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Transonic Flow on an Axially Symmetric Torus

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On a Riemannian manifold the existence (and uniqueness) of *subsonic* gas flows with prescribed circulation has been previously established (*Acta Math.* **125** (1970), 57–73). If the manifold is a torus of revolution then the gas dynamics equation reduces to a nonlinear ordinary differential equation and the flow can be described explicitly. We show that, as the circulations are increased, one obtains a complete family of solutions: smooth subsonic, smooth transonic, transonic with shocks, and smooth supersonic flows.

Contents: I. Gas Dynamics on a Manifold and Some Classical Problems; II. The Mass Flow Relation for the Speed; III. Description of "de Laval" Flows; IV. Local Theory; V. Global Solutions; VI. The Case $\gamma = 3$; VII. Description of Flows (Continued).

1. Gas Dynamics on a Manifold and Some Classical Problems

1.1. *Gas Dynamics on a Manifold*

If a steady flow on an orientable Riemannian manifold M is represented by a differential 1-form ω, whose components give the velocity components in the coordinate directions, then the requirement that the flow be irrotational (no circulation about curves homologous to zero) and the condition of conservation of mass lead to the first-order system of partial differential equations [8]

$$d\omega = 0, \tag{1.1a}$$

$$\delta(\rho\omega) = 0, \tag{1.1b}$$

where d is exterior differentiation, δ the adjoint of d, and the density ρ a scalar valued function which, by analogy with the flow of a polytropic gas, is given explicitly by

$$\rho = \left(1 - \frac{\gamma - 1}{2} Q\right)^{1/(\gamma-1)}, \qquad 0 \leqslant Q \leqslant \frac{2}{\gamma - 1} \tag{1.2}$$

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involving the adiabetic constant $\gamma > 1$. We shall call $Q = Q(\omega)$ the *speed* of the flow (strictly speaking, it is the square of the speed but this convention eliminates the frequent occurrence of radicals). Locally, if the Riemannian metric on M is given by the metric tensor g_{ij}, and the form ω is given in terms of local coordinates by $\omega = \omega_i\, dx^i$, then $Q = g^{ij}\omega_i\omega_j$. It is thus simply the square of the pointwise norm of the form ω, thought of as a section of the cotangent bundle.

1.2. *Related Classical Problems*

In order to put our results in some perspective we recall several classical problems of transonic flow. More detail and further references can be found in [1, 3, 5, 11].

The existence of smooth *subsonic* flows ($Q < 2/(\gamma + 1)$ everywhere) past a profile in the plane for a range $0 \leqslant Q_\infty < Q_c$ of prescribed speeds Q_∞ at ∞ was solved independently by Bers [2] and Shiffman [6]. As Q_∞ tends to Q_c (Q_c depends on the obstacle) the speed of the flow tends somewhere to the *sonic speed* $2/(\gamma + 1)$. Comparable results for *supersonic* flows ($Q > 2/(\gamma + 1)$ everywhere) and for *transonic* flows do not exist, and are not expected to, because of the nonexistence theorems of Morawetz [4].

A second problem, of even more relevance to us here, is that of channel flow. Consider a de Laval nozzle in which one has an *entry section* of decreasing cross section which, after attaining a minimum at the *throat*, increases in the exhaust section. Each section is assumed to be connected to a reservoir containing the gas at a constant pressure. Letting r denote the ratio of the exhaust reservoir pressure to the entry reservoir pressure, an approximate analysis (involving averaging over cross sections and assuming infinitely large reservoirs and entrance and exit cross sections) shows the following: If $r = 1$ then there is no flow (in the sense that $Q \equiv 0$). There exist two constants $0 < r_2 < r_1 < 1$ such that if $r_1 < r < 1$ then the flow is *subsonic* throughout the nozzle, having a maximum speed at the throat. If $r = r_1$ then the flow is *subsonic* in the entry and exhaust sections but *sonic* at the throat. For $r_2 < r < r_1$ the flow is the same in the entry section (as for $r = r_1$) but becomes *supersonic* entering the exhaust section and then undergoes a compression *shock*—the velocity decreasing (discontinuously) from supersonic to subsonic speed. If $r = r_2$ then the shock occurs at the exit. That the velocity decreases across a discontinuity is required for nondecreasing entropy.

1.3. *The Circulation Problem*

In recent work we have considered gas flows on compact manifolds and compact manifolds with boundary [7–9]. If circulations are prescribed on a homology basis (which amounts to prescribing the cohomology class of the form representing the flow) then the existence of *subsonic* flows is established. The results are, in some sense, a combination of classical Hodge theory and the results described above of Bers and Shiffman for subsonic flow pat a profile. With difficulties involving the singularity at ∞ avoided, the (finite) boundary, if nonempty, presents no problem.

We also showed [10] that the maximum speed of a subsonic flow on a surface must be attained at a point of nonpositive Gauss curvature. For a torus of revolution, given explicitly, one could hope to describe the subsonic solutions whose existence is guaranteed by the results of [7] and to locate the points at which the maximum speed is attained. Surprisingly perhaps, we will see that the analysis which follows leads to a description not only of subsonic flows but of supersonic and transonic flows as well.

In the following sections we will analyze completely global subsonic, supersonic, and transonic flows which are symmetric on an axially symmetric torus.

2. Mass Flow Relation for the Speed

Let T be the torus obtained by rotating about the z-axis the simple closed curve Γ of class C^2. If Γ is parametrized by $x = f(u)$ and $z = g(u)$ with $0 \leqslant u \leqslant 2\pi$ and $f(u) > 0$, then $(f')^2 + (g')^2 \neq 0$ and T is given parametrically by

$$\begin{aligned} x &= f(u) \cos v, \\ y &= f(u) \sin v, \\ z &= g(u). \end{aligned} \tag{2.1}$$

The metric tensor on T is given by $g_{11} = (f')^2 + (g')^2$, $g_{12} = g_{21} = 0$, and $g_{22} = f^2$. (For the "standard" torus with circular cross section, $f(u) = 1 + r \cos u$, $g(u) = r \sin u$ $(0 < r < 1)$, and $g_{11} = r^2$.)

The symmetry of T suggests the existence of ρ-harmonic forms ω (i.e., ω satisfying Eq. (1.1), where ρ is given by (1.2)) which are independent of the variable v. We write $\omega = \alpha\, du + \beta\, dv$, where α and β are thought of as periodic functions of u (period 2π). In fact, since ω is closed ($d\omega = 0$) we obtain immediately that β is constant. If in addition $\delta(\rho\omega) = 0$ then the components of ω must satisfy

$$\frac{\partial}{\partial u}\left(\left(\frac{g_{22}}{g_{11}}\right)^{1/2} \rho\alpha\right) + \frac{\partial}{\partial v}\left(\left(\frac{g_{11}}{g_{22}}\right)^{1/2} \rho\beta\right) = 0. \tag{2.2}$$

If, however, ω is independent of v then the same is true of the speed,

$$Q = \frac{\alpha^2}{g_{11}} + \frac{\beta^2}{g_{22}} = \frac{\alpha^2}{(f')^2 + (g')^2} + \frac{\beta^2}{f^2}, \tag{2.3}$$

and hence also of the density ρ. As a result the second term in (2.2) drops out. It follows that

$$\frac{g_{22}\rho^2\alpha^2}{g_{11}} = K$$

for some nonnegative constant K. Using (2.3) and setting $g_{22} = f^2$ we obtain the mass flow relation for Q

$$f^2(Q - \beta^2/f^2)\,\rho^2 = K > 0. \tag{2.4}$$

For fixed β this is a relation for Q as a function of u (and the parameter K). A solution $Q = Q(u; \beta, K)$ of (2.4) determines α by (2.3) and hence ω itself. Note that, from (2.3) and recalling that $Q \leqslant 2/(\gamma - 1)$, β must be in the range $0 \leqslant \beta \leqslant (2/(\gamma - 1))^{1/2} \min f$.

