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On action of diffeomorphisms of C*-algebras on derivations

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Abstract

In this paper we consider automorphisms of the domains of closed *-derivations of C*-algebras and show that they extend to automorphisms of C*-algebras, so we call them diffeomorphisms. The diffeomorphisms generate transformations of the sets of closed *-derivations of C*-algebras. In this paper we study the subgroups of diffeomorphisms that define "bounded" shifts of derivations and the subgroups of the stabilizers of derivations.

Keywords: Derivations; C*-algebras; Automorphisms; Diffeomorphisms

1. Introduction

Extensive development of non-commutative geometry requires elaborating of the theory of the domains of closed *-derivations of C*-algebras whose properties in many respects are analogous to the properties of algebras of differentiable functions. In this paper we consider automorphisms of the domains of derivations and show that they extend to automorphisms of C*-algebras, so we call them diffeomorphisms. The diffeomorphisms generate transformations of the sets of closed *-derivations of C*-algebras. In this paper we study the subgroups of diffeomorphisms that define "bounded" shifts of derivations and the subgroups of the stabilizers of derivations.

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Throughout the paper we denote by $(\mathfrak{A}, \|\cdot\|)$ a C*-algebra. A closed linear map δ from a dense *-subalgebra $D(\delta)$ of $\mathfrak{A}$ into $\mathfrak{A}$ is called a *closed *-derivation* if

$$\delta(AB) = A\delta(B) + \delta(A)B \quad \text{and} \quad \delta\left(A^*\right) = \delta(A)^* \quad \text{for } A, B \in D(\delta).$$

The subalgebra $D(\delta)$ is called the domain of δ; δ is bounded if and only if $D(\delta) = \mathfrak{A}$.

Let $\mathcal{A}$ be a dense *-subalgebra of $\mathfrak{A}$. Denote by $\mathrm{Der}(\mathcal{A})$ the set of all closed *-derivations δ on $\mathfrak{A}$ with $\mathcal{A} = D(\delta)$. We call $\mathcal{A}$ a *domain* if $\mathrm{Der}(\mathcal{A}) \neq \emptyset$. In Section 2 we show that all *-automorphisms of a domain $\mathcal{A}$ of $\mathfrak{A}$ extend to *-automorphisms of $\mathfrak{A}$. We call these extensions *diffeomorphisms* of $\mathfrak{A}$; they form a group that we denote by $\mathrm{Dif}(\mathcal{A})$. Each diffeomorphism ϕ defines a transformation T_ϕ of $\mathrm{Der}(\mathcal{A})$: for every $\delta \in \mathrm{Der}(\mathcal{A})$, the derivation $T_\phi(\delta) = \phi^{-1}\delta\phi$ also belongs to $\mathrm{Der}(\mathcal{A})$. The map $T: \phi \to T_\phi$ is an antirepresentation of the group $\mathrm{Dif}(\mathcal{A})$ into the set of all transformations of $\mathrm{Der}(\mathcal{A})$: $T_{\phi\theta} = T_\theta T_\phi$. We denote by $\mathcal{Z}(\delta)$ the stabilizer of δ:

$$\mathcal{Z}(\delta) = \left\{\phi \in \mathrm{Dif}(\mathcal{A}):\ \delta = T_\phi(\delta)\right\}$$

and by $B(\delta)$ the subgroup of $\mathrm{Dif}(\mathcal{A})$ of diffeomorphisms that define *bounded* shifts of δ:

$$B(\delta) = \left\{\phi \in \mathrm{Dif}(\mathcal{A}):\ \text{the derivation } T_\phi(\delta) - \delta \text{ is bounded on } \mathcal{A} \text{ in } \|\cdot\|\right\}.$$

Denote by $B(H)$ the algebra of all bounded operators on a Hilbert space H and by $C(H)$ the ideal of all compact operators. In this paper we study the structure of the groups $\mathcal{Z}(\delta)$ and $B(\delta)$, for $\delta \in \mathrm{Der}(\mathcal{A})$, when $\mathcal{A}$ are domains of C*-subalgebras $\mathfrak{A}$ of $B(H)$ that contain $C(H)$.

