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Algebraic geometric invariants for a class of one-relator groups

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Abstract

Given a finitely generated group H, the set $Hom(H, SL_2C)$ inherits the structure of an affine algebraic variety $R(H)$ called the *representation variety* of H. Let a one-relator group with presentation $G = \langle x_1, \ldots, x_n, y; W(\bar{x}) = y^k \rangle$ be given, where $W(\bar{x}) \neq 1$ is in the free group on the generators $\{\bar{x}\} = \{x_1, \ldots, x_n\}$, and $k \geq 2$. In this paper a theorem will be proven allowing the computation of $Dim\,(R(G))$ in terms of subvarieties of the representation variety of the free group on n generators, $R(F_n)$, arising from solutions to the equation $W(\bar{x}) = \pm I$ in SL_2C. Conditions are given guaranteeing the reducibility of $R(G)$. Finally, applications to the class of one-relator groups with non-trivial center are made. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

Given a finitely generated group G equipped with a set, $\{\bar{x}\} = \{x_1, \ldots, x_n\}$, of generators one defines $R(G)$ as the set of points $\{(\rho(x_1), \ldots, \rho(x_n))\} \in (SL_2C)^n$, where ρ is a representation of G in SL_2C. The points of $R(G)$ are in one-to-one correspondence with the representations of G in SL_2C; thus one can speak of representations of G as points of $R(G)$, and points of $R(G)$ as representations of G. It is well known that $R(G)$ is an affine variety in affine complex space of dimension $4n$. The defining equations of $R(G)$ arise from the defining relations of G relative to the generating set $\{\bar{x}\}$. The variety $R(G)$ is an invariant of the finitely generated presentation of G chosen. In other

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words, two different finite set of generators of G give rise to isomorphic representation varieties. The ideas involved in this invariant can be traced back to the work of Poincaré and others (see [8, 9]). Culler and Shalen have employed $R(G)$ in the study of the fundamental groups of three manifolds [4]. Baumslag and Shalen used a version of $R(G)$ in establishing inequalities derived from a presentation and guaranteeing that groups satisfying them are infinite [3].

The invariant $R(G)$ brings into *combinatorial group theory* the numerous invariants of *algebraic geometry* and *commutative algebra* associated with algebraic varieties. This abundance of structure deserves study. Invariants of particular interest are the dimension, and reducibility status of $R(G)$. In fact, with these naive invariants surprisingly succinct proofs of some traditional theorems of *combinatorial group theory* were given (see [7]). Dimension, for example, plays a central role in the work of Culler and Shalen [4], and Baumslag and Shalen [3].

Let F_n denote the free group of rank n; then $R(F_n)=(SL_2C)^n$. It is well known that SL_2C is an irreducible affine variety of dimension three, and consequently that $(SL_2C)^n$ is also irreducible and of dimension $3n$. Finitely presented, one-relator groups can be considered the closest finitely presented groups to free groups of finite rank [1], thus making the study of $R(G)$ for one-relator groups especially tractable. The justification for using SL_2C, besides being a relatively easy affine group to work with lies in the fact that one-relator groups share a host of properties with free groups and that finitely generated free groups embed into SL_2C.

Let G be a presentation with n generators and r relations; then the number of variables involved in the description of $R(G)$ is $4n$, and the number of polynomials $4r+n$. Since $R(G)\subseteq R(F_n)$ it follows that $Dim\,(R(G))\leq 3n$, and by using an elementary theorem involving the fibers of regular maps between algebraic varieties (see [11]) that $Dim\,(R(G))\geq 3(n-r)$. However, to establish the dimension of $R(G)$ precisely within those bounds can be impractical using existing computer-assisted techniques. Almost as difficult is to perform these computations for an arbitrary class of algebraic varieties. In this connection some examples illustrating the use of Theorem 0.2 and the sensitivity of $R(G)$ in discriminating within classes of groups will be given.

Let $W\neq 1$ be a freely reduced word in F_n involving all the generators $\{x_1,\dots,x_n\}$ of F_n. To the free group F_n add a new generator y, and now consider the relation $W=y^k$ of the one-relator group

$$G=\langle x_1,\dots,x_n,y;\,W=y^k\rangle, \tag{0.1}$$

where $k\geq 2$ is a positive integer.

Observation 0.1 The relation $W=y^k$ gives rise to an equation in SL_2C. Solutions to this equation are $(n+1)$-tuples of SL_2C matrices $(m_1,\dots,m_{n+1})$ such that the relation $W=y^k$ is satisfied when the $(n+1)$-tuple is evaluated in $W=y^k$ under the obvious assignment $x_i\to m_i$, for all $x_i\in\{x_1,\dots,x_n\}$, and $y\to m_{n+1}$. Further, observe that the relation can also be used to give rise to an equation $W=-y^k$ in SL_2C. Denote the SL_2C solutions to the equations $W=y^k$, and $W=-y^k$ by $\Sigma_n(W,k)$, and $\Sigma_n(-W,k)$, respectively. Notice that $\Sigma_n(W,k)$ and $\Sigma_n(-W,k)$ are affine algebraic varieties, also that the representation variety $R(G)$ of the one-relator group in (0.1) is $\Sigma_n(W,k)$.

