [LP], this corresponds to "eventually outgoing data.") The construction is completed by choosing a suitable real number t_0 and using the data furnished by v to solve a "backward" initial value problem of the wave equation for data on $\{t = t_0\}$.

It can also be shown that

$$u(z, t) = -2 \int E_\infty(y', s' - t, z)\, \partial_s w(y', s')\, |h(0, y')|^{1/2}\, |dy'\, ds'|. \tag{40}$$

The second member is well defined when $\partial_s w$ is a distribution with support contained in $\{s \leqslant c\}$ for some $c \in \mathbf{R}$. But then (39) is ill defined, unless one can add a condition that ensures that the second member of (40) (nominally extended as 0 to $\bar{Z}_a \times \mathbf{R}$) can be restricted to $\partial^+ M$ (to give the radiation field w^+). Should this be so, then $Wf(u)$ is the backward bicharacteristic flowout of $WF(w^+)$ lifted to char P^+, so that $WF(u)$ is the union of all bicharacteristics γ such that $\partial^+ \gamma \in WF(w^+)$. By reversing the time orientation, one can transfer the foregoing to *approaching waves*, that is to say $u=0$ for $t+1/x \ll 0$. These correspond to "eventually incoming" data.

To conclude, we remark that, following Lax and Philips in [LP], one can define an *energy space* H_E, as follows. For initial data φ_0, $\varphi_1 \in C_c^\infty(Z^0)$, the Hilbert norm

$$\|\varphi\|_E = \left((1/2) \int (|\varphi_1|^2 + (d\varphi_0, d\varphi_0)_g)\, d\mu_z \right)^{1/2} \tag{41}$$

is equal to $\|U(t)\,\varphi\|_E$ for all t, and it can be deduced from Proposition 2 that

$$\|\partial_s w^+\| \leqslant \|\varphi\|_E, \tag{42}$$

where w^+ is the forward radiation field of $u=(U(t)\,\varphi)_0$, and the norm in the first member is the $L^2(\partial Z \times \mathbf{R})$ norm, with respect to the invariant density induced on $\partial Z \times \mathbf{R}$. So, if one defines H_E as the completion of the vector space of $C_c^\infty(Z^0)$ data with respect to this norm, one can extend $U(t)$ to H_E and define a *forward radiation field map* $\mathscr{R}^+ \colon H_E \to L^2(\partial Z \times \mathbf{R})$ that sends φ to its radiation field (defined by taking the limit of the derivatives with respect to s of the radiation fields of an approximating sequence of C_c^∞ data). Then (42) implies that

$$\|\mathscr{R}^+ \varphi\| \leqslant \|\varphi\|_E. \tag{43}$$

(As the corresponding arguments in [F1] can be taken over, virtually as they stand, all proofs will be omitted.) Now let

$$H_E^1 = \{\varphi \in H_E : \|\mathscr{R}^+ \varphi\| = \|\varphi\|_E\}. \tag{44}$$

It can be shown that this subspace of H_E is the closure of eventually outgoing data (data for which $(U(t)\,\varphi)_0$ is a receding wave). Furthermore, H_E is the direct orthogonal sum of H_E^1 and $\ker \mathscr{R}^+$, and $H_E/\ker \mathscr{R}^+ \to L^2(\partial Z \times \mathbf{R})$ is a Hilbert isomorphism. It is very likely that, in fact, $\ker \mathscr{R}^+ = 0$, as the Laplacian of an almost Euclidean manifold has no point spectrum. (For scattering by an obstacle in R^n, this property ensures equality in (42).) Let us assume that $\ker \mathscr{R}^+ = 0$. Then $\mathscr{R}^+$ is a bijective Hilbert space isomorphism $H_E \to L^2(Z \times \mathbf{R})$, so that $\mathscr{R}^+$ has an inverse, which is equal to its adjoint. For C_c^∞ initial data φ, $\mathscr{R}^+\varphi$ is the derivative of the second member of (25), and it is not difficult to check that the adjoint then gives the initial data derived from (40). (One can also recover (40) itself from this, by using the identity $\mathscr{R}^+ U(c)\,\varphi = (\mathscr{R}^+\varphi)(y, s+c)$, $c \in \mathbf{R}$.) But in order to prove that both $\mathscr{R}^+$ and its inverse are given by (25) and (40), respectively, one would have to prove that at least one of (25) and (40) extends to H_E.

