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An accurate bound on tensor-to-scalar ratio and the scale of inflation

Sayantan Choudhury ^{a,*}, Anupam Mazumdar ^b^a *Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700 108, India*^b *Consortium for Fundamental Physics, Physics Department, Lancaster University, Lancaster LA1 4YB, UK*

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Abstract

In this paper we provide an accurate bound on primordial gravitational waves, i.e. tensor-to-scalar ratio (r) for a general class of single-field models of inflation where inflation occurs always below the Planck scale, and the field displacement during inflation remains sub-Planckian. If inflation has to make connection with the real particle physics framework then it must be explained within an effective field theory description where it can be trustable below the UV cut-off of the scale of gravity. We provide an analytical estimation and estimate the largest possible r , i.e. $r \leq 0.12$, for the field displacement less than the Planck cut-off.

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1. Introduction

If the primordial inflation [1] has to make connection to the observed world and be a *predictive science* then it has to be embedded within a particle theory [2], where the last 50–60 e-foldings of inflation must occur within a visible sector with a laboratory measured inflaton couplings to the Standard Model physics in order to create the right form of matter with the right abundance [4].¹

* Corresponding author.

¹ Note that after Planck there is no trace of isocurvature perturbations and there is a severe constraint on dark radiation [3]. Therefore the inflaton vacuum cannot be arbitrary as first pointed out in this review [2]. Models of inflation based on gauge invariant flat directions of Standard Model quarks and leptons naturally provides an ideal inflaton

This inevitably puts constraint on the vev of inflation, i.e. ϕ_0 , and the range of flatness of the potential during the observed 17 e-foldings of inflation, i.e. $\Delta\phi$. For the simplest single field dominated model of inflation, there are two important constraints which all the models must satisfy.

- $\phi_0 \ll M_p$ – vev of the inflaton must be bounded by the cut-off of the particle theory, where $M_p = 2.4 \times 10^{18}$ GeV. We are assuming that 4 dimensions M_p puts a natural cut-off here for any physics beyond the Standard Model.
- $|\Delta\phi| \ll M_p$ – the inflaton potential has to be flat enough during which a successful inflation can occur. Note that the flatness of the potential has to be fine tuned – there is no particle physics symmetry which can maintain the flatness [2]. We will assume $V''(\phi_0) \approx 0$, where $V(\phi)$ denotes the inflaton potential, and prime denotes derivative w.r.t. the ϕ field.

The aim of this paper is to impose these two conditions to obtain an improved bound on the tensor-to-scalar ratio, r . In Refs. [7–12], it was realised that it is possible to obtain large $r \sim \mathcal{O}(10^{-1}-10^{-2})$ for small field excursion characterised by, $\phi_0 \leq M_p$ and $\Delta\phi \leq M_p$. The bound on r was further improved in the recent work, see Ref. [6], where it was demonstrated that it is possible to saturate the Planck limit on tensor-to-scalar ratio, i.e. $r \leq 0.12$ [1].

Such a large tensor-to-scalar ratio can be obtained provided one deviates from a monotonic behaviour of slow roll parameter ϵ_V , which we will elaborately discuss below by incorporating the effects of higher order slow-roll corrections for generic class of sub-Planckian inflationary models in presence of a non-negligible and scale-dependent running of the scalar and tensor power spectrum [6–8,13].

In this respect we are improving on previously obtained bound on large r , i.e. $r \sim \mathcal{O}(0.1)$, where ϕ_0 and $|\Delta\phi|$ were taken beyond M_p , see [14], and for its most generalised updated version in presence of phase velocity at the horizon crossing also see [15]. Such a significant tensor to scalar ratio, can be obtained in the framework of large-field models of inflation, such as “chaotic inflation” [16,17], so-called “Higgs inflation” along with its conformal generalisation [18,19] and “axion monodromy inflation” [20]. In this class of models, slow-roll inflation occurs when the inflaton vacuum expectation value (VEV) exceeds the Planck scale, so that the large field excursion, $\Delta\phi > M_p$ is possible. However, in this paper the main goal is to provide an analytical expression for tensor-to-scalar ratio when $\Delta\phi < M_p$, as suggested in Refs. [7,8]. As it has been show recently [6], it is indeed possible to obtain large $r \leq 0.12$ for field values $\Delta\phi \leq M_p$, here we provide an analytical proof of the earlier results.

Our analytical results are important because any positive detection on large tensor-to-scalar ratio, i.e. $r \sim \mathcal{O}(0.01-0.1)$, in forthcoming experiments might not be able to conclusively favour high scale super-Planckian models of inflation.

