Hopf algebroids with bijective antipodes: 
axioms, integrals, and duals

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Abstract

Motivated by the study of depth 2 Frobenius extensions, we introduce a new notion of Hopf algebroid. It is a 2-sided bialgebroid with a bijective antipode which connects the two, left and right handed, structures. While all the interesting examples of the Hopf algebroid of J.H. Lu turn out to be Hopf algebroids in the sense of this paper, there exist simple examples showing that our definition is not a special case of Lu's. Our Hopf algebroids, however, belong to the class of $\times L$-Hopf algebras proposed by P. Schauenburg. After discussing the axioms and some examples, we study the theory of non-degenerate integrals in order to obtain duals of Hopf algebroids.

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1. Introduction

There is a consensus in the literature that bialgebroids, invented by Takeuchi [24] as $\times R$-bialgebras, are the proper generalizations of bialgebras to non-commutative base rings [5,13,17,20,21,25]. The situation of Hopf algebroids, i.e., bialgebroids with some sort of antipode, is less understood. The antipode proposed by J.H. Lu [13] is burdened by the need of a section for the canonical epimorphism $A \otimes A \rightarrow A \otimes R A$ the precise role of which remained unclear. The $\times R$-Hopf algebras proposed by P. Schauenburg in [18] have a clearcut categorical meaning. They are the bialgebroids $A$ over $R$ such that the

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forgetful functor $\mathcal{A} \mathcal{M} \to \mathcal{R} \mathcal{M}_\mathcal{R}$ is not only strict monoidal, which is the condition for $\mathcal{A}$ to be a bialgebroid over $\mathcal{R}$, but preserves the closed structure as well. In this very general quantum groupoid, however, antipode, as a map $A \to A$, does not exist.

Our proposal of an antipode, announced in [2], is based on the following simple observation. The antipode of a Hopf algebra $H$ is a bialgebra map $S : H \to H_{op}^{\text{Hop}}$. The opposite of a bialgebroid $A$, however, is not a bialgebroid in the same sense. In the terminology of [8] there are left bialgebroids and right bialgebroids; corresponding to whether $\mathcal{A} \mathcal{M}$ or $\mathcal{M} \mathcal{A}$ is given a monoidal structure. This suggests that the existence of antipode on a bialgebroid should be accompanied with a two-sided bialgebroid structure and the antipode should swap the left and right handed structures. More explicit guesses for what to take as a definition of the antipode can be obtained by studying depth 2 Frobenius extensions. In [8] it has been shown that for a depth 2 ring extension $N \subset M$ the endomorphism ring $A = \text{End}_N M_N$ has a canonical left bialgebroid structure over the centralizer $\mathcal{R} = C_H(A)$. If $N \subset M$ is also Frobenius and a Frobenius homomorphism $\psi : M \to N$ is given, then $A$ has a right bialgebroid structure, too. There is a candidate for the antipode $S : A \to A$ as transposition w.r.t. the bilinear form $m, m' \in M \mapsto \psi(mm')$. In fact, this definition of $S$ does not require the depth 2 property, so in this example antipode exists prior to comultiplication. Assuming the extension $M/N$ is either H-separable or Hopf–Galois, L. Kadison has shown [9,10] that the bialgebroid $A$ or its dual $B = (M \otimes N)N$ has an antipode in the sense of [13].

In a recent paper [7] B. Day and R. Street give a new characterization of bialgebroids in the framework of symmetric monoidal autonomous bicategories. They also introduce a new notion called Hopf bialgebroid. It is more restrictive than Schauenburg’s $\times_R$-Hopf algebra since it requires star autonomy [1] rather than to be closed. We will show in Section 4.2 that the categorical definition [7] of the Hopf bialgebroid is equivalent to our purely algebraic Definition 4.1 of Hopf algebroid, apart from the tiny difference that we allow $S^2$ to be nontrivial on the base ring. This freedom can be adjusted to the Nakayama automorphism of $\psi$ in case of the Frobenius depth 2 extensions. It is a new feature of our Hopf algebroids, compared to (weak) Hopf algebras, that the antipode is not unique. The various antipodes on a given bialgebroid were shown to be in one-to-one correspondence with the generalized characters (called twists) in [2].

It is encouraging that one can find ‘quantum groupoids’ in the literature that satisfy our axioms. Such are the weak Hopf algebras with bijective antipode, the examples of Lu–Hopf algebroids in [5], and the extended Hopf algebras in [11]. In particular, the Connes–Mosicvici algebra [6] is a Hopf algebra in this sense. We also present an example which is a Hopf algebroid in the sense of this paper but does not satisfy the axioms of [13]. This proves that the two notions of Hopf algebroid—the one in the sense of this paper and the one in the sense of [13]—are not equivalent. Until now we could neither prove nor exclude by examples the possibility that the latter was a special case of the former.

The left and right integrals in Hopf algebra theory are introduced as the invariants of the left and right regular module, respectively. In this analogy one can define left integrals in a left bialgebroid and right integrals in a right bialgebroid. Since a Hopf algebra has both left and right bialgebroid structures both left and right integrals can be defined.

The properties of the integrals in a (weak) Hopf algebra over a field $k$ carry information about its algebraic structure. For example, the Maschke’s theorem [3,12] states that it is
a semi-simple algebra if and only if it has a normalized integral. The Larson–Sweedler theorem [12,26] implies that it is finite dimensional over \( k \) if and only if it has a non-degenerate left (hence also a right) integral. In this case the \( k \)-dual also has a (weak) Hopf algebra structure.

Therefore, in Section 5, we analyze the consequences of the existence of a non-degenerate integral in a Hopf algebroid. We show that if there exists a non-degenerate integral in a Hopf algebroid \( \mathcal{A} \) over the base \( L \) then the ring extension \( L \rightarrow A \) is a Frobenius extension hence also finitely generated projective. We do not investigate, however, the opposite implication, i.e., we do not study the question under what conditions on the Hopf algebroid the existence of a non-degenerate integral follows.

If some of the \( L \)-module structures of a Hopf algebroid is finitely generated projective then the corresponding dual can be equipped with a bialgebroid structure [8]. There is no obvious way, however, how to equip it with an antipode in general. We show that in the case of Hopf algebroids possessing a non-degenerate integral the dual bialgebroids are all (anti-) isomorphic and they combine into a Hopf algebroid—depending on the choice of the non-degenerate integral. Therefore, we do not associate a dual Hopf algebroid to a given Hopf algebroid rather a dual isomorphism class to an isomorphism class of Hopf algebroids. The well-known \( k \)-dual of a finite (weak) Hopf algebra \( H \) over the commutative ring \( k \) turns out to be the unique (distinguished) (weak) Hopf algebra in the dual isomorphism class of the isomorphism class of the Hopf algebroid \( H \).

The paper is organized as follows. In Section 2 we introduce some technical conventions about bialgebroids that are used in this paper. Our motivating example, the Hopf algebroid corresponding to a depth 2 Frobenius extension of rings is discussed in Section 3. In Section 4 we give some equivalent definitions of Hopf algebroids. We prove that our definition is equivalent to the one in [7] hence gives a special case of the one in [18]. In the final subsection of Section 4 we present a collection of examples. In Section 5 we propose a theory of non-degenerate integrals as a tool for the definition of the dual Hopf algebroid.

2. Preliminaries on bialgebroids

In this technical section we summarize our notations and the basic definitions of bialgebroids that will be used later on. For more about bialgebroids we refer to the literature [5,8,18,19,21,22,24].

**Definition 2.1.** A *left bialgebroid* (or Takeuchi \( \times_L \)-bialgebra) \( \mathcal{A}_L \) consists of the data \((A, L, s_L, t_L, \gamma_L, \pi_L)\). The \( A \) and \( L \) are associative unital rings, the total and base rings, respectively. The \( s_L : L \rightarrow A \) and \( t_L : L^{op} \rightarrow A \) are ring homomorphisms such that the images of \( L \) in \( A \) commute making \( A \) an \( L-L \)-bimodule via

\[
l \cdot a \cdot l' := s_L(l)t_L(l')a.
\]  

(2.1)
The bimodule (2.1) is denoted by \( LAL \). The triple \((LAL, \gamma_L, \pi_L)\) is a comonoid in \( LM_L \), the category of \( L-L \)-bimodules. Introducing the Sweedler’s notation \( \gamma_L(a) \equiv a(1) \otimes a(2) \in A \otimes LA \) the identities

\[
\begin{align*}
    a_{(1)} t_L(l) \otimes a_{(2)} &= a_{(1)} \otimes a_{(2)} s_L(l), \\
    \gamma_L(1_A) &= 1_A \otimes 1_A, \\
    \gamma_L(ab) &= \gamma_L(a) \gamma_L(b), \\
    \pi_L(1_A) &= 1_L,
\end{align*}
\]

are required for all \( l \in L \) and \( a, b \in A \). The requirement (2.4) makes sense in the view of (2.2).

The \( L \) actions of the bimodule \( LAL \) in (2.1) are given by left multiplication. Using right multiplication there exists another \( L-L \)-bimodule structure on the total ring \( A \) of a left bialgebroid \( A_L \):

\[
l \cdot a \cdot l' := at_L(l)s_L(l').
\]

This \( L-L \)-bimodule is called \( LAL' \). This way \( A \) carries four commuting actions of \( L \).

If \( A_L = (A, L, s_L, t_L, \gamma_L, \pi_L) \) is a left bialgebroid then so is its co-opposite: \( A_L^{\text{cop}} = (A, L^{\text{op}}, t_L, s_L, \gamma_L^{\text{op}}, \pi_L) \). The opposite \( A_L^{\text{op}} = (A^{\text{op}}, L, t_L, s_L, \gamma_L, \pi_L) \) has a different structure that was introduced under the name right bialgebroid in [8].

**Definition 2.2.** A right bialgebroid \( A_R \) consists of the data \( (A, R, s_R, t_R, \gamma_R, \pi_R) \). The \( A \) and \( R \) are associative unital rings, the total and base rings, respectively. The \( s_R : R \to A \) and \( t_R : R^{\text{op}} \to A \) are ring homomorphisms such that the images of \( R \) in \( A \) commute making \( A \) an \( R-R \)-bimodule:

\[
r \cdot a \cdot r' := as_R(r')t_R(r).
\]

The bimodule (2.8) is denoted by \( RAR \). The triple \((RAR, \gamma_R, \pi_R)\) is a comonoid in \( RM_R \). Introducing the Sweedler’s notation \( \gamma_R(a) \equiv a(1) \otimes a(2) \in A \otimes RA \) the identities

\[
\begin{align*}
    s_R(r)a_{(1)} \otimes a_{(2)} &= a_{(1)} \otimes t_R(r)a_{(2)}, \\
    \gamma_R(1_A) &= 1_A \otimes 1_A, \\
    \gamma_R(ab) &= \gamma_R(a) \gamma_R(b), \\
    \pi_R(1_A) &= 1_R,
\end{align*}
\]

are required for all \( r \in R \) and \( a, b \in A \).
For the right bialgebroid $A_R$ we introduce the $R$-$R$-bimodule $R A_R$ via
\[ r \cdot a \cdot r' := s_R(r) t_R(r') a. \] (2.9)

This way $A$ carries four commuting actions of $R$.

Left (right) bialgebroids can be characterized by the property that the forgetful functor $A_M \to L A_L$ ($M_A \to R M_R$) is strong monoidal [17,20].

It is natural to consider the homomorphisms of bialgebroids to be ring homomorphisms preserving the comonoid structure. We do not want to make difference however between bialgebroids over isomorphic base rings. This leads to the following.

**Definition 2.3** [21]. A left bialgebroid homomorphism $A_L \to A'_L$ is a pair of ring homomorphisms $(\Phi : A \to A', \phi : L \to L')$ such that
\[
\begin{align*}
s'_L \circ \phi &= \Phi \circ s_L, \\
t'_L \circ \phi &= \Phi \circ t_L, \\
\pi'_L \circ \Phi &= \phi \circ \pi_L, \\
\gamma'_L \circ \Phi &= (\Phi \otimes \Phi) \circ \gamma_L.
\end{align*}
\]
The last condition makes sense since by the first two conditions $\Phi \otimes \Phi$ is a well-defined map $A \otimes_L A \to A \otimes_L A$.

The pair $(\Phi, \phi)$ is an isomorphism of left bialgebroids if it is a bialgebroid homomorphism such that both $\Phi$ and $\phi$ are bijective.

A right bialgebroid homomorphism (isomorphism) $A_R \to A'_R$ is a left bialgebroid homomorphism (isomorphism) $(A_R)^{op} \to (A'_R)^{op}$.

Let $A_L$ be a left bialgebroid. The equation (2.1) describes two $L$-modules $A_L$ and $L A$. Their $L$-duals are the additive groups of $L$-module maps $A^*_L := \{ \phi^*_L : A_L \to L \}$ and $^*_A := \{ ^*_\phi : L A \to L \}$, where $L L$ stands for the left regular and $L_L$ for the right regular $L$-module. Both $A^*_L$ and $^*_A$ carry left $A$ module structures via the transpose of the right regular action of $A$. For $\phi \in A^*_L$, $^*_\phi \in ^*_A$, and $a, b \in A$ we have
\[ (a \mapsto _\phi)(b) = _\phi(ab) \quad \text{and} \quad (a \mapsto ^*_\phi)(b) = ^*_\phi(ab). \]

Similarly, in the case of a right bialgebroid $A_R$—denoting the left and right regular $R$-modules by $R R$ and $R^R$, respectively—the two $R$-dual additive groups
\[ A^* := \{ \phi^* : A^R \to R \} \quad \text{and} \quad ^*_A := \{ ^*_\phi : ^R A \to R \} \]
carry right $A$-module structures:
\[ (\phi^* \leftarrow a)(b) = \phi^*(ab) \quad \text{and} \quad (^*_\phi \leftarrow a)(b) = ^*_\phi(ab). \]
The comonoid structures can be transposed to give monoid (i.e., ring) structures to the duals. In the case of a left bialgebroid \( \mathcal{A}_L \)

\[
(\phi \ast \psi)(a) = \psi (s_L \circ \phi \ast (a(1))a(2)) \quad \text{and} \quad (\ast \phi \ast \psi)(a) = \ast \psi (t_L \circ \ast \phi (a(2))a(1)) \tag{2.10}
\]

for \( \ast \phi, \ast \psi \in \ast \mathcal{A}, \phi, \psi \in \mathcal{A}_s \), and \( a \in \mathcal{A} \).

Similarly, in the case of a right bialgebroid \( \mathcal{A}_R \),

\[
(\phi \ast \psi)(a) = \phi (a(2) \circ t_R \circ \ast \psi (a(1))) \quad \text{and} \quad (\ast \phi \ast \psi)(a) = \ast \phi (a(1) \circ s_R \circ \ast \psi (a(2))) \tag{2.11}
\]

for \( \phi, \psi \in \mathcal{A}^*, \ast \phi, \ast \psi \in \ast \mathcal{A}^*, \) and \( a \in \mathcal{A} \).

In the case of a left bialgebroid \( \mathcal{A}_L \) also the ring \( \mathcal{A} \) has right \( \mathcal{A}_s \)- and right \( \ast \mathcal{A}_s \)-module structures:

\[
a \leftarrow \phi_s = s_L \circ \phi_s (a(1))a(2) \quad \text{and} \quad a \leftarrow \ast \phi = t_L \circ \ast \phi (a(2))a(1) \tag{2.12}
\]

for \( \phi_s \in \mathcal{A}_s, \ast \phi \in \ast \mathcal{A} \) and \( a \in \mathcal{A} \).

Similarly, in the case of a right bialgebroid \( \mathcal{A}_R \) the ring \( \mathcal{A} \) has left \( \mathcal{A}^* \)- and left \( \ast \mathcal{A}^* \)-structures:

\[
\phi^* \rightarrow a = a(2) \circ t_R \circ \phi^* (a(1)) \quad \text{and} \quad \ast \phi \rightarrow a = a(1) \circ s_R \circ \ast \phi (a(2)) \tag{2.13}
\]

for \( \phi^* \in \mathcal{A}^*, \ast \phi \in \ast \mathcal{A} \), and \( a \in \mathcal{A} \).

In the case when the \( L \) (\( R \)) module structure on \( \mathcal{A} \) is finitely generated projective then the corresponding dual has also a bialgebroid structure: if \( \mathcal{A}_L \) is a left bialgebroid such that the \( L \)-module \( \mathcal{A}_L \) is finitely generated projective then \( \mathcal{A}_s \) is a right bialgebroid over the base \( L \) as follows:

\[
(s_{\ast R}(l))(a) = \pi_L (a s_L (l)), \quad (t_{\ast R}(l))(a) = t \pi_L (a),
\]

\[
\gamma_{\ast R}(\phi_s) = b \rightarrow \ast \phi \otimes \beta_s^*, \quad \pi_{\ast R}(\phi_s) = \phi_1 \ast \Lambda.
\]

where \([b_i]\) is an \( L \)-basis in \( \mathcal{A}_L \) and \([\beta_s^i]\) is the dual basis in \( \mathcal{A}_s \).

Similarly, if \( \mathcal{A}_L \) is a left bialgebroid such that the \( L \)-module \( \mathcal{A}_L \) is finitely generated projective then \( \ast \mathcal{A} \) is a right bialgebroid over the base \( L \) as follows:

\[
(s_{\ast R}(l))(a) = \pi_L (a l), \quad (t_{\ast R}(l))(a) = \pi_L (a t L (l)),
\]

\[
\ast \gamma_{\ast R}(\phi) = \ast \beta^i \otimes b \rightarrow \ast \phi, \quad \pi_{\ast R}(\ast \phi) = \ast \phi_1 \ast \Lambda,
\]

where \([b_i]\) is an \( L \)-basis in \( \mathcal{A}_L \) and \([\ast \beta^i]\) is the dual basis in \( \ast \mathcal{A} \).

