Examples of non-archimedean Fréchet spaces without nuclear Köthe quotients

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Received 26 July 2007
Available online 26 January 2008
Submitted by B. Cascales

Abstract
Let $\mathbb{K}$ be a spherically complete non-archimedean valued field. We prove that the dual space $l_\infty$ of the Banach space $c_0$ has a total strongly non-norming subspace $M$. Using this subspace $M$ we construct a non-normable Fréchet space $F$ of countable type with a continuous norm such that its strong dual $F'_b$ is a strict LB-space. Next we show that $F$ has no nuclear Köthe quotient.

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Keywords: Strongly non-norming subspace in the dual of a non-archimedean Banach space; Strong dual of a non-archimedean Fréchet space; Strict non-archimedean LB-space

Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|: \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [9,12], and [11].

Any infinite-dimensional Banach space $E$ of countable type is isomorphic to the Banach space $c_0$ of all sequences in $\mathbb{K}$ converging to zero with the sup-norm and any closed subspace of $c_0$ is complemented [11, Theorem 3.16].

By a Köthe space we mean an infinite-dimensional Fréchet space with a Schauder basis and with a continuous norm. A space $E$ is nuclear if and only if $E$ has no quotient isomorphic to $c_0$ [19, Theorem 2]. Thus $E$ has a normable Köthe quotient if and only if $E$ is not nuclear.

It is known that $E$ has a quotient isomorphic to the nuclear Fréchet space $\mathbb{K}^\mathbb{N}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence [21, Theorem 3.1]. Clearly, $\mathbb{K}^\mathbb{N}$ has a Schauder basis but it is not a Köthe space, since it has no continuous norm. $E$ has a non-normable quotient with a continuous norm if and only if $E$ is not isomorphic to a countable product of Banach spaces [21, Theorem 3.7].
It arises a natural question whether any non-normable Fréchet space of countable type with a continuous norm has a non-normable Köthe quotient. Since any non-normable Köthe space has a nuclear Köthe quotient [19, Theorem 14], we ask whether any non-normable Fréchet space of countable type with a continuous norm has a nuclear Köthe quotient.

In this paper we prove that the answer to the above problem is negative if the field $\mathbb{K}$ is spherically complete.

Let $X$ be an infinite-dimensional Banach space and let $W$ be a subspace in the dual space $X'$ of $X$. We say that $W$ is total if it is dense in $(X', \sigma(X', X))$. As in the archimedean case one verifies that $W$ is total if and only if the following holds: if $x \in X$ is such that $f(x) = 0$ for all $f \in W$, then $x = 0$. By $W^1$ we denote the set of all elements $x' \in X'$ such that there exists a bounded net $(x'_n)$ in $W$ which converges to $x'$ in $(X', \sigma(X', X))$. Clearly $W^1$ is a subspace in $X'$. We put $W^0 = W$ and $W^n = (W^{n-1})'$ for $n \in \mathbb{N}$. We say that $W$ is strongly non-norming if $W^n \subsetneq X'$ for all $n \in \mathbb{N}$.

In this paper we prove that the dual $l_\infty$ of $c_0$ has a total strongly non-norming subspace $M$ if $\mathbb{K}$ is spherically complete (Theorem 1). Using this space $M$ we construct a non-normable Fréchet space $F$ of countable type with a continuous norm, such that the strong dual $F'_\mathfrak{d}$ of $F$ is a strict LB-space (Theorem 7). Next we show that $F$ has no infinite-dimensional Fréchet–Montel quotient with a continuous norm (Theorem 10). In particular, $F$ has no nuclear Köthe quotient.

In our paper we use some ideas of [6] and [8] (see also [1] and [2]).

### Preliminaries

The field $\mathbb{K}$ is spherically complete if any decreasing sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. We put $B_\mathbb{K} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$.

Let $E$ be a linear space. If $A \subset E$ then $\text{lin} A$ denotes the linear hull of $A$. A set $A \subset E$ is absolutely convex if for all $\alpha, \beta \in B_\mathbb{K}$ and $x, y \in E$ we have $\alpha x + \beta y \in A$. If $A \subset E$ then the set $\text{co} A = \{\sum_{i=1}^n \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_\mathbb{K}, a_1, \ldots, a_n \in A\}$ is the smallest absolutely convex subset of $E$ that contains $A$. Let $A$ be an absolutely convex set in $E$. We put $A^c = A$ if the valuation of $\mathbb{K}$ is discrete, and $A^c = \bigcap\{\alpha A : \alpha \in \mathbb{K}, |\alpha| > 1\}$ otherwise. We say that $A$ is edged if $A = A^c$.

We denote by $|\mathbb{K}|$ the closure of the set $|\mathbb{K}| = \{[\lambda] : \lambda \in \mathbb{K}\}$ in $\mathbb{R}$. A seminorm on a linear space $E$ is a function $p : E \to |\mathbb{K}|$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$ (see [12, p. 189]).

A seminorm $p$ on $E$ is a norm if ker $p = \{0\}$.