By I denote the 2×2 identity matrix, and let P_+ and P_- be the two algebraic subvarieties of $R(F_n)$ given by

$$P_+ = \{\rho \mid \rho \in R(F_n), \text{and}\, \rho(W) = I\}, \qquad P_- = \{\rho \mid \rho \in R(F_n), \text{and}\, \rho(W) = -I\}. \tag{0.2}$$

Notice that the correspondence between points of $R(F_n)$ and representations of F_n was used in (0.2).

Theorem 0.2. *If $k=2$, then*
(a) $Dim(\Sigma_n(W,k)) = \mathrm{Max}\{Dim(P_+), Dim(P_-)+2,\, 3n\} \le 3n+1$.
(b) $Dim(\Sigma_n(-W,k)) = \mathrm{Max}\{Dim(P_-), Dim(P_+)+2,\, 3n\} \le 3n+1$.
If $k\ge 3$, then
(a) $Dim(\Sigma_n(W,k)) = \mathrm{Max}\{Dim(P_+)+2,\, Dim(P_-)+2,\, 3n\} \le 3n+1$.
(b) $Dim(\Sigma_n(-W,k)) = \mathrm{Max}\{Dim(P_-)+2,\, Dim(P_+)+2, 3n\} \le 3n+1$.

Corollary 0.3. *If in Theorem* 0.2a $\mathrm{Max}\{Dim(P_+)+2, Dim(P_-)+2\} \ge 3n$, *then $\Sigma_n(W,k)$ is a reducible variety.*

Corollary 0.3 has been stated for the case $k\ge 3$; a slightly modified version of the corollary holds in the case when $k=2$. These versions and a similar corollary for $\Sigma_n(-W,k)$ are discussed in Section 2.

Observe that Theorem 0.2 reduces a question about the representation variety of an $n+1$ generated group into a question in the representation variety of the free group of rank n. This will prove quite useful. A further consequence of Corollary 0.3 is that an $n+1$ generated group G satisfying its conditions is not free. Examples of groups satisfying the conditions of Corollary 0.3 abound.

That the method used to obtain (0.1) can produce groups significantly different from free groups is easy to see. For example, consider groups of the type $G_{pt} = \langle x, y; x^p = y^t\rangle$, where p,t are integers greater than one. This infinite class of groups consists of groups having non-trivial center and with the property "NZ" (that G/G' is not isomorphic to the free abelian group of rank two). Incidentally, this class of groups contains an infinite number of groups satisfying the conditions of Corollary 0.3.

The next result follows easily from Theorem 0.2.

Theorem 0.4. *Let $G_{pt} = \langle a,b; a^p = b^t\rangle$, where $p,t\ge 2$, then $Dim(R(G)) = 4$, and $R(G)$ is reducible.*

A rather straightforward consequence obtained from Theorem 0.4 is that a one-relator group G with non-trivial center and with property NZ satisfies the inequality $3 \le Dim(R(G)) \le 4$. The NZ condition for one-relator groups with non-trivial center turns out to affect the invariant $R(G)$ in crucial ways. To illustrate this you will be shown an infinite class of one-relator groups having non-trivial center, failing the NZ property, such that for each G in the class $Dim(R(G)) > 5$.

Pietrowski, [13], showed that given a one-relator group G with non-trivial center and property NZ, that G can be presented as

$$G = \langle x_1, \dots, x_{m+1}; x_1^{p_1} = x_2^{q_1}, x_2^{p_2} = x_3^{q_2}, \dots, x_m^{p_m} = x_{m+1}^{q_m} \rangle, \tag{0.3}$$

where $p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$, and additionally that the isomorphism class of a group as in (0.3) is determined by the sequence $(p_1, q_1, p_2, q_2, \dots, p_m, q_m)$ together with its mirror image $(q_m, p_m, \dots, q_2, p_2, q_1, p_1)$. Call the first of these sequences *the p-sequence of* G, and define its *length* to be m. The next result follows rather effortlessly.

Theorem 0.5. (i) *If G is an one-relator group with non-trivial center and having the property that its p-sequence is of length greater or equal to two, and that some p_i and q_j are each greater than 2, then $Dim(R(G)) = 4$.*

(ii) *If the p-sequence of G is of length one, then $Dim(R(G)) = 4$.*

Corollary 0.6. *For G as in the previous theorem, $R(G)$ is reducible.*

Briefly, Section 1 contains elementary facts concerning solutions in SL_2C to equations of the type $X^p = A$, where $A \in SL_2C$, and several other lemmas playing a very important role in the proof of the main result. In Section 2, the main result, "Theorem 0.2" is proved. Finally, in the section titled "Examples" some calculations are made, as well as the applications already mentioned in the introduction.

1.

This section contains preliminary material necessary in the proof of Theorem 0.2; it is here merely for self-containment; with few exceptions results will be stated without proofs as those can be obtained simply by using basic facts in the theory of matrices and algebraic varieties appearing in, for example, [5, 11]. Some, like Lemma 1.1, follow directly from the Jordan normal form. Others, like Lemma 1.7 hold in more general settings, but only the SL_2C case is of interest here as that is the only affine group involved in this work. All algebraic varieties will be affine and not necessarily irreducible, unless otherwise stated. By $|V|$ denote the number of irreducible components of a zero-dimensional variety V.