If \( \mathcal{A}_R \) is a right bialgebroid such that the \( R \)-module \( \mathcal{A}_R \) is finitely generated projective then \( \mathcal{A}^* \) is a left bialgebroid over the base \( R \) as follows:
\[(s_L^*(r))(a) = \pi_R(r \pi_R(a)), \quad (t_L^*(r))(a) = \pi_R(s_R(r)a),\]

\[\gamma_L^*(\phi^*) = \phi^* \leftarrow b_i \otimes \beta^i, \quad \pi_L^*(\phi^*) = \phi^*(1_A),\]

where \(\{b_i\}\) is an \(R\)-basis in \(A^R\) and \(\{\beta^i\}\) is the dual basis in \(A^*\).

If \(A_R\) is a right bialgebroid such that the \(R\)-module \(R^A\) is finitely generated projective then \(^*A\) is a left bialgebroid over the base \(R\) as follows:

\[(^*s_L(r))(a) = \pi_R(t_R(r)a), \quad (^*t_L(r))(a) = \pi_R(a)r,\]

\[^*\gamma_L^*(\phi) = \phi^* \leftarrow \beta^i, \quad ^*\pi_L^*(\phi) = \phi(1_A),\]

where \(\{b_i\}\) is an \(R\)-basis in \(^R^A\) and \(\{\beta^i\}\) is the dual basis in \(^*A\).

### 3. The motivating example: D2 Frobenius extensions

#### 3.1. The forefather of antipodes

In this subsection \(N \rightarrow M\) denotes a Frobenius extension of rings. This means the existence of \(N\)-\(N\)-bimodule maps \(\psi : M \rightarrow N\) possessing quasibases. An element \(\sum_i u_i \otimes v_i \in M \otimes_N M\) is called the quasibasis of \(\psi\) [27] if

\[\sum_i \psi(mu_i) \cdot v_i = m = \sum_i u_i \cdot \psi(v_i m), \quad m \in M.\]  

As we shall see, already in this general situation there exist anti-automorphisms \(S\) on the ring \(A := \text{End}_N(M_N, 1)\) one for each Frobenius homomorphism \(\psi\). The \(S\) will become an antipode if the extension \(N \subset M\) is also of depth 2, so \(A\) also has coproduct(s).

The idea of writing the antipode as the difference of two Fourier transforms goes back to Radford’s paper [16] but plays important role in Szymanski’s treatment of finite index subfactors [23], too. If \(I\) is a non-degenerate left integral in a finite dimensional Hopf algebra \(H\) and \(\lambda \in H^*\) is the dual left integral, i.e., \(\lambda \rightarrow I = 1\), then the antipode can be written as \(S(h) = (\lambda \leftarrow h) \rightarrow I\). This formula generalizes also to weak Hopf algebras [3]. Here we will give the analogous formula for the two-step centralizer \(A\) of any Frobenius extension \(N \subset M\).

Instead of a dual algebra of \(A\) we have the second two-step centralizer \(B\) in the Jones tower of \(N \subset M\) which is the center of the \(N\)-\(N\)-bimodule \(M \otimes_N M\)

\[B := (M \otimes_N M)^N \equiv \{X \in M \otimes_N M \mid n \cdot X = X \cdot n \quad \forall n \in N\}.\]

It is a ring with multiplication \((b^1 \otimes b^2)(b'^1 \otimes b'^2) = b^1 b'^1 \otimes b^2 b'^2\) and unit \(1_B = 1_M \otimes 1_M\). Note that the ring structures of neither \(A\) nor \(B\) depend on the Frobenius structure. But if there is a Frobenius homomorphism \(\psi\) then Fourier transformation makes \(A\) and \(B\) isomorphic as additive groups.
Fixing a Frobenius homomorphism $\psi$ with quasibasis $\sum_i u_i \otimes v_i$, we can introduce convolution products on both $A$ and $B$ as follows. From now on we omit the summation symbol for summing over the quasibasis

$\alpha, \beta \in A \mapsto \alpha \ast \beta := \alpha(u_i)\beta(v_{i^-}) \in A,$

$\alpha, b \in B \mapsto a \ast b := a^1\psi(a^2b^1) \otimes b^2 \in B.$

The convolution product lends $A$ and $B$ new ring structures. The unit of $A$ is $\psi$ and the unit of $B$ is $u_i \otimes v_i$. The Fourier transformation is to relate these new algebra structures to the old ones. There are two natural candidates for a Fourier transformation:

$F : A \rightarrow B, \quad F(\alpha) := u_i \otimes \alpha(v_i),$  

$\dot{F} : A \rightarrow B, \quad \dot{F}(\alpha) := \alpha(u_i) \otimes v_i,$

$F^{-1} : B \rightarrow A, \quad F^{-1}(b) = \psi(b_1)b^2,$  

$\dot{F}^{-1} : B \rightarrow A, \quad \dot{F}^{-1}(b) = b_1\psi(b^1_2).$

They relate the convolution and ordinary products or their opposites as follows:

$F(\alpha \ast \beta) = F(\alpha)F(\beta), \quad \dot{F}(\alpha \ast \beta) = \dot{F}(\beta)\dot{F}(\alpha),$

$F(\alpha\beta) = F(\beta)\ast F(\alpha), \quad \dot{F}(\alpha\beta) = \dot{F}(\alpha)\ast \dot{F}(\beta).$

The difference between $F$ and $\dot{F}$ is therefore an anti-automorphism on both $A$ and $B$. This leads to the “antipodes”

$S_A : A \rightarrow A^{\text{op}}, \quad S_A := \dot{F}^{-1} \circ F, \quad S_A(\alpha) = u_i\psi(\alpha(v_i)_-),$  

$\dot{S}_A : B \rightarrow B^{\text{op}}, \quad \dot{S}_A := F \circ F^{-1}, \quad \dot{S}_A(b) = \psi(u_ib^1) b^2 \otimes v_i.$

with inverses

$S_A^{-1}(\alpha) = \psi(\alpha(u_i)_-)^{-1}v_i, \quad \dot{S}_A^{-1}(b) = u_i \otimes b^1\psi(b^2v_i).$

Notice that $S_A$ is just transposition w.r.t. the bi-$\text{N}$-linear form $(m, m') = \psi(mm')$ since

$\psi(mS_A(\alpha)(m')) = \psi(\alpha(m)m').$

These antipodes behave well also relative to the bimodule structures over the centralizer $C_M(\text{N}) := \{c \in M \mid cn = nc, \forall n \in \text{N}\}$. Let us consider $S_A$. The centralizer is embedded into $A$ twice: via left multiplications and right multiplications,

$L \xrightarrow{\lambda} A \xleftarrow{\rho} R,$
where \( L \) stands for \( C_M(N) \) and \( R \) for \( C_M(N)^{\text{op}} \). Clearly, \( \lambda(L) \subset C_M(\rho(R)) \). Introducing the Nakayama automorphism

\[
v : C_M(N) \rightarrow C_M(N), \quad v(c) := \psi(u_i c)v_i
\]
of \( \psi \) and using its basic identities

\[
\psi(mc) = \psi(v(c)m), \quad m \in M, \ c \in C_M(N),
\]
\[
u_i c \otimes v_i = u_i \otimes v(c)v_i, \quad c \in C_M(N),
\]
we obtain

\[
SA \circ \lambda = \rho \circ v^{-1}, \quad SA \circ \rho = \lambda
\]
and therefore

\[
SA(\lambda(l)\rho(r)\alpha) = SA(\alpha)\lambda(r)\rho(v^{-1}(l)), \quad l \in L, \ r \in R, \ \alpha \in A. \quad (3.7)
\]

In order to interpret the latter relation as the statement that \( SA \) is a bimodule map we define \( L \)- and \( R \)-bimodule structures on \( A \) by

\[
l_1 \cdot \alpha \cdot l_2 := s_L(l_1)t_L(l_2)\alpha,
\]
\[
r_1 \cdot \alpha \cdot r_2 := a t_R(r_1)s_R(r_2),
\]
where we introduced the ring homomorphisms

\[
s_L := L \xrightarrow{\lambda} A, \quad s_R := R \xrightarrow{\rho} A,
\]
\[
t_L := L^{\text{op}} \xrightarrow{\text{id}} R \xrightarrow{\rho} A, \quad t_R := R^{\text{op}} \xrightarrow{v} L \xrightarrow{\lambda} A. \quad (3.10)
\]

Also using the notation \( \theta \) for the inverse of the Nakayama automorphism when considered as a map

\[
\theta : L \xrightarrow{v^{-1}} R^{\text{op}},
\]
Eq. (3.7) can be read as

\[
SA(l_1 \cdot \alpha \cdot l_2) = \theta(l_2) \cdot SA(\alpha) \cdot \theta(l_1). \quad (3.11)
\]

**Remark 3.1.** The apparent asymmetry between \( t_L \) and \( t_R \) in (3.10) disappears if one repeats the above construction for the more general situation of a Frobenius \( N-M \)-bimodule \( X \) instead of the \( N \cdot M \) arising from a Frobenius extension of rings. As a matter of fact,
denoting by $X$ the (two-sided) dual of $X$ and setting $A = \text{End} X \otimes_M X$, $L = \text{End} X$, and $R = \text{End} X$, we find the obvious ring homomorphisms

$$s_L(l) = l \otimes X, \quad s_R(r) = X \otimes r;$$

but there is no distinguished map $L \rightarrow R^{\text{op}}$ like the identity is in the case of $X = NMM$. Instead we have two distinguished maps given by the left and right dual functors (transpositions). It is easy to check that in case of $X = NMM$ they are the identity and the $\nu^{-1}$, respectively, as we used in (3.10).

In order to restore the symmetry, let us introduce the counterpart of $\theta$ which is the identity as a homomorphism $\iota: R \xrightarrow{id} L^{\text{op}}$. Then, in addition to (3.11) the antipode satisfies also

$$S_A(r_1 \cdot \alpha \cdot r_2) = \iota(r_2) \cdot S_A(\alpha) \cdot \iota(r_1). \quad (3.12)$$

The most important consequence of (3.11) and (3.12) is the existence of a tensor square of $S_A$. In the case of Hopf algebras, one often uses expressions like $(S \otimes S) \circ \Sigma$, where $\Sigma$ is the symmetry $A \otimes A \rightarrow A \otimes A, x \otimes y \mapsto y \otimes x$ in the category of $k$-modules. Now we have bimodule categories $L_M$ and $R_M$ without braiding, so $\Sigma$ does not exist and neither do $S_A \otimes S_A$ nor $S_A^{-1} \otimes S_A^{-1}$ because $S_A$ is not a bimodule map. Instead we have the twisted bimodule properties (3.11) and (3.12) which guarantee the existence of the ‘composite of’ $S_A \otimes S_A$ and $\Sigma$ although individually they do not exist. More precisely, there exist twisted bimodule maps

$$S_{A \otimes L}: A \otimes L A \rightarrow A \otimes_R A, \quad \alpha \otimes_L \beta \mapsto S_A(\beta) \otimes_R S_A(\alpha),$$

$$S_{A \otimes R}: A \otimes_R A \rightarrow A \otimes_L A, \quad \alpha \otimes_R \beta \mapsto S_A(\beta) \otimes_L S_A(\alpha).$$

For later convenience let us record some useful formulas following directly from (3.2) and (3.10):

$$S_A \circ s_L = t_L \circ \nu^{-1}, \quad S_A \circ t_L = s_L, \quad t_L \circ \iota = s_R,$$

$$S_A \circ s_R = t_R \circ \nu^{-1}, \quad S_A \circ t_R = s_R, \quad t_R \circ \theta = s_L. \quad (3.13)$$

Notice also that $s_L(L)$ and $t_R(R)$ are the same subrings of $A$ and similarly $t_L(L) = s_R(R)$.

### 3.2. Two-sided bialgebroids

Recall from [8] that for depth 2 extensions $N \rightarrow M$ the $A$ has a canonical left bialgebroid structure over $L$ in which the coproduct $\gamma_L: A \rightarrow A \otimes L A$ is an $L$-$L$-bimodule map with respect to the bimodule structure (3.8). If $N \rightarrow M$ is also Frobenius then there is another right bialgebroid structure on $A$, canonically associated to a choice of $\psi$, in which $R$ is the base and $A$ is an $R$-$R$-bimodule via (3.9). Moreover, these two structures
are related by the antipode. This two-sided structure is our motivating example of a Hopf algebroid.

We start with a technical lemma on the left and right quasibases. $A$ and $B$ denotes the rings as before.

**Lemma 3.2.** Let $N \to M$ be a Frobenius extension and $\psi$, $u_i \otimes v_i$ be a fixed Frobenius structure. Let $n$ be a positive integer and $\beta_i, \gamma_i \in A$ and $c_i, b_i \in B$, for $i = 1, \ldots, n$. Assume they are related via $b_i = F(\gamma_i)$ and $c_i = \dot{F}(\beta_i)$. Then the following conditions are equivalent (summation symbols over $i$ suppressed):

(i) $b_1^i \otimes_N b_2^i \beta_i(m) = m \otimes_N 1_M, m \in M$;
(ii) $\gamma_i(m)c_1^i \otimes_N c_2^i = 1_M \otimes_N m, m \in M$;
(iii) $\gamma_i(m)\beta_i(m') = \psi(mm'), m, m' \in M$;
(iv) $b_1^i \otimes_N b_2^i c_1^i \otimes_N c_2^i = u_k \otimes_N 1_M \otimes_N v_k$.

If such elements exist the extension is called D2, i.e., of depth 2. The first two conditions are meaningful also in the non-Frobenius case and therefore $\{b_i, \beta_i\}$ was called in [8] a left D2 quasibasis and $\{c_i, \gamma_i\}$ a right D2 quasibasis. The equivalence of conditions (i) and (ii) was shown in [8, Proposition 6.4]. The rest of the proof is left to the reader.

For D2 extensions the map $\alpha \otimes \beta \mapsto \{m \otimes m' \mapsto \alpha(m)\beta(m')\}$ is an isomorphism

$$A \otimes LA \longrightarrow \text{Hom}_{N-N}(M \otimes NM, M),$$

see [8, Proposition 3.11]. Then the coproduct $\gamma_L : A \to A \otimes LA$ is the unique map $\alpha \mapsto \alpha_1 \otimes \alpha(1)$ which satisfies

$$\alpha(mm') = \alpha_1(m)\alpha(1)(m').$$

We can dualize this construction for D2 Frobenius extensions. We have the isomorphism

$$A \otimes RA \longrightarrow \text{Hom}_{N-N}(M, M \otimes NM), \quad \alpha \otimes R \beta \mapsto \alpha(-u_i) \otimes N \beta(v_i).$$

Then $\gamma_R : A \to A \otimes RA$ is defined as the unique map $\alpha \mapsto \alpha^{(1)} \otimes \alpha^{(2)}$ for which

$$\alpha(m)u_i \otimes v_i = \alpha^{(1)}(mu_i) \otimes N \alpha^{(2)}(v_i).$$  (3.14)

Explicit formulas for both coproducts, as well as their counits, are given in the corollary below. But even without these formulas we can find out how the two coproducts are related by the antipode.

**Theorem 3.3.** For a D2 Frobenius extension $N \to M$ of rings the endomorphism ring $A = \text{End}_N M_N$ is a left bialgebroid over $L = C_M(N)$ and a right bialgebroid over $R = L^{op}$ such that the antipode defined in (3.2) gives rise to isomorphisms
of left bialgebroids. That is to say,

\[ (A, R, s_R, t_R, \gamma_R, \pi_R) \xrightarrow{\text{op}} (A, L, s_L, t_L, \gamma_L, \pi_L) \]

\[ (A, L, s_L, t_L, \gamma_L, \pi_L) \xrightarrow{(S_A, \theta)} (A, R, s_R, t_R, \gamma_R, \pi_R) \xrightarrow{\text{cop}} \]

(3.15)

\[ S_A \circ s_R = s_L \circ \iota, \quad S_A \circ t_R = t_L \circ \iota, \]

(3.16)

\[ S_A \circ s_L = s_R \circ \theta, \quad S_A \circ t_L = t_R \circ \theta, \]

(3.17)

\[ \gamma_L \circ S_A = S_A \circ \gamma_R, \quad \gamma_R \circ S_A = S_A \circ \gamma_L, \]

(3.18)

\[ \pi_L \circ S_A = \iota \circ \pi_R, \quad \pi_R \circ S_A = \theta \circ \pi_L. \]

(3.19)

Proof. The left bialgebroid structure of \( A \) has been constructed in [8, Theorem 4.1]. The right bialgebroid structure will follow automatically after establishing the four properties of the antipode. The first two have already been discussed before. In order to prove (3.18) recall the definition (3.14) of \( \gamma_R \). Thus, (3.18) is equivalent to

\[ S_A^{-1}(S_A(\alpha)(1))(m u_i) \odot_N S_A^{-1}(S_A(\alpha)(1))(v_i) = \alpha(m) u_i \otimes_N v_i, \]

(3.20)

\[ S_A(S_A^{-1}(\alpha)(1))(m u_i) \odot_N S_A(S_A^{-1}(\alpha)(1))(v_i) = u_i \otimes_N v_i \alpha(m) \]

(3.21)

for all \( m \in M \). Expanding the left-hand side of (3.20) then using (3.6), then (3.4), then the definition of \( \gamma_L \), and finally (3.6) again we obtain

\[ S_A^{-1}(S_A(\alpha)(1))(m u_i) \psi(S_A^{-1}(S_A(\alpha)(1))(v_i) u_k) \odot_N v_k \]

\[ = S_A^{-1}(S_A(\alpha)(1))(m u_i) \psi(v_i S_A(\alpha)(1)(u_k)) \odot_N v_k \]

\[ = S_A^{-1}(S_A(\alpha)(1))(m S_A(\alpha)(1)(u_k)) \odot_N v_k \]

\[ = \psi(m S_A(\alpha)(1)(u_k) S_A(\alpha)(1)(u_j)) v_j \otimes_N v_k \]

\[ = \psi(m S_A(\alpha)(u_k u_j)) v_j \otimes_N v_k = \psi(\alpha(m) u_k u_j) v_j \otimes_N v_k \]

\[ = \alpha(m) u_k \otimes_N v_k \]

and analogously (3.21). Now it is easy to see that both \( \pi_L \circ S_A \) and \( \theta \circ \pi_L \circ S_A^{-1} \) are counits for \( \gamma_R \). Therefore, both are equal to the counit \( \pi_R \). This finishes the proof of the isomorphisms (3.15).