In this paper by a locally convex space (lcs) we mean a Hausdorff locally convex space. The set of all continuous seminorms on a lcs $E$ is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a base in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{B}$ with $q \preceq p$.

For any seminorm $p$ on a lcs $E$ the map $\overline{p} : E_p \to [0, \infty)$, $x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$.

A lcs $E$ is of countable type if for any $p \in \mathcal{P}(E)$ the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset.

Let $E$ be a lcs. The topological dual of $E$ we denote by $E'$. If $A \subset E$ and $M$ is a subspace of $E$ we set $A^0 = \{f \in E' : |f(x)| \leq 1 \text{ for } x \in A\}$ and $M^\perp = \{f \in E' : f(x) = 0 \text{ for } x \in M\}$. If $B \subset E'$ and $W$ is a subspace of $E'$ we put $\delta B = \{x \in E : |f(x)| \leq 1 \text{ for } f \in B\}$ and $\delta W = \{x \in E : f(x) = 0 \text{ for } f \in W\}$. It is easy to see that $M^\perp = M^0$ and $\delta W = \delta W^0$. A set $A \subset E$ is polar if $A = \delta(A^0)$.

A seminorm $p \in \mathcal{P}(E)$ is polar if the set $\{x \in E : p(x) \leq 1\}$ is polar.

If $A$ is an absolutely convex subset in a lcs $E$ then $\delta(A^0) = B^c$, where $B$ is the closure of $A$ in $(E, \sigma(E, E'))$ [12, Proposition 4.10].

A lcs $E$ is strongly polar if every $p \in \mathcal{P}(E)$ is polar, and polar if some family of polar seminorms forms a base in $\mathcal{P}(E)$. A lcs $E$ is strongly polar if and only if every absolutely convex closed edged subset in $E$ is polar [12, Theorem 4.7]. Any lcs of countable type is strongly polar [12, Theorem 4.4]. If $\mathbb{K}$ is spherically complete then any lcs over $\mathbb{K}$ is strongly polar [12, p. 196]. If $E$ is polar then $E'$ separates points of $E$ [12, Proposition 5.6].

A subset $B$ of a lcs $E$ is compactoid (or a compactoid) if for each neighbourhood $U$ of $0$ in $E$ there exists a finite subset $S$ of $E$ such that $B \subset U + \text{co } S$.

Let $E$ and $F$ be locally convex spaces. The space of all linear continuous maps from $E$ to $F$ is denoted by $L(E, F)$. An operator $T \in L(E, F)$ is an isomorphism if $T$ is injective, surjective and the inverse map $T^{-1}$ is continuous. $E$ is
isomorphic to $F$ ($E \cong F$) if there exists an isomorphism $T : E \to F$. A linear map $T : E \to F$ is compact if there exists a neighbourhood $U$ of 0 in $E$ such that $T(U)$ is compactoid in $F$. A map $T \in L(E, F)$ is semi-Fredholm if $\ker T$ is finite-dimensional and $T(E)$ is closed in $F$.

Let $E$ and $F$ be Banach spaces. If $T, S \in L(E, F)$, $T$ is semi-Fredholm and $S$ is compact then $T + S$ is semi-Fredholm [14, Corollary 3.3].

A lcs $E$ is nuclear if for any $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the map $\varphi_{p,q} : (E_q, \overline{\mathcal{P}}) \to (E_p, \overline{\mathcal{P}})$, $x + \ker q \to x + \ker p$ is compact.

Let $E$ be a lcs. We write $\mathcal{B}(E)$ for the family of all bounded subsets of $E$. The strong dual of $E$, that is the topological dual of $E$ with the strong topology $b(E', E)$, will be denoted by $E'_b$.

Any metrizable lcs $E$ possesses a non-decreasing base $(p_k)$ in $\mathcal{P}(E)$.

A Fréchet space is a metrizable complete lcs. Let $(x_n)$ be a sequence in a Fréchet space $E$. The series $\sum_{n=1}^{\infty} x_n$ is convergent in $E$ if and only if $\lim n x_n = 0$.

Let $E$ be a Fréchet space with a continuous norm. Then the topology of $E$ can be defined by a non-decreasing sequence $(\| \cdot \|_k)$ of norms. Denote by $F_k$ the completion of the normed space $E_k = (E, \| \cdot \|_k)$, $k \in \mathbb{N}$. The identity map $i_k : E_{k+1} \to E_k$ has a unique continuous extension $\phi_k : E_{k+1} \to F_k$, $k \in \mathbb{N}$. The space $E$ is said to be countably normed if the sequence of norms $(\| \cdot \|_k)$ can be chosen in such a way that each $\phi_k$ is injective.

A Fréchet space $E$ is a Fréchet–Montel space if every bounded set in $E$ is compactoid. A normable Fréchet space is a Banach space.

The Banach space of all bounded sequences in $\mathbb{K}$ with the sup-norm is denoted by $l_\infty$; it is isomorphic to the dual of $c_0$. For $S \subseteq \mathbb{N}$ we put $c_0(S) = \{ x = (x_n) \in c_0 : x_n = 0$ for $n \in (\mathbb{N} \setminus S) \}$ and $l_\infty(S) = \{ x = (x_n) \in l_\infty : x_n = 0$ for $n \in (\mathbb{N} \setminus S) \};$ clearly, $c_0(S)$ and $l_\infty(S)$ are closed subspaces of $c_0$ and $l_\infty$, respectively.

