Multi-axial failure of metallic strut-lattice materials composed of short and slender struts

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Abstract

Results of a parametric investigation on the failure of metallic strut lattices subjected to multi-axial loads are presented. The study involves two microscopic parameters related to the geometry of struts: strut-level strengthening and slenderness ratios. The strengthening procedure is designed such that minimum-strengthening represents the octettruss while maximum-strengthening represents the three-dimensional Warren truss. This way, the effects of both strut-level stretching and bending on the deformation responses together with coupled failure due to plastic yield and elastic buckling can be studied. The evaluated theoretical failure envelopes that include microscopic global and localized failure compare well with the numerical failure data obtained from finite element analysis. Among results, while the strengthening and slenderness ratios expectedly influence the sizes and shapes of the failure surfaces, they also dramatically alter microscopic deformation mechanisms leading to macroscopic failure.

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1. Introduction

Space-fillers have traditionally been used in architecture and civil engineering to construct large domes and shelters (e.g. Fuller and Applewhite, 1975; Yokoo et al., 1971), and in aerospace engineering to build outer-space systems such as orbiting spacecrafts and solar satellites (e.g. Davis, 1966). Space trusses are omni-directional in the way that they carry the applied loads and due to their polyhedra configurations, they mimic the manner in which natural materials are constructed at the atomic level. Typically, the truss cells possess either rotational or reflective symmetry which is a desirable characteristic for the development of mechanical models as it results in isotropic or orthotropic behavior. The new idea in materials engineering is to miniaturize space-filling systems to fit within the traditional materials length scales thereby creating materials whose properties can be directly manipulated to satisfy the ever-increasing demands of novel materials such as multi-functionality, smartness, and environmental responsibility.

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In structural engineering, the interest in strut lattices evolves from the desire to use them as structural materials in weight-efficient engineering systems as well as the need to save the material itself. Metallic foams and honeycombs are considered for such applications; but metallic trusses are now being pursued because they possess larger stiffness-to-density ratios (e.g. Wicks and Hutchinson, 2001; Evans et al., 2001). The main drawback of traditional cellular metals, particularly stochastic foams is that they are difficult to model because they are mainly anisotropic and exhibit wide scatter in their measured mechanical quantities due to foaming-induced material property gradients (Doyoyo and Mohr, in press), and are therefore not always ideal for reliable designs. On the other hand, due to a more direct way in which truss structures are manufactured, they may almost be immune from the data spread problem. However Zhou et al. (2004) and Sugimura (2004) have reported deviations in mechanical properties in practical truss lattices. Certainly with improvements on fabrication techniques, this problem can be minimized. Another advantage of strut lattices is that irrespective of the joint type, their methods of assembly are flexible and can range from scaffolding where the material is assembled in its actual location or form of usage or block manufacturing where the material is pre-assembled and then utilized. Manufacturing flexibility allows strut-based materials to be easily used in designs involving complex geometries while eliminating the problem of forming. For more details about some of the fabrication methods see Wallach and Gibson (2001), Sypeck and Wadley (2001) and Brittain et al. (2001).

In general, the deformation mode that dominates in a strut including stretching, bending, or twisting largely depends on its end or boundary conditions. Among what we have learnt about low-density metallic honeycombs is that energy absorption is sustained by the formation and propagation of plastic collapse bands in shell microstructures (Mohr and Doyoyo, 2004). For thin plates, the loading precursors of this type of failure mode are compression, bending, and twisting. That is, while stretching-dominated trusses are ideal for stiffness applications, energy-absorption can be enhanced by using strut systems that deform predominantly by bending and twisting. As for stiffness-based designs, bending and twisting deformations in addition to the stretching deformation at the strut level can increase the shear strength. Previous investigations have largely focused on stretching-dominated truss lattices. For example, the macroscopic stiffness of the octettruss has been derived from the micromechanical analysis of strut-level stretching (Nayfeh and Hefzy, 1978). Macroscopic failure of stretching-dominated truss structures has been investigated using strut-level analysis (Deshpande et al., 2001; Hutchinson et al., 2003) and homogenization (Mohr, 2005) revealing the complex nature of failure modes. The effective stiffness of open-cell foams has typically been approximated by analyzing the mechanical response of the tetrakaidecahedral cell whose effective deformation resistance depends largely on microscopic bending and twisting of constituting strut members (Warren and Kraynik, 1991; Zhu et al., 1997).

In this study, mechanical failure of strut-based structures that deform due to the combined strut-level stretching and bending is investigated. Critical parameters involving the geometry of struts such as strut-level strengthening and slenderness ratios are considered. Thus, we investigate the mechanical failure of strengthened strut-lattice materials composed of short and slender struts. The strengthening mechanism relies on connecting a cubic truss to an octettruss (Fuller, 1961) while gradually increasing the cross-sectional area of cubic struts until a regular three-dimensional Warren truss (Lake, 1992) is obtained. The cubic truss deforms by the combined stretching and bending of strut members (Janus-Michalska and Pecherski, 2003).

We view the strut-lattice material as a composite structure characterized by the microscopic strut-level deformations within closely-packed representative strut cells. The struts are perfectly straight with uniform cross-sections. The elastic constitutive model of the present material is readily derived by summing up microscopic axial and bending stiffnesses of strut members in the space-filling unit cell. Finite element analysis is used to observe the failure phenomenology and the theoretical failure surfaces are derived. The determined failure surfaces compare well with numerical failure data.

2. Analysis

Two strut-lattice plates that are composed of perfectly straight short and slender struts, are shown schematically in Fig. 1 with respect to the Cartesian co-ordinate system; \( i = X, Y, Z \). The lattice plates are constructed by the successive packing of a three-dimensional Warren truss that is composed of an octettruss connected to a cubic truss (see Fig. 2). The measure of shortness and slenderness of struts is established by comparing their slenderness ratios \( \lambda \) to the critical elastic buckling slenderness ratio \( \lambda_c \). (Aside: The slenderness ratio is given by
\[ \lambda = \frac{L_{\text{eff}}}{\sqrt{I/A}} \] where \( L_{\text{eff}}, I, \) and \( A \) are the effective strut length, strut’s moment of inertia, and strut’s cross-sectional area respectively. The effective length depends on the strut’s boundary and loading conditions. The critical buckling slenderness ratio is given by \( \lambda_s = \sqrt{\frac{E_s}{\sigma_s}} \) where \( E_s \) and \( \sigma_s \) are the strut’s Young’s modulus and longitudinal yield strength respectively.) Short struts have \( \lambda < \lambda_s \) and in general fail due to plastic yield, while slender struts have \( \lambda \geq \lambda_s \) and fail largely due to elastic buckling or due to plastic yield in stress regimes dominated by tensile loads. The perfectly straight struts in the spatially periodic medium are cylindrical in shape with uniform cross-sections of radii \( a_{\text{cub}} \) and \( a_{\text{oct}} \) for the cubic and octet lattices, respectively. Further, the cross-sectional area of struts in the cubic lattice is gradually increased until it is the same as that of the octet lattice. This defines the strengthening mechanism so that minimum-strengthening represents the octet-truss material while maximum-strengthening represents the three-dimensional Warren truss material. The strengthening ratio \( \eta \) is defined as the cross-sectional area of struts in the cubic lattice divided by the cross-sectional area of struts in the octet lattice

\[
\eta = \left( \frac{a_{\text{cub}}}{a_{\text{oct}}} \right)^2
\]  

Fig. 1. Truss plates composed of (a) short and (b) slender struts.