Corollary 3.4. Explicit formulas for the left and right bialgebroid structures can be given using the quasibases of Lemma 3.2 as follows:
\[ \gamma_L(\alpha) = \gamma_1 \otimes_L c_1^1 \alpha(c_{1,-}^1) = \alpha(b_1^1) b_1^2 \otimes L \beta_i = \gamma_1 \otimes_L \beta_i \ast \alpha = \alpha \ast \gamma_1 \otimes_L \beta_i, \]
\[ \pi_L(\alpha) = \alpha(1_M), \]
\[ \gamma_R(\alpha) = \alpha(\gamma_i(\gamma^1) c_{1,-}^2 \otimes_R \psi(v_k) \beta_i(v_k) \_ \_ \otimes_R b_1^1 \alpha(b_{2,-}^2) = \alpha \ast S_A(\beta_i) \otimes_R S_A(\gamma_i) \ast \alpha, \]
\[ \pi_R(\alpha) = u_i \psi \circ \alpha(v_i). \]

4. Hopf algebroids

4.1. The definition

The total ring of a Hopf algebroid carries eight canonical module structures over the base ring—modules of the kind (2.1), (2.7)–(2.9). In this situation the standard notation for the tensor product of modules, e.g., \( A \otimes_R A \), would be ambiguous. In order to avoid any misunderstandings, we therefore put marks on both modules, as in \( A \otimes_R A \) for example, that indicate the module structures taking part in the tensor product. Other module structures (commuting with those taking part in the tensor product) are usually unadorned and should be clear from the context.

For coproducts of left bialgebroids we use the Sweedler’s notation in the form \( \gamma_L(a) = a(1) \otimes a(2) \) and of right bialgebroids \( \gamma_R(a) = a(1) \otimes a(2) \).

**Definition 4.1.** The Hopf algebroid is a pair \((A_L, S)\) consisting of a left bialgebroid \( A_L = (A, L, s_L, t_L, \gamma_L, \pi_L) \) and an anti-automorphism \( S \) of the total ring \( A \) satisfying

\[ (i) \quad S \circ t_L = s_L \quad \text{and} \]
\[ (ii) \quad S^{-1}(a(2))_{(1)} \otimes S^{-1}(a(2))_{(2)} a(1) = S^{-1}(a) \otimes 1_A, \]
\[ S(a(1)) a(2) \otimes S(a(1))_{(2)} = 1_A \otimes S(a) \]

as elements of \( A_L \otimes L A \), for all \( a \in A \).

The axiom (4.3) implies that

\[ S(a(1)) a(2) = t_L \circ \pi_L \circ S(a) \]

for all \( a \in A \). Introduce the map \( \theta_L := \pi_L \circ S \circ s_L : L \to L \). Owing to (4.4), it satisfies

\[ t_L \circ \theta_L(l) = t_L \circ \pi_L \circ S \circ s_L(l) = S \circ s_L(l), \]
\[ \theta_L(l) \theta_L(l') = \pi_L \circ S \circ s_L(l) \pi_L \circ S \circ s_L(l') = \pi_L(1_L \circ \pi_L \circ S \circ s_L(l') S \circ s_L(l)) \]
\[ = \pi_L(S \circ s_L(l') S \circ s_L(l)) = \theta_L(l'). \]
In view of (4.5) $S$ is a twisted bimodule map $^R A^L \to L A_L$ where $R$ is a ring isomorphic to $L^{op}$ and the $R$-$R$-bimodule structure of $A$ is given by fixing an isomorphism $\mu : L^{op} \to R$:

$$r \cdot a \cdot r' := a s_L \circ \theta_L^{-1} \circ \mu^{-1}(r) t_L \circ \mu^{-1}(r'). \quad (4.6)$$

The usage of the same notation $^R A^L$ as in (2.8) is not accidental. It will turn out from the next Proposition 4.2 that there exists a right bialgebroid structure on the total ring $A$ over the base $R$ for which the $R$-$R$-bimodule (2.8) is (4.6).

It makes sense to introduce the maps

$$S_{A \otimes LA} : A_L \otimes L A \to A_R \otimes R A, \quad a \otimes b \mapsto S(b) \otimes S(a) \quad \text{and} \quad S_{A \otimes RA} : A_R \otimes R A \to A_L \otimes L A, \quad a \otimes b \mapsto S(b) \otimes S(a). \quad (4.7)$$

It is useful to give some alternative forms of the Definition 4.1.

Proposition 4.2. The following are equivalent:

(i) $(A_L, S)$ is a Hopf algebroid;

(ii) $A_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left bialgebroid and $S$ is an anti-automorphism of the total ring $A$ satisfying (4.1), (4.4), and

$$S_{A \otimes LA} \circ \gamma_L \circ S^{-1} = S_{A \otimes RA} \circ \gamma_L \circ S, \quad (4.8)$$

$$(\gamma_L \otimes id_A) \circ \gamma_R = (id_A \otimes \gamma_R) \circ \gamma_L, \quad (\gamma_R \otimes id_A) \circ \gamma_L = (id_A \otimes \gamma_L) \circ \gamma_R, \quad (4.9)$$

where we introduced the ring $R$ and the $R$-$R$-bimodule $^R A^L$ as in (4.6) and the map

$$\gamma_R := S_{A \otimes LA} \circ \gamma_L \circ S^{-1} \equiv S_{A \otimes RA} \circ \gamma_L \circ S : A \to A_R \otimes R A.$$

The equations in (4.9) are equalities of maps $A \to A_L \otimes L A_R \otimes R A$ and $A \to A_R \otimes R A_L \otimes L A$, respectively.

(iii) $A_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left bialgebroid and $A_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ is a right bialgebroid such that the base rings are related to each other via $R \simeq L^{op}$. $S$ is a bijection of additive groups and

$$s_L(L) = t_R(R), \quad t_L(L) = s_R(R) \quad \text{as subrings of } A, \quad (4.10)$$

$$(\gamma_L \otimes id_A) \circ \gamma_R = (id_A \otimes \gamma_R) \circ \gamma_L, \quad (\gamma_R \otimes id_A) \circ \gamma_L = (id_A \otimes \gamma_L) \circ \gamma_R, \quad (4.11)$$

$$S(t_L(l)at_L(l')) = s_L(l')s_L(l), \quad S(t_R(r')at_R(r)) = s_R(r)s_R(r'), \quad (4.12)$$

$$S(a(l_1)a(l_2)) = s_R \circ \pi_R(a), \quad a^{(1)}S(a^{(2)}) = s_L \circ \pi_L(a) \quad (4.13)$$

hold true for all $l, l' \in L, r, r' \in R$, and $a \in A$. 


\[ \text{721} \]
(iv) \( A_L \) is a left bialgebroid over \( L \) and \( A_R \) is a right bialgebroid over \( R \) such that the base rings are related to each other via \( R \cong L^{\text{op}} \) and Eqs. (4.10) and (4.11) hold true. Furthermore, the maps of additive groups

\[
\alpha : A^R \otimes_RA \to A_L \otimes_LA, \quad a \otimes b \mapsto a_{(1)} \otimes a_{(2)}b \quad \text{and} \\
\beta : A_R \otimes^R_A \to A_L \otimes_LA, \quad a \otimes b \mapsto b_{(1)}a \otimes b_{(2)}
\]

(4.14)

are bijective. (All modules appearing in (4.14) are the canonical modules introduced in Section 2.)

Each characterization of Hopf algebroids in Proposition 4.2 will be relevant in what follows. The one in (ii) is as similar to [13] as possible which will be useful in Section 4.3 both in checking that concrete examples of Lu–Hopf algebroids satisfy the axioms in Definition 4.1 and also in constructing Hopf algebroids in the sense of Definition 4.1 which do not satisfy the Lu axioms.

As it will turn out from the following proof of Proposition 4.2, the Definition 4.1 implies the existence of a right bialgebroid structure on the total ring of the Hopf algebroid. The characterization in (iii) uses the left and right bialgebroids underlying a Hopf algebroid in a perfectly symmetric way. This characterization will be appropriate in developing the theory of integrals in Section 5.

The characterization in (iv) is formulated in the spirit of [18], that is the bijectivity of certain Galois maps is required. The relevance of this form of the definition is that it shows that the Hopf algebroid in the sense of Definition 4.1 is a special case of Schauenburg’s \( \times_L \)-Hopf algebra.

Proof. (ii) \( \Rightarrow \) (iii). We construct a right bialgebroid \( A_R \) such that \((A_L, A_R, S)\) satisfies the requirements in (iii). Let \( R \) be a ring isomorphic to \( L^{\text{op}} \) and \( \mu : L^{\text{op}} \to R \) a fixed isomorphism. Set

\[
A_R = (A, R, s_R := t_L \circ \mu^{-1}, t_R := S^{-1} \circ t_L \circ \mu^{-1}, \gamma_R := S^{-1}_{A \otimes_R A} \circ \gamma_L \circ S, \\
\pi_R := \mu \circ \pi_L \circ S).
\]

(4.15)

(iii) \( \Rightarrow \) (iv). We construct the inverses of the maps (4.14):

\[
\alpha^{-1} : A_L \otimes_LA \to A^R \otimes_RA, \quad a \otimes b \mapsto a^{(1)} \otimes S(a^{(2)})b \quad \text{and} \\
\beta^{-1} : A_L \otimes_LA \to A_R \otimes^R_A, \quad a \otimes b \mapsto S^{-1}(b^{(1)})a \otimes b^{(2)}.
\]

(4.16)

(iv) \( \Rightarrow \) (i). Recall that the requirements in (iv) imply that both left bialgebroids \( A_L \) and \( A_{L^{\text{op}}} \) are \( \times_L \)-Hopf algebras in the sense of [18]. In particular, [18, Proposition 3.7] holds true for both. That is, denoting \( \alpha^{-1}(a \otimes 1_A) := a_+ \otimes a_- \) and \( \beta^{-1}(1_A \otimes a) := a_{[-]} \otimes a_{[+]} \), we have
(i) \( a_{+}(1) \otimes a_{+}(2)a_{-} = a \otimes 1_A \), \( a_{+}(1)a_{[-]} \otimes a_{+}(2) = 1_A \otimes a \),

(ii) \( a_{(1)+} \otimes a_{(1)-}a_{(2)} = a \otimes 1_A \), \( a_{(2)-}a_{(1)+} \otimes a_{(2)[+]1]} = 1_A \otimes a \),

(iii) \( (ab)_{+} \otimes (ab)_{-} = a_{+}b_{+} \otimes b_{-}a_{-} \), \( (ab)_{[-]} \otimes (ab)_{[+]1]} = b_{[-]}a_{[-]} \otimes a_{[+]1]}b_{[+]1]} \),

(iv) \( (1_A)_{+} \otimes (1_A)_{-} = 1_A \otimes 1_A \), \( (1_A)_{[-]} \otimes (1_A)_{[+]1]} = 1_A \otimes 1_A \),

(v) \( a_{+}(1) \otimes a_{+}(2) \otimes a_{-} = a_{(1)+} \otimes a_{(2)+} \otimes a_{(2)-} \),

\( a_{[-]} \otimes a_{[+]1]} \otimes a_{[+]2]} = a_{[+]1]} \otimes a_{[+]2]} \otimes a_{(2)+} \),

(vi) \( a_{+} \otimes a_{[-]}a_{[-]}a_{[-]} = a_{++} \otimes a_{-} \otimes a_{++} \),

\( a_{[-]} \otimes a_{[-]}a_{[-]} = a_{[-]} \otimes a_{[-]} \otimes a_{[+]2]} \),

(vii) \( a = a_{+}tL \circ \piL(a_{-}) \), \( a = a_{+}sL \circ \piL(a_{-}) \),

(viii) \( a_{+}a_{-} = sL \circ \piL(a) \), \( a_{[+]1]}a_{[-]} = tL \circ \piL(a) \). (4.17)

We define the antipode as

\[ S(a) := sR \circ \piR(a_{+})a_{-} \] (4.18)

and what is going to be its inverse as

\[ S'(a) := tR \circ \piR(a_{[-]}a_{-}) \] (4.19)

The maps (4.18) and (4.19) are well-defined due to the \( R \)-module map property of \( \piR \).

Since \( a_{1}A \otimes sL(l) = 1_A \otimes sL(l) \equiv tL(l) \otimes 1_A \), the requirement (4.1) holds true. By making use of (vi) and (i) of (4.17), one verifies

\[ S(a_{(1)})(1)_{2} \otimes S(a_{(1)})(2)_{2} = a_{(1)-}a_{(2)} \otimes sR \circ \piR(a_{(1)+})a_{(1)-(2)} = S(a) \]

and similarly

\[ S'(a_{(2)})(1)_{2} \otimes S'(a_{(2)})(2)_{2}a_{(1)} = S'(a) \otimes 1_A, \]

which becomes the requirement (4.2) once we proved \( S' = S^{-1} \). As a matter of fact by (vi) of (4.17)

\[ S(a)(1) \otimes S(a)(2) = a_{(1)-} \otimes sR \circ \piR(a_{(2)+})a_{(2)-} = a_{-} \otimes S(a) \]

hence, using (ii) of (4.17),

\[ \beta(a_{(2)} \otimes S(a_{(1)})) = a_{(1)+}a_{(2)} \otimes S(a_{(1)}+) = 1_A \otimes S(a), \]

so, by (viii) of (4.17) and (4.10),
\[ S' \circ S(a) = t_R \circ \pi_R \left( s_R \circ \pi_R(a_{(1)})a_{(1)} - a_{(2)} \right) = t_R \circ \pi_R(a_{(1)})a_{(1)} - a_{(2)} = s_L \circ \pi_L(a_{(1)})a_{(2)} = a. \]

In a similar way one checks that \( S \circ S' = \text{id}_A \).

The anti-multiplicativity of \( S \) is proven as follows. We have \( \alpha(t_R(r) \otimes 1_A) = t_R(r) \otimes 1_A \) and by \( \beta(1_A \otimes s_R(r)) = 1_A \otimes s_R(r) \) and \( S' = S^{-1} \) also \( S(t_R(r)a) = S(a)s_R(r) \) hence,

\[ S(ab) = s_R \circ \pi_R((ab)_+) \cdot (ab)_- = s_R \circ \pi_R(a_+ b_+ b_- a_-) = s_R \circ \pi_R \left( t_R \circ \pi_R(a_+ b_+ b_-) \right) a_- = S(t_R \circ \pi_R(a_+ b_+) a_- = S(b)s_R \circ \pi_R(a_+)a_- = S(b)S(a). \]

(i) \( \Rightarrow \) (ii). The requirements (4.1) and (4.4) hold obviously true. One easily checks that the maps \( \alpha \) and \( \beta \) in (4.14) are bijective with inverses

\[ \alpha^{-1}(a \otimes b) = S^{-1}(S(a)_{(2)}) \otimes S(a)_{(1)}b, \]
\[ \beta^{-1}(a \otimes b) = S^{-1}(b)_{(2)}a \otimes S^{-1}(b)_{(1)}. \]

This implies that the [18, Proposition 3.7] holds true both in \( \mathcal{A}_L \) and \( \mathcal{A}_{L_cop} \). In particular, introducing the maps

\[ \gamma_R : A \rightarrow A^R \otimes ^R A, \quad a \mapsto (\text{id}_A \otimes S^{-1}) \circ \alpha^{-1}(a \otimes 1_A), \]
\[ \gamma'_R : A \rightarrow A^R \otimes ^R A, \quad a \mapsto (S \otimes \text{id}_A) \circ \beta^{-1}(1_A \otimes a) \]

the part (v) of (4.17) reads as

\[ (\gamma_L \otimes \text{id}_A) \circ \gamma_R = (\text{id}_A \otimes \gamma_R) \circ \gamma_L. \quad (4.20) \]
\[ (\text{id}_A \otimes \gamma_L) \circ \gamma'_R = (\gamma'_R \otimes \text{id}_A) \circ \gamma_L. \quad (4.21) \]

This means that both (4.8) and (4.9) follow provided \( \gamma_R = \gamma'_R \). Using the Sweedler’s notation \( \gamma_R(a) = a^{(1)} \otimes a^{(2)} \) and \( \gamma'_R(a) = a^{(1)} \otimes a^{(2)} \) by the repeated use of (4.20) and (4.21), we obtain

\[ (\text{id}_A \otimes \gamma_R) \circ \gamma'_R(a) = a^{(1)} \otimes s_L \circ \pi_L(a^{(2)}_{(1)}) \otimes a^{(2)}_{(2)} = a^{(1)} \otimes s_L \circ \pi_L(a^{(2)}_{(1)}) \otimes a^{(2)}_{(2)} = a^{(1)} \otimes s_L \circ \pi_L(a^{(2)}_{(1)}) \otimes a^{(2)}_{(2)} = (\text{id}_A \otimes \gamma'_R) \circ \gamma_R(a). \]
Since by (viii) of (4.17) both \( a^{(1)}S(a^{(2)}) = \pi_L \circ s_L \circ \pi_L(a) \) and \( a^{(1)} \circ S(a^{(2)}) = s_L \circ a^{(1)} \circ \pi_L(a) \), we have

\[
\varepsilon(a) := S \circ s_L \circ \pi_L \circ S^{-1}(a) = S(a^{(1)})a^{(2)} = S^{-1} \circ s_L \circ \pi_L \circ S(a)
\]

and

\[
(m_A \otimes \id_A) \circ (\id_A \otimes \varepsilon \otimes \id_A) \circ (\id_A \otimes \gamma_R) \circ \gamma'_R(a) \\
= a^{(1)} \varepsilon\big(a^{(2)}\big) \otimes a^{(2)} = a^{(1)} \otimes a^{(2)}S^{-2} \circ s_L \circ \pi_L \circ S(a^{(2)}) \\
= S^{-1}\big(a^{(1)}\big)S \circ s_L \circ \pi_L \circ S^{-1}\big(a^{(1)}\big) \otimes a^{(2)} \\
= S\big(S^{-1}\big(a^{(1)}\big)\big) S \circ s_L \circ \pi_L \big(S^{-1}\big(a^{(1)}\big)\big) \otimes a^{(2)} = \gamma_R(a)
\]

hence by the equality of the left-hand sides \( \gamma_R = \gamma'_R \).