A strict LB-space is a lcs $(E, \tau)$ which is the inductive limit of an inductive sequence $((E_n, \tau_n))$ of Banach spaces such that $\tau_n + 1 \mid E_n = \tau_n$ for all $n \in \mathbb{N}$. For fundamentals of inductive limits of locally convex spaces we refer to [5].

A sequence $(x_n)$ in a lcs $E$ is a basis in $E$ if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n : E \to \mathbb{K}$, $x \to \alpha_n$ $(n \in \mathbb{N})$ are continuous, then $(x_n)$ is a Schauder basis in $E$. Let $(x_n) \subset E$. By $[x_n] : n \in \mathbb{N}$ we denote the closed linear span of the set $[x_n] : n \in \mathbb{N}$ in $E$. If $(x_n)$ is a (Schauder) basis in $[x_n]$, then it is called a (Schauder) basic sequence. Every infinite-dimensional Banach space has a Schauder basic sequence [11, Theorem 3.16].

Results

We start with the following result for Banach spaces (for the definition of strongly non-norming subspaces see Introduction).

Theorem 1. Assume that $\mathbb{K}$ is spherically complete. Then the dual $l_\infty$ of $c_0$ contains a total closed strongly non-norming subspace.

Proof. Let $X = c_0$ and let $i : X \to X''$ be the canonical injection. We will identify $X'$ with $l_\infty$.

(A) First we show inductively that for every $n \geq 0$ there is a total closed subspace $M$ of $X'$ with $M'' \subsetneq X'$. For $n = 0$ it is clear, since $c_0$ is a total closed subspace of $X'$ and $c_0 \subsetneq X'$. Assume that it is true for some $n \geq 0$. Let $N_1$ and $N_2$ be infinite disjoint subsets of $\mathbb{N}$ with $N_1 \cup N_2 = \mathbb{N}$. Let $h \in (i(c_0) + l_\infty(N_2)^\perp) \cap (l_\infty(N_1) + c_0(N_2))^\perp \subset l_\infty$. Then $h(x) = 0$ for $x \in l_\infty(N_1)$. There exist $f \in i(c_0)$ and $g \in l_\infty(N_2)^\perp$ with $f + g = h$. Since $f(x) = h(x) - g(x) = 0$ for $x \in c_0(N_2)$, we get $f \in i(c_0(N_1))$. Hence $h(x) = f(x) + g(x) = 0$ for $x \in l_\infty(N_2)$. Thus $h(x) = 0$ for all $x \in l_\infty$, so $h = 0$. We have shown that $i(c_0) + Y \subset Z = \{ 0 \}$ for $Y = c_0(N_1) \subset X$ and $Z = (l_\infty(N_1) + c_0(N_2))^\perp \subset X''$. Clearly $Y \cong c_0$. By the induction hypothesis, $Y'' = l_\infty(N_1)$ contains a total closed subspace $V$ with $V'' \subsetneq Y''$. Put $W = V + l_\infty(N_2)$; clearly, $W$ is a total closed subspace of $l_\infty$ and $W'' = V'' + l_\infty(N_2) \subsetneq l_\infty = X''$.

The unit closed ball $B$ of $V$ is a metrizable absolutely convex compactoid in $(Y', \sigma(Y', Y))$ [18, Propositions 3.1 and 6.1]. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Using [12, Proposition 8.2], we infer that there exists a linearly dense sequence $\{ y_k \}$ in $(Y', \sigma(Y', Y))$ with $\{ y_k \} \subset V$ such that $1 \leq \| y_k \| < |\alpha|$, $k \in \mathbb{N}$.

Clearly, $Z$ is an infinite-dimensional closed subspace of $X'' = l_\infty$. Let $(z_k)$ be a Schauder basic sequence in $Z$ with $\| z_k \| \to 0$. The linear continuous operator $S : X' \to Y'$, $S(x) = \sum_k z_k(x') y_k$ is compact, since $\| z_k \| \| y_k \| \to 0$ [11, Theorem 4.40]. It is easy to see that $S : Y'' \to X'', S'(y'') = \sum_k y''(y_k) z_k$.
Let \( j : Y \to X \) be the inclusion map. Then \( j'(X') \to Y', \) \( (j'(x'))(y) = y(x') \) and \( j'' : Y'' \to X'' \) \( (j''(y''))(x') = y''(j''(x')) \). Hence \( \ker j'' = Y'' \) \( j'(X') = Y' \), \( \ker j'' = \{0\} \) and \( j''(Y'') = (\ker j'') \perp = (Y'' \perp) \) [17, Proposition 6.7 and its proof]. Thus \( j'' \) is semi-Fredholm. Put \( T : X' \to Y', \) \( T = S + j' \). Since \( S' \) is compact [14, Proposition 5.7], the operator \( T' : X' \to X'', \) \( T' = S' + j'' \) is semi-Fredholm [14, Corollary 3.3]. Thus \( T'(Y'') \) is closed in \( X'' \). Using [7, Theorem 12], we infer that \( \ker T' \) is closed in \( Y' \). It follows that \( T'(Y'') \) is closed in \( (X'', \sigma(Y', X'')) \) and \( T'(Y'') = (\ker T') \perp \) (again by [17, Proposition 6.7 and its proof]). Put \( M = \ker T; \) then \( M = -T'(Y'') \) [12, Theorem 4.7].

