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Global Existence of Regular Solutions for the Vlasov–Poisson–Fokker–Planck System

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We study the global existence and uniqueness of regular solutions to the Cauchy problem for the Vlasov–Poisson–Fokker–Planck system. Two existence theorems for regular solutions are given under slightly different initial conditions. One of them completely includes the results of P. Degond (1986, *Ann. Sci. Ecole Norm. Sup.* **19**, 519–542).

Key Words: kinetic theory; Vlasov plasma physics; Vlasov–Poisson–Fokker–Planck system; regularity.

1. INTRODUCTION

Plasma means completely ionized gases. The Vlasov–Poisson–Fokker–Planck system, we often say VPFP for short, appears in Vlasov plasma physics and stems from the Liouville equation coupled with the Poisson equation for determining the self-consistent electrostatic or gravitational forces (see [7, 11]).

In this paper, we consider the global existence and uniqueness of regular solutions to the Cauchy problem for the VPFP system. Let $f(x, v, t)$ describe the microscopic density of particles located at position $x \in \mathbb{R}^N$ with velocity $v \in \mathbb{R}^N$ at time $t > 0$. Then, the VPFP system can be written as

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f - \Delta_v f = 0 \tag{1.1}$$

for $f = f(x, v, t)$, $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$, $t > 0$,

$$E(x, t) = \frac{\gamma}{S_{N-1}} \frac{x}{|x|^N} * \int f(x, v, t)\, dv \tag{1.2}$$

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with initial data

$$f(x, v, 0) = \phi(x, v),$$

where $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_N}), \nabla_v = (\partial_{v_1}, \ldots, \partial_{v_N}), \Delta_v$ is the Laplacian in the v variable, $\gamma = \pm 1, S_{N-1}$ is $(N-1)$-dimensional volume of the N-dimensional unit sphere, and the symbol $*$ is the convolution in the x variable. $E(x, t)$ is the force field (the electric field) acting on the particle.

Let $\rho(x, t)$ describe the macroscopic density of particles located at position $x \in \mathbb{R}^N$ at time $t > 0$; that is,

$$\rho(x, t) = \int f(x, v, t)\, dv.$$

Equation (1.2) can be written alternatively as the Poisson equation $E = -\nabla_x U$ with $-\Delta_x U = \gamma\rho$. Then, we see $U(x, t) = (2-N)^{-1} c_0 |x|^{2-N} * \rho(x, t)$ with $c_0 = \gamma / S_{N-1}$.

The sign $\gamma = +1$ represents electrostatic (repulsive) interaction between the particles of the same species, while $\gamma = -1$ represents gravitational (attractive) interaction. Note that if $\gamma = 0$, we have the linear Fokker–Planck equation, which describes the Brownian motion of particles in a surrounding bath.

Neunzert et al. [8] used the probabilistic method to prove the global existence of probability measure solutions for weak form of the two-dimensional VPFP system with the friction term. Degond [6] proved the global existence of a unique classical solution $f \in L^\infty(0, T; W^{1,1}_{x,v})$ for any $T > 0$ of the VPFP system, under the assumptions that $\phi \geq 0, \phi \in W^{1,1}_{x,v}$ (i.e., $\|\phi\|_{L^1_{x,v}} + \|D\phi\|_{L^1_{x,v}} < \infty$ with $D = \nabla_x$ and ∇_v), and $\langle v \rangle^m (|\phi| + |D\phi|) \in L^\infty_{x,v}$ for some $m > N$ in one- and two-dimensional cases. Moreover, the regularity problem was treated for smooth initial data, e.g., $\phi \in W^{k,1}_{x,v}$ and $\langle v \rangle^m (|\phi| + \cdots + |D^k \phi|) \in L^\infty_{x,v}$. To prove these results, the iterative scheme method was used. Here, we denote $\langle v \rangle = \sqrt{1 + |v|^2}$. We use the function space $W^{m,n,1}_{x,v}$ such that

$$\psi \in W^{m,n,1}_{x,v} \qquad \text{if and only if} \quad \sum_{|\alpha| \leq m, |\beta| \leq n} \|\nabla_x^\alpha \nabla_v^\beta \psi\|_{L^1_{x,v}} < \infty,$$

and we often use $W^{k,1}_{x,v}$ instead of $W^{k,k,1}_{x,v}$ for simplicity.