The following is a consequence of the proof of Proposition 4.2.

**Proposition 4.3.** Let \((A_L, S)\) be a Hopf algebroid and \(A_R\) a right bialgebroid such that \((A_L, A_R, S)\) satisfies the requirements of Proposition 4.2(iii). Then both \((S : A \rightarrow A^{\text{op}}, \nu := \pi_R \circ s_L : L \rightarrow R^{\text{op}})\) and \((S^{-1} : A \rightarrow A^{\text{op}}, \mu := \pi_R \circ t_L : L \rightarrow R^{\text{op}})\) are left bialgebroid isomorphisms \(A_L \rightarrow (A_R)^{\text{op}}\). In particular, \(A_R\) is unique up to an isomorphism of the form \((\id_A, \phi)\).

One easily checks that \(\mu^{-1} \circ \nu = \varepsilon_L\). For the sake of symmetry we introduce also \(\theta_R := \nu \circ \mu^{-1}\) with the help of which the right analogue of (4.5) holds true:

\[
S \circ s_R = t_R \circ \theta_R.
\]

Proposition 4.3 has an interpretation in terms of the forgetful functors \(\Phi_R : \mathcal{M}_A \rightarrow R\mathcal{M}_R\) and \(\Phi_L : \mathcal{M} \rightarrow R\mathcal{M}_R\) as follows. The antipode map defines two functors \(S\) and \(S' : \mathcal{M}_A \rightarrow \mathcal{M}\). They have object maps \((M, <) \mapsto (M, \varepsilon_S)\) and \((M, <) \mapsto (M, \langle S, <\rangle)\), respectively, and the identity maps on the morphisms. It is clear that \(S\) and \(S'\) are strict antimoidal equivalence functors. The ring automorphisms \(\mu\) and \(\nu\) define endo-functors \(\mu\) and \(\nu\) of \(R\mathcal{M}_R\). The object maps are \((M, \langle, <\rangle) \mapsto (M, \langle \varepsilon_S, \langle, <\rangle\rangle)\) and \((M, \langle, <\rangle) \mapsto (M, \langle \mu, \langle, <\rangle\rangle)\), respectively, and the identity map on the morphisms. The \(\mu\) and \(\nu\) are also strict antimoidal equivalence functors. We have then equalities of strong monoidal functors \(\Phi_L \circ S = \nu \circ \Phi_R\) and \(\Phi_L \circ S' = \mu \circ \Phi_R\).

Finally, we define the morphisms of Hopf algebroids.

**Definition 4.4.** A Hopf algebroid homomorphism (isomorphism) \((A_L, S) \rightarrow (A'_L, S')\) is a left bialgebroid homomorphism (isomorphism) \(A_L \rightarrow A'_L\). A Hopf algebroid homomorphism \((\Phi, \phi)\) is strict if \(S' \circ \phi = \phi \circ S\).
The existence of non-strict isomorphisms of Hopf algebroids—that is the non-uniqueness of the antipode in a Hopf algebroid—is a new feature compared to (weak) Hopf algebras. The antipodes making a given leftbialgebroid into a Hopf algebroid are characterized in [2].

In the following (in particular in Section 5) we are going to call a triple \((A_L, A_R, S)\) satisfying Proposition 4.2(iii) a **symmetrized form** of the Hopf algebroid \((A_L, S)\). The \(A_R\) is called the right bialgebroid underlying \((A_L, S)\). In the view of Proposition 4.3 the symmetrized form is unique up to the choice of the base ring \(R\) of \(A_R\) within the isomorphism class of \(L^{op}\)—the opposite of the base ring of \(A_L\)—and the isomorphism \(\mu : L^{op} \to R\) in (4.15).

Let us define the **homomorphisms** of symmetrized Hopf algebroids \(A = (A_L, A_R, S) \to A' = (A'_L, A'_R, S')\) as pairs of bialgebroid homomorphisms \((\Phi_L, \phi_L) : A_L \to A'_L\), \((\Phi_R, \phi_R) : A_R \to A'_R\). Then by Proposition 4.3 the Hopf algebroid homomorphisms \((\Phi, \phi) : (A_L, S) \to (A'_L, S')\) are injected into the homomorphisms of symmetrized Hopf algebroids \((A_L, A_R, S) \to (A'_L, A'_R, S')\) via \((\Phi, \phi) \mapsto (\Phi_L, \Phi_R, (\Phi, \phi)(S'^{-1} \circ \Phi \circ S, \mu' \circ \phi \circ \mu^{-1}))\)—where \(\mu = \pi_R \circ \iota_L\) and \(\mu' = \pi'_R \circ \iota'_L\) are the ring isomorphisms introduced in Proposition 4.3. This implies that two symmetrized Hopf algebroids \((A_L, A_R, S)\) and \((A'_L, A'_R, S')\) are isomorphic if and only if the Hopf algebroids \((A_L, S)\) and \((A'_L, S')\) are isomorphic.

A homomorphism \(((\Phi_L, \phi_L), (\Phi_R, \phi_R))\) of symmetrized Hopf algebroids \(A \to A'\) is **strict** if \(\Phi_L = \Phi_R\) as homomorphisms of rings \(A \to A'\). We leave it to the reader to check that this is equivalent to the requirement that \((\Phi_L, \phi_L)\) is a strict homomorphism of Hopf algebroids \((A_L, S) \to (A'_L, S')\), that is two symmetrized Hopf algebroids \((A_L, A_R, S)\) and \((A'_L, A'_R, S')\) are strictly isomorphic if and only if the Hopf algebroids \((A_L, S)\) and \((A'_L, S')\) are strictly isomorphic.

The usage of symmetrized Hopf algebroids allows for the definition of the opposite and co-opposite structures \(A^{op} = (A_R^{op}, A'_L^{op}, S^{-1})\) and \(A^{cop} = (A_L^{cop}, A_R^{cop}, S^{-1})\), respectively.

### 4.2. Relation to the Hopf bialgebroid of Day and Street

In [22] left bialgebroids over the base \(L\) have been characterized as opmonoidal monads \(T\) on the category \(L^\mathcal{M}_L\) such that their underlying functors \(T\) have right adjoints. The endofunctor \(T\) is given by tensoring over \(L^\mathcal{M}_L = L \otimes L^{op}\) with the \(L^\mathcal{M}_L\)-bimodule \(A\). The monad structure makes \(A\) into an \(L^\mathcal{M}_L\)-ring via an algebra map \(\eta : L^\mathcal{M}_L \to A\), while the opmonoidal structure comprises the coproduct and counit of the bialgebroid \(A\). In a recent preprint [7] Day and Street put this into the more general context of pseudomonoids in monoidal bicategories. Using the notion of \(*\)-autonomy [1] they propose a definition of **Hopf bialgebroid** as a \(*\)-autonomous structure on a bialgebroid. We are going to show in this section that their definition coincides with our Definition 4.1.

Working with \(k\)-algebras the base category \(V\) is the category \(\mathcal{M}_k\) of \(k\)-modules. Then the monoidal bicategory in question is \(\text{Mod}(\mathcal{V})\) having as objects the \(k\)-algebras and as hom-categories \(\text{Mod}(\mathcal{V})(A, B)\) the category \(B^\mathcal{M}_A\) of \(B\)-\(A\)-bimodules. The tensor product \(\otimes = \otimes_k\) of \(k\)-modules makes \(\text{Mod}(\mathcal{V})\) monoidal. A pseudomonoid in \(\text{Mod}(\mathcal{V})\) is a triple \((A, M, J)\) where \(A\) is an object, \(M : A \otimes A \to A\) and \(J : k \to A\) are 1-cells satisfying...
associativity and unitality up to invertible 2-cells that in turn satisfy the pentagon and the triangle constraints. If \((A, L, s, t, \gamma, \pi)\) is a left bialgebroid in \(V\) then we have a strong monoidal morphism

\[
\eta^* : (A, M, J) \rightarrow [L^e, m, j]
\]

(4.22)
of pseudomonoids where

\[
M = A \otimes_L A \quad \text{with actions} \quad a \cdot (x \otimes_L y) \cdot (a_1 \otimes a_2) = a_1 x a_1 \otimes_L a_2, \quad (4.23)
\]

\[
J = L \quad \text{with action} \quad a \cdot x = \pi(\alpha(s(x))), \quad (4.24)
\]

\[
m = L^e \otimes_L L^e \quad \text{with actions} \quad (l \otimes l') \cdot ((x \otimes x') \otimes_L (y \otimes y')) \cdot ((l_1 \otimes l_1') \otimes (l_2 \otimes l_2')) = (lx_l \otimes l'_x) \otimes_L (yl_2 \otimes l'_2 y'), \quad (4.25)
\]

\[
j = L \quad \text{with action} \quad (l \otimes l') \cdot x = l xl' \quad (4.26)
\]

and the bimodule \(\eta^* = L^e A\), induced by the algebra map \(\eta\), has a left adjoint \(\eta_* = A L^e\) with the counit \(\epsilon : \eta_* \circ \eta^* \rightarrow A\) induced by multiplication of \(A\). The connection of this bimodule picture with module categories can be seen by applying the monoidal pseudofunctor \(\text{Mod}(V)(k, -) : \text{Mod}(V) \rightarrow \text{Cat}\). Then pseudomonoids become monoidal categories, \(\eta^*\) becomes the strong monoidal forgetful functor \(\eta^* \otimes - : \Lambda M \rightarrow L^e M\), and \(\eta_*\) its opmonoidal left adjoint. An important piece of the structure is the bimodule morphism

\[
\psi : \eta_* \circ m \rightarrow M \circ (\eta_* \otimes \eta_*), \quad a \otimes (l \otimes l') \mapsto a(1) s(l) \otimes a(2) t(l'), \quad (4.27)
\]

which makes \(\eta_*\) opmonoidal and encodes the comultiplication \(\gamma : A \rightarrow A \otimes_L A\) of the bialgebroid.

In [7] a Hopf bialgebroid is defined to be a bialgebroid \(A\) over \(L\) together with a strong \(*\)-autonomous structure on the opmonoidal morphism \(\eta_* : L^e \rightarrow A\). The latter means:

(1) A \(*\)-autonomous structure on the pseudomonoid \(A\), i.e.,
   (a) a right \(A \otimes A\)-module \(\sigma\) defined on the \(k\)-module \(A\) in terms of an algebra isomorphism \(\xi : A \rightarrow A^{\text{op}}\) by
   \[
   x \cdot (a \otimes b) = \xi^{-1}(b) x a, \quad x, a, b \in A, \quad \text{and} \quad (4.28)
   \]
   (b) an isomorphism
   \[
   \Gamma : \sigma \circ (M \otimes A) \rightarrow \sigma \circ (A \otimes M).
   \]

(2) A “canonical” \(*\)-autonomous structure on the pseudomonoid \(L^e\) which consists of the following:
(a) A right $L^\ell \otimes L^\ell$-module $\sigma_0$ defined on $L^\ell$ using the isomorphism $\xi_0 : L^\ell \to (L^\ell)^{\text{op}}$, $l \otimes l' \mapsto \theta^{-1}(l') \otimes l$ for some algebra automorphism $\theta : L \to L$. Thus

$$(x \otimes x') \cdot ((l_1 \otimes l'_1) \otimes (l_2 \otimes l'_2)) = \theta^{-1}(l'_2)xl_1 \otimes l'_1x'l_2$$

for $x, x', l_1, l'_1, l_2, l'_2 \in L$ where juxtaposition is always multiplication in $L$ and never in $L^{\text{op}}$. (Note that allowing a non-trivial $\theta$ in the definition of $\sigma_0$ is a slight deviation from [7] which is, however, well motivated by Section 3.)

(b) The isomorphism

$$\Gamma_0 : \sigma_0 \circ (m \otimes L^\ell) \to \sigma_0 \circ (L^\ell \otimes m).$$

(3) The arrow $\eta_\ast$ is strongly $\ast$-autonomous in the following sense: There exists a 2-cell $\tau : \sigma_0 \circ (\eta_\ast \otimes \eta_\ast) \to \sigma$ such that

$$[\Gamma_0 \circ (\eta_\ast \otimes \eta_\ast \otimes \eta_\ast)] \bullet [\sigma \circ (\psi \otimes \eta_\ast)] \bullet [\tau \circ (L^\ell \otimes m)] \bullet \Gamma_0$$

and such that the map

$$\tau' : \sigma_0 \circ (\eta^\ast \otimes L^\ell) \to \sigma \circ (A \otimes \eta_\ast),$$

$$\tau' = [\sigma \circ (\eta \otimes \eta^\ast)] \bullet [\tau \circ (L^\ell \otimes m)] \bullet \Gamma_0$$

is an isomorphism. (We used $\circ$ and $\bullet$ to denote horizontal and vertical compositions in the bicategory $\text{Mod}(\mathcal{V})$.)

Now we are going to translate this categorical definition into simple algebraic expressions.

Lemma 4.5. $\ast$-autonomous structures on the pseudomonoid $\langle A, M, J \rangle$ are in one-to-one correspondence with the data $(\xi, \sum e_k \otimes f_k)$ where $\xi$ is an anti-automorphism of the ring $A$ and $\sum e_k \otimes f_k \in A \otimes_L A$ is such that for all $a \in A$

$$\sum_k \xi(a(1)_{(1)}e_k a(2)_{(1)}) e_k f_k = \sum_k e_k \otimes f_k \xi(a)$$

and such that there exists $\sum_j g_j \otimes h_j \in A \otimes_L A$ satisfying, for all $a \in A$:

$$\sum_j \xi^{-1}(a(2)_{(1)}g_j) \otimes \xi^{-1}(a(2)_{(2)}) h_j a(1) = \sum_j g_j \xi^{-1}(a) \otimes h_j,$$

$$\sum_{j,k} \xi(g_j_{(1)}e_k h_j) \otimes \xi(g_j_{(2)} f_k) = 1_A \otimes 1_A,$$

$$\sum_{j,k} \xi^{-1}(f_k_{(1)}g_j) \otimes \xi^{-1}(f_k_{(2)} h_j e_k) = 1_A \otimes 1_A$$

as elements of $A \otimes_L A$. 
Proof. In order to find explicit formulas for \( \Gamma \) and its inverse, we introduce the isomorphisms

\[
\varphi_+ : \sigma \circ (M \otimes A) \to A \otimes_L A, \tag{4.35}
\]

\[
x \otimes_{A \otimes A} (a \otimes_L b \otimes c) \mapsto (\xi^{-1}(c)x)_{(1)} a \otimes_L (\xi^{-1}(c)x)_{(2)} b, \tag{4.36}
\]

\[
\varphi_- : \sigma \circ (A \otimes M) \to A \otimes_L A, \tag{4.37}
\]

\[
x \otimes_{A \otimes A} (a \otimes b \otimes L c) \mapsto \xi(xa)_{(1)} b \otimes_L \xi(xa)_{(2)} c,
\]

with inverses

\[
\varphi_+^{-1}(a \otimes L b) = 1 \otimes_{A \otimes A} (a \otimes_L b \otimes 1), \quad \varphi_-^{-1}(b \otimes_L c) = 1 \otimes_{A \otimes A} (1 \otimes b \otimes L c).
\]

Then \( \Gamma' := \varphi_- \circ \Gamma \circ \varphi_+^{-1} \) is a twisted \( A^{\otimes 3} \)-automorphism of \( A \otimes_L A \) in the sense of

\[
\Gamma'(\xi^{-1}(c) \cdot (x \otimes_L y) \cdot (a \otimes b)) = \xi(a) \cdot \Gamma'(x \otimes_L y) \cdot (b \otimes c). \tag{4.38}
\]

Hence, \( \Gamma \) is uniquely determined by \( \sum_i e_i \otimes f_i = \Gamma'(1 \otimes 1) \) as

\[
\Gamma(x \otimes_{A \otimes A} (a \otimes_L b \otimes c)) = 1 \otimes_{A \otimes A} \left( x_{(1)} a \otimes \sum_k e_k x_{(2)} b \otimes_L f_k c \right) \tag{4.39}
\]

and \( \sum_k e_k \otimes f_k \) satisfies (4.31). Invertibility then implies that \( \Gamma'^{-1}(1 \otimes 1) = \sum_j g_j \otimes h_j \) satisfies the remaining equations. \( \square \)

We need also the expression for \( \Gamma_0 \). We leave it to the reader to check that

\[
\Gamma_0((x \otimes x') \otimes_{L' \otimes L'} (l_1 \otimes l'_1) \otimes (l_2 \otimes l'_2) \otimes (l_3 \otimes l'_3))
\]

\[
= (1 \otimes 1) \otimes_{L' \otimes L'} (xl_1 \otimes l'_1) \otimes (l_2 \otimes l'_2 x') \otimes_L (l_3 \otimes l'_3) \tag{4.40}
\]

is well-defined and is invertible.