Clearly, \( M \) is a closed subspace of \( X' \). We shall prove that \( M \) is total in \( X' \). Let \( x \in X \) with \( i(x) \in M = \overline{T}(Y'') \). For some \( y'' \in Y'' \) we have \( i(x) = S(y'') + j''(y'') \). Since \( S''(Y'') \subset Z \) and \( Z \cap (i(X) + (Y'' \perp)) = \{0\} \), we get \( \ker S''(Y'') = 0 \) and \( i(x) = j''(y'') \in (Y'' \perp) \); hence \( y''(y'') = 0 \), \( k \in \mathbb{N} \), and \( (i(x))Y'' \perp = 0 \), thus \( x = x'' \). Hence \( M = \ker T \) is total in \( Y' \).

Using [13, Theorem 1.4], we infer that \( \sigma(Y', X) = b(Y', Y) \); hence \( j'(x') \to j'(x') \) in \( Y' \), so \( y''(j'(x')) \to y''(j'(x')) \) for all \( y'' \in Y'' \). For \( y'' \in \ker S' \), \( \alpha \in A \) we have \( y''(j'(x')) = (j''(y''))(x') = 0 \), since \( j''(\ker S') \subset T'(Y'') \) and \( M = -T'(Y'') \). Hence \( y''(j'(x')) = 0 \) for all \( y'' \in \ker S' \); so \( j'(x') \in \ker S' = \left(\{y'_k : k \in \mathbb{N}\}\right) \perp = \{y'_k : k \in \mathbb{N}\} \subset C \). Thus \( x' \in (j''(V)) = W \). We have proved that \( M1 \subset W \). Hence \( M^{n+1} \subset W \subset X' \).

(B) Let \( (N_k) \) be a partition of \( A \) onto infinite subsets. Put \( X_k = (N_k) \) for \( k \in \mathbb{N} \). Denote by \( P_k \) the natural projection from \( l_\infty \) onto \( l_\infty(N_k) \); \( k \in \mathbb{N} \). Let \( M_k \) be a total closed subspace of \( X_k \) such that \( M_k \subset X_k \). Thus \( \sigma(Y', X) \) is absolutely convex, \( \sigma(Y', X) \) closed and form a neighbourhood base of zero in \( Y' \).

Using [13, Theorem 1.4], we infer that \( \sigma(Y', X) \) is absolutely convex, \( \sigma(Y', X) \) closed and form a neighbourhood base of zero in \( Y' \). Hence \( \sigma(Y', X) \) is absolutely convex, \( \sigma(Y', X) \) closed and form a neighbourhood base of zero in \( Y' \).

Suppose that \( x_k \in l_\infty \) \( (P_k(x_k)) \in \prod_{k=1}^{\infty} M_k \); clearly \( V = \bigcap_{k=1}^{\infty} P_k^{-1}(M_k) \), so \( V \) is a closed subspace of \( l_\infty = \bigcap_{k=1}^{\infty} M_k \). Suppose that \( x_k \in l_\infty \) \( (P_k(x_k)) = 0 \) for all \( v \in V \). Then \( x_k \in l_\infty \) \( (P_k(x_k)) = 0 \) for all \( x_k \in M_k \subset V \), \( k \in \mathbb{N} \). Since \( M_k \) is total in \( l_\infty(N_k) = c_0(N_k) \), we get \( P_k(x_k) = 0 \), \( k \in \mathbb{N} \). Thus \( x = 0 \), so \( V \) is total in \( X' \). It is easy to see that

\[
\left\{ x' \in l_\infty : (P_k(x')) \in \prod_{k=1}^{\infty} W_k \right\} \subset \left\{ x' \in l_\infty : (P_k(x')) \in \prod_{k=1}^{\infty} \bigcup_{k=1}^{n} M_k \right\},
\]

if \( W_k \) is a subspace of \( l_\infty(N_k) \) for all \( k \in \mathbb{N} \). Hence, by induction, we get for \( n \in \mathbb{N} \), \( V^n \subset \{x' \in l_\infty : (P_k(x')) \in \prod_{k=1}^{\infty} M_k^n \} \), where \( M_k^n \subset M_k \). Thus \( P_k(V^n) \subset M_k^n \subset l_\infty \), so \( V^n \subset l_\infty \), \( n \in \mathbb{N} \). We have shown that \( V \) is a total closed strongly non-norming subspace of \( l_\infty = c_0 \). \( \square \)

Using the \( p \)-adic Banach–Alaoglu theorem [9, Theorem 4.2] and [4, Lemma 2.4], we get the following:

**Proposition 2.** Let \( E \) be a locally convex space of countable type and let \( U \) be a neighbourhood of zero in \( E \). Then \( U^\circ \) is an absolutely convex complete metrizable compactoid in \( E' \) \( \sigma(E', E) \).