Also, Victory and O'Dwyer [13] obtained the similar results for the VPFP system with the friction term in one- and two-dimensional cases, together with a technique similar to that in [6].

Our aim of this paper is to prove the global existence and uniqueness of regular solutions of the VPFP system for initial data ϕ in a wider class than Degond's [6].

Our main result is as follows.

THEOREM 1.1. *Let* $N = 1, 2, 3$. *Suppose that* $\phi \geq 0$, $\langle v\rangle^m \phi \in L^\infty_{x,v}$ *for some* $m > N$, *and* $\phi \in L^1_{x,v}$ *if* $N = 1, 2$ *or* $\langle v\rangle^2 \phi \in L^1_{x,v}$ *if* $N = 3$. *Then, there exists a unique global solution* $f \geq 0$ *of the VPFP system belonging to* $C([0,\infty); L^1_{x,v}) \cap C((0,\infty); W^{m,n,1}_{x,v})$ *for any* $m, n \geq 0$, *with*

$$\sup_{0\leq t\leq T} \|E(t)\|_{L^\infty_x} < \infty \qquad \textit{for any } T > 0.$$

Moreover, we have

$$f \in C^\infty\big(\mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)\big) \qquad \textit{and} \qquad E \in C^\infty\big(\mathbb{R}^N \times (0,\infty)\big).$$

When $N = 1$ and 3, recently, in [9] we studied the global existence and uniqueness of regular solutions for the VPFP system together with the linear Fokker–Planck equation, under slightly better initial conditions than those of Theorem 1.1 (see Theorem 2.3 in Section 2). Thus, as a corollary of Theorem 2.3, we conclude Theorem 1.1 in one- and three-dimensional cases. (See the end of Section 2 for details.) Then, we need to focus the problem in two dimensions.

In Section 2 we will give the proof of Theorem 1.1. In Section 3, under other initial conditions related to Theorem 2.3, we will study the global existence and uniqueness of regular solutions of the two-dimensional VPFP system in the repulsive interaction case.

As for global existence in three dimensions, we refer to Bouchut [1, 2] and Castella [4]. They studied the existence of strong solutions under other initial conditions. As for the decay estimates of the force field E and the density ρ, and for the asymptotic behavior of the solutions with small initial data, we refer to [3, 5, 9, 10, 12] and the references cited therein.

Finally we fix some notation. The function spaces L^p_x, $L^p_{x,v}$, $L^p_x(L^q_v)$ mean $L^p(\mathbb{R}^N_x)$, $L^p(\mathbb{R}^N_x \times \mathbb{R}^N_v)$, $L^p(\mathbb{R}^N_x; L^q(\mathbb{R}^N_v))$, and so on. Positive constants will be denoted by C and will change from line to line; especially, C_T means constants depending on T.

2. REGULAR SOLUTIONS

First, we recall the local existence theorem, which is given by [9] (see Propositions 4.1 and 4.5 in [9]).

PROPOSITION 2.1. *Let* N *be any dimension. Suppose that* $\phi \geq 0$, $\phi \in L^1_{x,v}$, *and*

$$\sup_{\lambda\geq 0}\left\| \int \phi(x-\lambda v, v)\, dv \right\|_{L^\infty_x} < \infty \qquad (\textit{or, more simply, } \phi \in L^1_v(L^\infty_x)). \tag{2.1}$$

Then, for $m, n \geq 0$, there exists $T > 0$ such that the problem (1.1), (1.2) *admits the existence of a unique local solution $f \geq 0$ belonging to*

$$C\big([0, T); L^1_{x,v}\big) \cap C\big((0, T); W^{m,n,1}_{x,v}\big). \tag{2.2}$$

Moreover, if

$$\sup_{0 \leq t < T} \sup_{\lambda \geq 0} \left\| \int f(x - \lambda v, v, t)\, dv \right\|_{L^\infty_x} < \infty, \tag{2.3a}$$

then we can take $T = \infty$ in (2.2).

Furthermore, a sufficient condition for (2.3a) *is*

$$\sup_{0 \leq t < T} t^\delta \|E(t)\|_{L^\infty_x} < \infty \qquad \textit{for some } \delta \in [0, 1/2). \tag{2.3b}$$

The existence of the unique local solution f of the VPFP system is shown by using the contraction mapping principle.