**Lemma 4.6.** The \( \ast \)-autonomous property of \( \eta_\ast \) is equivalent to the existence of an element \( i \in A \) satisfying

\[
i \eta(\theta^{-1}(l') \otimes l) = \xi^{-1}(\eta(l \otimes l')) i, \quad l, l' \in L, \tag{4.41}
\]

\[
\sum_k e_k \otimes f_k \xi(i) = \xi(i)_{(1)} \otimes \xi(i)_{(2)}, \tag{4.42}
\]

while strong \( \ast \)-autonomy adds the requirement that \( i \) be invertible.
Proof. Since $1 \otimes 1$ is a cyclic vector in the $L^c \otimes L^c$-module $\sigma_0$, the module map $\tau$ is uniquely determined by $i := \tau(1 \otimes 1)$ as

$$\tau(x \otimes x') = i \eta(x \otimes x'), \quad (4.43)$$

Tensoring with $\eta_* = A A L^c$ from the right being the restriction via $\eta: L^c \rightarrow A$ the element $i$ is subject to condition (4.41) due to (4.28). Using the expressions (4.43), (4.39), (4.40), and (4.27) the $\ast$-autonomy condition (4.29) becomes equation (4.42). The mate of $\tau$ in (4.30) can now be written as

$$\tau^l((x \otimes x') \otimes_{L^c \otimes L^c} (a \otimes (l \otimes l'))) = i \eta(x \otimes x') a \otimes_{A \otimes L^c} (1 \otimes (l \otimes l')).$$

Hence $\tau^l$ is invertible iff the map $a \mapsto ia$ is, i.e., iff $i$ is invertible. $\blacksquare$

Theorem 4.7. A strong $\ast$-autonomous structure on the bialgebroid $A$ over $L$ in the sense of [7] is equivalent to a Hopf algebroid structure $(A_L, S)$ in the sense of Definition 4.1.

Proof. Using invertibility of $i \in A$ condition (4.41) has the equivalent form

$$\xi(i \eta(l' \otimes l)i^{-1}) = \eta(l \otimes \theta(l')) \quad (4.44)$$

and (4.42) can be used to express the element $\sum_k e_k \otimes f_k$ in terms of $i$,

$$e_i \otimes f_i = \xi(i(1) \otimes \xi(i(2))i^{-1}).$$

The conditions (4.31)–(4.34) are then equivalent to

$$g_j \otimes h_j = i_{(1)}i^{-1}_1 \otimes i_{(2)}, \quad (4.45)$$

$$[i^{-1}_2 \xi^{-1}(a_2) i]_{(1)} \otimes [i^{-1}_2 \xi^{-1}(a_2) i]_{(2)} a_{(1)} = i^{-1}_1 \xi^{-1}(a) i \otimes 1_A, \quad (4.46)$$

$$\xi(i a_{(1)} i^{-1})_{(1)} a_{(2)} \otimes \xi(i a_{(1)} i^{-1})_{(2)} = 1_A \otimes \xi(ai^{-1}). \quad (4.47)$$

This means that introducing $S: A \rightarrow A^{\text{op}}, a \mapsto \xi(ai^{-1})$, the conditions (4.44), (4.46), and (4.47) are identical to the axioms (4.1)–(4.3), respectively. $\blacksquare$

4.3. Examples

In addition to our motivating example in Section 3, let us collect some more examples of Hopf algebroids.

Example 4.8. Weak Hopf algebras with bijective antipode. Let $(H, \Delta, \varepsilon, S)$ be a weak Hopf algebra [3,4,14] (WHA) over the commutative ring $k$ with bijective antipode. This means that $H$ is an associative unital $k$ algebra, $\Delta: H \rightarrow H \otimes_k H$ is a coassociative coproduct.
It is an algebra map (i.e., multiplicative) but not unit preserving in general. In its stead we have weak comultiplicativity of the unit

\[ 1_{[1]} \otimes 1_{[2]} = 1_{[1]} \otimes 1_{[2]} \otimes 1_{[3]} = 1_{[1]} \otimes 1_{[2]} \otimes 1_{[1]'}, \]

where \( 1_{[1]} \otimes 1_{[2]} = \Delta(1) \). The map \( \varepsilon : H \to k \) is the counit of the coproduct \( \Delta \). Instead of being multiplicative, it is weakly multiplicative:

\[ \varepsilon(ab|[1])\varepsilon(h|[2])c = \varepsilon(abc) = \varepsilon(ab|[2])\varepsilon(h|[1])c \quad \text{for } a, b, c \in H. \]

The bijective map \( S : H \to H \) is the antipode, subject to the axioms

\[ h[1]S(h[2]) = \varepsilon(h[1])h[2], \quad S(h[1])h[2] = \varepsilon(h[1])h[2], \quad S(h[1])h[2]S(h[3]) = S(h) \]

for all \( h \in H \). (If \( H \) is finite over \( k \) then the assumption made about the bijectivity of \( S \) is redundant.) The WHA \((H, \Delta, \varepsilon, S)\) is a Hopf algebra if and only if \( \Delta \) is unit preserving.

The algebra \( H \) contains two commuting subalgebras: \( R \) is the image of \( H \) under the projection \( \cap R : h \mapsto 1_{[1]}\varepsilon(h[2]) \) and \( L \) under \( \cap L : h \mapsto \varepsilon(1_{[1]}h)1_{[2]} \)—generalizing the subalgebra of the scalars in a Hopf algebra. Both maps \( S \) and \( S^{-1} \) restrict to algebra anti-isomorphisms \( R \to L \). We have four commuting actions of \( L \) and \( R \) on \( H \):

\[ H^R : h \cdot r := hr, \quad RL : r \cdot h := hS^{-1}(r), \]
\[ HL : h \cdot l := S^{-1}(l)h, \quad LH : l \cdot h := lh. \]

Introduce the canonical projections \( p_R : H \otimes_k H \to H^R \otimes R H \) and \( p_L : H \otimes_k H \to HL \otimes LH \). There exists a left and a right bialgebroid structure corresponding to the weak Hopf algebra:

\[ \mathcal{H}_R := (H, R, \text{id}_R, S^{-1}|_R, p_R \circ \Delta, \cap R). \]
\[ \mathcal{H}_L := (H, L, \text{id}_L, S^{-1}|_L, p_L \circ \Delta, \cap L). \]

We leave it as an exercise to the reader to check that \((\mathcal{H}_L, \mathcal{H}_R, S)\) satisfies the requirements of Proposition 4.2(iii).

Notice that the examples of the above class are not necessarily finite dimensional and not even finitely generated over \( R \). To have a trivial counterexample, think of the group Hopf algebra \( kG \) of an infinite group.

**Example 4.9.** An example that does not satisfy the Lu-axioms [13]. Let \( k \) be a field the characteristic of which is different from 2. Consider the group bialgebra \( kZ_2 \) with presentation

\[ kZ_2 = \text{bialg}\left\{ t \mid t^2 = 1, \Delta(t) = t \otimes t, \varepsilon(t) = 1 \right\} \]
as a left bialgebroid over the base \( k \). That is to say, we set \( \mathcal{A}_L = (k \mathbb{Z}_2, k, \eta, \Delta, \varepsilon) \) where \( \eta \) is the unit map \( k \rightarrow k \mathbb{Z}_2, \lambda \mapsto \lambda 1 \). Introduce the would-be-antipode \( S : k \mathbb{Z}_2 \rightarrow k \mathbb{Z}_2, t \mapsto -t \).

**Proposition 4.10.** The pair \((\mathcal{A}_L, S)\) in Example 4.9 satisfies the axioms in Definition 4.1 but not the Lu-axioms.

**Proof.** One easily checks the conditions of Proposition 4.2(ii) on the single algebraic generator \( t \), proving that \((\mathcal{A}_L, S)\) is a Hopf algebroid in the sense of Definition 4.1.

Now, the base ring \( L \) being \( k \) itself, the canonical projection \( k \mathbb{Z}_2 \otimes_k k \mathbb{Z}_2 \rightarrow k \mathbb{Z}_2 \otimes_k k \mathbb{Z}_2 \) is the identity map leaving us with the only section \( k \mathbb{Z}_2 \otimes_k k \mathbb{Z}_2 \rightarrow k \mathbb{Z}_2 \otimes_k k \mathbb{Z}_2 \), the identity map. Since \( t(1)S(t(2)) = -1 \) and \( \eta \circ \varepsilon(t) = 1 \), this contradicts to that \((\mathcal{A}_L, S)\) is a Lu–Hopf algebroid. \( \square \)

In the Example 4.8 the left and right coproducts \( \gamma_L \) and \( \gamma_R \) are compositions of a coproduct \( \Delta : A \rightarrow A \otimes_k A \) with the canonical projections \( p_L \) and \( p_R \), respectively. Actually, many other examples can be found this way—by allowing for \( \Delta \) not to be counital.

**Proposition 4.11.** Let \( \mathcal{A}_L \) be a left bialgebroid such that \( A \) and \( L \) are \( k \)-algebras over some commutative ring \( k \), \( S \) an anti-automorphism of \( A \) such that the axioms (4.1) and (4.4) hold true. Suppose that \( \gamma_L = p_L \circ \Delta \), where \( p_L \) is the canonical projection \( A \otimes_k A \rightarrow \mathcal{A}_L \otimes_k A \) and \( \Delta : A \rightarrow A \otimes_k A \) is a coassociative (possibly non-counital) coproduct satisfying

\[ p_L \circ (S \otimes S) \circ \Delta^\text{op} = p_L \circ \Delta \circ S, \quad p_L \circ (S^{-1} \otimes S^{-1}) \circ \Delta^\text{op} = p_L \circ \Delta \circ S^{-1}. \]  \hspace{1cm} (4.48)

Then \((\mathcal{A}_L, S)\) is a Hopf algebroid in the sense of Definition 4.1.

**Proof.** We leave it to the reader to check that all the requirements of Proposition 4.2(ii) are satisfied. \( \square \)

**Example 4.12.** The groupoid Hopf algebroid. Let \( G \) be a groupoid that is a small category with all morphisms invertible. Denote the object set by \( G^0 \) and the set of morphisms by \( G^1 \). For a commutative ring \( k \) the groupoid algebra is the \( k \)-module spanned by the \( G^1 \) elements of \( G^1 \) with the multiplication given by the composition of the morphisms if the latter makes sense and 0 otherwise. It is an associative algebra and if \( G^0 \) is finite it has a unit \( 1 = \sum_{a \in G^0} a \). The groupoid algebra admits a left bialgebroid structure over the base subalgebra \( k G^0 \). The map \( s_L = t_L \) is the canonical embedding, \( \gamma_L \) is the diagonal map \( g \mapsto g \otimes_k g \), and \( \pi_L(g) := \text{target}(g) \). This left bialgebroid together with the antipode \( S(g) := g^{-1} \) is a Hopf algebroid in the sense of Definition 4.1. Actually this example is of the kind described in Proposition 4.11 with \( \Delta(g) := g \otimes_k g \).

**Example 4.13.** The algebraic quantum torus. Let \( k \) be a field and \( T_q \) the unital associative \( k \) algebra generated by two invertible elements \( U \) and \( V \) subject to the relation \( UV = q VU \) where \( q \) is an invertible element in \( k \). As it is explained in [11], the algebra \( T_q \) admits a
Lu–Hopf algebroid structure over the base subalgebra \( L \) generated by \( U \): the map \( s_L = t_L \) is the canonical embedding,
\[
\gamma_L(U^n V^m) := U^n V^m \otimes_L V^m \equiv V^m \otimes_L U^n V^m, \quad \pi_L(U^n V^m) := U^n,
\]
and the antipode \( S(U^n V^m) := V^{-m}U^n \). The section \( \xi \) of the canonical projection \( p_L : T_q \otimes_k T_q \to T_q \otimes_L T_q \) appearing in the Lu axioms is of the form
\[
\xi(U^n V^m \otimes_L U^k V^l) := U^{(n+k)} V^m \otimes_k V^l.
\]

The reader may check that these maps satisfy the Definition 4.1 as well. This example is also of the type considered in Proposition 4.11 with \( \Delta(U^n V^m) := U^n V^m \otimes_k V^m \).

**Example 4.14. Examples by Brzezinski and Militaru** [5]. In the paper [5] a wide class of examples of Lu–Hopf algebroids is described. Some other examples [13,15] turn out to belong also to this class.

The examples of [5] are Lu–Hopf algebroids of the type considered in Proposition 4.11. Let \( (H, \Delta_H, \varepsilon_H, \tau) \) be a Hopf algebra over the field \( k \) with bijective antipode \( \tau \) and the triple \( (L, \cdot, \rho) \) a braided commutative algebra in the category \( \mathcal{H} \mathcal{D} \mathcal{H} \) of Yetter–Drinfel’d modules over \( H \). Then the crossed product algebra \( L \# H \) carries a left bialgebroid structure over the base algebra \( L \):
\[
s_L(l) = l \# 1_H, \quad t_L(l) = \rho(l) := l_{(0)} \# l_{(1)}, \quad \gamma_L(l \# h) = (l \# h_{(1)}) \otimes_L (1_L \# h_{(2)}), \quad \pi_L(l \# h) = \varepsilon_H(h)l,
\]
where \( h_{(1)} \otimes_k h_{(2)} := \Delta_H(h) \). It is proven in [5] that the left bialgebroid (4.49) and the bijective antipode
\[
S(l \# h) := (\tau(h_{(2)}) \tau^2(l_{(1)})) \cdot l_{(0)} \# \tau(h_{(1)}) \tau^2(l_{(2)}) \quad (4.50)
\]
form a Lu–Hopf algebroid. It is obvious that \( \gamma_L \) is of the form \( p_L \circ \Delta \) with \( \Delta(l \# h) := (l \# h_{(1)}) \otimes_k (1_L \# h_{(2)}) \). The map \( \Delta \) is well-defined since \( L \# H = L \otimes_k H \) as a k-space and \( \Delta_H \) maps \( H \) into \( H \otimes_k H \). We leave it to the reader to check that \( \Delta \) satisfies (4.48) hence the left bialgebroid (4.49) and the antipode (4.50) form a Hopf algebroid in the sense of Definition 4.1.

The Example 4.9 is not of the type considered in Proposition 4.11. Although \( \gamma_L \) is of the form \( p_L \circ \Delta \), the \( \Delta \) does not satisfy (4.48). In [11] data \( (\mathcal{A}_L, S, \tilde{S}) \) satisfying compatibility conditions somewhat analogous to (4.48) were introduced under the name *extended Hopf algebra*. The next proposition states that extended Hopf algebras with \( S \) bijective (such as Example 4.9) provide examples of Hopf algebroids.

**Proposition 4.15.** Let \( (\mathcal{A}_L, S, \tilde{S}) \) be an extended Hopf algebra. This means that \( \mathcal{A}_L \) is a left bialgebroid such that \( \mathcal{A} \) and \( L \) are \( k \)-algebras over some commutative ring \( k \). The maps \( S \) and \( \tilde{S} \) are anti-automorphisms of the algebra \( \mathcal{A} \), \( \tilde{S}^2 = \text{id}_\mathcal{A} \) and both pairs \( (\mathcal{A}_L, S) \) and
\((A_L, \tilde{S})\) satisfy (4.1) and (4.4). The map \(\gamma_L\) is a composition of a coassociative coproduct \(\Delta : A \rightarrow A \otimes_k A\) and the canonical projection \(p_L : A \otimes_k A \rightarrow A \otimes_L A\). The compatibility relations

\[
\Delta \circ S = (S \otimes S) \circ \Delta^\text{op} \quad \text{and} \quad \Delta \circ \tilde{S} = (S \otimes \tilde{S}) \circ \Delta^\text{op}
\]

hold true. Then the pair \((A_L, \tilde{S})\) is a Hopf algebroid in the sense of Definition 4.1.

**Proof.** We leave to the reader to check that the condition (4.9)—hence all requirements of Proposition 4.2(ii)—hold true. \(\square\)

5. Integral theory and the dual Hopf algebroid

In this section we generalize the notion of non-degenerate integrals in (weak) bialgebras to bialgebroids. We examine the consequences of the existence of a non-degenerate integral in a Hopf algebroid. We do not address the question, however, under what conditions on the Hopf algebroid does the existence of a non-degenerate integral follow. That is we do not give a generalization of the Larson–Sweedler theorem on bialgebroids and neither of the (weaker) [3, Theorem 3.16] stating that a weak Hopf algebra possesses a non-degenerate integral if and only if it is a Frobenius algebra. (About the implications in one direction see however [2, Theorem 6.3] and Theorem 5.5 below, respectively.)

Assuming the existence of a non-degenerate integral in a Hopf algebroid we show that the underlying bialgebroids are finite. The duals of finite bialgebroids w.r.t. the base rings were shown to have bialgebroid structures [8] but there is no obvious way how to transpose the antipode to (either of the four) duals. As the main result of this section we show that if there exists a non-degenerate integral in a Hopf algebroid then the four dual bialgebroids are all (anti-) isomorphic and they can be made Hopf algebroids. This dual Hopf algebroid structure is unique up to isomorphism (in the sense of Definition 4.4).