2. Generic framework for sub-Planckian inflation

The tensor to scalar ratio can be defined by taking into account of the higher order corrections, see Refs. [13,21,22]:

$$r = 16\epsilon_H \frac{[1 - (\mathcal{C}_E + 1)\epsilon_H]^2}{[1 - (2\mathcal{C}_E + 1)\epsilon_H + \mathcal{C}_E\eta_H]^2}, \quad (2.1)$$

candidate embedded within a visible sector with well-known interactions, discussed in Ref. [4], and their recent update after Planck [5,6].

where $\mathcal{C}_E = 4(\ln 2 + \gamma_E) - 5$ with $\gamma_E = 0.5772$ is the *Euler–Mascheroni constant* [13]. In Eq. (2.1) the Hubble slow roll parameters (ϵ_H, η_H) are defined as:

$$\epsilon_H = -\frac{d \ln H}{d \ln a} = -\frac{\dot{H}}{H^2}, \quad \eta_H = -\frac{d \ln \dot{\phi}}{d \ln a} = -\frac{\ddot{\phi}}{H\dot{\phi}}, \tag{2.2}$$

where dot denotes time derivative with respect to the physical time. Now considering the effect from the leading order dominant contributions from the slow-roll parameters, the Hubble slow-roll parameters can be expressed in terms of the potential dependent slow-roll parameters, (ϵ_V, η_V) , as: $\epsilon_H \approx \epsilon_V + \dots$, and $\eta_H \approx \eta_V - \epsilon_V + \dots$, where \dots comes from the higher order contributions of (ϵ_V, η_V) . Here the slow-roll parameters (ϵ_V, η_V) are given by in terms of the inflationary potential $V(\phi)$, which can be expressed as:

$$\epsilon_V = \frac{M_p^2}{2} \left(\frac{V'}{V} \right)^2, \quad \eta_V = M_p^2 \left(\frac{V''}{V} \right). \tag{2.3}$$

We would also require two other slow-roll parameters, (ξ_V^2, σ_V^3) , in our analysis, which are given by:

$$\xi_V^2 = M_p^4 \left(\frac{V'V'''}{V^2} \right), \quad \sigma_V^3 = M_p^6 \left(\frac{V'^2V''''}{V^3} \right). \tag{2.4}$$

With the help above mentioned slow roll parameters, i.e. $\epsilon_V, \eta_V, \xi_V$ and σ_V , we can recast Eq. (2.1) as:

$$r \approx 16\epsilon_V \frac{[1 - (\mathcal{C}_E + 1)\epsilon_V]^2}{[1 - (3\mathcal{C}_E + 1)\epsilon_V + \mathcal{C}_E\eta_V]^2} \tag{2.5}$$

where we have neglected the contributions from the higher order slow-roll terms, as they are sub-dominant at the leading order. With the help of

$$\frac{d}{d \ln k} = -M_p \frac{\sqrt{2\epsilon_H}}{1 - \epsilon_H} \frac{d}{d\phi} \approx -M_p \frac{\sqrt{2\epsilon_V}}{1 - \epsilon_V} \frac{d}{d\phi}, \tag{2.6}$$

and Eq. (2.5), we can derive a simple expression for the tensor-to-scalar ratio, r , as:

$$r = \frac{8}{M_p^2} \frac{(1 - \epsilon_V)^2 [1 - (\mathcal{C}_E + 1)\epsilon_V]^2}{[1 - (3\mathcal{C}_E + 1)\epsilon_V + \mathcal{C}_E\eta_V]^2} \left(\frac{d\phi}{d \ln k} \right)^2. \tag{2.7}$$

Consequently, we can obtain a new bound on r in terms of the momentum scale (k):

$$\begin{aligned} \int_{k_e}^{k_{cmb}} \frac{dk}{k} \sqrt{\frac{r(k)}{8}} &= \frac{1}{M_p} \int_{\phi_e}^{\phi_{cmb}} d\phi \frac{(1 - \epsilon_V)[1 - (\mathcal{C}_E + 1)\epsilon_V]}{[1 - (3\mathcal{C}_E + 1)\epsilon_V + \mathcal{C}_E\eta_V]} \\ &\approx \frac{1}{M_p} \int_{\phi_e}^{\phi_{cmb}} d\phi (1 - \epsilon_V) [1 + \mathcal{C}_E(2\epsilon_V - \eta_V) + \dots] \\ &\approx \frac{\Delta\phi}{M_p} \left\{ 1 + \frac{1}{\Delta\phi} \left[(2\mathcal{C}_E - 1) \int_{\phi_e}^{\phi_{cmb}} d\phi \epsilon_V - \mathcal{C}_E \int_{\phi_e}^{\phi_{cmb}} d\phi \eta_V \right] + \dots \right\}, \end{aligned} \tag{2.8}$$

where note that $\Delta\phi \approx \phi_{cmb} - \phi_e$ is positive in Eq. (2.8), and this implies the left hand side of the integration over momentum within an interval, $k_e < k < k_{cmb}$, is also positive. Individual integrals involving ϵ_V and η_V were estimated in Appendix A, see Eqs. (A.1) and (A.2).

Here (ϕ_e, k_e) and (ϕ_{cmb}, k_{cmb}) represent inflaton field value and the corresponding momentum scale at the end of inflation and the Hubble crossing respectively. The imprints of the primordial gravitational waves can be directly measured in the CMB experiments via $r(k_{cmb})$. It is important to note that the recent observational constraint from Planck [1] only fixes the upper bound on $r(k_{cmb} \approx k_*)$ (≤ 0.12) by fixing the upper bound of the scale of inflation at the GUT scale ($V_* \lesssim 10^{16}$ GeV).