Given $M \in SL_2C$, denote by $Cent(M)$ the SL_2C centralizer of M. Given a positive integer p and $M \in SL_2C$, denote by $\Omega(p, M)$ the affine variety $\{A \mid A \in SL_2C, A^p = M\}$. By $\mathrm{Tr}(A)$ indicate the trace of the matrix A. Finally, by $Orb(A)$ designate the orbit under conjugation of an SL_2C matrix A. Two matrices A and B are said to be similar if one can be conjugated to the other. Any two matrices A and B in SL_2C similar over GL_2C are also similar over SL_2C.

Lemma 1.1. *Every matrix in SL_2C is similar to one of the following matrices:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

where, $a \neq \pm 1$.

Corollary 1.2. *If* $\mathrm{Tr}(A) \neq \pm 2$, *then A is diagonalizable.*

The coordinate ring of SL_2C is the algebra $A = C[t_1, \dots, t_4]/J$, where J is the ideal consisting of all polynomials in $A = C[t_1, \dots, t_4]$ which vanish in SL_2C as a subvariety of complex affine four-dimensional space C^4 given by the zeros of the polynomial $t_1t_4 - t_3t_2 - 1$.

Lemma 1.3. *Let B be a matrix of* trace b, *where $b \neq \pm 2$. A matrix $A \in SL_2C$ lies in $Orb(B)$ iff it lies in the SL_2C vanishing set of $t_1 + t_4 - b$, where the t_i's are as above.*

Corollary 1.4. *Let $A \in SL_2C$ be any matrix of a given* trace $b \neq \pm 2$. *Then any matrix B in SL_2C having trace b is similar to A.*

Corollary 1.5. *Let $A \in SL_2C$ be of trace $b \neq \pm 2$. Then the orbit of A under SL_2C conjugation is an irreducible algebraic variety of dimension two.*

Lemma 1.6. (i) *If $p = 2$, then $Dim\,\Omega(p,I) = 0$, and $|\Omega(p,I)| = 2$.*
(ii) *If $p > 2$, then $Dim\,\Omega(p,I) = 2$, and is reducible.*
(iii) *For $p \geq 2$, $Dim\,\Omega(p,-I) = 2$, and for $p \geq 3$ is reducible.*

Proof. (i) Suppose $p = 2$. Let $A \in SL_2C$ be given. Then $A^2 = I$ implies $A = A^{-1}$. Thus, the only matrices in $\Omega(p,I)$ are $\pm I$, and consequently, the reducibility and cardinality assertion follow.

(ii) Assume that $p > 2$ is given. Then any matrix A of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ such that $a \neq \pm 1$ is a pth root of unity is a solution to the equation $X^p = I$. Choose A such that $\mathrm{Tr}(A) \neq \pm 2$. Use Corollary 1.5 to obtain that $Orb(A)$ is an affine variety of dimension two. $Dim\,\Omega(p,I)$ is precisely two since SL_2C is a three-dimensional irreducible variety properly containing $\Omega(p,I)$. To deduce the reducibility apply Corollary 1.5 and Lemma 1.3.

(iii) For $p = 2$, $A = \left(\begin{smallmatrix} i & 0 \\ 0 & i^{-1} \end{smallmatrix}\right)$ is a solution to the equation $X^p = -I$; so are all matrices in $Orb(A)$. By Corollary 1.5 the dimension of $Orb(A)$ is two. Thus $Dim\,\Omega(p,-I) \geq 2$. That it is strictly smaller than three follows from the fact that $\Omega(p,-I) \neq SL_2C$. When $p \geq 2$ the proof follows in an analogous fashion to the proof of (ii) above. To show reducibility when $p \geq 3$ use Lemma 1.3 and Corollary 1.5. $\square$

The next Lemma shall be used to determine the dimension of the varieties $\Omega(p,M)$, for M a matrix in SL_2C other than $\pm I$.

Lemma 1.7. *Let*

$$C=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}-1&1\\0&-1\end{pmatrix},\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix},a\neq\pm 1\right\},$$

and suppose $A\in C$. *Then* $\Omega(p,M)$, *where* $M=TAT^{-1}$, *is given by*

$$S_c=\{TC_AA_rC_A^{-1}T^{-1}\mid C_A\in Cent(A),\ A_r\in\Omega(p,A)\}.$$

Lemma 1.8. (i) *Let* $p\geq 2$ *be any even integer,* $B=\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$, *and* $M\in Orb(B)$. *Then* $\Omega(p,M)=\emptyset$.

(ii) *Let* $p\geq 3$ *be any odd integer, and* M *as in* (i). *Then* $Dim\,\Omega(p,M)=0$, *and* $|\Omega(p,M)|=1$.

(iii) *Let* $p\geq 2$ *be any integer,* $B=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, *and* $M\in Orb(B)$. *Then* $Dim\,\Omega(p,M)=0$, *and* $|\Omega(p,M)|$ *is one if* p *is odd and two if* p *is even.*

(iv) *Let* $p\geq 2$ *be an integer, and* $M\in Orb(A)$, *where* $A=\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)$ *and* $a\neq\pm 1$; *then* $Dim\,\Omega(p,M)=0$, *and* $|\Omega(p,M)|=p$.

Proof. (i) The elementary divisor of M is the polynomial $(\lambda-(-1))^2$ and thus the elementary divisor for $X\in\Omega(p,M)$ is of the form $(\lambda-r_i)^2$, where $r_i^p=-1$ (see [5]). Since $X\in SL_2C$ such values r_i must also meet the condition that $r_i^2=1$. This together implies that only $r_i=-1$ is possible and only when p is an odd integer.