If \( X \) is a closed absolutely convex subset of a locally convex space \( E \) and \( Y \) is an absolutely convex complete metrizable compactoid in \( E \) then the set \( (X + Y)^\circ \) is closed in \( E \) [15, Theorem 1.4 and its proof]. Moreover for every absolutely convex subsets \( A, B \subset E \) we have \( (A + B)^\circ = (A^\circ + B)^\circ \) [5, Lemma 0.1]. Hence, using Proposition 2, we get (by induction) the following:

**Lemma 3.** Let \( D \) be a subset of a locally convex space \( E \) of countable type and let \( (U_k) \) be a sequence of neighbourhoods of zero in \( E \). Then the sets \( (D^n + \sum_{k=1}^{\infty} U_k^n)^\circ, n \in \mathbb{N} \), are closed in \( E \) \( \sigma(E', E) \).

A lcs \( E \) is a (DF)-space if it has a fundamental sequence \( (B_n) \) of bounded sets and for every sequence \( (V_n) \) of absolutely convex neighbourhoods of zero in \( E \) such that the set \( V = \bigcap_{n=1}^{\infty} V_n \) is bornivorous, \( V \) is a neighbourhood of zero in \( E \).

**Proposition 4.** Let \( E \) be a metrizable locally convex space of countable type. Then the strong dual \( E' \) of \( E \) is a (DF)-space.
Let \((U_n)\) be a decreasing base of polar neighbourhoods of zero in \(E\). We put \(W_n = U_n^\circ\) for \(n \in \mathbb{N}\). By [5, Lemma 2.5.4(i)], a subset of \(E_b^\circ\) is bounded if and only if it is equicontinuous; so \((W_n)\) is a fundamental sequence of bounded subsets in \(E_b^\circ\).

Let \((V_n)\) be a sequence of absolutely convex neighbourhoods of zero in \(E_b^\circ\) such that the set \(V = \bigcap_{n=1}^\infty V_n\) is bornivorous in \(E_b^\circ\). We shall prove that \(V\) is a neighbourhood of zero in \(E_b^\circ\). Let \(n \in \mathbb{N}\). We have \(\alpha_n W_n \subset V\) for some \(\alpha_n \in (\mathbb{K} \setminus \{0\})\) and \(B_{\alpha_n}^\circ \subset V_n\) for some bounded subset \(B_{\alpha_n}\) in \(E\). Put \(H_n = B_{\alpha_n}^\circ + \sum_{k=1}^n \alpha_k W_k\); clearly \(H_n \subset V_n\). By Lemma 3 the set \(H_n^\circ\) is closed in \((E', \sigma(E', E))\); thus \((H_n^\circ)' = H_n^\circ\). Put \(H = \bigcap_{n=1}^\infty H_n\). For every \(k \in \mathbb{N}\) there exists \(\gamma_k \in \mathbb{K}\) with \(|\gamma_k| = |\alpha_k|\) such that \(\gamma_k W_k \subset \bigcap_{n=1}^k B_{\alpha_n}^\circ\); then \(\gamma_k W_k \subset H\). It follows that \(^{\circ}H\) is bounded in \(E\), so \((^{\circ}H)'\) is a neighbourhood of zero in \(E_b^\circ\). On the other hand, using [12, Proposition 4.10], we have \((^{\circ}H)' = (H')' = \bigcap_{n=1}^\infty H_n^\circ = \bigcap_{n=1}^\infty \alpha_k W_k \subset aH \subset aV\) for every \(\alpha \in \mathbb{K}\) with \(|\alpha| > 1\); so \(V\) is a neighbourhood of zero in \(E_b^\circ\). □

(C) Perez-Garcia and W.H. Schikhof proved recently the following:

Any absolutely convex subset \(A\) of a metrizable lcs of countable type is contained in the closed absolutely convex hull of some countably subset \(X\) of \(A\) [10].

Using this result we obtain

**Proposition 5.** Let \(G\) be a dense subspace of a metrizable locally convex space \(E\) of countable type. Then \(G_b^\circ\) is isomorphic to \(E_b^\circ\).

**Proof.** Let \(j : G \to E\) be the inclusion map. We shall prove that its adjoint \(j^\prime : E_b^\circ \to G_b^\circ\) is an isomorphism. The map \(j^\prime\) is injective and continuous, since \(G\) is dense in \(E\) and every bounded subset of \(G\) is bounded in \(E\); density of \(G\) also implies that \(j^\prime\) is surjective. To prove that \(j^\prime\) is open it is enough to show that every bounded subset \(A\) of \(E\) is contained in the closure of some bounded subset \(B\) of \(G\).