In order to perform the momentum integration in the left hand side of Eq. (2.8), we have used $r(k)$ at any arbitrary momentum scale, which can be expressed as:

$$r(k) = r(k_*) \left(\frac{k}{k_*} \right)^{a + \frac{b}{2} \ln(\frac{k}{k_*}) + \frac{c}{6} \ln^2(\frac{k}{k_*}) + \dots}, \tag{2.9}$$

where

$$a = n_T - n_S + 1, \quad b = (\alpha_T - \alpha_S), \quad c = (\kappa_T - \kappa_S)$$

are explicitly defined in Ref. [6]. These parameterisation characterises the spectral indices, n_S, n_T , running of the spectral indices, α_S, α_T , and running of the running of the spectral indices, κ_S, κ_T . Here the subscript (S, T) represent the scalar and tensor modes.

It was earlier confirmed by the WMAP9+high- l +BAO+ H_0 combined constraints that: $\alpha_S = -0.023 \pm 0.011$ and $\kappa_S = 0$ within less than 1σ C.L. [23]. After the Planck release it is important to see the impact on $r(k_*)$ due to running, and running of the running of the spectral tilt by modifying the generic power law form of the parameterisation of tensor-to-scalar ratio. The combined Planck+WMAP9 constraint confirms that: $\alpha_S = -0.0134 \pm 0.0090$ and $\kappa_S = 0.020^{+0.016}_{-0.015}$ within 1.5σ statistical accuracy [1], which additionally includes $\kappa_S \neq 0$ possibility.

At the next to leading order, the simplest way to modify the power law parameterisation is to incorporate the effects of higher order Logarithmic corrections in terms of the presence of non-negligible running, and running of the running of the spectral tilt as shown in Eq. (2.9), which involves higher order slow-roll corrections.²

After substituting Eq. (2.9) in Eq. (2.8), we will show that additional information can be gained from our analysis: first of all it provides more accurate and improved bound on tensor-to-scalar ratio in presence of non-negligible running and running of the running of the spectral tilt. In our analysis super-Planckian physics doesn't play any role as the effective theory puts naturally an upper cut-off set by the Planck scale. Consequently the prescription only holds good for sub-Planckian VEVs, $\phi_0 < M_p$ and field excursion, $\Delta\phi < M_p$ for inflation. Both these outcomes open a completely new insight into the particle physics motivated models of inflation, which are valid below the Planck scale.

Further note that the momentum integral has non-monotonous behaviour of the slow-roll parameters (ϵ_V, η_V) within the interval, $k_e < k < k_{cmb}$, which implies that ϵ_V and η_V initially increase within an observable window of e-foldings (which we will define in the next section, see Eq. (3.1)), and then decrease at some point during the inflationary epoch when the observable scales had left the Hubble patch, and eventually increase again to end inflation [7,8].

² It is important to note that when Ref. [14] first derived a bound on large tensor-to-scalar ratio for super-Planckian inflationary models (with $\Delta\phi > M_p$), the above mentioned constraints on α_S, κ_S were not taken into account due to lack of observational constraints.

In the most general situation, in Eq. (2.9), the parameters a , b and c are all functions of arbitrary momentum scale [6]. After imposing the above mentioned non-monotonicity behaviour of the slow-roll parameters within this interval, we can easily express the parameters a , b and c at the pivot scale k_* , which is approximately close to the CMB scale, i.e. $k_{cmb} \approx k_*$. The computational details of the momentum integration appearing in Eq. (2.8) are elaborately discussed in Appendix B, see Eqs. (B.1), (B.2) and the subsequent discussion.

Let us now expand a generic inflationary potential around the vicinity of ϕ_0 where inflation occurs, and impose the flatness condition such that, $V''(\phi_0) \approx 0$. This yields a potential, see [24]:

$$V(\phi) = \alpha + \beta(\phi - \phi_0) + \gamma(\phi - \phi_0)^3 + \kappa(\phi - \phi_0)^4 + \dots, \quad (2.10)$$

where $\alpha \ll M_p^4$ denotes the height of the potential, and the coefficients $\beta \ll M_p^3$, $\gamma \ll M_p$, $\kappa \ll \mathcal{O}(1)$ determine the shape of the potential in terms of the model parameters. Typically, α can be set to zero by fine tuning, but here we wish to keep this term for generality.

Note that at this point, we do not need to specify any particular model of inflation for Eq. (2.10). However, not all of the coefficients are independent once we prescribe the model of inflation here. This is true *only* if the model is fully embedded within a particle theory such as that of MSSM [4]. We will always observe the crucial constraints: $\phi_0 < M_p$ and $\Delta\phi < M_p$. Then $\Delta\phi$ can be redefined as, $\Delta\phi = (\phi_{cmb} - \phi_0) - (\phi_e - \phi_0)$.