Fig. 2. The three-dimensional Warren truss composed of octet-truss and cubic truss.
In this paper, we only consider strengthening ratios equal to $\eta = 0, 0.25, 0.5, 1$ for the short strutted truss. Four different short strutted beams possessing the four different strengthening ratios are shown in Fig. 3. For the combined slender and short strutted lattice, only the strengthening parameters equal to $\eta = 0, 0.25, 1$ are investigated. Fig. 4 illustrates the detailed schematics of the present strut unit cell showing its octet and cubic parts. The octet struts are assumed to be pin-jointed at points labeled $O'$ and $P$. The cubic struts are connected to the octet lattice at pin-joint $P$ while they are fixed at $O$ or the origin of the cubic lattice. The length $O'P$ of struts in the octet lattice is $L^{\text{oct}} = \sqrt{2}L$, while the length $OP$ of struts in the regular cubic lattice is $L^{\text{cub}} = L$. For the lattice plate composed of short struts, the aspect ratio of the octet struts is kept fixed at $L^{\text{oct}}/a^{\text{oct}} = 10$ while that of cubic struts is varied such that $L^{\text{cub}}/a^{\text{cub}} = 10/\sqrt{2}\eta$. All cubic struts, despite their low values of the strengthening ratios are considered short in terms of the critical slenderness ratio. As for the combined slender and short strutted lattice, the octet struts’ aspect ratio is fixed at $L^{\text{oct}}/a^{\text{oct}} = 25$ and that

Fig. 3. Truss beams made up of struts with different strengthening ratios for short strutted lattice.

Fig. 4. Schematic of octetruss and cubic truss showing geometric details of the truss lattice.
of cubic struts is varied as $L_{\text{cub}}/a_{\text{cub}} = 25/\sqrt{2\pi}$. It shall later be shown that these aspect ratios allow for three different lattice slenderness situations: slender octetruss, combined slender (octetruss) and short (cubic truss) lattice, and slender octet and cubic lattices.

Each strut is modeled as an elasto rigid-plastic material with the following constitutive law:

$$\varepsilon = \sigma/\sigma_s \quad \text{for} \quad \sigma \leq \sigma_s$$

$$\sigma = \sigma_s \quad \text{for} \quad \sigma > \sigma_s$$

(2a)

(2b)

where $\varepsilon_s$ is the yield strain of the solid struts. For the present strut lattice $\sigma_s = 170 \text{ MPa}$, and $\varepsilon_s = 0.00243$. The Young’s modulus $E_s$ and elastic Poisson’s ratio $\nu_s$ of the struts are $E_s = 70 \text{ GPa}$ and $\nu_s = 0.29$, respectively. The critical slenderness ratio for elastic buckling of this material is $\lambda_s \approx 64$. The mass density of the solid struts is $\rho_s = 2700 \text{ kg/m}^3$. The above properties can easily be adapted to any material that exhibits negligible strain hardening.

Conceptually, we view the strut lattice as a composite structure where a regular repeating microstructure or the smallest periodic unit containing the contribution of individual struts to the overall response is taken as the representative volume element. The macroscopic stress and strain tensors of the entire composite are taken as volume averages of the corresponding quantities of this element. That is, at the macroscopic scale, it is possible to treat the strut lattice as a continuous solid whose constitutive response is characterized by a mechanically equivalent unit cell with the same volume fraction as the bulk structure. The analytical methodology is established keeping in mind that the strut cell contains two constituting lattices: octahedral and cubic lattices. Due to their pin–pin connections, the octet struts can only support longitudinal loads and are free to rotate with no possibility of bending. For the cubic lattice, the pin–fixed connections allow cubic struts to support axial loads in addition to microscopic reaction moments and transverse loads. In what follows, the unidirectional microscopic axial and bending stiffnesses of the strut unit cell are presented. Note that for the octetruss, there are six $n = 6$ groups of parallel struts defining directions along which external loads are carried by the stretching of struts. Each group of struts forms a unidirectional continuum that possesses only one non-zero equivalent stretching stiffness in the local longitudinal direction. For a pinned linear elastic strut of length $L$, the microscopic equivalent stretching modulus $(c_{\text{St}}^{\text{oct}})_n$ obtained by projecting the axial stiffness to the local longitudinal direction of the strut is then given by

$$(c_{\text{St}}^{\text{oct}})_n = \frac{2\pi E_s}{\sqrt{2}} \left(\frac{a_{\text{oct}}}{L_{\text{oct}}}\right)^2$$

The subscript “St” denotes the stretching deformation mode. For the cubic truss, there are two $n = 2$ groups of parallel directions that carry both axial and bending loads. Similarly as for the octetruss, for the $n$th group of struts, the microscopic equivalent stretching modulus $(c_{\text{St}}^{\text{cub}})_n$ of the cubic truss is given by

$$(c_{\text{St}}^{\text{cub}})_n = 2\pi E_s \eta \left(\frac{a_{\text{oct}}}{L_{\text{oct}}}\right)^2$$

(4)

The microscopic bending stiffness of groups of parallel directions in the cubic lattice is composed of two components, namely; (1) bending stiffness due to struts that deform like Bernoulli–Euler beams where deflections depend on bending moments and (2) bending stiffness due to struts that behave like Timoshenko beams whose deformation depends on transverse loads. For a built-in linear elastic strut of length $L$ subjected to a transverse force $T$, the transverse displacement $v$ is given as $v = (\tilde{l}^2/3EI + kL/GA)T$, where $k$ and $G$ are the cross-section form factor and shear modulus, respectively. Here $(\tilde{l}^4/3EI + kL/GA)^{-1}$ is the bending stiffness. Projecting the bending stiffness to the local direction and for the $n$th group of struts, it follows that the equivalent microscopic bending modulus $(c_{\text{Be}}^{\text{cub}})_n$ is given by:

$$(c_{\text{Be}}^{\text{cub}})_n = \left[\frac{1}{3E_s \pi \eta^2} \left(\frac{L_{\text{oct}}}{a_{\text{oct}}}\right)^4 + \frac{k(1 + \nu_s)}{E_s \pi \eta} \left(\frac{L_{\text{oct}}}{a_{\text{oct}}}\right)^2\right]^{-1}$$

(5)
The macroscopic linear elastic constitutive equation for a strut lattice is then given as follows:

\[ \rho_t = \rho_s \sum_n v_n \]  

(6)

where \( v_n \) is the volume fraction of all groups of parallel struts. The above equation can be simplified by noting that the unidirectional equivalent stretching modulus is a volume average of moduli of all groups of parallel struts or \( (c_{\text{Si}})_n = E_0 v_n \). Thus, for the present strut lattice, we can write

\[ \rho_t = \frac{\rho_s}{E_s} \sum_n (c_{\text{St}})^{\text{oct}}_n + \frac{\rho_s}{E_s} \sum_n (c_{\text{St}})^{\text{cub}}_n \]  

(7)

recalling that the octetruss has six groups of parallel members while the cubic truss has two such groups, the above equation can be solved to obtain the relative mass density \( \rho^* = \rho_t/\rho_s \)

\[ \rho^* = 6\pi(\sqrt{2} + \eta) \left( \frac{d^{\text{oct}}}{L^{\text{oct}}} \right)^2 \]  

(8)

Macroscopic stiffness for a strut lattice is obtained by transforming a summed contribution of microscopic unidirectional stiffnesses of all constituting struts in the space-filling truss system with respect to the global Cartesian coordinate system. For the octetruss, the macroscopic stiffness matrix \( (c_{\text{St}})^{\text{oct}}_{ijkl} \) is obtained by transforming and adding up microscopic stiffnesses of each of the six groups of parallel members

\[ (c_{\text{St}})^{\text{oct}}_{ijkl} = \sum_n (c_{\text{St}})^{\text{oct}}_n (x_{ij} x_{k} x_{l})_n \]  

(9)

Here, \( x_i \) are the direction cosines defining the orientation of the longitudinal axis of the struts relative to each of the global coordinate axes \((X, Y, Z)\). Similarly, the macroscopic stiffness matrix for the cubic truss lattice \( (c_{\text{cub}})^{\text{ijkl}} \) can be obtained with its stretching and bending contributions, respectively

\[ (c_{\text{cub}})^{\text{ijkl}} = \sum_n (c_{\text{cub}})_n (x_{ij} x_{k} x_{l})_n \]  