(ii) For B as in (i) let $X\in\Omega(p,B)$; then its elementary divisor has the form $(\lambda-r_i)^2$, where $r_i^p=-1$ and $r_i^2=1$; so only $r_i=-1$ is possible. Thus, $X=TBT^{-1}$. For p odd $B^p=\left(\begin{smallmatrix}-1&p\\0&-1\end{smallmatrix}\right)$. Thus, $X^p=TB^pT^{-1}$. It is easy to see that T is an upper triangular non-singular matrix. Thus, X is also upper triangular. So $X=\left(\begin{smallmatrix}-1&y\\0&-1\end{smallmatrix}\right)$, where $y\neq 0$. Now for p odd $X^p=\left(\begin{smallmatrix}-1&py\\0&-1\end{smallmatrix}\right)$. So $X=\left(\begin{smallmatrix}-1&p^{-1}\\0&-1\end{smallmatrix}\right)$ is the single element of $\Omega(p,B)$. It can be easily shown that $Cent(B)$ is commutative and that $\Omega(p,B)$ lies in $Cent(B)$. Thus, S_c of Lemma 1.7 consists only of $\Omega(p,B)$ which by the above consists of only one element. Thus, $Dim\,S_c=0$.

(iii) Proofs of the following assertions will be necessary:

(1) For $p\geq 2$ an even integer, $Dim\,\Omega(p,B)=0$, and $|\Omega(p,B)|=2$.

(2) For $p>2$ an odd integer, $Dim\,\Omega(p,B)=0$, and $|\Omega(p,B)|=1$.

Proof of (1). Let p be even and X in $\Omega(p,B)$. By a similar argument to the above involving elementary divisors one obtains that for some $T\in SL_2C$, possibly differing depending on the case, either *a:* $X=T\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)T^{-1}$ or *b:* $X=T\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)T^{-1}$. The cases will be treated separately. *Case a:* Then $X^p=T\left(\begin{smallmatrix}1&p\\0&1\end{smallmatrix}\right)T^{-1}$ implies that $X=\left(\begin{smallmatrix}1&p^{-1}\\0&1\end{smallmatrix}\right)$. So $Dim\,\Omega(p,B)=0$ and is of cardinality one. *Case b:* Then $X^p=T\left(\begin{smallmatrix}1&-p\\0&1\end{smallmatrix}\right)T^{-1}$ implies that $X=\left(\begin{smallmatrix}-1&-p^{-1}\\0&-1\end{smallmatrix}\right)$. Thus, $Dim\,\Omega(p,B)=0$, and its cardinality is one.

Proof of (2). Let p be odd and $X \in \Omega(p,B)$. Then by similar arguments to the preceding, involving elementary divisors, one gets that $X = T\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)T^{-1}$, for some $T \in SL_2C$; so $X^p = T\left(\begin{smallmatrix}1 & p\\ 0 & 1\end{smallmatrix}\right)T^{-1}$. Thus, $X = \left(\begin{smallmatrix}1 & p^{-1}\\ 0 & 1\end{smallmatrix}\right)$. Thus, $Dim\,\Omega(p,B) = 0$ and its cardinality is one. This concludes the proofs of (1) and (2).

It is easy to show that $Cent(B)$ is commutative and that $\Omega(p,B) \subset Cent(B)$. Thus, the set S_c of Lemma 1.7 has the same cardinality as $\Omega(p,B)$, and as a consequence $Dim(S_c) = 0$, concluding the proof of (iii).

(iv) Begin by finding solutions to the equation $X^p = A$. Notice that the elementary divisors of A are $(\lambda - a)$ and $(\lambda - a^{-1})$. Thus, the elementary divisors of X have to be of the form $(\lambda - \varepsilon_1)$, $(\lambda - \varepsilon_1^{-1})$, where $(\varepsilon_1)^p = a$. Thus, there are only a finite number of conjugacy classes of matrices X with $X^p = M$. $Cent(A)$ consists of diagonal non-singular matrices; so they commute with any diagonal matrix. Now applying Lemma 1.7 yields $Dim\,\Omega(p,M) = 0$. The cardinality of $\Omega(p,M)$ is p, by Corollary 1.4. □

Corollary 1.9. *For $B \in SL_2C$, $Dim\,\Omega(p,B) > 0$ only when $B = I$ and $p \geq 3$, or $B = -I$ and $p \geq 2$.*

Proof. This is a direct consequence of Lemmas 1.6 and 1.8. □

2.

In this section first an intuitive feeling is given for how the proof of Theorem 0.2 is put together; then some preliminary definitions are introduced, and finally the theorem is proved, followed by proofs of the lemmas and minor results cited in the demonstration and not previously addressed in Section 1.

Let V, W be two algebraic varieties (W is irreducible of known dimension larger than zero). Suppose that the dimension of V is desired, and that $V = \{(V - S) \cup (S)\}$ where S is some subvariety of V; then $Dim(V) = \mathrm{Max}\{Dim(V - S), Dim(S)\}$. If S is chosen in a clever way relative to some dominating regular map $\Phi : V \to W$ with nice properties, then one can determine the dimension of $\{V - S\}$ using a fibre theorem such as Proposition 2.5. Finally, to determine the dimension of V the dimension of S must be computed or bounded. In the proof of Theorem 0.2a, $\Sigma(W,K)$ will play the role of V and $R(F_n)$ that of W. Definition of the various parts involved is the goal of what follows.