An operator F on H with the dense domain $D(F)$ *implements* $\delta \in \mathrm{Der}(\mathcal{A})$ if

$$AD(F) \subseteq D(F) \quad \text{and} \quad \delta(A)|_{D(F)} = i[F, A]|_{D(F)} = i(FA - AF)|_{D(F)} \quad \text{for all } A \in \mathcal{A}. \tag{1.1}$$

Bratteli and Robinson proved in [2] that, if $C(H) \subseteq \mathfrak{A} \subseteq B(H)$ and $\mathcal{A}$ is a domain of $\mathfrak{A}$, then each $\delta \in \mathrm{Der}(\mathcal{A})$ has a symmetric implementation: a closed symmetric operator S on H that implements δ. The operator S can be chosen (see [5, Theorem 27.21]) to be a *minimal implementation*, that is, for each closed operator F that implements δ,

$$S + t\mathbf{1}|_{D(S)} \subseteq F \quad \text{for some } t \in \mathbb{C}.$$

With each closed symmetric operator S on H, we associate a *-subalgebra

$$\mathcal{A}_S = \left\{A \in B(H):\ AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S) \text{ and } [S, A]|_{D(S)} \text{ is bounded}\right\} \tag{1.2}$$

of $B(H)$. It is the domain (see [5]) of a closed *-derivation δ_S into $B(H)$ defined by

$$\delta_S(A) = i\overline{[S, A]} \quad \text{for } A \in \mathcal{A}_S,$$

where $\overline{[S, A]}$ is the closure of $[S, A]|_{D(S)} = (SA - AS)|_{D(S)}$. Furthermore, $\mathcal{A}_S = B(H)$ if and only if S is bounded. If S is unbounded, δ_S is unbounded and $\mathcal{A}_S$ is a Hermitian semisimple Banach *-algebra with respect to the norm

$$\|A\|_{\delta_S} = \|A\| + \left\|\delta_S(A)\right\| \quad \text{for } A \in \mathcal{A}_S.$$

Denote by $\mathcal{F}_S$ the closure in $\|\cdot\|_{\delta_S}$ of the set of all finite rank operators in $\mathcal{A}_S$ and set

$$\mathcal{J}_S = \{A \in \mathcal{A}_S \cap C(H)\colon\ \delta_S(A) \in C(H)\}.$$

Then (see [5]) $\mathcal{F}_S$ and $\mathcal{J}_S$ are domains of $C(H)$ and closed *-ideals of $\mathcal{A}_S$. The *-derivations

$$\delta_S^{\min} = \delta_S|\mathcal{F}_S \quad \text{and} \quad \delta_S^{\max} = \delta_S|\mathcal{J}_S$$

of $C(H)$ are closed; they are the *minimal* and the *maximal* closed *-derivations of $C(H)$ with minimal implementation S. It was proved in [6] that the closure of $(\mathcal{J}_S)^2$ in $\|\cdot\|_{\delta_S}$ coincides with $\mathcal{F}_S$ and that $\mathcal{J}_S = \mathcal{F}_S$ if S is selfadjoint.

In Section 3 we establish a link between minimal symmetric implementations of two derivations from $\mathrm{Der}(\mathcal{A})$. We prove that if S and T are such implementations, then the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ coincide and the norms $\|\cdot\|_{\delta_S}$ and $\|\cdot\|_{\delta_T}$ on them are equivalent. It was shown in [7] that if these norms are equal then $S - t\mathbf{1} = \pm UTU^*$ for some unitary operator U and $t \in \mathbb{R}$. In Section 3 we consider the general case and obtain some necessary conditions that S and T satisfy.