(10a)

\[ (c_{\text{Be}})^{\text{ijkl}} = \sum_n (c_{\text{Be}})_n (y_{ij} y_{k} y_{l})_n \]  

(10b)

Here, \( y_i \) are the direction cosines between the transverse axis of the struts and each of the global coordinate axes. The macroscopic linear elastic constitutive equation for a strut lattice is then given as follows:

\[ \Sigma_{ij} = \left[ (c_{\text{St}})^{\text{oct}}_{ijkl} + (c_{\text{cub}})^{\text{ijkl}} + (c_{\text{Be}})^{\text{ijkl}} \right] E_{kl} \]  

(11a)

\[ E_{ij} = \left[ (c_{\text{St}})^{\text{oct}}_{ijkl} + (c_{\text{cub}})^{\text{ijkl}} + (c_{\text{Be}})^{\text{ijkl}} \right]^{-1} \Sigma_{kl} \]  

(11b)

where \( \Sigma_{ij} \) and \( E_{kl} \) are the second-order macroscopic stress and strain tensors of the strut lattice respectively. The fourth order tensors \( (c_{\text{St}})^{\text{ijkl}}_{ijkl} + (c_{\text{cub}})^{\text{ijkl}}_{ijkl} + (c_{\text{Be}})^{\text{ijkl}}_{ijkl} = \hat{c}_{ijkl} \) and \( [(c_{\text{St}})^{\text{ijkl}}_{ijkl} + (c_{\text{cub}})^{\text{ijkl}}_{ijkl} + (c_{\text{Be}})^{\text{ijkl}}_{ijkl}]^{-1} = \hat{s}_{ijkl} \) are the stiffness and compliance matrices, respectively. For the present analysis, it is convenient to use a compact form which takes advantage of cubic symmetry arguments. That is, the stiffness matrix \( \hat{c}_{ijkl} \) which has 81 independent constants is reduced into a compact stiffness matrix \( \hat{C}_{ij} \) with 21 independent constants, so that

\[ \Sigma_i = \hat{C}_{ij} E_j \]  

(12)

where \( \Sigma_i \) and \( E_j \) are the independent six components of stress and strain. We can also write \( E_i = \hat{S}_{ij} \Sigma_j \), where \( \hat{S}_{ij} = \hat{C}_{ij}^{-1} \) is the compact compliance tensor. Particularly, each pair of the subscripts in the tensor equations can be substituted with a single subscript in the compact form such that \( 1 \leftrightarrow 11 \) \( 2 \leftrightarrow 22 \) \( 3 \leftrightarrow 33 \) \( 4 \leftrightarrow 23 \) \( 5 \leftrightarrow 13 \) \( 6 \leftrightarrow 12 \). For instance, the coefficient for the shearing stiffness is given as

\[ (\hat{C}_{\text{St}})_{14} = (\hat{C}_{\text{St}})_{2323} = \sum_n (c_{\text{St}})_n (x_2 x_3 x_2 x_3)_n \]  

(13)
We recognize that for the present material, the stiffness matrix $C_{ij}$ contains only three independent coefficients. Solving the above stiffness equations, we obtain for our strut lattice

$$
\begin{align*}
\Sigma_{XY} & \begin{bmatrix} AB + C & A & A & 0 & 0 & 0 \\
A & AB + C & A & 0 & 0 & 0 \\
A & A & AB + C & 0 & 0 & 0 \\
0 & 0 & 0 & A + C & 0 & 0 \\
0 & 0 & 0 & 0 & A + C & 0 \\
0 & 0 & 0 & 0 & 0 & A + C
\end{bmatrix}
\end{align*}
$$

where the quantities $A$, $B$, and $C$ are given as follows:

$$
\begin{align}
A &= \frac{\pi E_s}{\sqrt{2}} \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 \\
B &= 2 + 2\sqrt{2} \eta \\
C &= 2\pi E_s \left[ \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^4 + \frac{k(1 + \nu_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right]^{-1}
\end{align}
$$

The compliance matrix $\tilde{S}_{ij}$ which is obtained as the inverse of the stiffness matrix is given by

$$
\tilde{S}_{ij} = \begin{bmatrix}
\tilde{S}_{11} & -\tilde{S}_{12} & -\tilde{S}_{12} & 0 & 0 & 0 \\
-\tilde{S}_{12} & \tilde{S}_{11} & -\tilde{S}_{12} & 0 & 0 & 0 \\
-\tilde{S}_{12} & -\tilde{S}_{12} & \tilde{S}_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{S}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{S}_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{S}_{44}
\end{bmatrix}
$$

where longitudinal ($\tilde{S}_{11}$, $\tilde{S}_{12}$) and shear $\tilde{S}_{12}$ compliances are given by

$$
\begin{align}
S_{11} &= \frac{A + AB + C}{C^2 + AC + 2ABC + A^2B^2 + A^2B - 2A^2} \\
S_{12} &= \frac{A}{C^2 + AC + 2ABC + A^2B^2 + A^2B - 2A^2} \\
S_{44} &= \frac{2}{A + C}
\end{align}
$$

This completes our derivation of the elastic constitutive model of the present strut material. In the next section, the failure phenomenology of the material is determined.

3. Failure phenomenology

The failure of the strut lattice is governed by either plastic yield or elastic buckling at the level of struts or the microscopic yield and elastic buckling within the truss lattice depending on strengthening and slenderness ratios of constituting struts. Due to the omni-directional nature of the strut-based structure, we can define a failure criterion such that failure occurs if any strut or groups of struts within the lattice fail(s). When initial failure occurs, the load–displacement curve will deviate from linearity in the present case where struts deform in a linear elastic manner during the early deformation stages. In what follows, we use two different approaches to analyze the failure phenomenology: (1) we utilize finite element analysis to observe the strut-level failure mechanisms; (2) we derive the theoretical failure envelopes using micromechanical analysis at the level of struts.
3.1. Finite element observations

The finite element code used in the study is ABAQUS/standard 6.4-1 (HKS, 2003). Some of the features of the finite element model we employed are: (a) spatial discretization of struts with one three-noded quadratic beam element; (b) beam material represented by an elasto rigid plastic constitutive model in Eq. (2); (c) with the exception of the loading direction, all degrees of freedom are constrained at the vertices to prevent rigid body rotation of the truss lattice; (d) quasi-static proportional displacements with 200 static steps are applied at the restrained vertices; and (e) the following loading directions were investigated: $A_{XX}/A_{YY}$, $A_{XY}/A_{YY} = 0$, ±0.1, ±0.5, ±0.7, ±1, ±1.5, ±2, ±2.4, ∞, where $A_{XX}$, $A_{YY}$, $A_{XY}$ denote the applied longitudinal and shear displacements corresponding to the resulting normal and transverse forces $P_{XX}$, $P_{YY}$, $P_{XY}$, respectively. Our choice of biaxial loading is guided by the fact that the strut unit cell possesses cubic symmetry. That is, it is enough to consider only two stress spaces, namely normal–normal stress interactions: $\Sigma_{XX} - \Sigma_{YY} \equiv \Sigma_{XY} - \Sigma_{ZZ} \equiv \Sigma_{XX} = \Sigma_{ZZ} = 0$ and shear-normal stress interactions: $\Sigma_{XY} = 0$. Next, we summarize some of the finite element observations for three different cases of lattice slenderness, namely: short strutted lattice (both octet and cubic struts are short); slender and short strutted lattice (octet struts are slender while cubic struts are short); and slender strutted lattice (both octet and cubic struts are slender).