Now substituting the explicit form of the potential stated in Eq. (2.10), we evaluate the crucial integrals of the first and second slow-roll parameters (ϵ_V, η_V) appearing in the right hand side of Eq. (2.8). For details see appendix where the leading order results are explicitly mentioned.

3. Accurate bound on ‘ r ’ for small field values of inflation

At any arbitrary momentum scale the number of e-foldings, $\mathcal{N}(k)$, between the Hubble exit of the relevant modes and the end of inflation can be expressed as [1]:

$$\begin{aligned} \mathcal{N}(k) \approx & 71.21 - \ln\left(\frac{k}{k_0}\right) + \frac{1}{4} \ln\left(\frac{V_\star}{M_p^4}\right) + \frac{1}{4} \ln\left(\frac{V_\star}{\rho_{end}}\right) \\ & + \frac{1 - 3w_{int}}{12(1 + w_{int})} \ln\left(\frac{\rho_{rh}}{\rho_{end}}\right), \end{aligned} \quad (3.1)$$

where ρ_{end} is the energy density at the end of inflation, ρ_{rh} is an energy scale during reheating, $k_0 = a_0 H_0$ is the present Hubble scale, V_\star corresponds to the potential energy when the relevant modes left the Hubble patch during inflation corresponding to the momentum scale $k_\star \approx k_{cmb}$, and w_{int} characterises the effective equation of state parameter between the end of inflation and the energy scale during reheating.

Within the momentum interval, $k_e < k < k_{cmb}$, the corresponding number of e-foldings is given by, $\Delta\mathcal{N} = \mathcal{N}_e - \mathcal{N}_{cmb}$, as

$$\Delta\mathcal{N} = \ln\left(\frac{k_{cmb}}{k_e}\right) \approx \ln\left(\frac{k_\star}{k_e}\right) = \ln\left(\frac{a_\star}{a_e}\right) + \ln\left(\frac{H_\star}{H_e}\right) \approx \ln\left(\frac{a_\star}{a_e}\right) + \frac{1}{2} \ln\left(\frac{V_\star}{V_e}\right) \quad (3.2)$$

where (a_\star, H_\star) and $(a_e H_e)$ represent the scale factor and the Hubble parameter at the pivot scale and end of inflation, and we have used the fact that $H^2 \propto V$. We can estimate the contribution of the last term of the right hand side by using Eq. (2.10) as follows:

$$\begin{aligned}
 \ln\left(\frac{V_\star}{V_e}\right) &= \ln\left(\frac{\alpha + \beta(\phi_\star - \phi_0) + \gamma(\phi_\star - \phi_0)^3 + \kappa(\phi_\star - \phi_0)^4 + \dots}{\alpha + \beta(\phi_e - \phi_0) + \gamma(\phi_e - \phi_0)^3 + \kappa(\phi_e - \phi_0)^4 + \dots}\right) \\
 &\approx \ln\left(1 + \underbrace{M_p \frac{\beta}{\alpha} \left(\frac{\Delta\phi}{M_p}\right)}_{\ll 1} \left[1 + \underbrace{\dots}_{\ll 1}\right]\right) \\
 &\approx \ln(1 + \underbrace{\dots}_{\ll 1}),
 \end{aligned}
 \tag{3.3}$$

where $(\Delta\phi/M_p) \ll 1$, and we assume $(\beta M_p/\alpha) \ll 1$, consequently, Eq. (3.4) reduces to the following simplified expression:

$$\Delta\mathcal{N} \approx \ln\left(\frac{k_\star}{k_e}\right) \approx \ln\left(\frac{a_\star}{a_e}\right) \approx 17 \text{ e-folds.}
 \tag{3.4}$$

Within the observed limit of Planck, i.e. $\Delta\mathcal{N} \approx 17$, the slow-roll parameters, see Eqs. (A.1), (A.2) of Appendix A, show non-monotonic behaviour, where the corresponding scalar and tensor amplitude of the power spectrum remains almost unchanged.³

At the scale of Hubble crossing ($k_\star = a_\star H_\star$), the slow-roll parameter ϵ_V must be sufficiently large enough to generate an observable value of tensor-to-scalar ratio r_\star at the pivot/normalisation scale k_\star , and it must increase over the $\Delta\mathcal{N} \approx 17$ e-foldings, as first pointed out in Refs. [7,8]. After Hubble crossing ($k_\star \gg a_\star H_\star$), the slow-roll parameter ϵ_V must quickly decrease, which is necessary to generate enough e-folds of inflation. However instead of a quick decrement of ϵ_V if it decreases gradually, it will need to eventually decrease to a much smaller value because, $\epsilon_V \propto (\Delta\phi/M_p \Delta\mathcal{N}) < 1/17$, by imposing the constraint, $\Delta\phi < M_p$.