Remark 2.2. (i) Due to Lemma 2.6 below, we know that the condition (2.1) is fulfilled by initial data ϕ such that $\langle v \rangle^m \phi \in L^\infty_{x,v}$ for some $m > N$.

(ii) When $N = 1$, by the definition of the force field E, we see immediately the boundedness of E and hence we can take $T = \infty$ in (2.2). When $N = 3$, in the previous paper [9] we gave the a priori estimate under the conditions that $|v|^2 \phi \in L^1_{x,v}$ and $\phi \in L^\infty_{x,v}$ in addition to the assumptions of Proposition 2.1, and then we obtained the following.

THEOREM 2.3 (Ono and Strauss [9]). *Let $N = 1, 3$. Suppose that $\phi \geq 0$, $\sup_{\lambda \geq 0} \| \int \phi(x - \lambda v, v)\, dv \|_{L^\infty_x} < \infty$, and $\phi \in L^1_{x,v}$ if $N = 1$ or $\langle v \rangle^2 \phi \in L^1_{x,v}$ and $\phi \in L^\infty_{x,v}$ if $N = 3$. Then, there exists a unique global solution $f \geq 0$ of the VPFP system belonging to $C([0, \infty); L^1_{x,v}) \cap C((0, \infty); W^{m,n,1}_{x,v})$ for any $m, n \geq 0$, with $\sup_{0 \leq t \leq T} \|E(t)\|_{L^\infty_x} < \infty$ for any $T > 0$. Moreover, we have $f \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty))$ and $E \in C^\infty(\mathbb{R}^N \times (0, \infty))$.*

Let us focus on the two-dimensional case. The first result for the two-dimensional VPFP system is as follows.

THEOREM 2.4. *Let $N = 2$. Suppose that $\phi \geq 0$, $\phi \in L^1_{x,v}$, and $\langle v \rangle^m \phi \in L^\infty_{x,v}$ for some $m > 2$. Then, there exists a unique global solution $f \geq 0$ of the VPFP system belonging to $C([0, \infty); L^1_{x,v}) \cap C((0, \infty); W^{m,n,1}_{x,v})$ for any $m, n \geq 0$, with*

$$\sup_{0 \leq t \leq T} \|\langle v \rangle^m f(t)\|_{L^\infty_{x,v}} < \infty \qquad \textit{and} \qquad \sup_{0 \leq t \leq T} \|E(t)\|_{L^\infty_x} < \infty$$

for any $T > 0$. Moreover, we have $f \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty))$ and $E \in C^\infty(\mathbb{R}^N \times (0, \infty))$.

Proof. Let $N = 2$ and let f be a solution given by Proposition 2.1 on $[0, T)$. It is enough to derive a priori bound $\|E(t)\|_{L^\infty_x} < \infty$ on $[0, T)$.

Then, by a direct calculation we know that the solution f satisfies

$$\begin{aligned}\partial_t(\langle v\rangle^m f) + v\cdot\nabla_x(\langle v\rangle^m f)&\\ +\left\{E + 2m\frac{v}{\langle v\rangle^2}\right\}\cdot\nabla_v(\langle v\rangle^m f) - \Delta_v(\langle v\rangle^m f) &= F,\end{aligned} \tag{2.4}$$

where $F = F_1 + F_2$ with

$$F_1 = m\left\{(m+2)\frac{|v|^2}{\langle v\rangle^2} - N\right\}\langle v\rangle^{m-2} f$$

and

$$F_2 = mE\cdot v\langle v\rangle^{m-2} f.$$

We first estimate the $L^\infty_{x,v}$-norm of the terms F_1 and F_2. We freely use the fact that $\|f(t)\|_{L^p_{x,v}} \le \|\phi\|_{L^p_{x,v}}$ for $1 \le p \le \infty$. It is easy to see that

$$\|F_1(t)\|_{L^\infty_{x,v}} \le C\|\langle v\rangle^{m-2} f(t)\|_{L^\infty_{x,v}} \le C\|\langle v\rangle^m f(t)\|_{L^\infty_{x,v}}. \tag{2.5}$$