Denote the union of the SL_2C solutions to the equations $W = y^k$, $W = -y^k$ obtained from the relation in (0.1) by $\Sigma_n(\pm W,k)$; consult Observation 0.1. By F_n denote the free group on $\{x_1,\dots,x_n\}$. Let ρ be a representation of F_n in SL_2C. Define $\rho(\bar{x}) = \rho(x_1),\dots,\rho(x_n)$. A fruitful way of thinking of $\Sigma_n(\pm W,k)$ is as

$$\Sigma_n(\pm W,k) = \{(\rho(\bar{x}),\sigma) \mid \rho \in R(F_n), \sigma \in \Omega(k, \pm\rho(W))\}. \tag{2.1}$$

Clearly,

$$\Sigma_n(W,k)=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\ \sigma\in\Omega(k,\rho(W))\},$$

$$\Sigma_n(-W,k)=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\sigma\in\Omega(k,-\rho(W))\}.$$

Now define a map

$$\Phi:\Sigma_n(\pm W,k)\to R(F_n) \tag{2.2}$$

given by $\Phi(m_1,\dots,m_n,m_{n+1})=(m_1,\dots,m_n)$. Notice that the map Φ can be restricted to $\Sigma_n(W,k)$ or $\Sigma_n(-W,k)$. The context will render clear which restriction is being considered. Let $S\subset\Sigma_n(W,k)$ be defined as follows:

$$S=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\sigma\in\Omega(k,\rho(W)),\rho(W)=\pm I\}. \tag{2.3}$$

Clearly, $S=S_+\cup S_-$, where $S_+=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\sigma\in\Omega(k,\rho(W)),\rho(W)=+I\}$, and $S_-=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\sigma\in\Omega(k,\rho(W)),\rho(W)=-I\}$. Notice that S is a subvariety of $\Sigma_n(W,k)$.

Observation 2.1. $S_+=\Phi(S_+)\times\Omega(k,I)$, and $S_-=\Phi(S_-)\times\Omega(k,-I)$.

In the same way a subvariety S^- of $\Sigma_n(-W,k)$ can be defined: $S^-=\{(\rho(\bar{x}),\sigma) \mid \rho\in R(F_n),\sigma\in\Omega(k,-\rho(W)),\rho(W)=\pm I\}$. In an analogous fashion, $S^-=S_+^-\cup S_-^-$.

Observation 2.2. $S_+^-=\Phi(S_+)\times\Omega(k,-I)$, and $S_-^-=\Phi(S_-)\times\Omega(k,I)$.

Clearly, $\Phi(S_+)$, and $\Phi(S_-)$ are affine varieties contained in $2T$ and $-2T$, respectively, where $2T$ and $-2T$ are the following subvarieties of $R(F_n)$:

$$2T=\{\rho \mid \rho\in R(F_n),\mathrm{Tr}(\rho(W))=2\},\ \text{and}\ -2T=\{\rho \mid \rho\in R(F_n),\mathrm{Tr}(\rho(W))=-2\}. \tag{2.4}$$

Using Lemma 1.6 and Corollary 1.9, one can deduce that any point in the image of Φ not having zero-dimensional inverse image lies in $\{2T\cup -2T\}$; this was precisely the justification for the selection of S and $\pm 2T$.

Remark 2.3. Describe two quasi-affine varieties $\ddot{A}$ and $\ddot{O}$ of $R(F_n)$ relative to $\Sigma_n(W,k)$ and $\Sigma_n(-W,k)$, respectively. Their description will depend on the parity of k in $\Sigma_n(W,k)$ and $\Sigma_n(-W,k)$, respectively.

(a) For k even: $\ddot{A}=\{R(F_n)-\{\Phi(S)\cup -2T\}\}$, and $\ddot{O}=\{R(F_n)-\{\Phi(S^-)\cup 2T\}\}$.

(b) For k odd: $\ddot{A}=\{R(F_n)-\Phi(S)\}$, and $\ddot{O}=\{R(F_n)-\Phi(S^-)\}$.

Regardless of the parity of k, $\ddot{A}$ and $\ddot{O}$ are non-empty and consequently of dimension $3n$ (see Lemmas 2.7 and 2.9). The separation into "even" and "odd" is the result of the anomaly introduced by part (i) of Lemma 1.8.

The proof of Theorem 0.2 can now be given. So as to handle the various cases of the theorem systematically and compactly the following integer function is being introduced.

$$f(x)=\begin{cases}1 & \text{if } x\neq 2,\\ 0 & \text{if } x=2.\end{cases} \tag{2.5}$$

Proof of Theorem 0.2. (a) Notice that $\Sigma_n(W,k)=((\Sigma_n(W,k)-S)\cup S)$. Consequently, $Dim(\Sigma_n(W,k))=\mathrm{Max}\{Dim(\Sigma_n(W,k)-S), Dim(S)\}$. By Lemma 2.6 $Dim(\ddot{A})=Dim(\Sigma_n(W,k)-S)=3n$. In Corollary 2.8 it was established that $Dim(S)\leq 3n+1$. Thus, it is true that $Dim(\Sigma_n(W,k))=\mathrm{Max}\{Dim(\Sigma_n(W,k)-S), Dim(S)\}\leq 3n+1$. Notice that $\Phi(S_+)=P_+$, and that $\Phi(S_-)=P_-$. Now employing the function in (2.5) together with Observation 2.1 and Lemma 1.6 gives that $Dim(S)=\mathrm{Max}\{Dim(P_+)+2f(k), Dim(P_-)+2\}$. Thus, $Dim(\Sigma_n(W,k))=\mathrm{Max}\{\mathrm{Max}\{Dim(P_+)+2f(k), Dim(P_-)+2\}, 3n\}\leq 3n+1$.