By reversing the time orientation, one obtains a map $\mathscr{R}^-$: $H_E \to L^2(\partial Z \times \mathbf{R})$, whose restriction to C_c^∞ data φ sends φ to the backward radiation field of $\partial_t(U(t)\,\varphi)_0$, with similar properties. As it follows from (27) that

$$\ker \mathscr{R}^+ = 0 \qquad \text{if and only if} \quad \ker \mathscr{R}^- = 0,$$

$\mathscr{R}^-$ will again be a bijective Hilbert space isomorphism in the case under discussion, and one can define a *scattering operator*

$$S = \mathscr{R}^+(\mathscr{R}^-)^{-1}\colon L^2(\partial Z \times \mathbf{R}) \to L^2(\partial Z \times \mathbf{R})$$

which is a unitary operator. It corresponds to Lax and Phillips' "scattering operator" in an exterior domain [LP, 151]. (The *scattering matrix* $\mathscr{S}$ is obtained by conjugating S with the Fourier transform in s. It can, and usually is, defined directly for the associated Helmholtz equation; see [M], for example.)

Formally, the Schwartz kernel of S is

$$\begin{aligned} K_s &= \lim_{x\to 0}\,(x^{-m}\,\partial_t E_\infty(y', s+s'+1/x, x, y)) \\ &= \lim_{x\to 0}\ \lim_{x'\to 0}\,(2(xx')^{-m}\,\partial_t E(x, y, s-s'+1/x+1/x', x', y')) \end{aligned}$$

(the sign of the second member of (4) has to reversed to go forward from $-\infty$). In the basic example of $\mathbf{R}^n\backslash\{0\}$ in polar coordinates ($z=(1/x)\,y$, $y\in S^{n-1}$), this can easily be verified; in general, a good deal more work would be needed to establish it. It does suggest that Proposition 3 may extend to solutions of the wave equation with finite energy.

APPENDIX

By (35), the bicharacteristics of P^- satisfy the equations

$$\begin{aligned} &\partial x/\partial\alpha = \sigma - x^2\xi, \qquad \partial s/\partial\alpha = \xi, \qquad \partial y/\partial\alpha = h^*\eta \\ &\partial\xi/\partial\alpha = x\xi^2 + 1/2(\partial_x h^*\eta, \eta), \qquad \partial\sigma/\partial\alpha = 0, \qquad \partial\eta/\alpha = 1/2(\partial_y h^*\eta, \eta), \end{aligned} \tag{A.1}$$

where $h^* = (h^{ij}(x, y))$ here denotes the $(n-1)\times(n-1)$ matrix of the contravariant components of h, so that $h^c(x, y, \eta) = (h^*\eta, \eta)$. They must be supplemented by initial conditions, say

$$x = x_0,\quad s = s_0,\quad y = y^0,\quad \xi = \xi_0,\quad \sigma = \sigma_0,\quad \eta = \eta^0, \quad \text{when } \alpha = 0. \tag{A.2}$$

Here α is an affine parameter; recall that $(x, s, y, \alpha\xi, \alpha\sigma, \alpha\eta)$ are functions of $(x_0, y^0, s_0, \alpha\xi_0, \alpha\sigma_0, \alpha\eta^0)$. (The sub- and superscripts $-$ have been omitted, for simplicity.) In addition, the symbol of P^- vanishes on bicharacteristics, so that

$$2\xi\sigma = x^2\xi^2 + (h^*\eta, \eta). \tag{A.3}$$

Obviously, σ is constant,

$$\sigma = \sigma_0. \tag{A.4}$$

One solution of (A.1) is, for any h,

$$x = \sigma_0\alpha, \quad s = s_0, \quad y = y^0, \quad \xi = 0, \quad \sigma = \sigma_0 > 0, \quad \eta = 0.$$

These ("radial") bicharacteristics are the generators of the conormal bundle of the characteristic $\{s = s_0\}$; they will be of no concern in the sequel.