For the considerations of this section the “symmetric definition” of Hopf algebroids, i.e., the characterization of Proposition 4.2(iii) is the most appropriate. Throughout the section we use the symmetrized form of the Hopf algebroid introduced at the end of Section 4.1.

It is important to emphasize that although the Definitions 5.1 and 5.3 are formulated in terms of a particular symmetrized Hopf algebroid \(A = (A_L, A_R, S)\), actually they depend only on the Hopf algebroid \((A_L, S)\). That is to say, if \(\ell\) is a (non-degenerate) left integral in a symmetrized Hopf algebroid then it is one in any other symmetrized form of the same Hopf algebroid. Therefore, \(\ell\) can be called a (non-degenerate) left integral of the Hopf algebroid. Analogously, although the anti-automorphism \(\xi\) in Lemma 5.9 is defined in terms of a particular symmetrized Hopf algebroid, it is invariant under the change of the underlying right bialgebroid.

For a symmetrized Hopf algebroid \(A = (A_L, A_R, S)\) we use the notations of Section 2: the \(A_s\) and \(sA\) are the \(L\)-duals of \(A_L\), the \(A^*\) and \(\ast A\) the \(R\)-duals of \(A_R\). Also for the coproducts of \(A_L\) and \(A_R\) we write \(\gamma_L(a) = a^{(1)} \otimes a^{(2)}\) and \(\gamma_R(a) = a^{(1)} \otimes a^{(2)}\), respectively.
5.1. Non-degenerate integrals

**Definition 5.1.** The **left integrals** in a left bialgebroid $A_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ are the invariants of the left regular $A$ module

$$\mathcal{I}^L(A) := \left\{ \ell \in A \mid a \ell = s_L \circ \pi_L(a) \ell \forall a \in A \right\}.$$

The **right integrals** in a right bialgebroid $A_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ are the invariants of the right regular $A$ module

$$\mathcal{I}^R(A) := \left\{ \Upsilon \in A \mid \Upsilon a = T s_R \circ \pi_R(a) \forall a \in A \right\}.$$

The left/right integrals in a symmetrized Hopf algebroid $(A_L, A_R, S)$ are the left/right integrals in $A_L/A_R$.

**Lemma 5.2.** For a symmetrized Hopf algebroid $A$ the following properties of the element $\ell$ of $A$ are equivalent:

(i) $\ell \in \mathcal{I}^L(A)$;

(ii) $a \ell = t_L \circ \pi_L(a) \ell$ for all $a \in A$;

(iii) $S(\ell) \in \mathcal{I}^R(A)$;

(iv) $S^{-1}(\ell) \in \mathcal{I}^R(A)$;

(v) $S(a) \ell^{(1)} \otimes a \ell^{(2)} = \ell^{(1)} \otimes a \ell^{(2)}$ as elements of $A^R \otimes^R A$, for all $a \in A$.

**Proof.** Left to the reader. $\square$

**Definition 5.3.** The left integral $\ell$ in the symmetrized Hopf algebroid $A$ is **non-degenerate** if the maps

$$\ell_R : A^* \rightarrow A, \quad \phi^* \mapsto \phi^* \mapsto \ell, \quad \text{and}$$

$$\ell_R : A^* \rightarrow A, \quad \phi^* \mapsto * \phi \mapsto \ell$$

are bijective. The right integral $\Upsilon$ in the symmetrized Hopf algebroid $A$ is **non-degenerate** if $S(\Upsilon)$ is a non-degenerate left integral, i.e., if the maps

$$\Upsilon_L : A_k \rightarrow A, \quad \phi_k \mapsto \Upsilon \mapsto \phi_k, \quad \text{and}$$

$$\Upsilon : A_k \rightarrow A, \quad \phi \mapsto \Upsilon \mapsto \phi$$

are bijective.

**Remark 5.4.** If $\ell$ is a non-degenerate left integral in the symmetrized Hopf algebroid $A$ then so is in $A_{\text{cop}}$ and when replacing $A$ with $A_{\text{cop}}$ the roles of $\ell_R$ and $\ell$ become interchanged. Hence, any statement proven in a symmetrized Hopf algebroid possessing a non-degenerate left integral on $\ell_R$ implies that the co-opposite statement holds true on $\ell$. 

Theorem 5.5. Let $A$ be a symmetrized Hopf algebroid possessing a non-degenerate left integral. Then the ring extensions $s_R: R \to A$, $t_R: R^{op} \to A$, $s_L: L \to A$, and $t_L: L^{op} \to A$ are all Frobenius extensions.

Proof. Let $\ell$ be a non-degenerate left integral in $A$. With its help we construct the Frobenius system for the extension $s_R: R \to A$. It consists of a Frobenius map $\lambda^* := \ell^{-1}_R(1_A): RA^R \to R$ (5.3) and a quasi-basis (in the sense of (3.1)) for it $\ell(1) \otimes S(\ell(2)) \in AR \otimes RA$.

As a matter of fact the $\lambda^*$ is a right $R$-module map $AR \to R$ by construction. We claim that it is also a left $R$-module map $RA \to R$. Since $(\lambda^* \leftarrow S(a)) \mapsto \ell(2)_R(\lambda^*(S(a)^{(1)})) = a(\lambda^* \mapsto \ell) = a,$ (5.4) the inverse $\ell^{-1}_R$ maps $a \in A$ to $\lambda^* \leftarrow S(a)$. This implies that $\lambda^* \leftarrow s_R(r) = \ell^{-1}_R \circ t_R(r)$. Now for a given element $r \in R$ the map $\chi(r)^*: A^R \to R$, $a \mapsto r\lambda^*(a)$ is also equal to $\ell^{(2)}_R \circ t_R(r)$ as

$$\chi(r)^* \mapsto \ell = \ell^{(2)}_R(r\lambda^*(\ell^{(1)})) = (\lambda^* \mapsto \ell) t_R(r) = t_R(r).$$

Applying the two equal maps $\lambda^* \leftarrow s_R(r)$ and $\chi(r)^*$ to an element $a \in A$, we obtain

$$\lambda^*(s_R(r)a) = r\lambda^*(a).$$

(5.5)

This proves that $\lambda^*$ is an $R$-$R$-bimodule map $RA^R \to A$. Also

$$s_R \circ \lambda^*(a^{(1)}_R) S(\ell^{(2)}) = S(\lambda^* \mapsto \ell) a = a$$

and

$$\ell^{(1)} s_R \circ \lambda^*(S(\ell^{(2)})) a = a \ell^{(1)} s_R \circ \lambda^* \circ S(\ell^{(2)}).$$

(5.6)

Now we claim that $\ell^{(1)} s_R \circ \lambda^* \circ S(\ell^{(2)}) = \lambda^* \circ S \mapsto \ell$ is equal to $1_A$, which proves the claim. (Recall that by (5.5) $\lambda^* \in S$ is an element of $^*A$.) Since $\ell^{-1}_R(a) = \lambda^* \leftarrow S(a),$

$$\phi^*(a) = [\lambda^* \leftarrow S(\phi^* \mapsto \ell)](a) = \lambda^*(s_R \circ \phi^*(\ell^{(1)}) S(\ell^{(2)})) a$$

for all $\phi^* \in A^*$ and $a \in A$.

By the bijectivity of $r\ell$, we can introduce the element $^{*}\lambda := r\ell^{-1}(1_A) \in ^*A$. Analogously to (5.5) and (5.4), we have

$$^{*}\lambda(t_R(r)a) = ^*\lambda(a) r$$

and

$$r\ell^{-1}(a) = ^*\lambda \leftarrow S^{-1}(a).$$

(5.9)
Since both $R\ell$ and $S^{-1}$ are bijective, so is the map $A \rightarrow *A$, $a \mapsto ^*\lambda \leftarrow a$. Using the identities (5.9), (5.5), and (5.8), compute

$$(^*\lambda \leftarrow S^{-1}(\lambda^* \circ S \rightarrow \ell))(a) = ^*\lambda (t_R \circ \lambda^* \circ S(\ell^{(2)})S^{-1}(\ell^{(1)})a)$$

$$= \lambda^* (s_R \circ ^*\lambda \circ S^{-1}(\ell^{(1)})S(\ell^{(2)})S(a)) = ^*\lambda(a)$$

for all $a \in A$. This is equivalent to $\lambda^* \circ S \rightarrow \ell = 1_A$ that is $^*\lambda = \lambda^* \circ S$ proving that $(\lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$ is a Frobenius system for the extension $s_R : R \rightarrow A$.

By repeating the same proof in $A_{\text{cop}}$, we obtain the Frobenius system $(^*\lambda, \ell^{(2)} \otimes S^{-1}(\ell^{(1)}))$ for the extension $t_R : R \rightarrow A$.

It is straightforward to check that $(\mu^{-1} \circ \lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$ is a Frobenius system for the extension $t_L : L \rightarrow A$, and $(\nu^{-1} \circ ^*\lambda, \ell^{(2)} \otimes S^{-1}(\ell^{(1)}))$ is a Frobenius system for the extension $s_L : L \rightarrow A$. 

From now on, let $\ell$ be a non-degenerate left integral in the symmetrized Hopf algebroid $A$, set $\lambda^* = \ell^{-1}_R(1_A)$ and $^*\lambda = \rho^{-1}_L(1_A)$.

Theorem 5.5 implies that for a symmetrized Hopf algebroid $A$ possessing a non-degenerate integral the modules $A^R$, $A^L$, and $L^A$ are finitely generated projective. Hence, by the result of [8], their duals $A^*$ and $^*A$ carry left bialgebroid structures over the base $R$, and $A_*$ and $^*A_*$ carry right bialgebroid structures over the base $L$

$s^*_R(r)(a) = r \pi_R(a), \quad ^*s_L(r)(a) = \pi_R(t_R(r)a),$

$t^*_L(r)(a) = \pi_R(s_R(r)a), \quad ^*t_L(r)(a) = \pi_R(\ell_R(r)a),$

$\gamma^*_L(\phi^*) = \phi^* \leftarrow \ell^{(1)} \otimes \ell^{-1}_R(\ell^{(2)}), \quad ^*\gamma_L(\phi) = \rho^{-1}_L(\ell^{(1)} \otimes \phi \leftarrow \ell^{(2)},$

$\pi^*_L(\phi^*) = \phi^*(1_A), \quad ^*\pi_L(\phi) = \phi(1_A),$  

$s^*_R(l)(a) = \pi_L(as_L(l)), \quad ^*s_L(l)(a) = \pi_L(\ell_L(l)),$

$t^*_L(l)(a) = \pi_L(la), \quad ^*t_R(l)(a) = \pi_L(\ell_L(l)),$

$\gamma^*_R(\phi_*) = \ell(1) \rightarrow \phi_* \otimes \ell^{-1}_L(\ell(2)), \quad ^*\gamma_R(\phi) = L^{-1}(\ell(1) \otimes \ell(2) \rightarrow \phi,$

$\pi^*_R(\phi_*) = \phi_*(1_A), \quad ^*\pi_R(\phi) = \phi(1_A).$ (5.11)

**Lemma 5.6.** Let $\ell$ be a non-degenerate left integral in the symmetrized Hopf algebroid $A$. Then for $\lambda^* = \ell^{-1}_R(1_A)$, $^*\lambda = \rho^{-1}_L(1_A)$, and any element $a \in A$, the identities

$$\lambda^* \leftarrow a = s_R \circ \lambda^*(a), \quad \text{ (5.12)}$$

$$^*\lambda \leftarrow a = t_R \circ ^*\lambda(a) \quad \text{ (5.13)}$$

hold true.
Proof. One checks that
\[ \phi^* \lambda^* = \ell^{-1}_R(\phi^* \rightarrow 1_A) = s^*_L \circ \phi^*(1_A) \lambda^* \]
for all \( \phi^* \in A^* \). This implies that \( \phi^*(\lambda^* \rightarrow a) = \phi^*(s_R \circ \lambda^*(a)) \) for all \( \phi^* \in A^* \). Since \( A^R \) is finitely generated projective by Theorem 5.5, this proves (5.12). The identity (5.13) follows by Remark 5.4.

The left integrals in a Hopf algebroid were defined in Definition 5.1 as the left integrals in the underlying left bialgebroid. The non-degeneracy of the left integral was defined in Definition 5.3 using however the underlying right bialgebroid as well, that is it relies to the whole of the Hopf algebroid structure. Therefore, it is not obvious whether the non-strict isomorphisms of Hopf algebroids preserve non-degenerate integrals. In the rest of this subsection, we prove that this is the case:

**Proposition 5.7.** Let both \((A_L, S)\) and \((A_L, S')\) be Hopf algebroids. Then their non-degenerate left integrals coincide.

**Proof.** A left integral \( \ell \) in \((A_L, S)\) is a left integral in \((A_L, S')\) by definition.

Let \( A_R = (A, R, s_R, t_R, \gamma_R, \pi_R) \) and \( A'_R = (A, R', s'_R, t'_R, \gamma'_R, \pi'_R) \) be the right bialgebroids underlying the Hopf algebroids \((A_L, S)\) and \((A_L, S')\), respectively. It follows from the uniqueness of the maps \( \alpha^{-1} \) and \( \beta^{-1} \) in (4.16) that the coproducts \( \gamma_R(a) = a^{(1)} \otimes a^{(2)} \) of \( A_R \) and \( \gamma'_R(a) = a'^{(1)} \otimes a'^{(2)} \) of \( A'_R \) are related as

\[ a^{(1)} \otimes S'^{-1} \circ S(a^{(2)}) = a'^{(1)} \otimes a'^{(2)} = S' \circ S^{-1}(a^{(1)}) \otimes a^{(2)}. \]

(5.14)

With the help of the maps \( \mu = \pi_R \circ t_L, \mu' = \pi'_R \circ t'_L, v = \pi_R \circ s_L, \) and \( v' = \pi'_R \circ s'_L \), we can introduce the isomorphisms of additive groups:

\[ A^* \rightarrow A'^*, \quad \phi^* \mapsto \mu' \circ \mu^{-1} \circ \phi^*, \]

\[ ^*A \rightarrow ^*A', \quad ^*\phi \mapsto v' \circ v^{-1} \circ ^*\phi. \]

Then the canonical actions (2.13) of \( A'^* \) and \( A^* \) and of \( ^*A' \) and \( ^*A \) on \( A \) are related as

\[ \mu' \circ \mu^{-1} \circ \phi^* \rightarrow a = S'^{-1} \circ S(\phi^* \rightarrow a), \]

\[ v' \circ v^{-1} \circ ^*\phi \rightarrow a = S' \circ S^{-1}(^*\phi \rightarrow a) \]

what implies the non-degeneracy of the left integral \( \ell \) in \((A_L, S')\) provided it is non-degenerate in \((A_L, S)\).
5.2. Two-sided non-degenerate integrals

Proposition 5.7 above proves that the structure of the non-degenerate left integrals is the same within an isomorphism class of Hopf algebroids. In this subsection we prove that for a non-degenerate left integral \( \ell \) in the Hopf algebroid \((A_L, S)\) there exists a distinguished representative \((A_L, S)'_\ell\) in the isomorphism class of \((A_L, S)\) with the property that \( \ell \) is not only a non-degenerate left integral in \((A_L, S)'_\ell\) but also a non-degenerate right integral.

The Hopf algebroids with two-sided non-degenerate integral are of particular interest. Both the Hopf algebroid structure constructed on the dual of a Hopf algebroid in Section 5.3 and the one associated to a depth 2 Frobenius extension in Section 3 belong to this class.

Lemma 5.8. Let \( \ell \) be a non-degenerate left integral in a symmetrized Hopf algebroid \( A \). Set \( \lambda^* := \ell_R^{-1}(1_A) \) and \( ^*_\lambda := \ell_R^{-1}(1_A) \). Then any (not necessarily non-degenerate) left integral \( \ell' \in I_L(A) \) satisfies

\[
\ell_S \circ \lambda^*(\ell') = \ell' = \ell_R \circ {^*_\lambda}(\ell').
\]

Proof. Observe that for \( ^*_\phi \in ^*_A \) and \( \ell' \in I_L(A) \) we have \( ^*_\phi \curvearrowleft S^{-1}(\ell') = {^*_\ell} \circ {^*_\phi} \circ S^{-1}(\ell') \); hence

\[
\ell' = (^*_\lambda \curvearrowleft S^{-1}(\ell')) \rightarrow \ell = {^*_\ell} \circ ^*_\lambda \circ S^{-1}(\ell') \rightarrow \ell = \ell_S \circ \lambda^*(\ell').
\]

The identity \( \ell' = \ell_R \circ {^*_\lambda}(\ell') \) follows by Remark 5.4. \( \square \)

Lemma 5.9. Let \( \ell \) be a non-degenerate left integral in a symmetrized Hopf algebroid \( A \). Set \( \lambda^* := \ell_R^{-1}(1_A) \). Then the map \( \xi : A \rightarrow A; a \mapsto S((\lambda^* \curvearrowleft \ell) \rightarrow a) \) is a ring anti-automorphism.

Proof. Using Lemma 5.8 one checks that

\[
[(\lambda^* \curvearrowleft \ell) \rightarrow a][(\lambda^* \curvearrowleft \ell) \rightarrow b] = a(2) b(2) t_R \circ \lambda^* (\ell a(1) b(1)) = a(2) b(2) t_R \circ \lambda^* (\ell b(1) a(1)) = a(2) b(2) t_R \circ \lambda^* (\ell a(1) b(1)) = (\lambda^* \curvearrowleft \ell) \rightarrow ab
\]

for \( a, b \in A \), hence the map \( \xi \) is anti-multiplicative. By analogous calculations the reader may check that it is bijective with inverse \( \xi^{-1}(a) = S^{-1}(\ell \rightarrow a) \), where \( ^*_\lambda := R \ell_R^{-1}(1_A) = \lambda^* \circ S \). \( \square \)

Proposition 5.10. Let \( \ell \) be a non-degenerate left integral in a symmetrized Hopf algebroid \( A \). Then the following maps are bijective:
\[ \ell_L : A \to A, \quad \phi_s \mapsto \ell \leftarrow \phi_s, \]
\[ L \ell : A \to A, \quad \phi \mapsto \ell \leftarrow \phi. \]

**Proof.** We claim that with the help of the ring isomorphism \( \nu \) (introduced in Proposition 4.3) we have \( \ell \leftarrow \phi = \xi^{-1} \circ \ell_R(\nu \circ \phi \circ S^{-1}) \), which implies the bijectivity of \( L \ell \).