Substituting the results obtained from Eq. (A.1), Eq. (A.2) and Eq. (B.2) (see Appendix A and Appendix B), and with the help of Eq. (3.4), up to the leading order, we obtain:

$$\begin{aligned}
 &\sqrt{\frac{r(k_\star)}{8}} \left\{ \left(\frac{a}{4} - \frac{b}{16} + \frac{c}{48} - \frac{1}{2} \right) \left[1 - \frac{k_e^2}{k_\star^2} \right] + \dots \right\} \\
 &\approx \left\{ \underbrace{\left(1 + \sum_{m=0}^{10} \mathbf{A}_m \left(\frac{\phi_e - \phi_0}{M_p} \right)^m \right)}_{\ll 1} \frac{\Delta\phi}{M_p} \right. \\
 &\quad \left. + \underbrace{\sum_{m=0}^{10} \frac{m \mathbf{A}_m}{2} \left(\frac{\phi_e - \phi_0}{M_p} \right)^{m-1} \left(\frac{\Delta\phi}{M_p} \right)^2}_{\ll 1} + \dots \right\},
 \end{aligned}
 \tag{3.5}$$

where $(k_e/k_\star) \approx \exp(-\Delta\mathcal{N}) = \exp(-17) \approx 4.13 \times 10^{-8}$ and we have defined a new dimensionless binomial expansion coefficient (\mathbf{A}_m) as:

$$\mathbf{A}_m = M_p^{m+2} \left[\left(\mathcal{C}_E - \frac{1}{2} \right) \mathbf{C}_m - 6\mathcal{C}_E \mathbf{D}_m \right] \quad (\forall m = 0, 1, 2, \dots, 10)
 \tag{3.6}$$

³ In this paper we fix $\Delta\mathcal{N} \approx 17$ e-foldings as within this interval the combined Planck+WMAP9 constraints on the amplitude of power spectrum $\ln(10^{10} P_S) = 3.089^{+0.024}_{-0.027}$ (within 2σ C.L.), spectral tilt $n_S = 0.9603 \pm 0.0073$ (within 2σ C.L.), running of the spectral tilt $\alpha_S = -0.0134 \pm 0.0090$ (within 1.5σ C.L.) and running of running of spectral tilt $\kappa_S = 0.020^{+0.016}_{-0.015}$ (within 1.5σ C.L.) are satisfied [1].

with an additional requirement $\mathbf{D}_m = 0$ for $m = 0$ and $m > 6$ obtained from the binomial series expansion obtained from the leading order results of the slow-roll integrals stated in Appendix A.⁴ Additionally it is also important to note that the expansion coefficient $\mathbf{A}_m(\forall m)$ are suppressed by the various powers of the scale of inflation, α , which is the leading order term in generic expansion of the inflationary potential as shown in Eq. (2.10) (see Eq. (A.3) in Appendix A). Consequently we can expand the left side of Eq. (3.5) in the powers of $\Delta\phi/M_p$, using the additional constraint $\Delta\phi < (\phi_e - \phi_0) < M_p$. This clearly implies that the highlighted terms by $\underbrace{\dots}$ are sufficiently smaller than unity for which we can easily neglect the higher order terms of $\Delta\phi/M_p$.

To the first order approximation – we can take $k_\star \approx k_{cmb}$ within 17 e-foldings of inflation, and neglecting all the higher powers of $k_e/k_\star \approx \mathcal{O}(10^{-8})$ from the left hand side of Eq. (3.5). Consequently, Eq. (3.5) reduces to the following compact form for $r(k_\star)$:

$$\frac{3}{25} \sqrt{\frac{r(k_\star)}{0.12}} \left\{ \frac{3}{400} \left(\frac{r(k_\star)}{0.12} \right) - \frac{\eta_V(k_\star)}{2} - \frac{1}{2} - \left(6\mathcal{C}_E + \frac{14}{3} \right) \epsilon_V^2(k_\star) - \frac{\eta_V^2(k_\star)}{6} + \left(\mathcal{C}_E - \frac{1}{4} \right) \frac{\xi_V^2(k_\star)}{2} - \left(2\mathcal{C}_E - \frac{5}{12} \right) \eta_V(k_\star) \epsilon_V(k_\star) - \frac{\sigma_V^3(k_\star)}{24} + \dots \right\} \approx \frac{|\Delta\phi|}{M_p}, \quad (3.7)$$

provided at the pivot scale, $k = k_\star \approx k_{cmb} \gg k_e$, here $\eta_V \gg \{\epsilon_V^2, \eta_V^2, \xi_V^2, \sigma_V^3, \dots\}$ approximation is valid for which at the leading order, the first three terms dominate over the other higher order contributions appearing in the right hand side of Eq. (3.7).