3.1.1. Short strutted lattice

Longitudinal $\hat{C}_{11}$ and shear $\hat{C}_{44}$ moduli obtained from finite element analysis and the constitutive theory of Eq. (14) are compared in Table 1 for different values of the strengthening ratio in the case of the short strutted lattice. It can be seen from the table that the finite element moduli as functions of $\eta$ are rather equal to those predicted by the theory. This fully verifies our theoretical constitutive model presented in the previous section. Fig. 5 illustrates plots of the Mises stress* during cubic failure of the short strutted lattice under biaxial longitudinal loading. (Aside: The Mises stress distribution in struts is better viewed in color. The color red denotes the case when the Mises stress within the strut is greater than the yield strength or $\sigma \geq \sigma_y$. For the black and white version, the yielded cubic struts are labeled with a single hatch “/” while the yielded octet struts are labeled with double hatch “//”.) Initial failure of the strut lattice at different loading directions is characterized by the plastic yield of some pairs of cubic struts while other cubic struts and all octet struts continue to deform elastically. This is so, except for the special case of equi compression–compression and equi tension–tension loading or $A_{XX}/A_{YY} = 1$; where pairs of cubic and octet struts lying in the $XY$ plane undergo plastic yield at the same time. For the equi tension–compression and equi compression–tension loading or $A_{XX}/A_{YY} = -1$; the two pairs of cubic struts orientated along the $X$- and $Y$-directions undergo plastic yield while none of the octet struts yield. For unidirectional loading or $A_{XX}/A_{YY} = 0$, ∞, only a pair of cubic struts parallel to the loading direction yields. When the applied loads are increased, yield of the octet struts takes place. For equi compression–compression and equi tension–tension loading, pairs of octet struts lying along the $XY$ plane undergo plastic yield as already established above. For equi tension–compression and equi compression–tension loading; pairs of octet struts lying along the $ZX$ and $ZY$ planes undergo plastic yield. For general longitudinal loading directions or $A_{XX}/A_{YY} \neq \pm 1, 0, \infty$; only a pair of cubic or octet struts undergo plastic yield depending on the dominance of either $A_{XX}$ or $A_{YY}$ displacements. In the case of shear-normal loading

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Theory $\hat{C}_{11}$ (GPa)</th>
<th>FEA $\hat{C}_{11}$ (GPa)</th>
<th>Theory $\hat{C}_{44}$ (GPa)</th>
<th>FEA $\hat{C}_{44}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.07</td>
<td>2.05</td>
<td>1.56</td>
<td>1.59</td>
</tr>
<tr>
<td>0.25</td>
<td>3.38</td>
<td>3.41</td>
<td>1.56</td>
<td>1.51</td>
</tr>
<tr>
<td>0.5</td>
<td>4.64</td>
<td>4.68</td>
<td>1.57</td>
<td>1.52</td>
</tr>
<tr>
<td>1</td>
<td>7.11</td>
<td>7.16</td>
<td>1.62</td>
<td>1.54</td>
</tr>
</tbody>
</table>
or $\Delta_{XY}/\Delta_{YY}$, we observe that pairs of octet and cubic struts yield at the same time with each specific mode depending on the dominance of either longitudinal or transverse displacements as illustrated in Fig. 6.

### 3.1.2. Slender and short strutted lattice

This is the case when the octet struts are slender while the cubic struts are short as defined by the critical slenderness ratio for elastic buckling. The Mises plots during failure are presented in Fig. 7(a) and (b) under biaxial longitudinal loading and shear-longitudinal loading, respectively. Only two cases of the strengthening ratio are considered, namely: minimum ($\eta = 0$) and maximum ($\eta = 1$) strengthening. For the case of minimum-strengthening under biaxial longitudinal loading, global elastic buckling of pair(s) of octet struts occur under compression–compression and tension–compression loading, while plastic yield of pair(s) of octet struts occur under tension–tension loading with the exception of a small tension–tension regime in the neighborhood of unidirectional loading (global buckling will be labeled “i” in general; see Fig. 7(a) for this particular case). For the maximum-strengthened lattice under longitudinal biaxial loading, global elastic buckling of pair(s) of octet struts occurs under compression–compression and tension–compression loading followed by the plastic yield of cubic struts (see Fig. 7(a)). A special case is that of equi compression–compression loading where both the octet and cubic struts buckle and yield at the same time. In the case of tension–tension loading, only plastic yielding of either octet or cubic struts is observed. For minimum-strengthening under shear-normal loading, local elastic buckling of pair(s) of octet struts occur under shear-compressive loading, while plastic yield of octet struts occur under mainly shear-tensile loading (local buckling will in general be labeled “ii”, see Fig. 7(b) for this case). However, under simple shear loading and a small loading regime in the neighborhood of simple shear, global buckling of a pair of octet struts occurs (see Fig. 6). Under shear-longitudinal loading
for the maximum strengthening case, the failure modes are the same as those of the minimum-strengthening case.

3.1.3. Slender strutted lattice

This is the case when both cubic and octet struts are slender while also possessing equal elastic buckling stresses and a strengthening ratio equal to $\eta = 0.25$. Their failure modes under biaxial longitudinal loading are shown in Fig. 7(c) through the Mises plots of the deformed lattice. Under biaxial longitudinal loading, global buckling of cubic struts occur under compression–compression loading and a larger part of the tension–compression loading regimes. However, plastic yield of cubic struts occur under tension–tension loading and a small part of the tension–compression loading regimes. A special case of global buckling occurs under equi compression–compression loading when both octet and cubic struts undergo global buckling at the same time (see the case $\Delta_{XY}/\Delta_{YY} = 1$ in Fig. 7(c)). As for the shear-longitudinal loading case, plastic yield of a combination of pairs of octet and cubic struts occur and no buckling is observed for this case.
3.2. Theoretical failure envelopes

The finite element observations of the previous section illustrate the complexities of the failure modes. In what follows, we evaluate the theoretical failure envelopes under biaxial longitudinal loading and longitudinal-shear loading for short and slender strutted lattices. Please note a basic assumption about the failure of cubic struts: if the struts were considered thick, failure of the cubic struts could occur at O due to the combined effects of shear stresses and reaction moments. However, for thinner struts that characterize the responses of practical sandwich strut cores, then this type of failure can be neglected without loss of accuracy. However, our failure analysis will be pursued keeping this in mind. The failure phenomenology is described in terms of the local longitudinal strain acting on the strut as follows: for the short-strutted lattice, if the local

---

Fig. 7. Buckling of the truss lattice: (a) global buckling (denoted by “i”) of octet struts for slender (octet) and short (cubic) strutted lattice followed by cubic yield (denoted as “j”) under longitudinal biaxial loading; (b) local buckling (denoted by “ii”) of octet struts for slender (octet) and short (cubic) strutted lattice under shear-longitudinal loading followed by cubic yield; and (c) global buckling of cubic and octet struts for slender strutted lattice under longitudinal biaxial loading; no buckling is observed at all for shear-normal loading in this case.
longitudinal strain in any pair(s) of cubic struts is less than the yield strain of the strut or \( \varepsilon_{\text{cub}} < \varepsilon_s \) then no failure occurs, and the constitutive model for the truss lattice is given by Eq. (14) as \( \Sigma_i = \hat{C}_{ij}E_j \). On the other hand, if the longitudinal strain in any pair(s) of cubic struts is \( \varepsilon_{\text{cub}} = \varepsilon_s \), then cubic failure occurs and the macroscopic failure stress field \( \Sigma_i^{(\text{cub})} \) at this point is given by

\[
\Sigma_i^{(\text{cub})} = \hat{C}_{ij}E_j^{(\text{cub})}
\]

where \( E_j^{(\text{cub})} \) is the macroscopic strain field at the cubic failure point. For the post-cubic yield case or \( \varepsilon_{\text{cub}} > \varepsilon_s \), we assume that the constitutive model takes the form

\[
\Sigma_i = \Sigma_i^{(\text{cub})} + \hat{C}_{ij}E_j^{(\text{oct})}
\]