While we have, for simplicity, been speaking about a torus of revolution, the mass flow equation (2.4) has been obtained assuming only that g_{ij} is independent of v and $g_{12} = g_{21} = 0$. Since any (abstractly defined) torus T on which the circle group acts as a group of isometries with an invariant circle can be parametrized by coordinates u and v in such a way that the metric g_{ij} satisfies these assumptions, the derivation of (2.4) and our subsequent results hold without modification for such a torus (with $f = g_{22}$).

3. Description of "de Laval" Flows

3.1. *The Basic Problem*

Consider the torus of revolution T given by Eqs. (2.1). Here, and again in Section 5, we make the additional "de Laval" assumption on f,

$$\begin{aligned} f' &< 0 \qquad \text{for } 0 < u < \pi, \\ f' &> 0 \qquad \text{for } \pi < u < 2\pi, \end{aligned} \tag{3.1}$$

in which case, f has a *unique* maximum at $u = 0$ and a *unique* minimum at $u = \pi$. This results in considerable simplification of the analysis which follows, since the extrema of f play an important role. In Section 7 we will discuss the necessary modifications if this assumption is dropped. That the extrema occur at 0 and π is, of course, just an arbitrary, but convenient, choice of parametrization.

Given circulations c_1 and c_2 in the u direction ($v =$ constant) and the v direction ($u =$ constant), respectively, we seek a differential 1-form ω satisfying (1.1) and (1.2) which is independent of the coordinate v. We have seen that the speed Q of such a form is given by (2.3) and satisfies the mass flow equation (2.4). Computing c_2 over a curve along which u is constant gives $c_2 = \int_0^{2\pi} \beta\, dv = 2\pi\beta$. This incidentally gives, by the last sentence of Section 2, an upper bound on c_2.

We begin by disposing of some cases. If $c_1 = 0$, then a solution is given by $\alpha = 0$, $\beta = c_2/2\pi$, in which case $Q = \beta^2/f^2$ and $K = 0$. The flow is described by streamlines in the v direction along which the speed is constant. The maximum speed is attained on the "inner" circle where $u = \pi$. For c_2 not too large the flow is subsonic and this is in agreement with the maximum principle of [10] mentioned above. As c_2 is increased the speed eventually becomes sonic on this circle and then a smoothly transonic flow develops with a supersonic band bounded by sonic lines ($u = \pi \pm \theta$, where θ is an increasing function of c_2). For c_2 still larger, the flow (in general) becomes supersonic everywhere. (See, however, the remarks at the end of Section 3.5.)

There is, in some sense, another flow for which $K = 0$. The density ρ vanishes for Q equal to the maximum speed $2/(\gamma - 1)$. For any admissible β one can then find α from (2.3). For this degenerate case the speed is, of course, constant on T.

Before proceeding to a discussion of the flows in the general case of prescribed circulations c_1 and c_2 we would like first to describe in detail the flows for the case $c_2 = 0$ (in which case $\beta = 0$). Although these results are, strictly speaking, contained in the general case, they are already nontrivial (as was *not* the case for $c_1 = 0$). Our main reason for describing this case separately, however, is a strong analogy with the classical problem of flow in a de Laval nozzle mentioned in Section 1.

By the remarks following (2.4), it suffices to find solutions $Q = Q(u)$ of Eq. (2.4). First we seek solutions of the initial value problem at $u = \pi$ and then, of the circulation problem.

3.2. *The Initial Value Problem for Q ($c_2 = 0$)*

We will see (Theorem 5.1) that corresponding to any value $Q_\pi \neq 2/(\gamma + 1)$, $0 < Q_\pi < 2/(\gamma - 1)$, there exists a unique solution of (2.4) with $Q(\pi) = Q_\pi$ which is everywhere subsonic or supersonic accordingly as Q_π is less than or greater than $2/(\gamma + 1)$. The curves are sketched as $Q^+(u)$ and $Q^-(u)$, respectively, in Fig. 1.

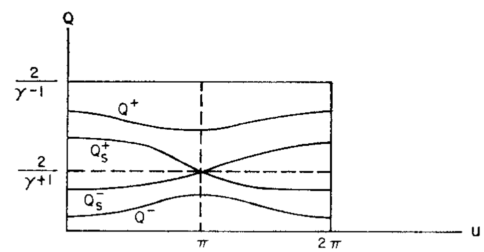

FIG. 1. Smooth solutions ($C_2 = 0$).

If Q_π is the sonic value $2/(\gamma + 1)$, then by Theorem 5.2 there is one solution which is subsonic everywhere except at π, where it is sonic, and a second solution which is supersonic everywhere except at π, where it is sonic. These curves Q_s^+ and Q_s^- are also shown in Fig. 1. In general, these solutions will have a discontinuity in the derivative dQ/du at $u = \pi$. However (Corollary 5.4), the left- and right-hand derivatives of the two solutions will always match in such a way that the curves meet smoothly as shown. It is important to note that both solutions correspond to the same value of K. This means that a *piecewise conti-*

nuous curve which agrees everywhere with either Q_s^+ or Q_s^- will still be a solution of Eq. (2.4). Such discontinuities correspond to shocks in the flow and the usual shock condition requiring an increase in entropy implies a decrease in velocity across the shock. It is thus clear that: (i) A solution can pass from the Q_s^- curve to the Q_s^+ curve only at $u = \pi$; (ii) only one discontinuity is possible from the Q_s^+ curve to the Q_s^- curve; and (iii) no discontinuous solution can be formed from a pure subsonic–pure supersonic pair of curves with the same K. The location of the Q_s^+ to Q_s^- discontinuity thus parametrizes the set of all transonic solutions with shocks (see Fig. 2). There are *no* smooth transonic flows if $c_2 = 0$. Since the

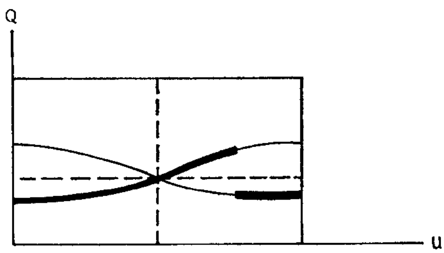

FIG. 2. Shock solutions ($C_2 = 0$).

streamlines are orthogonal to the shock lines, the shocks in this case are normal (compression) shocks. In fact, since there is no component of the flow in the v direction, the analogy with the flow behavior in a de Laval nozzle is seen to be hardly surprising. If we consider the subsurface $\tilde{T}$ of T cut off by the curves $v = v_1$ and $v = v_2$, then the flow restricts to $\tilde{T}$ which we recognize as a curvilinear two-dimensional de Laval nozzle in space.