Now, it is also possible to recast $a(k)$, $b(k)$, $c(k)$, in terms of $r(k)$, and the slow roll parameters by using the relation, Eq. (2.5), to write:

$$\begin{aligned} a(k_\star) &\approx \left[\frac{r(k_\star)}{4} - 2\eta_V(k_\star) - 4 \left(2\mathcal{C}_E + \frac{1}{3} \right) \epsilon_V(k_\star) \eta_V(k_\star) - 4 \left(6\mathcal{C}_E + \frac{11}{3} \right) \epsilon_V^2(k_\star) \right. \\ &\quad \left. + 2\mathcal{C}_E \xi_V^2(k_\star) - \frac{2}{3} \eta_V^2(k_\star) + \dots \right], \\ b(k_\star) &\approx [16\epsilon_V^2(k_\star) - 12\epsilon_V(k_\star) \eta_V(k_\star) + 2\xi_V^2(k_\star) + \dots], \\ c(k_\star) &\approx [-2\sigma_V^3 + \dots], \end{aligned} \quad (3.8)$$

where “...” involves higher order slow-roll contributions which are negligibly small in the leading order approximation. The additional constraint $a \gg b \gg c$ defined in Eq. (2.9) is always satisfied by the general class of inflationary potentials, for instance the saddle or the inflection point models of inflation do satisfy this constraint [4].

The recent observations from *Planck* puts an upper bound on gravity waves via tensor-to-scalar ratio as $r(k_\star) \leq 0.12$ at the pivot scale, $k_\star = 0.002 \text{ Mpc}^{-1}$ [1]:

$$V_\star \leq (1.96 \times 10^{16} \text{ GeV})^4 \frac{r(k_\star)}{0.12}. \quad (3.9)$$

⁴ In Eq. (3.6), and Eqs. (A.1), (A.2) (see Appendix A), \mathbf{C}_p and \mathbf{D}_q are Planck suppressed dimensionful (mass dimension $[M_p^{-(m+2)}]$) binomial series expansion coefficient which are expressed in terms of the generic model parameters ($\alpha, \beta, \gamma, \kappa, \dots$) as presented in Eq. (2.10). These coefficients follow another additional restriction on the indices appearing in the subscript as, $p = 0, 1, \dots, 10$, and $q = 1, 2, \dots, 6$. Instead of using two indices (p, q) if we generalise them by a single index m as mentioned in Eq. (3.6), the above mentioned requirement on \mathbf{D}_m naturally appears in the present context.

Combining Eqs. (3.7) and (3.9), we have obtained a closed relationship between V_* and $\Delta\phi$, as:

$$\frac{|\Delta\phi|}{M_p} \leq \frac{\sqrt{V_*}}{(2.20 \times 10^{-2} M_p)^2} \left| \left\{ \frac{V_*}{(2.78 \times 10^{-2} M_p)^4} - \frac{\eta_V(k_*)}{2} - \frac{1}{2} - \left(6\mathcal{C}_E + \frac{14}{3}\right) \epsilon_V^2(k_*) \right. \right. \\ \left. \left. - \frac{\eta_V^2(k_*)}{6} + \left(\mathcal{C}_E - \frac{1}{4}\right) \frac{\xi_V^2(k_*)}{2} - \left(2\mathcal{C}_E - \frac{5}{12}\right) \eta_V(k_*) \epsilon_V(k_*) \right. \right. \\ \left. \left. - \frac{\sigma_V^3(k_*)}{24} + \dots \right\} \right|, \tag{3.10}$$

where $\eta_V \gg \{\epsilon_V^2, \eta_V^2, \xi_V^2, \sigma_V^3, \dots\}$ are satisfied, and at the leading order first three terms dominate over the other higher order contributions, therefore

$$\frac{|\Delta\phi|}{M_p} \leq \frac{\sqrt{V_*}}{(2.20 \times 10^{-2} M_p)^2} \left| \frac{V_*}{(2.78 \times 10^{-2} M_p)^4} - \frac{\eta_V(k_*)}{2} - \frac{1}{2} \right|. \tag{3.11}$$

The above Eqs. (3.10), (3.11) are new improved bounds on $\Delta\phi$ during a slowly rolling single field ϕ within an effective field theory treatment, where the vev of an inflaton remains sub-Planckian, i.e. $\phi_0 < M_p$ and $\Delta\phi \ll M_p$. From Eq. (3.7), we can see that large $r \sim 0.1$ can be obtained for models of inflation where inflation occurs below the Planck cut-off. Our conditions, Eqs. (3.7), (3.10), provide new constraints on model building for inflation within particle theory, where the inflaton potential is always constructed within an effective field theory with a cut-off. Note that $\eta_V(k_*) \geq 0$ can provide the largest contribution, in order to satisfy the current bound on $r \leq 0.12$, the shape of the potential has to be *concave*.

4. Summary and discussion

To summarise, in this paper we have presented an accurate bound on tensor to scalar ratio, r , and $\Delta\phi$ for a sub-Planckian models of inflation in presence of higher order slow-roll correction, see Eqs. (3.7), (3.10), (3.11). The bounds obtained here satisfy the numerical estimations made for inflation models based on saddle or inflection points with sub-Planckian VEVs.

Further, we have shown that it is indeed possible to realise large tensor-to-scalar ratio for sub-Planckian vevs of inflation by assuming the non-monotonicity of the slow-roll parameter ϵ_V . Our constraints would help inflationary model builders and perhaps would enable us to reconstruct the inflationary potential [25–28] for a single field model of inflation. We have also analysed the fact that the additional constraint on slow-roll parameter, $\eta_V(k_*) \geq 0$, at the pivot scale of momentum, k_* , also restricts the shape of the potential to be a *concave* one.