Denote by $\mathcal{U}_S$ the group of all unitary operators in the algebra $\mathcal{A}_S$ and set

$$\mathcal{Z}_S = \{U \in \mathcal{U}_S\colon\ \delta_S(U) = \lambda U \text{ for some } \lambda \in \mathbb{C}\}.$$

We show in Section 4 that if $C(H) \subseteq \mathfrak{A} \subseteq B(H)$ and $\mathcal{A}$ is a domain of $\mathfrak{A}$, then each $\phi \in \mathrm{Dif}(\mathcal{A})$ is implemented by a unitary operator U_ϕ: $\phi(A) = U_\phi A U_\phi^*$ for all $A \in \mathfrak{A}$. Moreover, if $\delta \in \mathrm{Der}(\mathcal{A})$ then $\phi \in B(\delta)$ if and only if $U_\phi \in \mathcal{U}_S$, and $\phi \in \mathcal{Z}(\delta)$ if and only if $U_\phi \in \mathcal{Z}_S$, where S is a minimal symmetric implementation of δ. Identifying $\mathrm{Dif}(\mathcal{A})$, $B(\delta)$ and $\mathcal{Z}(\delta)$ with the corresponding subgroups of unitary operators, we have

$$B(\delta) = \mathrm{Dif}(\mathcal{A}) \cap \mathcal{U}_S \quad \text{and} \quad \mathcal{Z}(\delta) = \mathrm{Dif}(\mathcal{A}) \cap \mathcal{Z}_S.$$

Section 5 is devoted to the investigation of the structure of the groups $\mathcal{Z}_S$. In Section 6 we study the problem of constructing domains of C*-algebras that extend the domains $\mathcal{J}_S$. Let $\mathcal{A}$ be a domain of a C*-subalgebra $\mathfrak{A}$ of $B(H)$ and let $C(H) \nsubseteq \mathfrak{A}$. Assume that there is a derivation in $\mathrm{Der}(\mathcal{A})$ implemented by a symmetric operator S. Then $\mathcal{A} + \mathcal{J}_S$ is a dense *-subalgebra of the C*-algebra $\mathfrak{A} + C(H)$ and $\delta = \delta_S|(\mathcal{A} + \mathcal{J}_S)$ is a *-derivation of $\mathfrak{A} + C(H)$. We provide some sufficient conditions for δ to be a closed derivation which implies that $\mathcal{A} + \mathcal{J}_S$ is a domain of $\mathfrak{A} + C(H)$. Numerous examples of such domains can be obtained by considering the **-commutant*

$$\mathcal{C}_S = \operatorname{Ker}\delta_S = \{A \in \mathcal{A}_S\colon\ \delta_S(A) = 0\}$$

of S. It is a W*-algebra and we prove that, for each C*-subalgebra $\mathfrak{A}$ of $\mathcal{C}_S$ satisfying some simple conditions, the algebra $\mathfrak{A} + \mathcal{J}_S$ is a domain of the C*-algebra $\mathfrak{A} + C(H)$. In particular, $\mathcal{C}_S + \mathcal{J}_S$ is a domain of the C*-algebra $\mathcal{C}_S + C(H)$. Finally, we show that, for each symmetric operator S,

$$B(\delta_S^{\min}) = B(\delta_S^{\max}) = \mathcal{U}_S \quad \text{and} \quad \mathcal{Z}(\delta_S^{\min}) = \mathcal{Z}(\delta_S^{\max}) = \mathcal{Z}(\delta) = \mathcal{Z}_S \quad \text{where } \delta = \delta_S|(\mathcal{C}_S + \mathcal{J}_S).$$

All symmetric operators in this paper are assumed to be closed.