(b) Notice that $\Sigma_n(-W,k)=((\Sigma_n(-W,k)-S^-)\cup S^-)$. Consequently, $Dim(\Sigma_n(-W,k))=\mathrm{Max}\{Dim(\Sigma_n(-W,k)-S^-), Dim(S^-)\}$. Lemma 2.6, established that $Dim(\ddot{O})=Dim(\Sigma_n(-W,k)-S^-)=3n$. In Corollary 2.8 it was established that $Dim(S^-)\leq 3n+1$. Thus, it is true that $Dim(\Sigma_n(-W,k))=\mathrm{Max}\{Dim(\Sigma_n(-W,k)-S^-), Dim(S^-)\}\leq 3n+1$. Observe that $\Phi(S_+^-)=P_+$, and $\Phi(S_-^-)=P_-$. Now by employing the function in (2.5) together with Observation 2.2 and Lemma 1.6, it follows that $Dim(S^-)=\mathrm{Max}\{Dim(P_-)+2f(k), Dim(P_+)+2\}$. Thus, $Dim(\Sigma_n(-W,k))=\mathrm{Max}\{\mathrm{Max}\{Dim(P_-)+2f(k), Dim(P_+)+2\}, 3n\}\leq 3n+1$. $\square$

The next proposition is used in the proof of Proposition 2.5. Its proof can be found in [11]. Recall that a map between varieties $\Phi: V\to W$ is said to be almost surjective if the closure of $\Phi(V)$ in W is W.

Proposition 2.4. *Suppose that $\Phi: V\to W$ is an almost surjective regular map between irreducible varieties. Let $r=Dim(V)-Dim(W)$, then there exists an open set $O\subset W$ such that*

(i) $O\subset\Phi(V)$.

(ii) *For all irreducible closed subsets $q\subset W$ such that $q\cap O\neq\emptyset$ and for all components p of $\Phi^{-1}(q)$ such that $p\cap\Phi^{-1}(O)\neq\emptyset$, $Dim(p)=Dim(q)+r$.*

Proposition 2.5. *Let $\phi: V\to W$ be a regular map between two algebraic varieties, where W is irreducible and $Dim(W)=n>0$. Let V_1 and W_1 be two proper closed subvarieties of V and W, respectively, such that the restricted map $\phi: V^0\to W^0$, where $V^0=V-V_1$ and $W^0=W-W_1$, is such that*

(1) *$\phi: V^0\to W^0$ is onto.*

(2) *ϕ has zero-dimensional fiber above each point of W^0.*

(3) *$\phi^{-1}(W^0)=V^0$.*

then $Dim(Cl(W^0))=Dim(Cl(V^0))=n$, where $Cl(W^0)$ denotes the Zariski closure of W^0.

Proof. Let $V = \mathscr{V}_1 \cup \mathscr{V}_2 \cup \cdots \cup \mathscr{V}_m$ be the unique expression of V as a union of maximal irreducible components, and let $V_0^i = \mathscr{V}_i - (V_1 \cap \mathscr{V}_i)$, for $1 \leq i \leq m$. Notice $\phi^{-1}(W^0) = \{V_0^j \cup \cdots \cup V_0^k\}$. Let V_0^i be an element of $\{V_0^j \cup \cdots \cup V_0^k\}$ of maximal dimension; $Dim(V_0^i) \geq n$ since V^0 maps onto W^0. Restrict ϕ to V_0^i and notice that ϕ has finite fiber in the image of its restriction. Also $Cl(V_0^i) = \mathscr{V}_i$. Notice $Cl(\phi(Cl(V_0^i))) = W_0^i \subset W$. Clearly, W_0^i is an irreducible subvariety of W. Let $E = \phi(Cl(V_0^i))$. By Proposition 2.4 there exists a non-empty Zariski open set $O \subseteq E$. Let $O' = \{O \cap (W_0^i - W_1)\}$; clearly $O' \subseteq O$. Also $\phi^{-1}(O')$ is dense in $Cl(V_0^i)$. Notice that for a point $u \in O'$, $Dim(\phi^{-1}(u)) = 0$. Set, $\Delta = \phi^{-1}(O')$, and notice that $Dim(\Delta) = Dim(Cl(V_0^i))$. For any point p in Δ, $Dim(\phi^{-1}(\phi(p))) = 0$. Let $\phi(p) = q \in O'$. Now q is a single point in O' and, by Proposition 2.4, $Dim(\phi^{-1}(q)) = Dim(q) + Dim(Cl(V_0^i)) - Dim(W_0^i)$. Since $\phi^{-1}(q)$ is a finite number of points, and q is a single point, it follows that $0 = 0 + Dim(Cl(V_0^i)) - Dim(W_0^i)$. Thus, $Dim(Cl(V_0^i)) = Dim(W_0^i)$. Thus, $Dim(W_0^i) \geq n$. But W_0^i is a closed subvariety of W which happens to be an irreducible variety of dimension n. It follows thus that $W_0^i = W$, and consequently that $Dim(Cl(V_0^i)) = n$. □

Lemma 2.6. *If $\ddot{A}$, and $\ddot{B}$ are as in Remark 2.3 then*

(i) $\Phi(\Sigma_n(W,k) - S) = \ddot{A}$, *and for any point* $r \in \ddot{A}$, $Dim(\Phi^{-1}(r)) = 0$.