Since it follows from Lemma 2.6 below that

$$\begin{aligned}\|E(t)\|_{L^\infty_x} &\le C\|\rho(t)\|^{1/2}_{L^1_x}\|\rho(t)\|^{1/2}_{L^\infty_x} = C\|f(t)\|^{1/2}_{L^1_{x,v}}\left\|\int f\,dv\right\|^{1/2}_{L^\infty_x}\\ &\le C\|f(t)\|^{1/2}_{L^1_{x,v}}\left(\|f(t)\|^{1-2/m}_{L^\infty_{x,v}}\|\langle v\rangle^m f(t)\|^{2/m}_{L^\infty_{x,v}}\right)^{1/2}\\ &\le C\|\langle v\rangle^m f(t)\|^{1/m}_{L^\infty_{x,v}} \qquad \text{for } m > 2,\end{aligned} \tag{2.6}$$

we observe

$$\begin{aligned}\|F_2(t)\|_{L^\infty_{x,v}} &\le C\|E(t)\|_{L^\infty_x}\|\langle v\rangle^{m-1} f(t)\|_{L^\infty_{x,v}}\\ &\le C\|\langle v\rangle^m f(t)\|^{1/m}_{L^\infty_{x,v}}\left(\|f(t)\|^{1/m}_{L^\infty_{x,v}}\|\langle v\rangle^m f(t)\|^{1-1/m}_{L^\infty_{x,v}}\right)\\ &\le C\|\langle v\rangle^m f(t)\|_{L^\infty_{x,v}} \qquad \text{for } m > 2.\end{aligned} \tag{2.7}$$

Therefore, from Proposition 2.5 below together with (2.5) and (2.7), we have

$$\begin{aligned}\|\langle v\rangle^m f(t)\|_{L^\infty_{x,v}} &\le \|\langle v\rangle^m \phi\|_{L^\infty_{x,v}} + C\int_0^t \|F(s)\|_{L^\infty_{x,v}}\,ds\\ &\le C + C\int_0^t \|\langle v\rangle^m f(s)\|_{L^\infty_{x,v}}\,ds,\end{aligned} \tag{2.8}$$

and from Gronwall's inequality, we deduce

$$\sup_{0\le t<T}\|\langle v\rangle^m f(t)\|_{L^\infty_{x,v}} \le C_T,$$

and hence from (2.6) we observe

$$\sup_{0\le t<T} \|E(t)\|_{L^\infty_x} \le C_T,$$

which is the desired a priori estimate. Thus the function f is the global solution of the two-dimensional VPFP system. Moreover, using Eq. (1.1), we see that the solution f and the force field E are smooth for $t > 0$. ∎

Next, we prove the inequality (2.8) which we used in the proof of Theorem 2.4 (cf. Degond [6]).

PROPOSITION 2.5. *Let f be a solution on $[0, T)$ given by Proposition 2.1. Suppose that the assumptions of Theorem 2.4 are fulfilled. Then, the solution f satisfies*

$$\|\langle v\rangle^m f(t)\|_{L^\infty_{x,v}} \le \|\langle v\rangle^m \phi\|_{L^\infty_{x,v}} + \int_0^t \|F(s)\|_{L^\infty_{x,v}}\, ds \tag{2.9}$$

for $0 \le t < T$.

Proof. Putting

$$g(x, v, t) = \langle v\rangle^m f(x, v, t) \qquad \text{and} \qquad H(x, v, t) = E(x, t) + 2m\frac{v}{\langle v\rangle^2},$$

we have from (2.4) that

$$L[g] \equiv \partial_t g + v\cdot\nabla_x g + H\cdot\nabla_v g - \Delta_v g = F \tag{2.10}$$

with $g|_{t=0} = \langle v\rangle^m \phi$.