Finally, we observe that we have chosen to prescribe the value of Q at $u = \pi$, the point at which $f(u)$ is minimum. From Fig. 1, one sees that if another value of u had been chosen it would not always be possible to find a *global* solution having a prescribed value at that point. On the other hand we will see (Proposition 4.1) that a local solution always exists. It may happen, however, that it cannot be continued to a global solution on the whole interval $[0, 2\pi]$ but instead terminates at some value of u at which $Q = 2/(\gamma + 1)$ and dQ/du becomes infinite.

3.3. *The Circulation Problem* ($c_2 = 0$)

We have seen that there is a one-parameter family of curves $Q_t(u)$, encompassing the subsonic, transonic with shock, and supersonic solutions, whose integrals depend continuously on the parameter t and range from 0 to $4\pi/(\gamma - 1)$. But

$$c_1(t) =: \int_0^{2\pi} \alpha\, du = \int_0^{2\pi} Q_t^{1/2}[(f')^2 + (g')^2]^{1/2}\, du$$

so that values of $c_1(t)$ vary continuously from 0 to $L(2/(\gamma - 1))^{1/2}$ where L is the length of the curve v = constant. It follows that there exist polytropic flows having prescribed circulations c_1 $(0 \leqslant c_1 \leqslant L(2/(\gamma - 1))^{1/2})$ in the u direction and $c_2 = 0$ in the v direction. If c_1 is small the flow is subsonic and if c_2 is large the flow is supersonic. In the intermediate range it is transonic with one normal compression shock.

3.4. *The Initial Value Problem* ($c_2 \neq 0$)

The most important fact to be observed if $c_2 \neq 0$ is that the sonic value $2/(\gamma + 1)$ loses significance. In its place (see Eq. 4.4) is a critical curve $Q = \hat{Q}(u)$ which depends explicitly on the metric, the circulation c_2, and the point u. Moreover, if $c_2 \neq 0$ then $\hat{Q}(u) > 2/(\gamma + 1)$. For fixed $c_2 = 2\pi\beta$ we will say that a solution $Q(u)$ is subcritical, critical, or supercritical at u, accordingly, as $Q(u)$ is less than, equal to, or greater than $\hat{Q}(u)$. By Theorems 5.1 and 5.2, there exists, for $\beta^2/f^2(u) < Q_\pi < 2/(\gamma - 1)$, $Q_\pi \neq \hat{Q}(\pi)$, an everywhere subcritical (respectively, supercritical) solution $Q(u)$ if Q_π is subcritical (respectively, supercritical). The curves $Q^-(u)$ and $Q^+(u)$ in Fig. 3 depict such solutions. In addition, there is

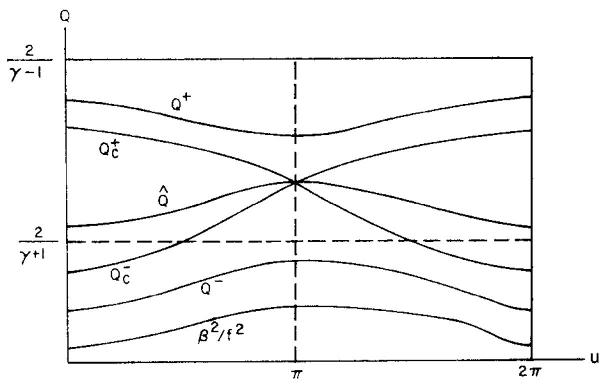

FIG. 3. Smooth solutions ($C_2 \neq 0$).

a subcritical–critical solution $Q_c^-(u)$ and a supercritical–critical solution $Q_c^+(u)$. These are also drawn in Fig. 3. Their differentiability behavior at π is similar to that in the case $c_2 = 0$ already discussed. Again, a one-parameter family of solutions with a single compression shock occurs (see Fig. 4). A similar argument to that in Section 3.2 shows that such a solution must pass smoothly from the curve $Q_c^-(u)$ to the curve $Q_c^+(u)$ at $u = \pi$ and then must undergo a single jump discontinuity from the $Q_c^+(u)$ curve to the $Q_c^-(u)$ curve.

The speed in front of the shock must be supersonic (since it is supercritical) but may be supersonic or subsonic behind the shock. Note that the shocks in this case are *oblique* compression shocks.

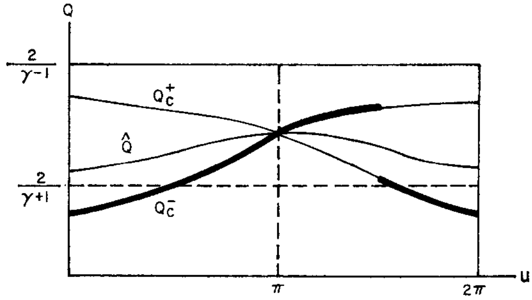

FIG. 4. Shock solutions ($C_2 \neq 0$).

If data are prescribed at a point $u \neq \pi$ then a local solution might terminate at a point u at which $Q(u) = \hat{Q}(u)$ and dQ/du becomes infinite.

3.5. *The Circulation Problem* ($c_2 \neq 0$)

As in Section 3.3 one sees that, for fixed c_2, there exist polytropic flows for a range of circulation c_1 ($0 \leqslant c_1 \leqslant c_1^*$) in the u direction (where c_1^* depends on c_2). If c_1 is small the flow is subcritical; if it is large the flow is supercritical (and hence supersonic). In the intermediate range the flow has a single oblique compression shock. Subcritical does not, however, imply subsonic so we should perhaps emphasize that if the circulation c_2 in the v direction satisfies $0 < c_2 < \min\,(2\pi f_0(2/(\gamma+1))^{1/2}, 2\pi f_\pi(2/(\gamma-1))^{1/2})$ then *there is a range of circulation c_1 in the u direction for which the flow is smoothly transonic.*

A graph showing the dependence of the character of the solutions for prescribed circulations c_1 and c_2 is given in Fig. 5. The region corresponding to

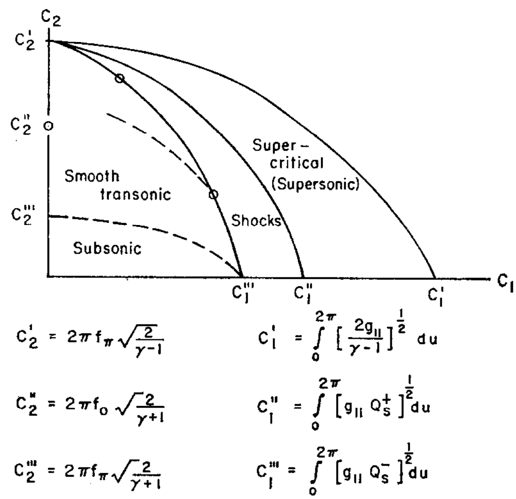

FIG. 5. The circulation problem.

supercritical (and hence supersonic) solutions and the region of solutions with shocks (supersonic–supersonic or supersonic–subsonic) are indicated. The region bounded by the c_1 and c_2 axes and the curve c_1'''–c_2' corresponds to subcritical solutions. This always contains a (proper) subregion of subsonic solutions (under the dotted curve c_1'''–c_2'''). There are also always smooth transonic solutions as shown and in addition, there *may* (depending on the geometry determined by f) be a region of supersonic–subcritical solutions. The dividing curve between these two regions would have as one endpoint a point on the curve c_1'''–c_2' (but *not* on the c_1 axis). The other endpoint could be either on the c_2 axis between c_2' and c_2''' or on the curve c_1'''–c_2', again depending on f. We have indicated this by an incomplete dotted curve.