where \( \Sigma_i^{(\text{cub})} \) is the macroscopic stress field representing the plastic and elastic* contribution of cubic struts to the overall macroscopic stress in the post-cubic failure case and \( \hat{C}_{ij}^{\text{oct}} \) is the stiffness of the octetruss. (*Note that not all cubic struts yield at the same time at all times, so that while some cubic struts behave plastically others behave elastically.) This assumed failure phenomenology relies heavily on the assumption of elastic rigid-plastic behavior at the strut level. At the same time, the tensor \( \Sigma_i^{(\text{cub})} \) is not easily determined for general loading situations. It can be obtained for some specific loading conditions where the yield point can be readily evaluated such as: (1) equibiaxial longitudinal loading (all cubic struts lying along the \( XY \) plane yield), (2) unidirectional loading (a pair of cubic struts parallel to the unidirectional load yield), (3) and simple shear loading (all octet and cubic struts lying along the shear plane yield). Thus, for these three cases, we will assume that \( \Sigma_i^{(\text{cub})} \approx \Sigma_i^{(\text{cub})} \). Octet failure occurs when the longitudinal strain in any pair(s) of the octet struts is equal to the yield strain of the strut or \( \varepsilon_{\text{oct}} = \varepsilon_s \), then the macroscopic failure stress field \( \Sigma_i^{(\text{oct})} \) at this point is given by

\[
\Sigma_i^{(\text{oct})} = \Sigma_i^{(\text{cub})} + \hat{C}_{ij}E_j^{(\text{oct})}
\]

where \( E_j^{(\text{oct})} \) is the macroscopic strain field at the octet failure point. When \( \varepsilon_{\text{oct}} > \varepsilon_s \), most struts in the lattice would have undergone plastic yield, eventually leading to a non-linear mechanical response of the whole lattice. In this paper, we do not analyze such a case; although it would have to be evaluated to understand the energy absorption capabilities of the present strut lattice. As for the slender and short strutted lattice, we employ the same failure phenomenology as described above with plastic yielding replaced by elastic buckling. The above discussions define our approach to evaluate the theoretical failure envelopes for the strut lattice.

We will determine failure strains under a prescribed stress state. Since we are only interested in plane stress failure surfaces, we will only determine yield points under equibiaxial longitudinal loading, simple shear loading, and unidirectional loading. The failure surfaces should then pass through these points. That is, for a given applied biaxial force field (combinations of \( P_{XX}, P_{YY}, P_{XY} \)), the problem reduces into determining the failure displacements, namely \( A_{XX}, A_{YY}, A_{XZ}, A_{YX}, A_{YZ} \) and hence the macroscopic failure strains \( E_{ij}^f = A_{ij}^f/L, i, j = X, Y, Z \) or \( E_{XX}, E_{YY}, E_{XX}, E_{YZ}, E_{XY}, E_{YZ} \) that are then substituted into the constitutive model in Eq. (18) to obtain the corresponding macroscopic failure stresses.

### 3.2.1. Short strutted lattice

The relations between force and displacement for given degrees of freedom (DOF) along the global axes can be obtained using Maxwell's reciprocal theorem (Maxwell, 1864) and the transformation tensor. For the three DOF involving only longitudinal displacements, it can be shown that the force–displacement relation for the combined octetruss and cubic truss is given by

\[
\begin{bmatrix}
E_A^f a^f
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 1
1 & 2 & 1
1 & 1 & 2
\end{bmatrix}
\frac{1}{L^f}
\begin{bmatrix}
A_{XX}
A_{YY}
A_{ZZ}
\end{bmatrix}
\begin{bmatrix}
P_{XX}
P_{YY}
P_{ZZ}
\end{bmatrix}= \begin{bmatrix}
P_{XX}
P_{YY}
P_{ZZ}
\end{bmatrix}
\]

We now proceed to solve for the failure displacements and hence macroscopic failure strains for equi tension–tension or equi compression–compression case \( (P_{XX}/P_{YY} = 1) \); equi tension–compression or equi compression–tension case \( (P_{XX}/P_{YY} = -1) \); and unidirectional case \( (P_{XX}/P_{YY} = 0, \infty) \). For the present problem involving plane stress loading conditions with \( P_{ZZ} = 0 \), Eq. (19) becomes
\begin{align}
(2 + \sqrt{2}\eta)A_{XX} + A_{YY} + A_{ZZ} &= \frac{P_{XX}L_{\text{oct}}}{E_{v}A_{\text{oct}}} \tag{20a} \\
A_{XX} + (2 + \sqrt{2}\eta)A_{YY} + A_{ZZ} &= \frac{P_{YY}L_{\text{oct}}}{E_{v}A_{\text{oct}}} \tag{20b} \\
A_{XX} + A_{YY} + (2 + \sqrt{2}\eta)A_{ZZ} &= 0 \tag{20c}
\end{align}

For the equi tension–tension case or \( P_{XX}/P_{YY} = 1 \), we obtain the following displacement relations by solving Eq. (20): \( \Delta_{XX}/\Delta_{YY} = 1 \) and \( \Delta_{ZZ} = -\Delta_{XX}/(1 + \sqrt{2}\eta/2) \). The applied biaxial force field at yield \( P_{XX}^{f} = P_{YY}^{f} = P_{s} \), results in the corresponding yield displacements \( \Delta_{XX}^{f} = \Delta_{YY}^{f} = \Delta_{s} = \varepsilon_{s}L = \sigma_{s}E_{s} \), and \( \Delta_{ZZ}^{f} = -\Delta_{s}/(1 + \sqrt{2}\eta/2) \). Therefore, the macroscopic strains at yield for the case \( \Delta_{XX}/\Delta_{YY} = 1 \) are \( \varepsilon_{XX}^{f} = \varepsilon_{YY}^{f} = \varepsilon_{s} \), and \( \varepsilon_{ZZ}^{f} = -\varepsilon_{s}/(1 + \sqrt{2}\eta/2) \). Substituting the above strains in the constitutive relation in Eq. (18b), we obtain the yield point A (see Fig. 8) under equi tension–tension or equi compression–compression loading

\[
\Sigma_{XX}^{f} = \Sigma_{YY}^{f} = \left[ \frac{\pi}{\sqrt{2}} \left( 3 - \frac{2}{2 + \sqrt{2}\eta} \right) \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^{2} + 2\pi\eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^{2} + 2\pi \left( \frac{1}{3\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^{4} + \frac{k(1 + \varepsilon_{s})}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^{2} \right) \right]^{2} \sigma_{s} \tag{21}
\]

Note in Fig. 8 that the cubic and octet failure surfaces intersect at A as expected. (Recall that this loading condition denotes the case when pairs of octet and cubic struts parallel to the \( XY \) plane undergo plastic yield at the same time). For the case of equi tension–compression or equi compression–tension loading, that is \( P_{XX}/P_{YY} = -1 \), we obtain the following displacement relations by solving Eq. (20): \( \Delta_{XX} = -\Delta_{YY} \) and \( \Delta_{ZZ} = 0 \). The applied biaxial force field at yield, \( P_{XX}^{f} = -P_{YY}^{f} = P_{s} \), results in the corresponding yield

![Fig. 8. A representative graph of typical failure modes for the strut lattice showing different intersection points lying along specific loading directions, namely; \( \Sigma_{XX}/\Sigma_{YY} = \pm 1,0,\infty \). The dashed surface denotes \textit{cubic failure envelope} while the solid line denotes \textit{octet failure envelope}.](image-url)
displacements \( A_{XX}^t = -A_{YY}^t = A_s = \varepsilon_s L \) and \( A_{ZZ}^t = 0 \), so that the failure strains are \( E_{XX}^t = \varepsilon_s \), \( E_{YY}^t = -\varepsilon_s \), and \( E_{ZZ}^t = 0 \). Substituting the above strains into the constitutive equation in Eq. (18b), we obtain the cubic yield point \( B_{\text{cub}} \) (see Fig. 8) along \( P_{XX}/P_{YY} = -1 \)

\[
\Sigma_{XX}^B(\text{cub}) = -\Sigma_{YY}^B(\text{cub}) = \left[ \frac{\pi}{\sqrt{2}} \left( \frac{a_{\text{cub}}}{L_{\text{cub}}} \right)^2 + 2\pi\eta \left( \frac{a_{\text{cub}}}{L_{\text{cub}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{cub}}}{a_{\text{cub}}} \right)^4 + \frac{k(1 + \nu_s)}{\eta} \left( \frac{L_{\text{cub}}}{a_{\text{cub}}} \right)^2 \right) \right] \sigma_s \tag{22}
\]