Step 1. We will prove that if $g|_{t=0} \ge 0$ and $F \ge 0$, then $g \ge 0$. Putting $G(x, v, t) = e^{-t} g(x, v, t)$ and

$$\widetilde{L}[G] \equiv \partial_t G + G + v\cdot\nabla_x G + H\cdot\nabla_v G - \Delta_v G,$$

we observe

$$\widetilde{L}[G] = e^{-t} L[g] = e^{-t} F \ge 0. \tag{2.11}$$

We will argue by contradiction. Assume that there exists a point $(x_0, v_0, t_0) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$ such that $g(x_0, v_0, t_0) < 0$. Then, we have

$$G(x_0, v_0, t_0) < 0,$$

and we put $k = -G(x_0, v_0, t_0)$, $k > 0$. For $A > 0$ and $\varepsilon > 0$ which are fixed below, we define a function

$$G_\varepsilon(x, v, t) = G\big(x, v, t\big) + \varepsilon\big(At + |x|^2 + 2|v|^2\big).$$

Then,

$$\widetilde{L}[G_\varepsilon] = \widetilde{L}[G] + \varepsilon\big(A + At + |x|^2 + 2x \cdot v + 2|v|^2 + 4H \cdot v - 4N\big),$$

and by Young's inequality and (2.11),

$$\widetilde{L}[G_\varepsilon] \geq \varepsilon A - \varepsilon\big(-|v|^2 + 4\|H\|_{L^\infty_{x,v}}|v| + 4N\big).$$

Since $t_0 < T$, it follows from (2.3b) with $\delta = 0$ in Proposition 2.1 that $\|H(t)\|_{L^\infty_{x,v}} < \infty$ for any $0 \leq t \leq t_0$. We take $A > 0$ such that

$$A \geq \sup\big\{-|v|^2 + 4\|H(t)\|_{L^\infty_{x,v}}|v| + 4N \mid v \in \mathbb{R}^N \text{ and } t \in [0, t_0]\big\},$$

and then

$$\widetilde{L}[G_\varepsilon] \geq 0. \tag{2.12}$$

Moreover, we take $\varepsilon > 0$ such that

$$G_\varepsilon(x_0, v_0, t_0) = G(x_0, v_0, t_0) + \varepsilon(At_0 + |x_0|^2 + 2|v_0|^2) \leq -\frac{k}{2},$$

and we set

$$\mathscr{S}_{x,v} = \bigg\{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N \mid |x|^2 + 2|v|^2 > \varepsilon^{-1}\Big(-\inf_{(x,v,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,t_0]} G(x, v, t)\Big)\bigg\}.$$

Since $G|_{t=0} = \langle v\rangle^m \phi \geq 0$ and $G \in C(\mathbb{R}^N \times \mathbb{R}^N \times (0, t_0])$, we see that $\mathscr{S}_{x,v}$ is not empty and

$$G_\varepsilon > 0 \qquad \text{on } \mathscr{S}_{x,v} \times (0, t_0].$$

On the other hand, since $G_\varepsilon|_{t=0} = \langle v\rangle^m \phi + \varepsilon(|x|^2 + 2|v|^2) \geq 0$, $G_\varepsilon(x_0, v_0, t_0) < 0$, and $G_\varepsilon \in C(\mathbb{R}^N \times \mathbb{R}^N \times (0, t_0])$, we observe that there exists a point $(x_1, v_1, t_1) \in ((\mathbb{R}^N \times \mathbb{R}^N)\backslash\mathscr{S}_{x,v}) \times (0, t_0]$ such that

$$G_\varepsilon(x_1, v_1, t_1) = \inf_{(x,v,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,t_0]} G_\varepsilon(x, v, t).$$

Then, we see $\partial_t G_\varepsilon(x_1, v_1, t_1) \leq 0$, $\nabla_x G_\varepsilon(x_1, v_1, t_1) = 0$, $\nabla_v G_\varepsilon(x_1, v_1, t_1) = 0$, and $\Delta_v G_\varepsilon(x_1, v_1, t_1) \geq 0$, and hence

$$\widetilde{L}[G_\varepsilon(x_1, v_1, t_1)] \leq G_\varepsilon(x_1, v_1, t_1) \leq G_\varepsilon(x_0, v_0, t_0) < 0$$

which is a contradiction to (2.12). Therefore we conclude $g \geq 0$.

Step 2. We will prove that for $t > 0$,

$$\sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,t) \le \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,0) + \int_0^t \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} F(x,v,s)\,ds \tag{2.13}$$

and

$$\inf_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,t) \ge \inf_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,0) + \int_0^t \inf_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} F(x,v,s)\,ds. \tag{2.14}$$

Putting

$$u = -g + \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,0) + \int_0^t \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} F(x,v,s)\,ds,$$

we observe

$$L[u] = -L[g] + \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} F(x,v,t) \ge 0$$

and $u|_{t=0} = -g|_{t=0} + \sup_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g|_{t=0} \ge 0$. Thus, from Step 1 we have $u \ge 0$, which gives (2.13).