4. Local Theory

4.1. *Local Existence*

We seek a (local) solution $Q = Q(u)$ of the initial value problem

$$f^2(Q - \beta^2/f^2)\,\rho^2 = K = \text{positive constant}, \tag{4.1a}$$

$$Q(u_1) = Q_1\,, \qquad \beta^2/f_1 < Q_1 < 2/\gamma - 1, \tag{4.1b}$$

where $\rho = (1 - ((\gamma - 1)/2)\,Q)^{1/(\gamma-1)}$ and we have written $f_1 = f(u_1)$. Let K_1 be the constant obtained by evaluating Eq. (4.1a) at $u = u_1$, $Q = Q_1$, and set

$$F(u, Q) = f^2(Q - \beta^2/f^2)\,\rho^2 - K_1\,. \tag{4.2}$$

The existence of a solution of (4.1) would follow from the implicit-function theorem if $F_Q(u_1, Q_1) \neq 0$. A computation yields

$$F_Q(u_1, Q_1) = f_1^2\left(1 - \frac{\gamma - 1}{2}Q_1\right)^{(3-\gamma)/(\gamma-1)}(\hat{Q}(u_1) - Q_1), \tag{4.3}$$

where the function $\hat{Q}(u)$ is given by

$$\hat{Q}(u) = \frac{2}{\gamma + 1}(1 + \beta^2/f^2). \tag{4.4}$$

We shall call $\hat{Q}(u)$ the *critical function* or *critical curve* and it will play a prominent role in what follows for reasons which shall become evident. We have

Proposition 4.1. *If $\beta^2/f_1^2 < Q_1 < 2/(\gamma - 1)$ and $Q_1 \neq \hat{Q}(u_1)$ then, in some interval containing u_1, there exists a unique continuously differentiable solution of the initial value problem* (4.1).

Note 1. $\hat{Q}(u) > 2/(\gamma + 1)$ unless $c_2 = 0$ in which case $\hat{Q}(u)$ is the constant $2/(\gamma + 1)$.

Note 2. In this section we do *not* make the de Laval assumption (3.1) on f.

Note 3. Our computations show, in fact,

COROLLARY 4.1. *For* $u = u_1$, $(d/dQ)\{(Q - \beta^2/f_1^2)\rho^2\}$ *is positive for* $Q < \hat{Q}(u_1)$, *zero for* $Q = \hat{Q}(u_1)$, *and negative for* $Q > \hat{Q}(u_1)$.

PROPOSITION 4.2. *Let* $Q = Q(u)$ *be a solution of the initial value problem* (4.1) *in a neighborhood* N *of* u_1. *Then*

(i) *If* $Q_1 < \hat{Q}(u_1)$ *then* $Q(u)$ *is an increasing* (*decreasing*) *function of* u *if* $f' < 0$ ($f' > 0$) *in* N.

(ii) *If* $Q_1 > \hat{Q}(u_1)$ *then* $Q(u)$ *is a decreasing* (*increasing*) *function of* u *if* $f' < 0$ ($f' > 0$) *in* N.

Proof. Again by the implicit-function theorem

$$\frac{dQ}{du} = -\frac{F_u}{F_Q} = -\frac{4}{\gamma + 1}\frac{f'}{f}\frac{Q(1 - ((\gamma - 1)/2)Q)}{\hat{Q} - Q}. \tag{4.5}$$

The conclusions follow by observing the signs of f' and of $\hat{Q}$–Q in the various cases.

4.2. *Dependence of K on Q*

The quantity K in (4.1a) gives the (square of the) mass flow in the u direction as a function of u and Q (where $\beta^2/f^2 < Q < 2/(\gamma - 1)$). For a fixed $u = u_1$ we write $K = K(Q)$ and observe first that $K = 0$ for $Q = \beta^2/f_1^2$ and for $Q = 2/(\gamma - 1)$. Moreover, at $u = u_1$, dK/dQ is (by Corollary 4.1) greater than, equal to, or less than zero, accordingly, as Q is less than, equal to, or greater than $\hat{Q}(u_1)$. Hence K attains a maximum at $Q = \hat{Q}(u_1)$ which we denote by $K_c(u_1)$ (see Fig. 6).

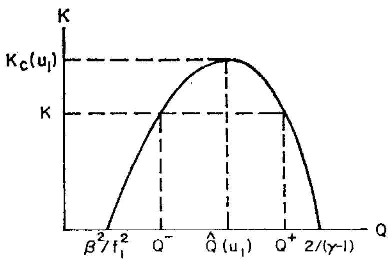

FIG. 6. Dependence of K on Q.

Summarizing, we have

PROPOSITION 4.3. *For $u = u_1$ and $0 \leqslant K < K_c(u_1)$ there exist two values of Q (which we denote by Q^- and Q^+) such that*

(i) $K = f_1^2(Q^\pm - \beta^2/f_1^2)(1 - ((\gamma - 1)/2) Q^\pm)^{2/(\gamma-1)}$,

(ii) $\beta^2/f_1^2 < Q^- < \hat{Q}(u_1) < Q^+ < 2/(\gamma - 1)$,

(iii) *Q^- (resp. Q^+) is a monotone increasing (resp. decreasing) function of K.*

(iv) *$Q^- \nearrow \hat{Q}(u_1)$ and $Q^+ \searrow \hat{Q}(u_1)$ as $K \to K_c(u_1)$.*

We will also need to know how the maximum mass flow $K_c(u)$ that can pass through a point u is controlled by the value of f at u.

PROPOSITION 4.4. *If $f(u_1) \leqslant f(u_2)$ then $K_c(u_1) \leqslant K_c(u_2)$ with equality holding only if $f(u_1) = f(u_2)$.*

Proof. If $f_1 \leqslant f_2$ then $\hat{Q}(u_1) \geqslant \hat{Q}(u_2)$ but computations show that $f_1^2\hat{Q}(u_1) \leqslant f_2^2\hat{Q}(u_2)$ and, since ρ is a decreasing function of Q, that $\rho^2(\hat{Q}(u_1)) \leqslant \rho^2(\hat{Q}(u_2))$. However, $K_c(u_i) = K(\hat{Q}(u_i)) = f_i^2(\hat{Q}(u_i) - \beta^2/f_i^2)\,\rho^2(\hat{Q}(u_i))$ and the conclusion follows.

4.3. *Dependence on the Initial Value*

If a solution $Q(u)$ satisfying $Q(u_1) = Q_1$ can be continued to a point $u = u_2$ we will be interested (see Section 5.3, Theorem 5.2) in the dependence of the functional value $Q_2 = Q(u_2)$ upon the initial value Q_1.

PROPOSITION 4.5. *Let either* (a) $Q(u; K) < \hat{Q}(u)$ *or* (b) $Q(u; K) > \hat{Q}(u)$ *be a family of solutions parametrized by K, each solution defined in an interval containing u_1 and u_2. Let $Q_1 = Q(u_1; K)$ and $Q_2 = Q(u_2; K)$ denote their functional values at u_1 and u_2, respectively. Then Q_2 is an increasing function of Q_1.*

Proof. In case (a), K is, by Proposition 4.3, an increasing function of Q_1, and Q_2 is an increasing function of K. In case (b), both functions are decreasing. Thus, the conclusion holds in either case.