Larger forces have to be applied for the octet struts to undergo plastic yield along \( P_{XX}/P_{YY} = -1 \). From compatibility, the relations \( A_{XX} = -A_{YY} \) and \( A_{ZZ} = 0 \) still hold in the post-cubic yield case. If \( A_{\text{cub}} \) represents the total local displacement acting on the octet strut, then it can easily be shown from geometry arguments that \( A_{\text{cub}} = A_{XX}^t = A_{YY}^t = 2\varepsilon_s L = 2A_s \) and \( A_{ZZ}^t = 0 \). That is, the applied force field \( P_{XX}^t = P_{YY}^t = P_{\text{cub}}^t + P_{\text{oct}}^t \) when the octet struts yield generates macroscopic yield strains given by:

\[
E_{XX}^t = -E_{YY}^t = 2\varepsilon_s \quad \text{and} \quad E_{ZZ}^t = 0. \tag{23}
\]

Substituting these failure strains in the post-cubic yield constitutive model of Eq. (18c), we obtain the octet yield point \( C_{\text{oct}} \) (see Fig. 8) along \( P_{XX}/P_{YY} = -1 \)

\[
\Sigma_{XX}^C = \left[ \frac{2\pi}{\sqrt{2}} \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi\eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^4 + \frac{k(1 + \nu_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right) \right] \sigma_s \tag{24}
\]

For unidirectional loading, for example \( P_{XX}/P_{YY} = 0 \) and \( P_{ZZ} = 0 \), Eq. (19) becomes

\[
\begin{align*}
(2 + \sqrt{2}\eta)A_{XX} + A_{YY} + A_{ZZ} &= 0 \tag{24a} \\
A_{XX} + (2 + \sqrt{2}\eta)A_{YY} + A_{ZZ} &= \frac{P_{YY}L_{\text{oct}}}{E_s A_{\text{oct}}} \tag{24b} \\
A_{XX} + A_{YY} + (2 + \sqrt{2}\eta)A_{ZZ} &= 0 \tag{24c}
\end{align*}
\]

Solving the above equation for displacements, we obtain \( A_{XX} = A_{ZZ} \) and \( A_{YY} = -(3 + \sqrt{2}\eta)A_{XX} \). Using arguments similar to the equi tension–compression case, a unidirectional load at yield \( P_{YY}^t = P_s \) results in the following displacement field: \( A_{YY}^t = 3A_s \) and \( A_{XX}^t = A_{ZZ}^t = -3A_s/(3 + \sqrt{2}\eta) \) at the point of octet yield; so that the corresponding macroscopic failure strains for this unidirectional loading case are \( E_{XX}^t = 3\varepsilon_s \) and \( E_{ZZ}^t = E_{YY}^t = -3\varepsilon_s/(3 + \sqrt{2}\eta) \). Substituting the above strains in the post-cubic yield constitutive model of Eq. (18c), we obtain the unidirectional octet yield point \( C \) (see Fig. 8)

\[
\Sigma_{YY}^C = \left[ \frac{2\pi}{\sqrt{2}} \left( 3 - \frac{2}{2 + \sqrt{2}\eta} \right) \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi\eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^4 + \frac{k(1 + \nu_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right) \right] \sigma_s \tag{25}
\]

Note that it can also be shown that \( \Sigma_{XX}^C = 0 \) and that the other unidirectional loading directions (including \( P_{XX}/P_{YY} = \infty \)) posses the same magnitude of the yield stress. Based on failure data collected from finite element analysis, we assume that the yield envelopes for the biaxial longitudinal loading case are enclosed by linear functions in stress space passing through the stress points evaluated in the previous paragraphs and also schematically presented in Fig. 8. We also introduce a new coordinate axis, namely the \( X'Y' \)-axis. This axis is orientated at \( \theta = \tan^{-1}(\Sigma_{YY}^t/\Sigma_{XX}^t) = \pi/4 \) relative to the \( XY \)-axis. The schematic shows that the yield envelopes for cubic and octet failure are symmetric with respect to the \( X'Y' \)-axis. The cubic yield envelope involving plastic yield of cubic struts is given by

\[
\frac{\Sigma_{XX}^t}{\sqrt{(\Sigma_{XX}^t)^2 + (\Sigma_{YY}^t)^2}} \pm \sqrt{\left( \Sigma_{XX}^C \right)^2 + (\Sigma_{YY}^C)^2} = \pm 1 \tag{26}
\]

The above equation could easily have been simplified by noting that \( \Sigma_{XX}^t = \Sigma_{YY}^t \) and \( \Sigma_{XX}^C = \Sigma_{YY}^C \) as presented in Eqs. (21) and (22), respectively. However, we decide not to simplify as the form of Eq. (26) could
be approximately applicable for some cases of minor cubic asymmetry of the truss lattice. We observe that the stresses in the \( X'Y' \) plane are related to those in the \( XY \) plane through the rotation angle \( \theta \) such that

\[
\begin{pmatrix}
\Sigma_{X'Y'}^\prime \\
\Sigma_{Y'Y'}^\prime
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\Sigma_{XX} \\
\Sigma_{YY}
\end{pmatrix}
\]

(27)

Therefore, the yield envelope for cubic failure in the \( X-Y \) plane is given by

\[
\frac{\Sigma_{XX} + \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^A)^2 + (\Sigma_{YY}^A)^2}} \pm \frac{\Sigma_{XX} - \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^B)^2 + (\Sigma_{YY}^B)^2}} = \pm \sqrt{2}
\]

(28)

The above equation denotes four different modes of initial failure in \( \Sigma_{YY} - \Sigma_{XX} \) stress space as shown schematically in Fig. 8. Several modes of failure are identified when the octet struts undergo plastic yield. The 1st and 3rd modes of octet failure shown in Fig. 8 are given by

\[
\frac{\Sigma_{XX} + \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^A)^2 + (\Sigma_{YY}^A)^2}} = \pm \sqrt{2}
\]

(29)

The 2nd and 4th modes for octet failure also shown schematically in Fig. 8 are given by

\[
\Sigma_{YY} = \frac{\Sigma_{YY}^B - \Sigma_{XX}^B}{\Sigma_{XX}^B} \Sigma_{XX} \pm \Sigma_{YY}^C
\]

\[
\Sigma_{XX} = \frac{\Sigma_{YY}^C - \Sigma_{XX}^C}{\Sigma_{XX}^C} \Sigma_{YY} \mp \Sigma_{XX}^C
\]

(30)

(31)

For the DOF involving a state of only transverse displacements, it can be shown by applying Maxwell’s reciprocal theorem and the transformation matrix that the force–displacement relations are given by

\[
E_s \begin{bmatrix}
A_x^\text{oct} \\
L_x^\text{oct}
\end{bmatrix} + \left\{ \frac{2k(1 + \nu_s)L_x^\text{oct}}{\sqrt{2}A_x^\text{oct} L_x^\text{oct}} + \frac{2\pi L_x^\text{oct}^3}{3\sqrt{2}\eta^2(A_x^\text{oct})^2} \right\}^{-1} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\Delta_{XY} \\
\Delta_{XZ} \\
\Delta_{YZ}
\end{bmatrix} = \begin{bmatrix}
P_{XY} \\
P_{XZ} \\
P_{YZ}
\end{bmatrix}
\]

(32)