Next, putting

$$v = g - \inf_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} g(x,v,0) - \int_0^t \inf_{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N} F(x,v,s)\,ds,$$

by the same argument above, we have $v \ge 0$, which gives (2.14).

From Step 2 we immediately deduce

$$\|g(t)\|_{L^\infty_{x,v}} \le \|g(0)\|_{L^\infty_{x,v}} + \int_0^t \|F(s)\|_{L^\infty_{x,v}}\,ds$$

which implies the desired estimate (2.9). ∎

Next, we state a useful lemma.

LEMMA 2.6. *If* $\langle v\rangle^m \psi \in L^\infty_{x,v}$ *for some* $m > N$, *then* $\psi \in L^1_v(L^\infty_x)$ *and hence* $\psi \in L^\infty_x(L^1_v)$, *and we have*

$$\|\psi\|_{L^\infty_x(L^1_v)} \le \|\psi\|_{L^1_v(L^\infty_x)} \le C\|\psi\|^{1-N/m}_{L^\infty_{x,v}} \|\langle v\rangle^m \psi\|^{N/m}_{L^\infty_{x,v}}. \tag{2.15}$$

Proof. The first inequality in (2.15) is clear. For $R > 0$, we have

$$\int \|\psi(\cdot, v)\|_{L_x^\infty}\, dv = \left(\int_{|v|\geq R} + \int_{|v|\leq R}\right) \|\psi(\cdot, v)\|_{L_x^\infty}\, dv$$
$$\leq CR^{-(m-N)} \|\langle v\rangle^m \psi\|_{L_{x,v}^\infty} + CR^N \|\psi\|_{L_{x,v}^\infty}.$$

Making the optimal choice $R = (\|\langle v\rangle^m \psi\|_{L_{x,v}^\infty} / \|\psi\|_{L_{x,v}^\infty})^{1/m}$, we conclude the second inequality in (2.15). ∎

As a corollary of Theorems 2.3 and 2.4 together with Lemma 2.6, we have the following global existence theorem, which completely includes Degond's results [6].

COROLLARY 2.7. *Let $N = 1, 2, 3$. Suppose that $\phi \geq 0$, $\langle v\rangle^m \phi \in L_{x,v}^\infty$ for some $m > N$, and $\phi \in L_{x,v}^1$ if $N = 1, 2$ or $\langle v\rangle^2 \phi \in L_{x,v}^1$ if $N = 3$. Then, there exists a unique global solution $f \geq 0$ of the VPFP system belonging to $C([0, \infty); L_{x,v}^1) \cap C((0, \infty); W_{x,v}^{m,n,1})$ for any $m, n \geq 0$. Moreover, we have $f \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty))$ and $E \in C^\infty(\mathbb{R}^N \times (0, \infty))$.*

3. REPULSIVE INTERACTION CASE

In the repulsive interaction case (i.e., $\gamma = +1$ in (1.2)), under the different conditions from Theorem 2.4, we consider the global existence and uniqueness of regular solutions for the two-dimensional VPFP system. Here, we denote the initial force field at time $t = 0$ by

$$E_0(x) \equiv E(x, 0) = \frac{+1}{S_{N-1}} \frac{x}{|x|^N} * \int \phi(x, v)\, dv, \tag{3.1}$$

where the symbol $*$ is the convolution in the x variable.

Another global existence theorem for the two-dimensional VPFP system is written as follows.

THEOREM 3.1 (Repulsive interaction). *Let $N = 2$ and $\gamma = +1$ in (1.2). Suppose that $\phi \geq 0$, $\langle v\rangle^2 \phi \in L_{x,v}^1$, $\phi \in L_{x,v}^\infty$, $\sup_{\lambda\geq 0} \|\int \phi(x - \lambda v, v)\, dv\|_{L_x^\infty} < \infty$, and $E_0 \in L_x^2$. Then, there exists a unique global solution $f \geq 0$ of the VPFP system belonging to $C([0, \infty); L_{x,v}^1) \cap C((0, \infty); W_{x,v}^{m,n,1})$ for any $m, n \geq 0$, with*

$$\sup_{0\leq t\leq T} \|E(t)\|_{L_x^\infty} < \infty \qquad \textit{for any } T > 0.$$

Moreover, we have $f \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty))$ and $E \in C^\infty(\mathbb{R}^N \times (0, \infty))$.