5. PROOF OF THE EXISTENCE THEOREM AND THE CONVERGENCE TO THE SONIC SOLUTIONS

5.1. *Preliminaries*

In this section, we seek global solutions $Q(u)$ of the mass flow equation for polytropic flow with $\rho = (1 - ((\gamma - 1)/2) Q)^{1(\gamma-1)}$:

$$f^2\rho^2\left(Q - \frac{\beta^2}{f^2}\right) = f^2\left(1 - \frac{\gamma - 1}{2} Q\right)^{2/(\gamma-1)}\left(Q - \frac{\beta^2}{f^2}\right) = K. \qquad (5.1)$$

Here, K is the mass-flow constant, and f is a positive, periodic, twice continuously differentiable function satisfying assumption (3.1). The constant K is uniquely determined from (5.1) by the value of Q at any point u. In particular,

$$K = f(\pi)^2 \left(1 - \frac{\gamma - 1}{2} Q(\pi)\right)^{2/(\gamma-1)} \left(Q(\pi) - \frac{\beta^2}{f(\pi)^2}\right). \tag{5.2}$$

We shall omit the limiting cases for which $K = 0$ by always assuming that

$$\frac{\beta^2}{f(\pi)^2} < Q(\pi) < \frac{2}{\gamma - 1}.$$

LEMMA 5.1. *Suppose Q is a solution of* (5.1) *on an interval containing π. Then, $K \neq 0$ there and*

(a) $$\frac{\beta^2}{f(u)^2} < Q(u) < \frac{2}{\gamma - 1},$$

(b) $$Q(u) - \frac{\beta^2}{f(u)^2} > \frac{K}{f(0)^2},$$

(c) $$1 - \frac{\gamma - 1}{2} Q(u) > \left(\frac{\gamma - 1}{2} \frac{K}{f(0)^2}\right)^{(\gamma-1)/2}.$$

Proof. From (5.2) and the assumption on the initial value at π, $K > 0$ and neither ρ^2 nor $Q - \beta^2/f(u)^2$ can vanish. Since ρ^2 never vanishes, neither does $1 - ((\gamma - 1)/2) Q$, and since $1 - ((\gamma - 1)/2) Q(\pi) > 0$, it follows that $1 - ((\gamma - 1)/2) Q(u) > 0$. This proves (a).

From (a),

$$\left(1 - \frac{\gamma - 1}{2} Q\right)^{2/(\gamma-1)} < 1,$$

and therefore,

$$\frac{K}{f(0)^2} \leqslant \frac{K}{f^2} < Q - \frac{\beta^2}{f^2},$$

which proves (b). Finally,

$$\frac{K}{f(0)^2} \leqslant \frac{K}{f^2} < \frac{2}{\gamma - 1} \left(1 - \frac{\gamma - 1}{2} Q\right)^{2/(\gamma-1)}.$$

Since $1 - ((\gamma - 1)/2) Q > 0$, raising this inequality to the power $(\gamma-1)/2$ proves (c).

In the following, we make extensive use of the auxiliary function:

$$\hat{Q}(u) = \frac{2}{\gamma + 1} \left(1 + \frac{\beta^2}{f(u)^2}\right).$$

We wish to study the behavior of solutions of the differential equation

$$\frac{dQ}{du} = -\frac{4}{\gamma+1}\frac{f'}{f}Q\left(\frac{1-((\gamma-1)/2)Q}{\hat{Q}-Q}\right). \tag{5.3}$$

In particular, consider the pair of initial value problems:

$(5.3)^{\pm}$ Find solutions $Q^{\pm}(u)$ of Eq. (5.3) with prescribed initial values $Q^{+}(\pi)$ (respectively, $Q^{-}(\pi)$) satisfying

(+) $Q^{+}(\pi) > \hat{Q}(\pi)$,

(−) $Q^{-}(\pi) < \hat{Q}(\pi)$.

Differentiable solutions of (5.1) satisfy (5.3). Conversely, solutions of $(5.3)^{\pm}$, in some neighborhood of $u = \pi$, satisfy the equation

$$\frac{d}{du}\left(f^2\rho^2\left(Q - \frac{\beta^2}{f^2}\right)\right) = 0$$

and hence, the mass-flow relation (5.1). Therefore, we have shown

PROPOSITION 5.1. *Let $Q(u)$ be continuously differentiable in some interval containing π with $Q(\pi) \neq \hat{Q}(\pi)$. Then, Q is a solution of* (5.1) *if and only if Q is a solution of* $(5.3)^{+}$ *or* $(5.3)^{-}$.

COROLLARY 5.1. *Solutions $Q^{\pm}(u)$ of $(5.3)^{\pm}$ defined in some interval containing π satisfy the conclusions of Lemma* 5.1 *there.*

PROPOSITION 5.2. *Let $Q(u)$ be a solution of* $(5.3)^{+}$ *or* $(5.3)^{-}$ *on some interval containing π for which the function $\varphi(u) = (\hat{Q}(u) - Q(u))^2 > 0$. Then, φ has a unique minimum at $u = \pi$ and for $u \neq \pi$ on that interval, $\varphi(u) > \varphi(\pi)$.*

To prove the proposition we will need the following important

LEMMA 5.2. (a) *The function $(\hat{Q} - Q)^2$ satisfies the differential equation*

$$\frac{d}{du}(\hat{Q} - Q)^2 = \frac{8f'(u)}{f(u)}\, g(u, Q(u)), \tag{5.4}$$

where

$$g(u, Q(u)) = \frac{1}{\gamma+1}\left\{Q(u)\left(1 - \frac{\gamma-1}{2}Q(u)\right) + \frac{\beta^2}{f(u)^2}(Q(u) - \hat{Q}(u))\right\}.$$

(b) *If $(\hat{Q}(u) - Q(u))^2 > 0$ in an interval containing π, then*

$$g(u, Q(u)) > C(K) > 0, \tag{5.5}$$

where

$$C(K) = \frac{2}{(\gamma+1)^2}\left(1 + \frac{\beta^2}{f(0)^2}\right) \min\left(\frac{K}{f(0)^2}, \left(\frac{\gamma-1}{2}\frac{K}{f(0)^2}\right)^{(\gamma-1)/2}\right)$$

is a monotone increasing function of the mass-flow constant K.

To prove (a),

$$\begin{aligned}\frac{d}{du}(\hat{Q}-Q)^2 &= 2(\hat{Q}-Q)\left(\frac{d\hat{Q}}{du} - \frac{dQ}{du}\right)\\ &= \frac{8f'}{f}\frac{1}{\gamma+1}\frac{\beta^2}{f^2}(Q(u)-\hat{Q}(u)) - 2(\hat{Q}-Q)\frac{dQ}{du}\\ &= \frac{8f'}{f}\frac{1}{\gamma+1}\frac{\beta^2}{f^2}(Q-\hat{Q}) + \frac{8f'}{f}\frac{1}{\gamma+1}Q\left(1-\frac{\gamma-1}{2}Q\right).\end{aligned}$$

To prove (b), if $Q(\pi) < \hat{Q}(\pi)$ then $Q(u) < \hat{Q}(u)$ and

$$\begin{aligned}g(u, Q(u)) &> \frac{1}{\gamma+1}\left\{Q(u)\left(1-\frac{\gamma-1}{2}\hat{Q}(u)\right) + \frac{\beta^2}{f^2}(Q-\hat{Q})\right\}\\ &= \frac{1}{\gamma+1}\hat{Q}\left(Q - \frac{\beta^2}{f^2}\right) > \frac{2}{(\gamma+1)^2}\left(1+\frac{\beta^2}{f(0)^2}\right)\frac{K}{f(0)^2}.\end{aligned}$$

If $Q(\pi) > \hat{Q}(\pi)$ then $Q(u) > \hat{Q}(u)$ and

$$\begin{aligned}g(u, Q(u)) &> \frac{1}{\gamma+1}\hat{Q}(u)\left(1-\frac{\gamma-1}{2}Q(u)\right)\\ &> \frac{2}{(\gamma+1)^2}\left(1+\frac{\beta^2}{f(0)^2}\right)\left(\frac{\gamma-1}{2}\frac{K}{f(0)^2}\right)^{(\gamma-1)/2}.\end{aligned}$$

This proves the lemma.