For the case of shear loading in one direction, that is \( P_{XY}, P_{YZ} = P_{XZ} = 0 \), it follows from Eq. (32) that the only non-vanishing displacement is \( \Delta_{XY} \), while \( \Delta_{XZ} = \Delta_{YZ} = 0 \). Once again, suppose that \( A_x^\text{oct} \) denotes the total local displacement of the octet strut. From geometry arguments it follows that \( \Delta_{XY} = \sqrt{2} A_x^\text{oct} \) and the strain in the octet strut is given by \( \varepsilon_{XY}^\text{oct} = A_x^\text{oct} / \sqrt{2} L \). When yield of the octet strut occurs at \( \varepsilon_{XY}^\text{oct} = \varepsilon_s \), then the corresponding yield displacement in the octet strut is \( \Delta_{s} = \varepsilon_s \sqrt{2} L \). Thus the macroscopic transverse displacements at the yield point are \( \Delta_{XY} = 2\varepsilon_s L \), \( \Delta_{XZ} = \Delta_{YZ} = 0 \) and the macroscopic yield strains are \( E_{XY}^\text{oct} = 2\varepsilon_s \), \( E_{YZ}^\text{oct} = E_{XZ}^\text{oct} = 0 \). Substituting the above strains in the constitutive model of Eq. (18b), we obtain

\[
\Sigma_{XY}^C = \frac{2\pi}{\sqrt{2}} \left( \frac{L_x^\text{oct}}{A_x^\text{oct}} \right)^2 + 2\pi \left[ \frac{1}{\eta^4} \left( \frac{L_x^\text{oct}}{A_x^\text{oct}} \right)^4 + \frac{k(1 + \nu_s)}{\eta} \left( \frac{L_x^\text{oct}}{A_x^\text{oct}} \right)^2 \right]^{-1} \sigma_s
\]

(33)

From the behavior of the data in finite element simulation, we assume a parabolic yield envelope in \( \Sigma_{XY} - \Sigma_{XY}^C \) space, given by

\[
\Sigma_{XY} = \pm \frac{\Sigma_{XY}^C}{\Sigma_{YY}^C} (\Sigma_{XX} - \Sigma_{YY}^C) (\Sigma_{XX} + \Sigma_{YY}^C)
\]

(34)

Figs. 9–12 show the comparison of the above theoretical failure envelopes to the numerical failure data obtained from finite element analysis for the short-strutted lattice. Also shown in the figures are the schematics of the different modes of strut-level yield mechanisms, expectedly showing the dependence of the yield modes on the state of loading. Yielded struts are denoted by dashed lines. Also note that the general trend of yield failure is the same in the shear-normal stress space irrespective of the strengthening ratio. Observe that the
strengthening ratio does influence the shear strength: the shear strength increases by about a factor of 2 in the range $0 \leq \eta \leq 1$. However, in the same normal-shear space, it can be seen that the longitudinal strength

Fig. 9. Comparison of theoretical yield envelopes with numerical data for short-strutted lattice with $\eta = 0$: (a) in biaxial longitudinal loading space and (b) in shear-longitudinal loading space.
Fig. 10. Comparison of theoretical yield envelopes for octet and cubic failure with numerical data for short-strutted lattice with \( \eta = 0.25 \): (a) in biaxial longitudinal loading space and (b) in shear-longitudinal loading space.
Fig. 11. Comparison of theoretical yield envelopes for octet and cubic failure with numerical data for short-strutted lattice with $\eta = 0.5$: (a) in biaxial longitudinal loading space and (b) in shear-longitudinal loading space.
Fig. 12. Comparison of theoretical yield envelopes for octet and cubic failure with numerical data for short-strutted lattice with $\eta = 1$: (a) in biaxial longitudinal loading space and (b) in shear-longitudinal loading space.
remains almost the same over the range \(0 \leq \eta \leq 1\). In order to provide a better qualitative comparison, we recall that the relative density also increases with the strengthening ratio. That is, in effect, we can conclude that the strengthened configuration, though it might increase the shear properties, does in fact lower the longitudinal yield strength. In the case of biaxial longitudinal loading, we observe that both the cubic and octet yield envelopes are anisotropic, while they do in fact intersect along \(\Sigma_{XX} = \Sigma_{YY}\). Again, the magnitudes of the failure loads increase with the strengthening ratio.

### 3.2.2. Slender and short strutted lattices

Slender struts are expected to fail by elastic buckling in deformation regimes dominated by compressive loads. The critical Euler buckling load of the pinned octet struts is given by \(P_{E}^{\text{oct}} = \pi^2 E_s (a_{\text{oct}}^2 / L_{\text{oct}}^2)^2\), while those of cubic struts which are pinned at one end and then fixed at the other end is given by \(P_{E}^{\text{cub}} = 2\pi^2 E_s (a_{\text{cub}}^2 / L_{\text{cub}}^2)^2\). The corresponding critical elastic buckling stresses for the octet and cubic struts are given by \(\sigma_{E}^{\text{oct}} = \pi^2 E_s (a_{\text{oct}}^2 / 4L_{\text{oct}}^2)^2\) and \(\sigma_{E}^{\text{cub}} = \pi^2 E_s (a_{\text{cub}}^2 / L_{\text{oct}}^2)^2\), respectively. As discussed in the earlier sections, in order to investigate the slender strutted lattice, both the octet and cubic slenderness ratios should be selected such that \(\lambda_{\text{oct}} = 2L_{\text{oct}}^2 / a_{\text{oct}} \geq \lambda_s\) and \(\lambda_{\text{cub}} = L_{\text{oct}}^2 / a_{\text{cub}} \geq \lambda_s\). Recall that the critical slenderness ratio for the solid struts is \(\lambda_s = \sqrt{\pi^2 E_s / \sigma_s} = 64\). For the present struts, \(L_{\text{oct}} = 10\sqrt{2} \text{mm}\) and \(a_{\text{oct}} = 0.4 \text{mm}\), so that \(\lambda_{\text{oct}} \approx 71\) for \(\eta = 0\), \(\lambda_{\text{cub}} \approx 35\) for \(\eta = 1\), and \(\lambda_{\text{cub}} \approx 71\) for \(\eta = 0.25\). That is, it is expected that for the case \(\eta = 0\), octet struts will buckle; for the case \(\eta = 1\), octet struts will buckle while cubic struts will yield; and finally for the case \(\eta = 0.25\), both cubic and octet struts will buckle. Thus, we can classify two different states of lattice slenderness based on our choice of the strengthening ratios, namely: (1) combined slender and short strutted lattice, \(\eta = 1\); and (2) slender strutted lattice, \(\eta = 0.25\). Note that the octetruss is not left out as it represents the special case for either state (1) or (2); and its buckling surfaces can easily be obtained by removing the cubic truss. In what follows, we evaluate the buckling envelopes for the above two cases of lattice slenderness. Elastic buckling is a type of collapse that occurs below the yield point. That is, we can deduce the elastic buckling envelopes from the previous section by introducing the stress reduction factor \(R\). The stress reduction factors for octet and cubic yield are given as

\[
R_{\text{oct}} = \frac{\pi^2 (a_{\text{oct}}^2 / L_{\text{oct}}^2)^2 E_s}{4(L_{\text{oct}}^2)^2 \sigma_s} \quad (35)
\]

\[
R_{\text{cub}} = \frac{\pi^2 \eta (a_{\text{cub}}^2 / L_{\text{oct}}^2)^2 E_s}{(L_{\text{oct}}^2)^2 \sigma_s} \quad (36)
\]