(ii) $\Phi(\Sigma_n(-W,k) - S^-) = \ddot{O}$, *and for any point* $r \in \ddot{O}$, $Dim(\Phi^{-1}(r)) = 0$.

(iii) $Dim(\Sigma_n(W,k) - S) = Dim(\ddot{A}) = 3n$.

(iv) $Dim(\Sigma_n(-W,k) - S^-) = Dim(\ddot{O}) = 3n$.

Proof. (i) and (ii) Bearing in mind the parity of k, (i) and (ii) are consequences of Corollary 1.9, the definition of the subvarieties S and S^-, and the fact that matrices M other than $-I$ with $\mathrm{Tr}(M) = -2$ have the property that $\Omega(p,M) = \emptyset$, whenever p is even (see Lemma 1.8).

(iii) and (iv) Parts (iii) and (iv) follow directly from parts (i) and (ii) above, Proposition 2.5, the fact that $Dim(R(F_n)) = 3n$, and that $\ddot{A}$, $\ddot{O}$ are quasi-affine subvarieties of the irreducible variety $R(F_n)$. □

Lemma 2.7. (i) $Dim(\Phi(S)) < 3n$ *and* $\Phi(S_+)$ *is non-empty.*

(ii) $Dim(\Phi(S^-)) < 3n$ *and* $\Phi(S_+^-)$ *is non-empty.*

Proof. (i) W is a non-trivial word in the free group on $\{x_1, \ldots, x_n\}$. Since a non-cyclic free group has a trivial center and embed into SL_2C (see [14]) there exists some representation $\rho \in R(F_n)$ with $\rho(W) \neq \pm I$. But $R(F_n)$ is an irreducible variety of dimension $3n$, and $\Phi(S)$ a proper subvariety; it must be then that $Dim(\Phi(S)) \leq 3n - 1$. Observe that the trivial representation $(I, I, \ldots, I)$ lies in $\Phi(S_+)$ and, that consequently, it is not empty.

(ii) The proof of (ii) is essentially the same once Observation 2.2 is made. □

Corollary 2.8. (i) $Dim(S) \leq 3n+1$.
(ii) $Dim(S^-) \leq 3n+1$.

Proof. (i) and (ii) These are direct consequences of Lemma 2.7, Observations 2.1 and 2.2, together with Lemma 1.6. □

Lemma 2.9. (i) $Dim(2T) \leq 3n-1$.
(ii) $Dim(-2T) \leq 3n-1$.

Proof. (i) and (ii) The trace conditions on $2T$ and $-2T$, respectively, give rise to two polynomial functions on $R(F_n)$ which are restrictions of polynomial functions in complex affine $4n$-dimensional space. The varieties $2T$ and $-2T$ are the respective vanishing sets of these polynomial functions. By [2, p. 98] there exists a $\rho \in R(F_n)$ with the property $\mathrm{Tr}(\rho(W)) \neq 2$. Thus, $2T$ does not contain all of $R(F_n)$. Notice that $2T \cap R(F_n) \neq \emptyset$ since the trivial representation lies in $2T$. Thus, $Dim(2T) \leq 3n-1$. In a similar fashion, $-2T$ does not contain all of $R(F_n)$, since the trivial representation is not in it. It follows then that $Dim(-2T) \leq 3n-1$. □

Proposition 2.10. *Let S be a proper subvariety of some algebraic variety V. If $Dim(V-S)=n$ and $Dim(S) \geq n$, where $n \geq 0$ then V is reducible.*

Proof. Assume that V is irreducible. Let $Dim(V-S)=n$ and $Dim(S)=n$. This implies that S is properly contained in V and has the same dimension as the presumedly irreducible variety V, a contradiction. On the other hand, If $Dim(V-S)=n$, and $Dim(S)>n$ then V is reducible which is a contradiction. □

For the sake of compactness and generality, the statement and the proof of the next corollary will be made using the function introduced in (2.5).

Corollary 0.3. *If in Theorem* 0.2a $\mathrm{Max}\{Dim(P_+)+2f(k), Dim(P_-)+2\} \geq 3n$, *then $\Sigma_n(W,k)$ is a reducible variety.*

Proof. Using Observation 2.1 and the fact that $\Phi(S_+)=P_+$, and $\Phi(S_-)=P_-$, it follows that $Dim(S)=\mathrm{Max}\{Dim(P_+)+2f(k), Dim(P_-)+2\}$, see (2.5). By Lemma 2.6, $Dim(\Sigma_n(W,k)-S)=3n$. Now, if $Dim(S) \geq 3n$ an application of Proposition 2.10 yields the result. □

Again, in the statement of the next corollary the function in (2.5) is used.