As a matter of fact,
\[
\ell \leftarrow \phi = \xi^{-1} \circ \ell_R(\nu \circ \phi \circ S^{-1}) \]
\[
= \ell_L \circ (\nu^{-1} \circ (\nu \leftarrow S(\phi))) \circ S(\ell(2)) \ell(1) = \ell_R^{-1} \circ \ell(1) \circ \ell(1) \circ S_R \circ S_\lambda(\ell(2) a(2)) \]
\[
= \ell_R^{-1} \circ \ell(1) \circ S_R \circ S_\lambda(\ell(2) a(2)) = \ell_R^{-1}(\ell(1) S_R \circ S_\lambda(\ell a(2))) \]
\[
= \ell_R^{-1}(\ell(1) S_R \circ S_\lambda(\ell a(2))) = \xi^{-1}(a). \tag{5.15}
\]

Similarly, by the application of (5.15) to \( A_{\text{cop}} \),

\[
\ell \leftarrow \phi_s = \xi \circ \ell R(\mu \circ \phi_s \circ S), \tag{5.16}
\]

hence \( L \ell \) is also bijective. \( \square \)

Using (5.16) we have an equivalent form of the anti-automorphism \( \xi \) introduced in Lemma 5.9:

\[
\xi(a) = \ell \leftarrow (a \mapsto \ell^{-1}_L(1_A)). \tag{5.17}
\]

**Lemma 5.11.** Let \( \ell \) be a non-degenerate left integral in a symmetrized Hopf algebroid \( A \). Then for all elements \( a, b \in A \) we have the identities

\[
\ell_R^{-1}(b) \mapsto a = \ell^{-1}(a) \mapsto b, \tag{5.18}
\]
\[
a \leftarrow \ell_L^{-1}(b) = b \leftarrow \ell^{-1}(a), \tag{5.19}
\]
\[
\ell_R^{-1}(b) \mapsto a = a \leftarrow \ell_L^{-1}(b), \tag{5.20}
\]
\[
\ell^{-1}(b) \mapsto a = a \leftarrow \ell^{-1}(b). \tag{5.21}
\]

**Proof.** We illustrate the proof on (5.18). Use Lemma 5.6 to see that
\[
\ell_R^{-1}(a) \mapsto b = b(1) \lambda^* \mapsto S(b(2)) a = S_\lambda \circ \pi_L(b(1)) a(2) \ell_R \circ \lambda^* (S(b(2)) a(1)) \]
\[
= \ell_R^{-1}(b) \mapsto a,
\]
where \( \lambda^* = \ell_R^{-1}(1_A) \). The rest of the proof is analogous. \( \square \)
Lemma 5.12. Let \( \ell \) be a non-degenerate left integral in a symmetrized Hopf algebroid \( A \). Set \( \lambda^* = \ell_R^{-1}(1_A) \). Then the map \( \kappa : R \to R, r \mapsto \lambda^*(\ell_T R(r)) \) is a ring automorphism.

Proof. It follows from Lemma 5.8 that \( \kappa \) is multiplicative: for \( r, r' \in R \)

\[
\kappa(r)\kappa(r') = \lambda^*(\ell_T R(r))\lambda^*(\ell_T R(r')) = \lambda^*(\ell S_R \circ \lambda^*(\ell_T R(r'))\ell_T R(r)) = \lambda^*(\ell_T R(r'))\ell_T R(r) = \kappa(rr').
\]

In order to show that \( \kappa \) is bijective, we construct the inverse \( \kappa^{-1} : r \mapsto \lambda^*(\ell_S R(r)) \) where \( \lambda^* = \ell_{R^{-1}}(1_A) = \lambda \circ S. \)

Proposition 5.13. Let \( \ell \) be a non-degenerate left integral in the Hopf algebroid \( (A_L, S) \). Then there exists a unique Hopf algebroid \( (A'_L, S'_L) \) such that \( \ell \) is a two-sided non-degenerate integral in \( (A_L, S'_L) \).

Proof. Uniqueness. Suppose that \( (A_L, S'_L) \) is a Hopf algebroid of the required kind. Denote the underlying right bialgebroid by \( A'_R = (A, R', s'_R, t'_R, y'_R, \pi'_R) \). Define \( \lambda'^* \in A'^* \) with the property that \( \lambda'^* \circ \ell = 1_A \) (where \( \circ \) denotes the canonical action (2.13) of \( A'^* \) on \( A \)). Introducing the notation \( \gamma'_R(a) = a^{(1)} \otimes a^{(2)} \), one checks that

\[
S'^{-1}(\ell) = (\lambda'^* \circ \ell) \circ \ell = \ell t'_R \circ \lambda'^* (\ell \ell^{(1)}) = \ell t'_R \circ \lambda'^* (\ell s'_R \circ \pi'_R (\ell^{(1)})) = \ell t'_R \circ \lambda'^* = \ell.
\]

With the help of the element \( \pi_L \circ S^{-1} \circ S' \in A_s \), we have

\[
S(a \leftarrow \pi_L \circ S^{-1} \circ S') = S \circ S'^{-1}(S'(a)^{(1)}) S_R \circ \pi_R (S'(a)^{(2)}) = S'(a)
\]

for all \( a \in A \), where in the last step the relation (5.14) has been used. Then the condition \( S'(\ell) = \ell \) is equivalent to

\[
S'(a) = S(a \leftarrow \ell^{-1}_L \circ S^{-1}(\ell)).
\]

This proves the uniqueness of \( S' \).

Existence. Let \( \xi \) be the anti-automorphism of \( A \) introduced in Lemma 5.9. We claim that \( (A_L, \xi) \) is a Hopf algebroid of the required kind. Introduce the right bialgebroid \( A'_R \) on the total ring \( A \) over the base \( R \) with structural maps

\[
s'_R = s_R, \quad t'_R = \xi^{-1} \circ s_R, \quad y'_R = \xi^{-1} A@_{A_R} \circ S_A \circ y_R \circ S^{-1} \circ \xi, \quad \pi'_R = \pi_R \circ S^{-1} \circ \xi,
\]

where \( A_R = (A, R, s_R, t_R, y_R, \pi_R) \) is the right bialgebroid underlying \( (A_L, S) \). First, we check that the triple \( (A_L, A'_R, \xi) \) satisfies Proposition 4.2(iii). Since

\[
\xi^{-1} \circ S \circ s_R = \xi^{-1} \circ \theta_R = S^{-1} \circ t_R \circ \theta_R = s_R \quad \text{and} \quad \xi^{-1} \circ S \circ t_R = \xi^{-1} \circ s_R,
\]

we obtain

\[
\xi^{-1} \circ S \circ s_R = \xi^{-1} \circ \theta_R = S^{-1} \circ t_R \circ \theta_R = s_R \quad \text{and} \quad \xi^{-1} \circ S \circ t_R = \xi^{-1} \circ s_R.
\]
the $\mathcal{A}'_{R}$ is a right bialgebroid isomorphic to $\mathcal{A}_{R}$ via the isomorphism $(\xi^{-1} \circ S, \text{id}_{R})$.

The requirement $s'_{R}(R) \equiv s_{R}(R) = t_{L}(L)$ obviously holds true. Since

$$t'_{R}(r) = \xi^{-1} \circ s_{R}(r) = S^{-1} \circ s_{R} \circ \lambda(S_{R}(r)) = t_{R} \circ \kappa^{-1}(r),$$

also $t'_{R}(R) \equiv t_{R}(R) = s_{L}(L)$. Since

$$\gamma'_{R}(a) \equiv a^{(1)} \otimes a^{(2)} = \xi^{-1} \circ \gamma_{L} \circ \xi(a) = a^{(1)} \otimes \xi^{-1} \circ S(a^{(2)})$$

$$= a^{(1)} \otimes S^{-1}(R\ell \epsilon^{-1} \circ S(\ell) \rightarrow S(a^{(2)})) = a^{(1)} \otimes S^{-1}(S(a^{(2)}) \leftarrow L\ell \epsilon^{-1} \circ S(\ell))$$

$$= a^{(1)} s_{R} \circ \kappa \circ \mu \circ L\ell \epsilon^{-1} \circ S(\ell) \circ S(a^{(2)}) \otimes a^{(3)}.$$

we have

$$(\gamma_{L} \otimes \text{id}_{A}) \circ \gamma'_{R}(a) = a^{(1)} \otimes a^{(2)} \otimes \xi^{-1} \circ S(a^{(2)}) = a^{(1)} \otimes a^{(2)} \otimes \xi^{-1} \circ S(a^{(2)})$$

$$= \text{id}_{A} \otimes \gamma'_{R}(a),$$

$$(\text{id}_{A} \otimes \gamma_{L}) \circ \gamma'_{R}(a) = a^{(1)} s_{R} \circ \kappa \circ \mu \circ L\ell \epsilon^{-1} \circ S(\ell) \circ S(a^{(2)}) \otimes a^{(3)}$$

$$= a^{(1)} s_{R} \circ \kappa \circ \mu \circ L\ell \epsilon^{-1} \circ S(\ell) \circ S(a^{(1)}) \otimes a^{(3)} \otimes a^{(2)}$$

$$= \text{id}_{A} \otimes \gamma'_{R}(a).$$

By Lemma 5.9, the $\xi$ is an anti-automorphism of the ring $A$. The identity $\xi \circ t'_{R} = s'_{R}$ is obvious and also

$$\xi \circ t_{L} = \xi \circ s_{R} \circ \mu = S \circ s_{R} \circ \mu = s_{L}.$$

Finally,

$$\xi(a^{(1)})a^{(2)} = s_{R} \circ \pi_{R}(\lambda^{*} \leftarrow \ell) \rightarrow a = s_{R} \circ \pi_{R} \circ S^{-1} \circ \xi(a) = s'_{R} \circ \pi'_{R}(a),$$

$$\xi^{a^{(1)}}a^{(2)} = a^{(1)} \otimes \xi^{-1} \circ S(a^{(2)}) = s_{L} \circ \pi_{L}(a).$$

This proves that $\mathcal{A}'_{L} = (\mathcal{A}_{L}, \mathcal{A}'_{R}, \xi)$ satisfies the conditions in Proposition 4.2(iii); hence $(\mathcal{A}_{L}, \xi)$ is a Hopf algebroid. Since

$$\xi(\ell) = S(\lambda^{*} \leftarrow \ell) \rightarrow \ell = S \circ S^{-1}(\ell) = \ell,$$

the $\ell$ is a two-sided non-degenerate integral in $\mathcal{A}'_{L}$.  
\[\Box\]
5.3. Duality

It follows from Theorem 5.5 that for a symmetrized Hopf algebroid \( A \) possessing a non-degenerate left integral \( \ell \) the dual rings (with respect to the base ring) carry bialgebroid structures. These bialgebroids (5.11) are independent of the particular choice of the non-degenerate integral. In this subsection we analyze these bialgebroids. We show that the four bialgebroids (5.11) are all (anti-) isomorphic and can be equipped with an \( \ell \)-dependent Hopf algebroid structure. Because of the \( \ell \)-dependence of this Hopf algebroid structure the duality of Hopf algebroids is sensibly defined on the isomorphism classes of Hopf algebroids.

**Lemma 5.14.** Let \( \ell \) be a non-degenerate left integral in the symmetrized Hopf algebroid \( A \). Then with the help of the anti-automorphism \( \xi \) of Lemma 5.9 we have the equalities

\[
\xi^{-1}(\ell^{(2)}) \otimes S^{-1}(\ell^{(1)}) = \ell(1) \otimes \ell(2) = S(\ell^{(2)}) \otimes \xi(\ell^{(1)})
\]  

(5.22)

in \( AL \otimes LA \).

**Proof.** The element \( \xi^{-1}(\ell^{(2)}) \otimes S^{-1}(\ell^{(1)}) \) is in \( AL \otimes LA \) since \( S^{-1} \circ s_R = t_R = s_L \circ v^{-1} \) and \( \xi^{-1} \circ t_R = S^{-1} \circ t_R = t_L \circ v^{-1} \). Using (5.15), in \( AL \otimes LA \) we have

\[
\xi^{-1}(\ell^{(2)}) \otimes S^{-1}(\ell^{(1)}) = \ell(1) \otimes t_R \circ \ast(\ell^{(2)}) S^{-1}(\ell^{(1)}) = \ell(1) \otimes \ell(2).
\]

The other equality follows by repeating the proof in \( A_{\text{cop}} \). \( \square \)

**Corollary 5.15.** For a non-degenerate left integral \( \ell \) in the symmetrized Hopf algebroid \( A \) the maps \( \ell_L \) and \( L \ell \) satisfy the identities

\[
\ell_L(a \rightarrow \phi) = \ell_L(\phi \ast)(a),
\]  

(5.23)

\[
L \ell(a \rightarrow \ast) = L \ell(\ast \phi)(a),
\]  

(5.24)

where \( \xi \) is the anti-automorphism of \( A \) introduced in Lemma 5.9.

**Theorem 5.16.** Let \( \ell \) be a non-degenerate left integral in the symmetrized Hopf algebroid \( A = (AL, AR, S) \). Then the left bialgebroids \( A^{*L}, \ast AL, (A_{R})^{\text{op}}_{L}, \ast (A_{R})^{\text{op}}_{L} \) in (5.11) are isomorphic via the isomorphisms

\[
(A_{R})^{\text{op}}_{L} \rightarrow (A_{R})^{\text{op}}_{L}(\ell^{-1} \circ \ell^{-1} \circ t_{L}, id_R)
\]

\[
(A_{R})^{\text{op}}_{L} \rightarrow (A_{R})^{\text{op}}_{L}(\ell^{-1} \circ \ell^{-1} \circ \theta_{R}),
\]

\[
A^{*L} \rightarrow \ast AL
\]

where \( \xi \) is the anti-automorphism of \( A \) introduced in Lemma 5.9 and the maps \( \mu, v, \) and \( \theta_{R} \) are the ring isomorphisms introduced in Proposition 4.3.
Proof. By Proposition 4.3, the map \( v \) is a ring isomorphism \( L^{op} \to R \). By Proposition 5.10, \( \ell_R^{−1} \circ \ell_L \) is bijective. Its anti-multiplicativity follows from (5.20). The comultiplicativity follows by the successive use of the identity \( \ell_R(\phi^∗ \leftarrow a) = S^{−1}(a)\ell_R(\phi^∗) \), the integral property of \( \ell \), (5.22), and (5.23):

\[
\gamma^∗_L \circ \ell^{-1}_R \circ \ell_L(\phi_a) = \ell^{-1}_R \circ \ell_L(\phi_a) \leftarrow \ell^{(1)} \otimes \ell^{-1}_R(\ell^{(2)}) \\
= \ell^{-1}_R(S^{−1}(\ell^{(1)})) \otimes \ell^{-1}_R(\ell^{(2)}) \\
= \ell^{-1}_R(\ell^{(2)}) \otimes \ell^{-1}_R(\ell_L(\phi_a)\ell^{(2)}) \\
= \ell^{-1}_R(\ell^{(2)}) \otimes \ell^{-1}_R(\ell_L(\phi_a)\xi(\ell^{(1)})) \\
= \ell^{-1}_R \circ \ell_L(\ell^{−1}(\ell^{(2)})) \otimes \ell^{-1}_R \circ \ell_L(\ell^{(1)} \rightarrow \phi_a) \\
= (\ell^{-1}_R \circ \ell_L \otimes \ell^{-1}_R \circ \ell_L) \circ \gamma^*_R(\phi_a).
\]

One checks also

\[
(\ell^{-1}_R \circ \ell_L \circ s_R(l))(a) = \lambda(S^{−1}(a)s_L \circ \pi_L[\ell^{(1)}s_L(l)])(\ell^{(2)}) = \nu(l)\pi_R(a) = (s^*_L \circ \nu(l))(a), \\
(\ell^{-1}_R \circ \ell_L \circ s_R(l))(a) = \lambda(S^{−1}(a)s_L[\pi_L(\ell^{(1)})\ell^{(2)}]) = \pi_R(s_R \circ \nu(l)a) = (t^*_L \circ \nu(l))(a), \\
\pi^*_L \circ \ell^{-1}_R \circ \ell_L(\phi_a) = \nu \circ \phi_a(\ell_L \circ \nu^{-1} \circ \lambda(\ell^{(2)})\ell^{(1)}) = \nu \circ \phi_a(1_A) = \nu \circ \pi_{sR}(\phi_a).
\]

This proves that \((\ell^{-1}_R \circ \ell_L, \nu)\) is a bialgebroid isomorphism \((A_{sR}^{op})^{\cop} \to A^{\ast}_{L}^{op} \). By Remark 5.4, \((\ell^{-1}_R \circ \ell_L, \mu)\) is a bialgebroid isomorphism \((A^\ast_{\mathcal{A}})^{op} \to \mathcal{A}^\ast_{L}^{op} \).

By (5.16), \( R \ell^{-1} \circ \xi^{-1} \circ \ell_R = \mu \circ \ell^{-1}_L \circ \ell_R(\phi^∗) \circ S \), hence we have to prove that \((\mu \circ S \circ \phi^∗ \circ S, \mu)\) is a bialgebroid isomorphism \((A_{sR}^{op})^{\cop} \to \mathcal{A}^\ast_{L}^{op} \).