To prove Proposition 5.2, we observe that $d\varphi/du = (8f'(u)/f(u))\, g(u, Q(u))$ vanishes only at π and

$$\frac{d^2\varphi}{du^2}(\pi) = \frac{8f''(\pi)}{f(\pi)} g(\pi, Q(\pi)) > 0.$$

COROLLARY 5.2. (a) *If* Q^- *is a solution of* (5.3)$^-$ *in some interval containing* π *in which* $\hat{Q}(u) - Q^-(u) > 0$, *then, on that interval,*

$$\hat{Q}(u) - Q^-(u) > \hat{Q}(\pi) - Q^-(\pi) > 0. \tag{5.6$^-$}$$

(b) *If* Q^+ *is a solution of* (5.3)$^+$ *in some interval containing* π *in which* $Q^+(u) - \hat{Q}(u) > 0$, *then,*

$$Q^+(u) - \hat{Q}(u) > Q^+(\pi) - \hat{Q}(\pi) > 0. \tag{5.6$^+$}$$

5.2. Solution of the Initial Problem for Non Critical Initial Data

THEOREM 5.1. *For each choice of the initial values* $Q^+(\pi)$ *or* $Q^-(\pi)$ *there exists a unique continuously differentiable periodic solution of* $(5.3)^+$ *or* $(5.3)^-$, *or alternately* (5.1). *The solution satisfies* $(5.6)^+$ *or* $(5.6)^-$ *as well as conditions* (a), (b), *and* (c) *of Lemma* 5.1.

Remark. It follows from the theorem that the Q^- solutions lie below the critical curve $\hat{Q}$ and the Q^+ solutions lie above the critical curve $\hat{Q}$. In addition, from Proposition 4.2, $Q^+(u)$ has a minimum at $u = \pi$. Therefore, $Q^+(u) \geqslant Q^+(\pi) > \hat{Q}(\pi) \geqslant 2/(\gamma + 1)$, or $Q^+(u)$ is always supersonic. If $\beta = 0$, $\hat{Q}(u) = 2/(\gamma + 1)$ and the Q^- solutions are subsonic. Otherwise, they may be subsonic, supersonic, or transonic.

Proof of Theorem 5.1. We shall solve the initial value problem $(5.3)^-$, the solution of $(5.3)^+$ being completely analogous.

Let $Q^-(\pi)$ be the given initial value. By Proposition 4.1, $(5.3)^-$ has a unique continuously differentiable solution in some interval containing π. By continuity, in some possibly smaller interval, $Q^-(u) < \hat{Q}(u)$. From Corollary 5.2, $(5.6)^-$ is satisfied on this interval. Conditions (a), (b), and (c) follow from Corollary 5.1. Therefore, the problem has a unique local solution in some interval about π.

Let $S = (u_1, u_2)$ be the largest open interval containing π on which the solution $Q^-(u)$ exists, is continuously differentiable, and satisfies the conditions $(5.6)^-$, (a), (b), and (c). We will show that $S = (0, 2\pi)$.

If this is not the case, then, say, $u_1 > 0$. Let $u_n \in S$ with $u_n \searrow u_1$. The sequence $Q^-(u_n)$ is then monotonically decreasing and bounded below so that $Q^-(u_n)$ tends to a limit $Q^-(u_1)$. It follows that $Q^-(u)$ is continuous on the closed interval $[u_1, \pi]$ and satisfies the mass-flow relation (5.1) there. By Lemma 5.1, conditions (a), (b), and (c) are valid on that interval. By continuity, $\hat{Q}(u) - Q^-(u) \geqslant \hat{Q}(\pi) - Q^-(\pi) > 0$ on the closed interval $[u_1, \pi]$. But then by Corollary 5.2, $(5.6)^-$ holds on that interval.

Again, by Proposition 4.1, the initial value problem

$$\frac{dQ}{du} = -\frac{4}{\gamma+1}\frac{f'}{f}Q\left(\frac{1-((\gamma-1)/2)Q}{\hat{Q}-Q}\right), \qquad Q(u_1) = Q^-(u_1),$$

has a unique solution in some interval containing u_1. By uniqueness, the solution agrees with $Q^-(u)$ for $u \geqslant u_1$. Hence, it is a continuously differentiable extension of $Q^-(u)$ in some interval to the left of u_1. Since $\hat{Q}(u_1) - Q^-(u_1) > 0$, by continuity, $\hat{Q}(u) - Q^-(u) > 0$ in some interval to the left of u_1, and therefore $(5.6)^-$ is satisfied there by Corollary 5.2. By Corollary 5.1, (a), (b), and (c) are also satisfied. But this implies that S was not the largest interval on which $Q^-(u)$ exists. Therefore u_1 must be zero. A similar analysis at the right endpoint shows that $u_2 = 2\pi$.

By Proposition 4.2, $Q^-(u)$ has limiting values at $u = 0$ and $u = 2\pi$, and is therefore continuous on the closed interval $[0, 2\pi]$.

From the differential equation (5.3), dQ^-/du has a limit at $u = 0$, namely,

$$\frac{dQ^-}{du} = -\frac{4}{\gamma+1}\frac{f'(0)}{f(0)}Q^-(0)\left(\frac{1-((\gamma-1)/2)Q^-(0)}{\hat{Q}(0)-Q^-(0)}\right) = 0,$$

since $f'(0) = 0$.

Also, $(dQ^-/du)(2\pi) = 0$, so that the curves have the same slope at 0 and 2π.

We next show that since f is periodic, Q^- is periodic. From (5.1),

$$\rho^2(Q^-(0))\left(Q^-(0) - \frac{\beta^2}{f(0)^2}\right) = \rho^2(Q^-(2\pi))\left(Q^-(2\pi) - \frac{\beta^2}{f(2\pi)^2}\right).$$

As in Section 4, Corollary 4.1,

$$\frac{d}{dQ}\left(\rho^2\left(Q - \frac{\beta^2}{f^2}\right)\right) = \frac{\gamma+1}{2}\left(1 - \frac{\gamma-1}{2}Q\right)^{(3-\gamma)/(\gamma-1)}, \qquad (\hat{Q} - Q) > 0.$$

Therefore,

$$\rho^2(Q_1)\left(Q_1 - \frac{\beta^2}{f^2}\right) = \rho^2(Q_2)\left(Q_2 - \frac{\beta^2}{f^2}\right)$$

implies $Q_1 = Q_2$, or, in this case, $Q^-(0) = Q^-(2\pi)$. This completes the proof of Theorem 5.1.