Let \(A\) be a bounded subset of \(E\). Then \(co A\) is bounded and has a countable subset \(X = \{x_n\} : n \in \mathbb{N}\) such that \(A\) is contained in the closed absolutely convex hull of \(X\) [10, Theorem 8.6.5]. Let \((U_n)\) be a decreasing base of absolutely convex neighbourhoods of zero in \(E\). Then for all \(n, k \in \mathbb{N}\) there exists \( z_{n,k} \in U_{n+k} \) such that \(x_n + z_{n,k} \in G\). Put \(Y = \{x_n + z_{n,k} : n, k \in \mathbb{N}\}\). Clearly, \(Y \subset G\) and \(X\) is contained in the closure of \(Y\) in \(E\). The set \(Z = \{z_{n,k} : n, k \in \mathbb{N}\}\) is bounded in \(E\). Indeed, for every \(m \in \mathbb{N}\) the set \(Z_m = \{z_{n,k} : n + k < m\}\) is finite and \(S_m = \{z_{n,k} : n + k \geq m\} \subset U_m\); so \(Z \subset \alpha_m U_m\) for some \(\alpha_m \in \mathbb{K}\). Thus \(X + Z\) is bounded in \(E\). Hence \(Y\) is bounded in \(G\), since \(Y \subset X + Z\). Clearly, \(A\) is contained in the closed absolutely convex hull of \(Y\) in \(E\). Then \(B = co Y\) meets the requirements. □

It is not hard to check the following:

**Remark 6.** Let \(\tau_1\) and \(\tau_2\) be locally convex topologies on a linear space \(X\). If \(\tau_1 \subset \tau_2\), then \(X_{\tau_1} \subset X_{\tau_2}\) and \(b(X_{\tau_2} \times X_{\tau_1}) X_{\tau_1} \subset b(X_{\tau_1} \times X_{\tau_1})\), where \(X_{\tau_i} = (X, \tau_i)\) for \(i = 1, 2\).

Now we can prove our next theorem.

**Theorem 7.** Assume that \(\mathbb{K}\) is spherically complete. Then there exists a non-normable countably normed Fréchet space \(F\) of countable type such that the strong dual \(F_b^\circ\) of \(F\) is a strict LB-space.

**Proof.** Let \(X = c_0\) and let \(M\) be a total closed strongly non-norming subspace of \(X'\). Denote by \(B\) the unit closed ball of \(X'\). We have \(M^k \subset M^k+1\) for \(k \in \mathbb{N}\). Indeed, if \(M^k = M^k+1\) for some \(k > 0\), then

\[
B \cap M^k \subset B \cap M^k + \sigma(X', X) \subset B \cap M^k+1 = B \cap M^k.
\]

Hence, by the \(p\)-adic Krein–Šmulian theorem [16, Corollary 5.2], we infer that \(M^k\) is closed in \((X', \sigma(X', X))\); a contradiction, because \(M^k\) is a proper \(\sigma(X', X)\)-dense subspace of \(X'\).

Put \(B_k = B \cap M^k\) for \(k > 0\). Let \(k \in \mathbb{N}\) and let \(|x|_k = \sup\{|f(x)| : f \in B_{k-1}\}\) for \(x \in X\). Clearly, \(|\cdot|_k\) is a norm on \(X\) and \(|x|_k \leq |x|\), \(x \in X\). Hence \(X_k \subset X'\), where \(X_k = (X, |\cdot|_k)\). It is easy to see that \((^{\circ}B_{k-1})^\circ = D_k^\circ\) and \(M^k = \text{lin} D_k\), where \(D_k\) is the closure of \(B_{k-1}\) in \(\sigma(X', X)\). It follows that \(X_k^\circ = M^k\).
Denote by $F_k$ the completion of $X_k$. Since the identity map $i_k : X_{k+1} \to X_k$ is continuous, we can extend $i_k$ to a continuous linear map $\phi_k : F_{k+1} \to F_k$. We shall prove that $\phi_k$ is injective. Let $(x_n)$ be a Cauchy sequence in $X_{k+1}$ which converges to 0 in $X_k$. Then $\|x_n\|_{k+1} \to a$ for some $a \geq 0$. Suppose that $a > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \exists f_n \in B_k : |f_n(x_n)| > 2^{-1} a.$$ 

Since $\|x_n\| \to 0$ we get $f(x_n) \to 0$ for any $f \in M_k$. Moreover

$$\exists n_1 \in \mathbb{N} : \|x_n - x_m\|_{k+1} \leq 2^{-1} a \quad \text{for all } n > m \geq n_1.$$ 

Hence $|f_n(x_n) - f_n(x_m)| \leq 2^{-1} a$ for all $n, m \geq n_1$. Let $n \geq n_1$; tending $m$ to infinity we get $|f_n(x_n)| \leq 2^{-1} a$ for every $n \geq n_1$; a contradiction. Thus $\|x_n\|_{k+1} \to 0$. It follows that $\phi_k$ is injective.

Therefore we can assume that $X \subset F_{k+1} \subset F_k$ and $\phi_k+1$ is the inclusion map. It is not hard to see that $F = \bigcap_{k=1}^{\infty} F_k$ with the linear topology generated by the sequence of norms $(\|\cdot\|_k|F)$ is a Fréchet space of countable type. Clearly, $F$ is a countably normed Fréchet space.