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Appendix A. Slow-roll integration

The crucial integrals of the first and second slow-roll parameters (ϵ_V, η_V) appearing in the right hand side of Eq. (2.8), which can be written up to the leading order as:

$$\begin{aligned}
 \int_{\phi_e}^{\phi_{cmb}} d\phi \epsilon_V &= \frac{M_p^2}{2} \int_{\phi_e}^{\phi_{cmb}} d\phi \frac{[\beta + 3\gamma(\phi - \phi_0)^2 + 4\kappa(\phi - \phi_0)^3 + \dots]^2}{[\alpha + \beta(\phi - \phi_0) + \gamma(\phi - \phi_0)^3 + \kappa(\phi - \phi_0)^4 + \dots]^2} \\
 &\approx \frac{1}{2} \sum_{p=0}^{10} \frac{M_p^{p+2} \mathbf{C}_p}{(p+1)} \left(\frac{\phi_e - \phi_0}{M_p}\right)^{p+1} \left\{ \left(1 + \frac{\Delta\phi}{M_p} \left(\frac{\phi_e - \phi_0}{M_p}\right)^{-1}\right)^{p+1} - 1 \right\} \\
 &\quad + \dots,
 \end{aligned} \tag{A.1}$$

$$\begin{aligned}
 \int_{\phi_e}^{\phi_{cmb}} d\phi \eta_V &= 6M_p^2 \int_{\phi_e}^{\phi_{cmb}} d\phi \frac{[\gamma(\phi - \phi_0) + 2\kappa(\phi - \phi_0)^2 + \dots]}{[\alpha + \beta(\phi - \phi_0) + \gamma(\phi - \phi_0)^3 + \kappa(\phi - \phi_0)^4 + \dots]} \\
 &\approx 6 \sum_{q=1}^6 \frac{M_p^{q+2} \mathbf{D}_q}{(q+1)} \left(\frac{\phi_e - \phi_0}{M_p}\right)^{q+1} \left\{ \left(1 + \frac{\Delta\phi}{M_p} \left(\frac{\phi_e - \phi_0}{M_p}\right)^{-1}\right)^{q+1} - 1 \right\} \\
 &\quad + \dots,
 \end{aligned} \tag{A.2}$$

where we have used the $(\phi - \phi_0) < M_p$ (including at $\phi = \phi_{cmb}$ and $\phi = \phi_e$) around ϕ_0 . The leading order dimensionful Planck scale suppressed expansion coefficients (\mathbf{C}_p) and (\mathbf{D}_q) are given in terms of the model parameters $(\alpha, \beta, \gamma, \kappa)$, which determine the height and shape of the potential in terms of the model parameters as:

$$\begin{aligned}
 \mathbf{C}_0 &= \frac{\beta^2}{\alpha^2}, & \mathbf{C}_1 &= -\frac{2\beta^3}{\alpha^3}, \\
 \mathbf{C}_2 &= \frac{6\beta\gamma}{\alpha^2}, & \mathbf{C}_3 &= \frac{8\beta\kappa}{\alpha^2} - \frac{14\beta^2\gamma}{\alpha^3}, \\
 \mathbf{C}_4 &= \frac{9\gamma^2}{\alpha^2} - \frac{18\beta^2\kappa}{\alpha^3}, & \mathbf{C}_5 &= \frac{24\gamma\kappa}{\alpha^2} - \frac{30\gamma^2\beta}{\alpha^3} - \frac{48\gamma^2\beta\kappa}{\alpha^3}, \\
 \mathbf{C}_6 &= \frac{16\kappa^2}{\alpha^2} - \frac{32\kappa^2\beta}{\alpha^3} - \frac{28\beta\gamma\kappa}{\alpha^3}, \\
 \mathbf{C}_7 &= -\frac{18\gamma^3}{\alpha^3} - \frac{16\beta\kappa^2}{\alpha^3}, & \mathbf{C}_8 &= -\frac{66\gamma^2\kappa}{\alpha^3}, \\
 \mathbf{C}_9 &= -\frac{80\gamma\kappa^2}{\alpha^3}, & \mathbf{C}_{10} &= -\frac{32\kappa^3}{\alpha^3}, \\
 \mathbf{D}_1 &= \frac{\gamma}{\alpha}, & \mathbf{D}_2 &= \frac{2\kappa}{\alpha} - \frac{\beta\gamma}{\alpha^2}, & \mathbf{D}_3 &= -\frac{2\kappa\beta}{\alpha^2}, \\
 \mathbf{D}_4 &= -\frac{\gamma^2}{\alpha^2}, & \mathbf{D}_5 &= -\frac{3\kappa\gamma}{\alpha^2}, & \mathbf{D}_6 &= -\frac{2\kappa^2}{\alpha^2}.
 \end{aligned} \tag{A.3}$$