Corollary 5.3. *The solutions of Theorem* 5.1 *satisfy the integral equations*

$$Q^\pm(u) = \hat{Q}(u) \pm \left((\hat{Q}(\pi) - Q^\pm(\pi))^2 + \int_\pi^u \frac{8f'}{f} g(t, Q^\pm(t))\, dt\right)^{1/2}, \qquad (5.7)^\pm$$

where $g(t, Q^\pm(t)) > C(K)$ *as defined in Lemma* 5.2.

The corollary follows by integrating (5.4) and taking square roots.

5.3. *The Critical Solutions*

Next, we study the behavior of the solutions as the initial values tend to the critical value $\hat{Q}(\pi)$.

We will need the following

Lemma 5.3. *Suppose* $h(u)$ *is continuously differentiable and increasing in a neighborhood of* π *with* $h(\pi) = 0$. *Then the function*

$$H(u) = \frac{h(u)}{(\int_\pi^u h(t)\, dt)^{1/2}}$$

is bounded and

(i) $\lim_{u \nearrow \pi} H(u) = -(2h'(\pi))^{1/2}$,

(ii) $\lim_{u \searrow \pi} H(u) = (2h'(\pi))^{1/2}$.

Proof. From L'Hospital's Rule,

$$\lim_{u \to \pi} \frac{h^2(u)}{\int_\pi^u h(t)\,dt} = 2h'(\pi).$$

Taking square roots with appropriate signs proves the lemma.

THEOREM 5.2. *Let $Q_n^{\pm}(u)$ be sequences of solutions of* $(5.3)^{\pm}$ *for which $Q_n^{\pm}(\pi)$ tend monotonically to $\hat{Q}(\pi)$. Then, $Q_n^{\pm}(u)$ converge uniformly on $[0, 2\pi]$ to solutions $Q_c^{\pm}(u)$ of* (5.1) *with $K = K_c$ and $Q_c^{+}(\pi) = Q_c^{-}(\pi) = \hat{Q}(\pi)$. For every $u \neq \pi$, $Q_c^{-}(u) < \hat{Q}(u)$, $Q_c^{+}(u) > \hat{Q}(u)$, $Q_c^{\pm}(u)$ are continuously differentiable, and there exists a constant M independent of u such that $| dQ_c^{\pm}/du | \leqslant M$.*

Proof of Theorem 5.2. Recall (Proposition 4.5) that if $Q_n^{-}(\pi)$ is monotone increasing, then $Q_n^{-}(u)$ is also monotone increasing, and bounded above by $\hat{Q}(\pi)$. We will show that the sequence $| dQ_n^{-}/du |$ is uniformly bounded. By Arzela's theorem, a subsequence converges uniformly. By monotonicity, the original sequence must converge.

Differentiating $(5.7)^{-}$, we see that a solution of $(5.3)^{-}$ satisfies

$$\frac{dQ^-}{du} = \frac{d\hat{Q}}{du} - \frac{1}{2} \frac{8(f'(u)/f(u))\, g(u, Q^-(u))}{((\hat{Q}(\pi) - Q^-(\pi))^2 + \int_\pi^u 8(f'/f)\, g(t, Q^-(t))\,dt)^{1/2}}.$$

Therefore

$$\left| \frac{dQ^-}{du} \right| \leqslant \left| \frac{d\hat{Q}}{du} \right| + \frac{|(2f'(u)/f(u))\, g(u, Q^-(u))|}{(\int_\pi^u (2f'/f)\, g(t, Q^-(t))\,dt)^{1/2}}.$$

Also, recall (Proposition 4.3) that K_n is a monotone increasing function of $Q_n^{-}(\pi)$, and $C(K)$ is an increasing function of K. Therefore, $g(t, Q_n^{-}(t)) > C(K_n) > C > 0$ for large n.

Since $f'(u)$ is increasing near π, and $f'(\pi) = 0$,

$$\int_\pi^u \frac{2f'}{f} g(t, Q_n^-(t))\,dt > C \int_\pi^u \frac{2f'}{f}\,dt.$$

Moreover, $g(u, Q_n^{-}(u))$ is bounded above. Therefore,

$$\left| \frac{dQ_n^-}{du} \right| \leqslant K_1 + K_2 \frac{|2f'(u)/f(u))|}{(\int_\pi^u (2f'/f)\,dt)^{1/2}}.$$

Applying Lemma 5.3, we see that the right-hand side is bounded, as was to be shown. It follows that $Q_n^-(u)$ converges uniformly to a continuous function $Q_c^-(u)$ with $Q_c^-(\pi) = \mathring{Q}(\pi)$.

In a similar fashion, if $Q_n^+(\pi)$ is monotone decreasing, then $Q_n^+(u)$ is also monotone decreasing and bounded below by $\mathring{Q}(\pi)$ (Proposition 4.5). In this case (Proposition 4.3), the mass flows K_n increase as $Q_n(\pi)$ decreases. Thus, $g(t, Q_n^+(t)) > C(K_n) > C > 0$, and one obtains a uniform bound on $| dQ_n^+/du |$. The sequence $Q_n^+(u)$ converges uniformly to a limit function $Q_c^+(u)$ with $Q_c^+(\pi) = \mathring{Q}(\pi)$.

From the uniform convergence of $Q_n^\pm(u)$,

$$g(t, Q_c^\pm(t)) > C(K_c) > 0,$$

and

$$Q_c^\pm(u) = \mathring{Q}(u) \pm \left(\int_\pi^u \frac{8f'}{f} g(t, Q_c^\pm(t))\, dt\right)^{1/2}. \tag{5.8)$^\pm$ (}$$

Therefore, $Q_c^+(u) > \mathring{Q}(u)$ and $Q_c^-(u) < \mathring{Q}(u)$ for $u \neq \pi$.

If $u \neq \pi$, the right-hand side is differentiable and

$$\frac{dQ_c^\pm}{du} = \frac{d\mathring{Q}}{du} \pm \frac{(2f'(u)/f(u))\, g(u, Q_c^\pm(u))}{(\int_\pi^u (2f'/f)\, g(t, Q_c^\pm(t))\, dt)^{1/2}}. \qquad (5.9)^\pm$$

As in the preceding argument, this is bounded independently of u. This completes the proof of Theorem 5.2.

Corollary 5.4. *The critical solutions $Q_c^\pm(u)$ have left- and right-handed derivatives at $u = \pi$ given by*

$$\text{(i)}\quad \lim_{u \nearrow \pi} \frac{dQ_c^\pm}{du} = \mp 4 \left(\frac{f''(\pi)}{f(\pi)(\gamma+1)^3}\left(1 + \frac{\beta^2}{f(\pi)^2}\right)\left(1 - \frac{\gamma-1}{2}\frac{\beta^2}{f(\pi)^2}\right)\right)^{1/2},$$

$$\text{(ii)}\quad \lim_{u \searrow \pi} \frac{dQ_c^\pm}{du} = \pm 4 \left(\frac{f''(\pi)}{(\gamma+1)^3 f(\pi)}\left(1 + \frac{\beta^2}{f(\pi)^2}\right)\left(1 - \frac{\gamma-1}{2}\frac{\beta^2}{f(\pi)^2}\right)\right)^{1/2}.$$

Therefore,

$$\text{(iii)}\quad \lim_{u \nearrow \pi} \frac{dQ_c^-}{du} = \lim_{u \searrow \pi} \frac{dQ_c^+}{du}$$

and

$$\lim_{u \nearrow \pi} \frac{dQ_c^+}{du} = \lim_{u \searrow \pi} \frac{dQ_c^-}{du}.$$

Remark. It follows from Corollary 5.4 that the functions

$$Q_1(u) = Q_c^-, \qquad \text{for } u \leqslant \pi,$$
$$ = Q_c^+, \qquad \text{for } u \geqslant \pi,$$

and

$$Q_2(u) = Q_c^+, \qquad \text{for } u \leqslant \pi,$$
$$= Q_c^-, \qquad \text{for } u \geqslant \pi,$$

are continuously differentiable.