Since $X_k = M_k \neq M_k^{k+1} = X_k^{k+1}$, we have that the norms $\|\cdot\|_k$ and $\|\cdot\|_{k+1}$ are not equivalent on $X$ for any $k \in \mathbb{N}$; so $F$ is a non-normable Fréchet space.

We shall prove that $F_b'$ is a strict $LB$-space.

Let $X_0$ be the metrizable lcs $(X, (\|\cdot\|_k))$; clearly, $X_0$ is a dense subspace of $F$. By Proposition 5 the strong duals of $F$ and $X_0$ are isomorphic, so it suffices to prove that the strong dual of $X_0$ is a strict $LB$-space.

Denote by $G_k$ the closure of $M^k$ in $X'$. We shall prove that

$$B \cap G_k \subset \{0\} \subset B \cap G_{k+1}.$$ 

If $f \in B \cap G_k$ and $f \neq 0$, then there exists $(f_n) \subset M^k$ with $\|f_n - f\| \to 0$. Hence $\|f_n\| = \|f\|$ for almost all $n \in \mathbb{N}$; so $f \in B \cap M^k = \overline{B_k} \subset \{0\}$. If $f \in (B_k)^0$, then $f \in M_{k+1}$ and $|f(x)| \leq \|x\|_{k+1} \leq \|x\|$ for all $x \in X$, so $f \in B \cap G_{k+1}$.

It follows that $G_k \subset X_{k+1}' \subset G_{k+1}$ and $b(X_{k+1}', X_{k+1})|G_k = b(X', X)|G_k$.

By Remark 6 we get $X_{k+1}' \subset X_0' \subset X'$, $b(X_0', X_0)|X_{k+1}' = b(X_{k+1}', X_{k+1})$ and $b(X', X)|X_0' \subset b(X', X)$. Hence we get

$$b(X_0', X_0)|G_k = b(X_{k+1}', X_{k+1})|G_k = b(X', X)|G_k \subset b(X_0', X_0)|G_k.$$ 

Thus $\bigcup_{k=1}^{\infty} G_k = \bigcup_{k=1}^{\infty} X_k' = X_0'$ and $b(X_0', X_0)|G_k = b(X', X)|G_k$ for all $k \in \mathbb{N}$.

Denote by $\tau$ the inductive limit topology on $X_0'$ generated by the strict inductive sequence $(G_k)$. We shall prove that $\tau = b(X_0', X_0')$. Clearly, $\tau \supset b(X_0', X_0')$. By [5, Theorem 1.4.7], we have $\tau|G_k = b(X', X)|G_k = b(X_0', X_0)|G_k$, $k \in \mathbb{N}$.

Let $U$ be an absolutely convex neighbourhood of zero in $X_0$. Then there exists a sequence $(U_k)$ of absolutely convex neighbourhoods of zero in $X_0 = (X_0', b(X_0', X_0'))$ such that $U_k \cap G_k \subset U \cap G_k$, $k \in \mathbb{N}$. Let $\mu \in \mathbb{K}$ with $|\mu| > 1$. Put $V_k = \mu^{-k}(B_k)$ and $S_k = V_k^0$, for $k \in \mathbb{N}$. The sequence $(V_k)$ is a decreasing base of polar neighbourhoods of zero in $X_0$. Hence, by the proof of Proposition 4, $(S_k)$ is an increasing fundamental sequence of bounded sets in $X_0'$. Since $S_k \subset X_k' \subset G_k$, we have $U_k \cap S_k \subset U \cap S_k$, $k \in \mathbb{N}$. The sets $W_k = S_k \cap U + U_k$, $k \in \mathbb{N}$, are absolutely convex neighbourhoods of zero in $X_0'$. Clearly, $S_k \cap W_k \subset S_k \cap U$ for $k \in \mathbb{N}$. The set $W = \bigcap_{k=1}^{\infty} W_k$ is bornivorous. Indeed, for every $k \in \mathbb{N}$ there exists $\alpha \in \mathbb{K}$ with $|\alpha| > 1$ such that $S_k \subset \alpha U_n$ for $1 \leq n \leq k$, then $S_k \subset \alpha (S_k \cap U)$, $\alpha = \alpha (S_k \cap U)$ for $n \geq k$, so $S_k \subset \alpha W_n$, for all $n \in \mathbb{N}$; hence $S_k \subset \alpha W$. By Proposition 4, $X_0'$ is a (DF)-space, so $W$ is a neighbourhood of zero in $X_0'$. Clearly, $X_0' = \bigcup_{k=1}^{\infty} S_k$ and $S_k \subset W \subset S_k \subset U$ for all $k \in \mathbb{N}$, so $W \subset U$. Thus $U$ is a neighbourhood of zero in $X_0'$. Hence $b(X_0', X_0') = \tau$.