Equations (A.1) are defined, in the first instance, above $(0, a) \times \partial Z \times \mathbf{R}$, but extend by continuity uniquely to $T^*(\partial^- M)$. Now $T^*(\partial^- M_-)$ is the lift of the characteristic $\partial^- M$, and is the union of "boundary bicharacteristics"

$$x = 0, \quad s = s_0 + \xi_0\alpha, \quad y = y^0, \quad \xi = \xi_0 \neq 0, \quad \sigma = 0, \quad \eta = 0. \tag{A.5}$$

The cancellation of the factor x^2 in

$$x^{-m} \square (x^m v) = x^2 P^- v^- = 0$$

gives rise to a discontinuity: bicharacteristics above M^- that tend to $\partial T^*(M^-)$ as $\sigma_0/|\eta_0| \to 0$ do not tend to a boundary bicharacteristic (A.5). One can see this from two special cases, in each of which (A.1) can be integrated explicitly or by quadratures.

The first of these is the *product case*: *h independent of x*, so that $h^* = h^*(y)$. Then (A.1) splits into groups, the equations for (y, η), and the equations for (x, s, ξ, σ). The (y, η) equations give the geodesics on ∂Z equipped with the metric $h(0, y, dy)$. So $(h^*\eta, \eta)$ is constant, and one can choose the geodesic distance measured along geodesics as affine parameter α, whence

$$(h^*\eta, \eta) = 1.$$

Then (A.3) becomes

$$2\xi\sigma_0 = x^2\xi^2 + 1.$$

From this and (A.1) it follows that $\partial^2 x/\partial\alpha^2 + x = 0$. Taking $x_0 = 0$ in (A.2) (so that $\xi_0 = 1/2\sigma_0$), it is easy to show that on the bicharacteristic γ with the other initial values as in (A.2),

$$x = \sigma_0 \sin\alpha, \quad s = s_0 + (1/\sigma_0)\tan(1/2\alpha), \quad \xi = 1/2\sigma_0 \cos^2(1/2\alpha). \tag{A.6}$$

Clearly, $\partial^- \gamma = (y^0, s_0)$, and $0 < x < a$ for $0 < \alpha < \pi$, $x \to 0$ as $\alpha \to \pi$. But then $s \to \infty$, so that one must go over to M^+ to determine $\partial^+ \gamma$. As (restoring the subscripts)

$$s_+ = s\_2/x, \qquad \xi_+ = \xi\_2\sigma_0/x^2,$$

it is not difficult to deduce from (A.6) that $x \to 0$ as $\alpha \to \pi$, and that $\partial^+ \gamma = (y^1, s_0)$ where y^1 is the end point of the geodesic on ∂Z given by (A.2) and the (t, η) group of equations in (A.1) that is of length π. So

$\partial^- \gamma$ and $\partial^+ \gamma$ project to points on ∂Z

that are the end points of a geodesic of length π.

The other elementary example (the *layered medium case*) is that of *h independent of y*, $h^* = h^*(x)$. Observe first that it follows from (A.1) and (A.3) that

$$(\partial x/\partial \alpha)^2 = \sigma_0^2 - x^2(h^*(x, y)\,\eta, \eta). \tag{A.8}$$

Now it is obvious from (A.1) that η is constant when h^* is independent of y. So one can obtain the bicharacteristics by quadratures in this case. It turns out that, taking $x_0 = 0$ again, x increases with α to a maximum x^* which is the smallest zero of $\sigma_0^2 - x^2(h^*(x)\,\eta_0, \eta_0)$. Clearly $x^* < a$ if $\sigma_0/|\eta_0|$ is sufficiently small, and in this case x then decreases to 0 as α increases further. One can go over to M^+ as before, and determine $\partial^+ \gamma$. One finds that (A.7) *holds in the limit as* $\sigma_0/|\eta| \to 0$.

It is to be expected that this holds in general; it may not be too difficult to deduce it from (A.1) in view of (A.8). If this is so, it provides a link between the asymptotic behaviour of solutions of the wave equation considered in this Note, and the elliptic wave case discussed comprehensively in [MZ].

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