It may be concluded from the above stress reduction factors that the octet struts are more susceptible to elastic buckling than cubic struts. For the case \(\eta = 1\), cubic yield is followed by octet buckling. We can deduce the octet buckling point for equi compression–compression loading by introducing the stress reduction factor into Eq. (21) obtaining

\[
\Sigma_{\text{XX}}^{4b} = \Sigma_{\text{YY}}^{4b} = \left[ \frac{\pi}{2} \left( 3 - \frac{2}{\sqrt{2} + 2\eta} \right) \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 + \frac{k(1 + v_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right) \right] R_{\text{oct}}^2 \sigma_s \quad (37)
\]

and the octet buckling point for equi compression–tension loading is similarly deduced from Eq. (23) as

\[
\Sigma_{\text{XX}}^{Bb(\text{oct})} = -\Sigma_{\text{YY}}^{Bb(\text{oct})} = \left[ \frac{2\pi}{\sqrt{2}} \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 R_{\text{oct}}^2 + 2\pi \eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 + \frac{k(1 + v_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right) \right] \sigma_s \quad (38)
\]

while the octet buckling stress for unidirectional compressive loading is also deduced from Eq. (25) by

\[
\Sigma_{\text{YY}}^{\text{CB}} = \left[ \frac{2\pi}{\sqrt{2}} \left( 3 - \frac{2}{\sqrt{2} + 2\eta} \right) \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \eta \left( \frac{a_{\text{oct}}}{L_{\text{oct}}} \right)^2 + 2\pi \left( \frac{1}{3\eta^2} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 + \frac{k(1 + v_s)}{\eta} \left( \frac{L_{\text{oct}}}{a_{\text{oct}}} \right)^2 \right) \right] \sigma_s \quad (39)
\]
From the linear trend of numerical failure data in finite element simulations, we assume linear yield interaction surfaces under compression–compression loading

$$\frac{\Sigma_{XX} + \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^{\text{sh}})^2 + (\Sigma_{YY}^{\text{sh}})^2}} = -\sqrt{2}$$  \hspace{1cm} (40)

while the elastic buckling modes in compression–tension stress space are given by

$$\Sigma_{YY} = \left(\frac{\Sigma_{YY}^{\text{sh}(\text{oct})} - \Sigma_{YY}^{\text{sh}(\text{cub})}}{\Sigma_{XX}^{\text{sh}(\text{cub})}}\right) \Sigma_{XX} \pm \Sigma_{YY}^{\text{sh}(\text{cub})}$$  \hspace{1cm} (41)

and

$$\Sigma_{XX} = \left(\frac{\Sigma_{YY}^{\text{sh}(\text{cub})} - \Sigma_{YY}^{\text{sh}(\text{oct})}}{\Sigma_{XX}^{\text{sh}(\text{cub})}}\right) \Sigma_{YY} \mp \Sigma_{YY}^{\text{sh}(\text{cub})}$$  \hspace{1cm} (42)

Similarly, the macroscopic octet buckling stress under shear loading is given by

$$\Sigma_{AX}^{\text{sh}(\text{oct})} = \left(\frac{2\pi}{\sqrt{2}} \left(\frac{a^{\text{oct}}}{L^{\text{oct}}}\right)^2 R^{\text{oct}} + \left[\frac{1}{3} \eta \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^4 + \frac{k(1 + \nu_s)}{\eta} \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^2\right]^{-1}\right) \sigma_s$$  \hspace{1cm} (43)

Under shear-compressive loading, the octet struts collapse by local buckling rather than global buckling as is the case under biaxial longitudinal loading. However, global buckling of the octet struts is observed under pure shear loading. Assuming linear yield interaction surfaces, we obtain the octet buckling envelopes in shear-normal stress space as

$$\Sigma_{YY} \pm \Sigma_{YY}^{\text{sh}(\text{oct})} = -1$$  \hspace{1cm} (44)

$$\Sigma_{YY} \pm \Sigma_{YY}^{\text{sh}(\text{oct})} = 1$$  \hspace{1cm} (45)

Two stress points are required in order to describe the elastic buckling failure of cubic truss. Slender cubic strusses undergo global elastic buckling under biaxial longitudinal loading while they fail by plastic yield under shear-normal loading. For equi compression–compression loading, the buckling point for the cubic struts is given by

$$\Sigma_{XX}^{\text{sh}(\text{cub})} = \Sigma_{YY}^{\text{sh}(\text{cub})} = \left[\frac{\pi}{\sqrt{2}} \left(3 - \frac{2}{2 + \sqrt{2}}\right) \left(\frac{a^{\text{oct}}}{L^{\text{oct}}}\right)^2 + 2\pi \eta \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^2 + 2\pi \left(\frac{1}{3} \eta^2 \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^4 + \frac{k(1 + \nu_s)}{\eta} \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^2\right)\right]^{-1} R^{\text{cub}} \sigma_s$$  \hspace{1cm} (46)

and the buckling point under equi tension–compression loading is given by

$$\Sigma_{XX}^{\text{sh}(\text{cub})} = -\Sigma_{YY}^{\text{sh}(\text{cub})} = \left[\frac{\pi}{\sqrt{2}} \left(\frac{a^{\text{oct}}}{L^{\text{oct}}}\right)^2 + 2\pi \eta \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^2 + 2\pi \left(\frac{1}{3} \eta^2 \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^4 + \frac{k(1 + \nu_s)}{\eta} \left(\frac{L^{\text{oct}}}{a^{\text{oct}}}\right)^2\right)\right]^{-1} R^{\text{cub}} \sigma_s$$  \hspace{1cm} (47)

The cubic buckling interaction curves are given by

$$\frac{\Sigma_{XX} + \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^{\text{sh}(\text{cub})})^2 + (\Sigma_{YY}^{\text{sh}(\text{cub})})^2}} \pm \frac{\Sigma_{XX} - \Sigma_{YY}}{\sqrt{(\Sigma_{XX}^{\text{sh}(\text{cub})})^2 + (\Sigma_{YY}^{\text{sh}(\text{cub})})^2}} = -\sqrt{2}$$  \hspace{1cm} (48)
The above theoretical buckling envelopes are compared with numerical finite element data in Figs. 13–15, showing a very close prediction. The schematics of local and global buckling modes at different loading conditions are illustrated in Figs. 13(a) and 13(b). Fig. 13(a) depicts global buckling under biaxial longitudinal loading, while Fig. 13(b) shows local and global buckling under shear-longitudinal loading.
conditions are also included in the figures. As observed in the yield case, the addition of the cubic truss does not necessarily change the overall failure modes particularly for shear-normal loading. For example, for

Fig. 14. Failure envelopes during cubic yield followed by octet buckling for the maximum strengthened truss lattice (dashed lines denote buckling, solid lines denote yield): (a) global buckling under biaxial longitudinal loading, (b) local and global buckling under shear-longitudinal loading.
minimum-strengthening, the octet struts failed by local buckling which is exactly the same failure mode observed in the maximum strengthening case that included the cubic truss. Also observe that for loading regimes dominated by the shear stress that global buckling occurs under combined normal and shear loads (see Fig. 13(b) and Fig. 14(b)).

4. Conclusions

A parametric investigation was conducted to understand the multi-axial failure of strut lattices composed of slender and short struts. Microscopic parameters including the newly-introduced strengthening ratio and the classical slenderness ratio at the strut level were studied for a three-dimensional Warren truss that can be partitioned into a stretching-dominated octetruss and a combined stretching and bending-dominated cubic truss. The theoretical failure envelopes were derived based on the micromechanics of the truss lattice at the level of struts. The theory was compared to numerical data from finite element analysis and a close comparison was obtained between the theory and numerical simulation. The failure surfaces that include plastic yield, local and global buckling are mainly linear in biaxial longitudinal loading and shear-normal loading spaces. The only exception is that of plastic yield under shear-normal loading where parabolic failure surfaces are observed. The results of this study are of significance in the design of strut lattices because both strengthening and slenderness ratios are key design parameters for the strut lattice.

Fig. 15. Failure envelopes during cubic buckling for the slender (octet and cubic struts) strutted case with \( \eta = 0.25 \) (dashed lines denote buckling, solid lines denote yield).
References


Doyoyo, M., Mohr, D., in press. Experimental determination of the mechanical effects of mass density gradients under multiaxial loading, Mechanics of Materials.