Corollary 2.11. *If in Theorem* 0.2b $\mathrm{Max}\{Dim(P_-)+2f(k), Dim(P_+)+2\} \geq 3n$, *then $\Sigma_n(-W,k)$ is a reducible variety.*

Proof. Use the above proof with minor modifications. □

3. Examples

Theorem 0.4. *Let G have presentation as in $G = \langle a, b; a^p = b^t \rangle$, where $p, t \geq 2$; then $Dim(R(G)) = 4$, and $R(G)$ is reducible.*

Proof. Let $\Phi(S_+) = \{\rho \mid \rho \in R(F_1)$ and $\rho(a^p) = I\}$, $\Phi(S_-) = \{\rho \mid \rho \in R(F_1)$ and $\rho(a^p) = -I\}$. By Lemma 1.6, $Dim(\Phi(S_+)) = 2$ if $p > 2$, else if $p = 2$, $Dim(\Phi(S_+)) = 0$. By Lemma 1.6, $Dim(\Phi(S_-)) = 2$. Using the notation $P_+ = \Phi(S_+)$, $P_- = \Phi(S_-)$ via Theorem 0.2 together with the function introduced in (2.5) it follows that $Dim(R(G)) = \mathrm{Max}\{\mathrm{Max}\{Dim(P_+) + 2f(t), Dim(P_-) + 2\}, 3\}$ (see (2.5)). Thus, $Dim(R(G)) = \mathrm{Max}\{4, 3\}$, in other words. The reducibility follows from Corollary 0.3. □

If necessary, refer to the Introduction for the definitions of the "*NZ property*" and the "*p-sequence*" for a one-relator group with non-trivial center.

Theorem 3.1. *Let G be a one-relator group with non-trivial center having the NZ property; then $3 \leq Dim(R(G)) \leq 4$.*

Proof. By a result of Meskin et al. [10] a group such as in (0.3) can also be presented as

$$\begin{aligned} H &= \langle x_1, x_{m+1} \mid x_1^{p_1 p_2 \cdots p_m} = x_{m+1}^{q_1 q_2 \cdots q_m}, [x_1^{p_1 \cdots p_{k-1}}, x_{m+1}^{q_k \cdots q_m}] \\ &= 1, \ldots, \text{ where } k = 2, \ldots, m \rangle. \end{aligned} \tag{3.1}$$

Notice that the group (3.1) is a proper quotient of a group G, that by Theorem 0.4 has $Dim(R(G)) = 4$. So H is isomorphic to G/N where N is a normal subgroup of G. It follows then that $Dim(R(H)) \leq Dim(R(G)) = 4$. That $3 \leq Dim(R(G))$ follows from a result of Murasugi asserting that a one-relator group with non-trivial center (other than a finite cyclic group) is torsion free and two-generated (see [12]). Thus, the deficiency of G is one, and consequently, $3 \leq Dim(R(G))$ (see [11]). □

Next, an example will be given of infinite class of groups that are one-relator groups with non-trivial center failing to have the NZ property and with $Dim(R(G)) \geq 5$. Let

$$G_m = \langle a, b; ab^m a^{-1} = b^m \rangle, \tag{3.2}$$

where $m \geq 3$ is an integer. These non-abelian groups are one-relator groups with non-trivial center, and they are not isomorphic for different values of m.

Consider the following class of groups

$$P_m = \langle x \rangle * Z_m, \tag{3.3}$$

consisting of the free product of an infinite cyclic group with a cyclic group of order m. Since the variety of representations of a free product is the product of the varieties of representations of the factors, $R(P_m) = (SL_2C) \times \Omega(m, I)$, and consequently, since

$\Omega(m,I)$ is reducible, $R(P_m)$ is reducible of dimension five (see Lemma 1.6). In fact, the varieties are all non-isomorphic for different values of m.

Observe that given $\rho \in R(P_m)$ that ρ lies also in $R(G_m)$. Thus, $R(P_m)$ injects in $R(G_m)$, and consequently, $Dim(R(G_m)) \geq 5$; that it is precisely five follows from the fact that a free group of rank two has an irreducible representation variety of dimension six, and consequently, no non-free group generated by two elements can have a representation variety of dimension six (see [7]).

In the next theorem it will be shown that for the overwhelming majority of one-relator groups with non-trivial center and having the NZ property, the dimension of their representation variety is precisely four.

Theorem 3.5. (i) *If G is an one-relator group with non-trivial center and having the property that its p-sequence is of length greater or equal to two, and that some p_i and q_j are each greater than* 2, *then $Dim(R(G)) = 4$.*

(ii) *If the p-sequence of G is of length one, then $Dim(R(G)) = 4$.*

Proof. (i) By [10] G has a presentation as in (3.1). Consider the quotient of this presentation (3.1) by the normal subgroup N generated by the two elements $x_i^{p_i}$ and $x_{m+1}^{q_j}$, where p_i and q_j are as in the theorem. Then, $G/N \cong \langle x_1, x_{m+1} \mid x_1^{p_i} = 1, x_{m+1}^{q_j} = 1 \rangle$. $R(G/N) \cong \Omega(p_i, I) \times \Omega(q_j, I)$ and is consequently a reducible variety of dimension four (see Lemma 1.6). The fact that G maps onto a group with representation variety of dimension four yields that $Dim(R(G)) \geq 4$, but by Theorem 3.1, $Dim(R(G)) \leq 4$. Thus, $Dim(R(G)) = 4$.

(ii) If the p-sequence is of length one apply Theorem 0.4. □

Corollary 0.6. *For G as in Theorem* 0.5, *$R(G)$ is reducible.*

Proof. G maps onto a group G/N (see proof of Theorem 0.5) with a reducible algebraic variety of the same dimension as $R(G)$; thus, $R(G)$ must be reducible. □

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