Appendix B. Momentum integration

The momentum integral appearing in the left hand side of the Eq. (2.8) is computed as:

$$\begin{aligned}
 & \int_{k_e}^{k_{cmb}} \frac{dk}{k} \sqrt{\frac{r(k)}{8}} \\
 &= \sqrt{\frac{r(k_\star)}{8}} \int_{k_e}^{k_{cmb}} \frac{dk}{k} \left(\frac{k}{k_\star}\right)^{a+\frac{b}{2} \ln\left(\frac{k}{k_\star}\right) + \frac{c}{6} \ln^2\left(\frac{k}{k_\star}\right) + \dots} \\
 &= \begin{cases} \sqrt{\frac{r(k_\star)}{8}} \left\{ \left(\frac{a}{4} - \frac{b}{16} + \frac{c}{48} - \frac{1}{2}\right) \left[\frac{k_{cmb}^2}{k_\star^2} - \frac{k_e^2}{k_\star^2}\right] \right. \\ \quad + \left(\frac{b}{8} - \frac{c}{24}\right) \left[\frac{k_{cmb}^2}{k_\star^2} \ln\left(\frac{k_{cmb}}{k_\star}\right) - \frac{k_e^2}{k_\star^2} \ln\left(\frac{k_e}{k_\star}\right)\right] \\ \quad \left. + \frac{c}{24} \left[\frac{k_{cmb}^2}{k_\star^2} \ln^2\left(\frac{k_{cmb}}{k_\star}\right) - \frac{k_e^2}{k_\star^2} \ln^2\left(\frac{k_e}{k_\star}\right)\right] + \dots \right\} & \text{for } a, b, c \neq 0 \text{ \& } a > b > c \\ \\ \sqrt{\frac{r(k_\star)}{8}} \left\{ \left(\frac{a}{4} - \frac{b}{16} - \frac{1}{2}\right) \left[\frac{k_{cmb}^2}{k_\star^2} - \frac{k_e^2}{k_\star^2}\right] \right. \\ \quad \left. + \frac{b}{8} \left[\frac{k_{cmb}^2}{k_\star^2} \ln\left(\frac{k_{cmb}}{k_\star}\right) - \frac{k_e^2}{k_\star^2} \ln\left(\frac{k_e}{k_\star}\right)\right] + \dots \right\} & \text{for } a, b \neq 0, c = 0 \text{ \& } a > b \\ \\ \frac{1}{a+1} \sqrt{\frac{r(k_\star)}{8}} \left[\left(\frac{k_{cmb}}{k_\star}\right)^a - \left(\frac{k_e}{k_\star}\right)^a \right] & \text{for } a \neq 0 \text{ \& } b, c = 0 \\ \\ \sqrt{\frac{r(k_\star)}{8}} \ln\left(\frac{k_{cmb}}{k_e}\right) & \text{for } a, b, c = 0. \end{cases} \tag{B.1}
 \end{aligned}$$

Further using $k_{cmb} \approx k_\star$ and $(k_e/k_\star) \approx \exp(-\Delta\mathcal{N}) = \exp(-17) \approx 4.13 \times 10^{-8}$, within 17 e-foldings Eq. (B.1) can be simplified to the following expression:

$$\int_{k_e}^{k_{cmb}} \frac{dk}{k} \sqrt{\frac{r(k)}{8}} \approx \begin{cases} \sqrt{\frac{r(k_\star)}{8}} \left(\frac{a}{4} - \frac{b}{16} + \frac{c}{48} - \frac{1}{2}\right) + \dots & \text{for } a, b, c \neq 0 \text{ \& } a > b > c \\ \\ \sqrt{\frac{r(k_\star)}{8}} \left(\frac{a}{4} - \frac{b}{16} - \frac{1}{2}\right) + \dots & \text{for } a, b \neq 0, c = 0 \text{ \& } a > b \\ \\ \frac{1}{a+1} \sqrt{\frac{r(k_\star)}{8}} + \dots & \text{for } a \neq 0 \text{ \& } b, c = 0 \\ \\ \mathcal{O}(17) \times \sqrt{\frac{r(k_\star)}{8}} & \text{for } a, b, c = 0, \end{cases} \tag{B.2}$$

where the last possibility $a, b, c = 0$ surmounts to the Harrison & Zeldovich spectrum, which is completely ruled out by Planck+WMAP9 data within 5σ C.L. Similarly the next to last possibility $a \neq 0 \text{ \& } b, c = 0$ is also tightly constrained by the WMAP9 and Planck+WMAP9 data within 2σ C.L.

The second possibility $a, b \neq 0, c = 0$, and $a > b$ is favoured by WMAP9 data and tightly constrained within 2σ window by Planck+WMAP9 data.

Finally, $a, b, c \neq 0$, and $a > b > c$ case is satisfied by both WMAP9 and Planck+WMAP9 data within 2σ C.L. Here $a > b > c$ is the only criterium which is always satisfied by a general class of inflationary potentials. In this article, we have only focused on the first possibility, i.e. $a > b > c$, from which we have derived all the constraint conditions for a generic model of sub-Planckian inflationary potentials.

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