We have shown that $X_0'$ is a strict $LB$-space. $F_b'$ is isomorphic to $X_0'$, so $F_b'$ is a strict $LB$-space. \(\square\)

**Proposition 8.** Let $X$ be a closed subspace of a Fréchet space $E$. If the quotient $(E / X)$ is a Fréchet–Montel space, then its strong dual $(E / X)_b'$ is isomorphic to the closed subspace $X^\perp$ of $E_b'$.
Proof. Let $\pi : E \to (E/X)$ be the quotient map. It is easily seen that the adjoint operator $\pi' : (E/X)' \to E_b'$ is a continuous linear injective map whose image is equal to $X^\perp$. Let $A$ be a bounded subset of $(E/X)$. Then $A$ is a compactoid in $(E/X)$, so there exists a compactoid $B$ in $E$ such that $\pi(B) = A$ [3, Proposition 2.5]. It is not hard to check that $(\pi(B))^\circ = (\pi')^{-1}(B^\circ)$, so $\pi'(A^\circ) = B^\circ \cap X^\perp$. Thus the map $\pi' : (E/X)' \to X^\perp$ is an isomorphism. □

To get our main result we need the following:

**Proposition 9.** If the strong dual $E_b'$ of a Fréchet space $E$ is a strict LB-space then $E$ has no infinite-dimensional Fréchet–Montel quotient with a continuous norm.

**Proof.** Let $((E_n, \tau_n))$ be a strict inductive sequence of Banach spaces such that $\lim(E_n, \tau_n) = E'_b$. Then $b(E', E)E_n = \tau_n$ for $n \in \mathbb{N}$ [5, Theorem 1.4.7]. Suppose that $E$ has a closed subspace $X$ such that the quotient $(E/X)$ is a Fréchet–Montel space with a continuous norm. By [20, Lemma 3 and its proof], the space $(E/X)'$ contains a linearly dense bounded subset $A$. Using Proposition 8 we infer that $E'_b$ has a bounded subset $B$ such that its closed linear span $Y$ in $E'_b$ is equal to $X^\perp$. Since the inductive sequence $((E_n, \tau_n))$ is regular [5, Theorem 1.4.13], there exists $n \in \mathbb{N}$ with $B \subset E_n$. Hence $X^\perp \subset E_n$, so $X^\perp$ is a Banach space. By Proposition 8, $(E/X)'_b$ is a Banach space. The space $(E/X)'$ is reflexive [12, Theorem 10.3], so it is a Banach–Montel space. Thus $(E/X)'$ is finite-dimensional [5, Proposition 0.3]. □

By Theorem 7 and Proposition 9 we obtain

**Theorem 10.** Assume that $\mathbb{K}$ is spherically complete. Then the countably normed Fréchet space $F$ of Theorem 7 has no infinite-dimensional Fréchet–Montel quotient with a continuous norm. In particular, $F$ has no nuclear Köthe quotient.

Finally we show the following:

**Proposition 11.** The strong dual $E'_b$ of a (DF)-space $E$ is a Fréchet space.

**Proof.** Let $(B_n)$ be a fundamental sequence of bounded sets in $E$. Then $(B_n^\circ)$ is a base of neighbourhoods of zero in $G = E'_b$. Thus $G$ is a metrizable locally convex space; we shall prove that $G$ is complete. Let $(f_n)$ be a Cauchy sequence in $G$. Then

$$(*) \quad \forall k \in \mathbb{N} \exists M_k \in \mathbb{N}: f_n - f_m \in B_k^\circ \text{ for all } n, m \geq M_k.$$

Let $x \in X$. Then for every $\alpha \in (\mathbb{K} \setminus \{0\})$ there exists $k(\alpha) \in \mathbb{N}$ such that $\alpha^{-1}x \in B_k(\alpha)$; hence $|f_n(x) - f_m(x)| \leq |\alpha|$ for all $n, m \geq M_k(\alpha)$. It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{K}$ for every $x \in E$. Thus there exists a linear functional $f$ on $E$ such that $f_n(x) \to f(x)$ for all $x \in E$. We shall prove that $f$ is continuous. The sets $V_n = \{x \in E: |f_n(x)| \leq 1\}$, $n \in \mathbb{N}$, are absolutely convex neighbourhoods of zero in $G$. The sequence $(f_n)$ is bounded in $G$, so for every $k \in \mathbb{N}$ there exists $\alpha_k \in \mathbb{K}$ with $\alpha_k \neq 0$ such that $(f_n) \subset (\alpha_k B_k)^\circ$. Hence $\alpha_k B_k \subset V_n$ for all $k, n \in \mathbb{N}$; so the set $V = \bigcap_{n=1}^\infty V_n$ is bornivorous in the (DF)-space $E$. Therefore $V$ is a neighbourhood of zero in $E$. Thus $f$ is continuous, since $|f(x)| \leq 1$ for all $x \in V$.

By $(*)$ we have $f_n - f \in B_k^\circ$ for every $k \in \mathbb{N}$ and every $n \geq M_k$. This means that $f_n \to f$ in $G$. We have shown that $G$ is complete. Thus $G$ is a Fréchet space. □

By Propositions 4 and 11 we obtain the following:

**Corollary 12.** The strong bidual of a metrizable locally convex space of countable type is a Fréchet space.

**Acknowledgment**

The author wishes to thank the referee for useful remarks and suggesting many improvements.
References