To get the above theorem, we use the following lemma and proposition, which have been given in the previous paper [9] (cf. [1, 4]).

LEMMA 3.2 [9, Proposition 3.5 (ii)]. *If* $\psi \in L^1_{x,v} \cap L^\infty_{x,v}$ *and* $|v|^k\psi \in L^1_{x,v}$ *for some* $k > 0$, *then* $\psi(x - \lambda v, v) \in L^p_x(L^1_v)$ *for any* $\lambda \in \mathbb{R}$ *and* $p \in [1, 1 + k/N]$, *and we have*

$$\sup_{\lambda \geq 0} \left\| \int |\psi(x - \lambda v, v)|\, dv \right\|_{L^p_x} \leq C\|\psi\|^{1/p-(N/k)(1-1/p)}_{L^1_{x,v}} \|\psi\|^{1-1/p}_{L^\infty_{x,v}} \||v|^k\psi\|^{(N/k)(1-1/p)}_{L^1_{x,v}}.$$

PROPOSITION 3.3 [9, Proposition 5.5]. *Let* $N \geq 2$. *Suppose that* $\phi \geq 0$, $\phi \in L^1_{x,v} \cap L^\infty_{x,v}$, $\sup_{\lambda \geq 0} \| \int \phi(x - \lambda v, v)\, dv\|_{L^\infty_x} < \infty$, *and*

$$K_r \equiv \sup_{0 \leq t < T} \|E(t)\|_{L^r_x} < \infty \qquad \textit{for some } r \in (2N, \infty]. \tag{3.2}$$

Then, for $p \in (l, \infty]$,

$$S_p \equiv \sup_{0 \leq t < T} \sup_{\lambda \geq 0} \left\| \int f(x - \lambda v, v, t)\, dv \right\|_{L^p_x} \tag{3.3}$$

satisfies

$$S_p \leq C + C_T S_l^{l/p} K_r^{r(1-l/p)}.$$

Proof of Theorem 3.1. Let $N = 2$ and $\gamma = +1$, and let f be a solution on $[0, T)$. Since a direct calculation leads to the energy identity

$$\frac{d}{dt}\left\{\||v|^2 f\|_{L^1_{x,v}} + \|E(t)\|^2_{L^2_x}\right\} = 4\|\phi\|_{L^1_{x,v}},$$

we obtain

$$\||v|^2 f\|_{L^1_{x,v}} \leq \||v|^2\phi\|_{L^1_{x,v}} + \|E_0\|^2_{L^2_x} + 4\|\phi\|_{L^1_{x,v}} T$$

for $0 \leq t < T$, where E_0 is given by (3.1).

Applying Lemma 3.2 with $p = q$, $k = 2$, we observe

$$\left\| \int f(x - \lambda v, v, t)\, dv \right\|_{L^q_x} \leq C\||v|^2 f\|^{1-1/q}_{L^1_{x,v}} \leq C_T$$

for $1 \leq q \leq 2$, and hence

$$S_q \leq C_T \qquad \text{for} \quad 1 \leq q \leq 2,$$

where S_q is given by (3.3).

From the Hardy–Littlewood–Sobolev inequality, it follows that

$$K_p \leq C \sup_{0 \leq t < T} \left\| \int f(x, v, t)\, dv \right\|_{L^q_x} \leq CS_q$$

for $1 < q < p < \infty$ with $1/p = 1/q - 1/2$, where K_p is given by (3.2), and hence

$$K_p \leq C_T \qquad \text{for} \quad 2 < p < \infty.$$

Finally, applying Proposition 3.3 with $l = 2$, $r = 5$, $p = \infty$, we conclude

$$S_\infty = \sup_{0 \leq t < T} \sup_{\lambda \geq 0} \left\| \int f(x - \lambda v, v, t)\, dv \right\|_{L_x^\infty} \leq C + C_T K_5^5 \leq C_T, \tag{3.4}$$

which is the desired a priori estimate (2.3a), and by the same way as in [9], we obtain from (3.4) that $K_\infty \leq C_T$. ∎

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