The map \( \mu \) is a ring isomorphism \( L^{op} \to R \) by Proposition 4.3. The map \( \phi_a \mapsto \mu \circ \phi_a \circ S \) is bijective. Its anti-multiplicativity is obvious. The anti-comultiplicativity follows from (5.16) and (5.22):

\[
\gamma^*_L(\mu \circ \phi_a \circ S) = R \ell^{−1}(\ell^{(1)}) \otimes \mu \circ \phi_a \circ S \leftarrow \ell^{(2)} \\
= \mu \circ \ell^{−1}_L \circ \xi(\ell^{(1)}) \circ S \otimes \mu \circ (S(\ell^{(2)}) \rightarrow \phi_a) \circ S \\
= \mu \circ \phi_a(2) \circ S \otimes \mu \circ \phi_a(1) \circ S.
\]

Finally, by Proposition 4.3

\[
(\mu \circ s_R(l) \circ S)(a) = \mu \circ \pi_L(\pi_R(S(a)s_L(l))) = \pi_R(s_R \circ \mu(l)a) = (s^*_L \circ \mu(l))(a), \\
(\mu \circ t_R(l) \circ S)(a) = \mu(\pi_L(\pi_R(S(a))) = \pi_R(\mu(l)) \mu(l) = (t^*_L \circ \mu(l))(a), \\
\pi^*_L(\mu \circ \phi_a \circ S) = \mu \circ \phi_a \circ S(1_A) = \mu \circ \pi_{sR}(\phi_a).
\]

This proves the theorem. \(\square\)
Theorem 5.17. Let $\ell$ be a non-degenerate left integral in the symmetrized Hopf algebroid $A$. Then the left bialgebroid $A'_{sL} = (A_s, R, s_L, t_L, y_{sL}, \pi_{sL})$ where
\[
\begin{align*}
s_{sL}(r)(a) &= \mu^{-1}(r)\pi_L(a), \\
t_{sL}(r)(a) &= \pi_L(at_R \circ \kappa^{-1}(r)), \\
y_{sL}(\phi_s) &= \xi^{-2}(\ell_2) \rightarrow \phi_s \otimes \xi^{-1}(\ell_1), \\
\pi_{sL}(\phi_s) &= \lambda^s(\ell \mapsto \phi_s),
\end{align*}
\]
the right bialgebroid $A_{sR}$ in (5.11), and the antipode $S^e_s := \ell^{-1}_L \circ \xi \circ \ell_L$ form a symmetrized Hopf algebroid denoted by $A'_L$. 

Proof. We show that the triple $A'_s := (A'_{sL}, A_{sR}, S^e_s)$ satisfies the conditions in Proposition 4.2(iii).

The $A'_{sL}$ is a left bialgebroid isomorphic to $(A_{sR})^{op}$ via the isomorphism $(\ell \circ \ell^{-1}_L \circ \ell_L, \mu)$. Also
\[
\begin{align*}
s_{sL}(R) &= t_{sR}(L) \quad \text{and} \quad t_{sL}(R) &= s_{sR}(L)
\end{align*}
\]
hold obviously true. Making use of the identities (5.22) and (5.23), one checks that
\[
\begin{align*}
(id_{A_s} \otimes y_{sL}) \circ y_{sL}(\phi_s) &= \xi^{-2}(\ell_2) \rightarrow \phi_s \otimes \xi^{-1}(\ell_1) \rightarrow \xi^{-1}(\ell_1) \otimes \xi^{-1}(\ell_2) \\
&= \xi^{-2}(\ell_2) \rightarrow \phi_s \otimes \xi^{-1}(\ell_1) \otimes \ell^{-1}_L(\ell_2) \\
&= \xi^{-2}(\ell_2) \rightarrow \phi_s \otimes \xi^{-1}(\ell_1) \otimes \ell^{-1}_L(\ell_2) \\
&= (y_{sL} \otimes id_{A_s}) \circ y_{sL}(\phi_s).
\end{align*}
\]
\[
\begin{align*}
(y_{sR} \otimes id_{A_s}) \circ y_{sL}(\phi_s) &= \ell(1) \xi^{-2}(\ell_2) \rightarrow \phi_s \otimes \xi^{-1}(\ell_2) \otimes \xi^{-1}(\ell_1) \\
&= \ell(1) \rightarrow \phi_s \otimes \xi^{-1}(\ell_2) \otimes \xi^{-1}(\ell_1) \\
&= (id_{A_s} \otimes y_{sL}) \circ y_{sR}(\phi_s).
\end{align*}
\]
The $S^e_s = (\ell^{-1}_L \circ \xi \circ \ell_L) \circ (\ell \circ \ell^{-1}_L \circ \ell_L)$ is a composition of ring isomorphisms $\ell \circ \ell^{-1}_L \circ \ell_L : (A_s)^{op} \rightarrow A$ and $\ell \circ \ell^{-1}_L \circ \ell_L : A \rightarrow A_s$, hence it is an anti-automorphism of the ring $A_s$. Also
\[
\begin{align*}
S^e_s \circ t_{sL}(l)(a) &= \mu^{-1} \circ R \circ \ell^{-1} \circ \ell_L \circ t_{sR}(l) \circ S^{-1}(a) \\
&= \mu^{-1} \circ \lambda (S^{-1}[S_L(\pi_L(\ell_1))]\ell_2)]S^{-1}(a)) \\
&= \mu^{-1} \circ \pi_R \circ S^{-1}(a sl(l)) = \pi_L(a sl(l)) = s_{sR}(l)(a), \\
S^e_s \circ t_{sL}(t)(a) &= \mu^{-1} \circ R \circ \ell^{-1} \circ \ell_L \circ t_{sL}(t) \circ S^{-1}(a) \\
&= \mu^{-1} \circ \lambda^s(asl \circ \pi_L(\ell_1) \circ \kappa^{-1}(r)) \ell_2) \\
&= \mu^{-1} \circ \lambda^s((l \circ \pi_L(a) \ell_R \circ \kappa^{-1}(r)) = \mu^{-1}(r)\pi_L(a) = s_{sL}(r)(a).
\end{align*}
\]
Since
\[ \gamma_L \circ \xi^{-1}(a) = \xi^{-1}(a^{(2)}) \otimes S^{-1}(a^{(1)}) \quad \text{and} \quad \gamma_R \circ \xi^{-1}(a) = \xi^{-1}(a^{(2)}) \otimes S^{-1}(a^{(1)}), \]
we have \( \gamma_L \circ \xi^{-2}(a) = \xi^{-1} \circ S^{-1}(a^{(1)}) \otimes S^{-1} \circ \xi^{-1}(a^{(2)}). \) Then we can compute
\[
\left[ S^\ell_s(\phi_s(2))\psi_s(1) \right](a) = \left[ \ell_L^\ell \circ \xi^{-1}(\ell(1)) \xi^{-2}(\ell(2)) \rightarrow \phi_s \right](a) \\
= \phi_s \left[ \left[ a \leftarrow \ell_L^\ell \circ \xi^{-1}(\ell(1)) \xi^{-2}(\ell(2)) \right] \right] \\
= \phi_s \left[ \left[ \xi^{-1}(\ell(1)) \leftarrow \ell \xi^{-2}(\ell(2)) \right] \right] \\
= \phi_s \circ \xi^{-2}(\ell(2)) \xi^{-2}(\ell(1)) \\
= \phi_s \circ \xi^{-2}(\ell(2)) \xi^{-2}(\ell(1)) = \phi_s(1)\pi_L(\ell(1))\ell^{-1}(a)(\ell(2)) = \phi_s(1)\pi_L(\ell(1))\ell^{-1}(a)(\ell(2)).
\]

This proves that \( A^L_s = (A^L_s, A^R_s, S^L_s) \) is a symmetrized Hopf algebroid. \( \square \)

Obviously, the strong isomorphism class of the Hopf algebroid \( (A^L_s, S^L_s) \) depends only on the Hopf algebroid \( (A_L, S) \) and the non-degenerate left integral \( \ell \) of it. It is insensitive to the particular choice of the underlying right bialgebroid \( A_R \).

The antipode \( S^L_s \) has a form analogous to (5.17):
\[
S^L_s(\phi_s)(a) = \left[ \ell \leftarrow \phi_s \rightarrow \ell_L^{-1}(1_A) \right](a). \tag{5.25}
\]

Using the left bialgebroid isomorphisms of Theorem 5.16, also the dual left bialgebroids \( (*,A_R)^{op}, A^L_s \), and \( A^L_L \) can be made Hopf algebroids, all strictly isomorphic to the above Hopf algebroid \( (A^L_s, S^L_s) \). They have the antipodes
\[
* S^L = L \ell^{-1} \circ \xi \circ L \ell, \tag{5.26}
\]
\[
S^L_s = \ell_R^{-1} \circ \xi \circ \ell_R, \tag{5.27}
\]
\[
S^L_s = \ell_R^{-1} \circ \xi \circ \ell_R. \tag{5.28}
\]

Let us turn to the interpretation of the role of the Hopf algebroid \( (A^L_s, S^L_s) \). As \( A^L_s \) is isomorphic to \( (*,A_R)^{op} \) and the right bialgebroid underlying \( (A^L_s, S^L_s) \) is \( A^R_s \), on the
first sight it seems to be natural to consider it as some kind of a dual of \((A_L, S)\). There are however two arguments against this interpretation. First, the Hopf algebra \((A^{L*}_L, S^*)\) depends on \(\ell\). Second, as it is proven in Proposition 5.19, \((A^{L*}_L, S^*)\) belongs to a special kind of Hopf algebroids: it possesses a two-sided non-degenerate integral.

**Lemma 5.18.** Let \(A\) be a symmetrized Hopf algebra such that the \(R\)-module \(A^R\) in (4.6) is finitely generated projective. Then a left integral \(\ell \in I^L(A)\) is non-degenerate if and only if the map \(\ell \in I^L(A)\) is bijective.

**Proof.** The **only if** part is trivial. In order to prove the **if** part, recall that by the proof of Lemma 5.6 for \(\lambda^* := \ell^{-1}_R(1_A)\) and all \(a \in A\) the identity \(\lambda^* \rightarrow a = s_R \circ \lambda^*(a)\) holds true, \(\lambda^*\) is an \(R\)-\(R\)-bimodule map \(R A^R \rightarrow R\), and the inverse of \(\ell\) reads as \(\ell^{-1}_R(a) = \lambda^* \rightarrow S(a)\). Then

\[
\phi^*(a) = \ell^{-1}_R \circ \ell_R(\phi^*(a)) = \lambda^*(s_R \circ \phi^*(\ell_R(1_A))) = \lambda(R(\phi^*(\ell_R(1_A)))) = \phi^*(a)_R(\phi^*(\ell_R(1_A)))
\]

for all \(\phi^* \in A^*_L\) and \(a \in A\). Using the finitely generated projectivity of \(A^R\), we have \(\ell_R(1_A) = \lambda^* \circ S = 1_A\). Since \(\lambda^* \circ S \in \mathfrak{s}_A\), the inverse \(\ell^{-1}_R(a) = \lambda^* \circ S = 1_A\). \(\square\)

**Proposition 5.19.** Let \(A\) be a symmetrized Hopf algebra possessing a non-degenerate left integral \(\ell\). Then the element \(\ell^{-1}_L(1_A)\) in \(A_A\) is a two-sided non-degenerate integral in the symmetrized Hopf algebra \(A^L_A\) (constructed in Theorem 5.17).

**Proof.** It follows from (5.16) that \(\ell^{-1}_L(1_A) = \mu^{-1} \circ \lambda^*\) where \(\lambda^* := \ell^{-1}_R(1_A)\) and \(\mu\) is the ring isomorphism introduced in Proposition 4.3. For all \(\phi_a \in A_A\) and \(a \in A\), we have

\[
[(\mu^{-1} \circ \lambda^*)\phi_a](a) = \phi_a \circ S(\lambda^* \rightarrow S^{-1}(a)) = \phi_a(1_A)\mu^{-1} \circ \lambda^*(a),
\]

hence

\[
[(\mu^{-1} \circ \lambda^*)\phi_a](a) = \phi_a \circ \pi_a(\mu^{-1} \circ \lambda^*(a)) = \phi_a(1_A)\mu^{-1} \circ \lambda^*(a),
\]

which proves that \(\mu^{-1} \circ \lambda^*\) is a right integral. Using (5.17),

\[
S^*_A(\mu^{-1} \circ \lambda^*) = \ell^{-1}_L \circ \xi \circ \ell_L(\mu^{-1} \circ \lambda^*) = \ell^{-1}_L \circ \xi^2(1_A) = \ell^{-1}_L(1_A) = \mu^{-1} \circ \lambda^*.
\]

hence \(\mu^{-1} \circ \lambda^*\) is also a left integral.

As it is proven in [8], since \(A_L\) is finitely generated projective, so is the left \(L_A \equiv L\)-module \(L_A(A_A)\). The corresponding dual bialgebroid \(\star(A_A)_L\) is isomorphic to \(A_L\) via the isomorphism \((\iota, \text{id}_L)\) of left bialgebroids where

\[
\iota: A \rightarrow \star(A_A), \quad \iota(a)(\phi_a) := \phi_a(a).
\]  

(5.29)

Since
\[
\begin{align*}
(\iota(a) \to \phi_s) & = \pi_L \left([b \leftarrow (\ell^{(1)} \to \phi_s)] s_L \circ \xi^{-1}_L[\ell^{(2)}](a)\right) \\
& = \phi_s(b s_R \circ \lambda^* \circ a \ell^{(1)} S(\ell^{(2)})) = (a \to \phi_s)(b),
\end{align*}
\]

the map \(L_*(\mu^{-1} \circ \lambda^*) : \ast(A_s) \to A_s, \iota(a) \mapsto \iota(a) \leftarrow \mu^{-1} \circ \lambda^* \equiv a \to \mu^{-1} \circ \lambda^*\) is bijective with inverse
\[
L_*(\mu^{-1} \circ \lambda^*)^{-1} : \phi_s \mapsto \iota \circ \ell_L \circ \xi^{-1} = \iota \circ R(\mu \circ \phi_s \circ S).
\]

The application of Lemma 5.18 finishes the proof. \(\square\)

In the view of Proposition 5.7, the following definition makes sense.

**Definition 5.20.** The dual of the isomorphism class of a Hopf algebroid \((A_L, S)\) possessing a non-degenerate left integral \(\ell\) is the isomorphism class of the Hopf algebroid \((A_s^L, S_s^L)\) (constructed in the Theorem 5.17).

The next proposition shows that this notion of duality is involutive.

**Proposition 5.21.** Let \(\ell\) be a non-degenerate left integral in the Hopf algebroid \((A_L, S)\). Then the Hopf algebroid
\[
\left(\left(A_s^L, S_s^L\right)_{\xi^{-1}(1_A)}, \left(S_s^L\right)_{\xi^{-1}(1_A)}\right)
\]
(5.30)
is strictly isomorphic to \((A_s^L, S_s^L)\)—the Hopf algebroid constructed in Proposition 5.13. In particular, the Hopf algebroid (5.30) is isomorphic to \((A_L, S)\).

**Proof.** Since the Hopf algebroids
\[
\left(\left(A_s^L, S_s^L\right)_{\xi^{-1}(1_A)}, \left(S_s^L\right)_{\xi^{-1}(1_A)}\right) \quad \text{and} \quad \left(\ast(A_s)_{L}, \ast(S_s^L)_{\xi^{-1}(1_A)}\right)
\]
are strictly isomorphic, it suffices to show that the isomorphism \((\iota, \text{id}_L)\) of left bialgebroids \(A_L \to \ast(A_s)_{L}\) in (5.29) extends to a strict isomorphism of Hopf algebroids \((A_s^L, S_s^L) \to \ast(A_s)_{L}, \ast(S_s^L)_{\xi^{-1}(1_A)}\).

By (5.28) for a non-degenerate left integral \(\ell\) in the Hopf algebroid \((A_L, S)\), the antipode \(\ast S_\ell\) of the Hopf algebroid \((A_L, \ast S_\ell)\) reads as
\[
\ast S_\ell(\ast \phi) = \ast \lambda \leftarrow (\ast \phi \to S^{-1}(\ell)).
\]

Applying it to the non-degenerate integral \(\xi^{-1}(1_A)\) in \((A_s^L, S_s^L)\), we obtain
\[
\ast(S_s^L_{\xi^{-1}(1_A)}(\iota(a)) = \iota(\ell) \leftarrow [\iota(a) \to S^{-1}_s \circ \xi^{-1}_L(1_A)]
\]
\[
= \iota(\ell \leftarrow [a \to \xi^{-1}_L(1_A)]) = \iota \circ \xi(a). \quad \square
\]
The duality of (weak) Hopf algebras is re-obtained from Definition 5.20 as follows. Let $H$ be a finite weak Hopf algebra over a commutative ring $k$. Let $(H_L, S)$ be the corresponding Hopf algebroid (introduced in the Example 4.8) and let $\ell$ be a non-degenerate left integral in $H$. Recall that, in order to reconstruct the weak Hopf algebra from the Hopf algebroid in Example 4.8, one needs a distinguished separability structure on the base ring. The dual weak Hopf algebra is the unique weak Hopf algebra in the isomorphism class of $(H^\ell_{*L}, S^\ell_{*})$ corresponding to the same separability structure on $L$ as $H$ corresponds to.

If $H$ is a Hopf algebra over $k$ then—since the separability structure on $k$ is unique—the dual Hopf algebra is the only Hopf algebra in the isomorphism class of $(H^\ell_{*L}, S^\ell_{*})$.

References