Remark 2. If $f''(\pi) = 0$, the solutions $Q_c^\pm$ are continuously differentiable at π.

To prove the corollary, we will compute the limits in $(5.9)^\pm$ as $u \to \pi$. First observe that $d\hat{Q}/du = 0$ at $u = \pi$. Next, let $h(u) = (2f'(u)/f(u))\, g(u, Q_c^\pm(u))$. The function $h(u)$ is increasing in a neighborhood of π with $h(\pi) = 0$. We must check that $h'(u)$ is continuous. Differentiating,

$$h'(u) = \frac{2f''}{f} g(u, Q_c^\pm(u)) + \frac{f'}{f}\left(\frac{\partial g}{\partial u} + \frac{\partial g}{\partial Q}\frac{dQ_c^\pm}{du} - \frac{2f'}{f} g(u, Q(u)\right).$$

For $u \neq \pi$, this expression is continuous. Since $dQ_c^\pm/du$ are bounded functions and $f'(\pi) = 0$, $h'(\pi) = (2f''(\pi)/f(\pi))\, g(\pi, Q_c^\pm(\pi))$ exists.

Therefore, $h(u)$ satisfies the hypotheses of Lemma 5.3.

Now,

$$g(\pi, Q_c^\pm(\pi)) = g(\pi, \hat{Q}(\pi)) = \frac{1}{\gamma+1}\hat{Q}(\pi)\left(1 - \frac{\gamma-1}{2}\hat{Q}(\pi)\right)$$
$$= \frac{4}{(\gamma+1)^3}\left(1 + \frac{\beta^2}{f(\pi)^2}\right)\left(1 - \frac{\gamma-1}{2}\frac{\beta^2}{f(\pi)^2}\right).$$

Therefore,

$$2h'(\pi) = \frac{16}{(\gamma+1)^3}\frac{f''(\pi)}{f(\pi)}\left(1 + \frac{\beta^2}{f(\pi)^2}\right)\left(1 - \frac{\gamma-1}{2}\frac{\beta^2}{f(\pi)^2}\right).$$

Applying Lemma 5.3 in each of the four cases now gives the conclusions of the corollary.

6. The Case $\gamma = 3$

If $\gamma = 3$, the mass flow relation (5.1) in the preceding section is quadratic in Q; namely,

$$(1 - Q)\left(Q - \frac{\beta^2}{f^2}\right) = \frac{K}{f^2} \tag{6.1}$$

or

$$Q^2 - \left(1 + \frac{\beta^2}{f^2}\right)Q + \frac{\beta^2 + K}{f^2} = 0. \tag{6.2}$$

Since $\hat{Q} = \frac{1}{2}(1 + \beta^2/f^2)$ we find that

$$Q = \hat{Q} \pm (\hat{Q}^2 - (\beta^2 + K)/f^2)^{1/2}. \tag{6.3$^\pm$}$$

On the other hand, recall the integral identities $(5.7)^\pm$:

$$Q(u) = \hat{Q}(u) \pm \left((\hat{Q}(\pi) - Q(\pi))^2 + \int_\pi^u \frac{8f'}{f} g(t, Q(t))\, dt\right)^{1/2}.$$

Since $(\hat{Q}(\pi) - Q(\pi))^2 = \hat{Q}(\pi)^2 - (\beta^2 + K)/f(\pi)^2$ and

$$\int_\pi^u \frac{8f'}{f} g\, dt = \frac{\beta^2 + K}{f(\pi)^2} - \frac{\beta^2 + K}{f^2} + \hat{Q}^2 - \hat{Q}(\pi)^2,$$

we see that $(5.7)^\pm$ is the same as $(6.3)^\pm$ for $\gamma = 3$.

The explicit nature of the formula $(6.3)^\pm$ allows us to read off all properties of the solutions directly.

7. Description of Flows (Continued)

In the description (Section 3) of the global solutions $Q(u)$ we have assumed (3.1) that f has only one minimum between 0 and 2π. The local theory of Section 4 makes no use of this whatsoever. In Section 5 the fact that we have arranged that f has a minimum at $u = \pi$ is, as we have pointed out, merely a technical convenience. Continuing the solution to the left to $u = 0$ depends solely on the fact that this is the closest maximum point of f to the left of π. (The corresponding observation holds for $u = 2\pi$.)

Note that, by Eq. (4.5) for an arbitrary $f \in C^2$, if $f' = 0$ on an interval then any smooth solution $Q(u)$ is constant on the interval. With no loss of generality then, we assume that f has isolated extrema with relative minima at u_i, $i = 1,..., n$, and interlaced maxima at u_i', say with $u_i' < u_i < u_{i+1}'$. The arguments of Section 5 then lead to a family of solutions on the interval $[u_i', u_{i+1}']$ corresponding to mass-flow constants K (see Section 4.2) with $0 < K \leqslant K_c(u_i)$. Each solution has vanishing derivatives at the endpoints as follows from the proof of Theorem 5.1. But Q is determined by u and K (as usual, this is all for fixed c_2) so that two solutions, one on each of adjacent intervals, will meet (smothly) at the common endpoint if and only if they correspond to the same value of K (and both are either subcritical or both supercritical). As we have just seen, the range of K is *not* the same for all intervals but depends on u_i. More precisely, since $K_c(u_i) = K(\hat{Q}(u_i))$, it depends on the value of f at u_i. Let $K_c = \min K_c(u_i)$ over all relative minima u_i of f. By Proposition 4.4 this minimum is assumed at an absolute minimum of f.

THEOREM 7.1. *Let $f \in C^2$ be an arbitrary (periodic) function with isolated extrema. Then for $0 < K < K_c$ there exist a smooth subcritical solution $Q^-(u)$ and a smooth supercritical solution $Q^+(u)$, both defined on the interval $[0, 2\pi]$. For $K = K_c$ there exist smooth solutions $Q_c^-(u)$ and $Q_c^+(u)$ which are subcritical (respectively, supercritical) everywhere except at an absolute minimum of f, where they are critical.*

Flows with compression shocks can be constructed as in Section 3.4. As regards the circulation problem, we have

THEOREM 7.2. *Suppose that f has a unique absolute minimum at $u = u^*$ (but it may have other relative minima). Let $c_2^* = 2\pi(2/(\gamma - 1))^{1/2} f(u^*)$ be the upper bound on circulation in the v-direction and let $c_2 < c_2^*$. Then there is a constant c_1^* depending on c_2 (see Section 3.5) such that for $c_1 < c_1^*$ there exists a unique flow on T having circulations c_1 and c_2. For appropriate choices of c_1 and c_2 the flow is smoothly subsonic, smoothly transonic, transonic with one compression shock, or smoothly supersonic.*

Remark. If the absolute minimum of f is attained at several values of u, then flows with several compression shocks are possible. In this case the solution of the circulation problem need *not* be unique.

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