2. Extension of automorphisms from subalgebras of C*-algebras

Let $\mathcal{A}$ be a dense *-subalgebra of a unital C*-algebra $\mathfrak{A}$. It is called a Q-subalgebra of $\mathfrak{A}$ if

$$\mathbf{1} \in \mathcal{A} \quad \text{and} \quad \mathrm{Sp}_{\mathcal{A}}(A) = \mathrm{Sp}_{\mathfrak{A}}(A) \quad \text{for all } A \in \mathcal{A}. \tag{2.1}$$

If $\mathcal{A}$ is a dense *-subalgebra of a non-unital C*-algebra $\mathfrak{A}$, consider the unitizations $\widetilde{\mathfrak{A}} = \mathfrak{A} + \mathbb{C}\mathbf{1}$ of $\mathfrak{A}$ and $\widetilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}\mathbf{1}$ of $\mathcal{A}$. The algebra $\mathcal{A}$ is a Q-subalgebra of $\mathfrak{A}$ if

$$\mathrm{Sp}_{\widetilde{\mathcal{A}}}(A) = \mathrm{Sp}_{\widetilde{\mathfrak{A}}}(A) \quad \text{for all } A \in \mathcal{A}.$$

The domains of closed *-derivations of $\mathfrak{A}$ are Q-subalgebras of $\mathfrak{A}$ (see [2,5]).

Proposition 2.1. *Let $\mathcal{A}$ be a Q-subalgebra of a C*-algebra $\mathfrak{A}$ and let ϕ be a *-automorphism of $\mathcal{A}$. Then $\|\phi\| = 1$, so ϕ extends to a *-automorphism of $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be unital. Since $\mathrm{Sp}_{\mathcal{A}}(A) = \mathrm{Sp}_{\mathcal{A}}(\phi(A))$, for $A \in \mathcal{A}$, we have

$$\mathrm{Sp}_{\mathfrak{A}}(A) = \mathrm{Sp}_{\mathcal{A}}(A) = \mathrm{Sp}_{\mathcal{A}}\bigl(\phi(A)\bigr) = \mathrm{Sp}_{\mathfrak{A}}\bigl(\phi(A)\bigr). \tag{2.2}$$

If $A = A^* \in \mathcal{A}$ then $\phi(A)^* = \phi(A^*) = \phi(A)$ and, by (2.2),

$$\|A\| = \sup_{\lambda \in \mathrm{Sp}_{\mathfrak{A}}(A)} |\lambda| = \sup_{\lambda \in \mathrm{Sp}_{\mathfrak{A}}(\phi(A))} |\lambda| = \bigl\|\phi(A)\bigr\|.$$

Hence, for $B \in \mathcal{A}$,

$$\|B\|^2 = \bigl\|B^*B\bigr\| = \bigl\|\phi\bigl(B^*B\bigr)\bigr\| = \bigl\|\phi(B)^*\phi(B)\bigr\| = \bigl\|\phi(B)\bigr\|^2.$$

For non-unital $\mathfrak{A}$, we have the proof by replacing in the above argument $\mathcal{A}$ by $\widetilde{\mathcal{A}}$ and $\mathfrak{A}$ by $\widetilde{\mathfrak{A}}$. □

For a Q-subalgebra $\mathcal{A}$ of a C*-algebra $\mathfrak{A}$, denote by $\mathrm{Der}(\mathcal{A})$ the set of all closed unbounded *-derivations δ on $\mathfrak{A}$ with $\mathcal{A} = D(\delta)$. We call $\mathcal{A}$ a *domain* if $\mathrm{Der}(\mathcal{A}) \neq \emptyset$. We call a *-automorphism ϕ of $\mathfrak{A}$ a *diffeomorphism*, if it preserves a domain $\mathcal{A}$ in $\mathfrak{A}$ and denote by $\mathrm{Dif}(\mathcal{A})$ the group of all diffeomorphisms of $\mathfrak{A}$ that preserve $\mathcal{A}$. Proposition 2.1 yields

Corollary 2.2. *$\phi \to \phi|_{\mathcal{A}}$ is an isomorphism from* $\mathrm{Dif}(\mathcal{A})$ *onto the set of all *-automorphisms of $\mathcal{A}$.*

Any domain $\mathcal{A}$ is a Hermitian semisimple Banach *-algebra (see [5]) with respect to each norm

$$\|A\|_\delta = \|A\| + \bigl\|\delta(A)\bigr\| \quad \text{for } A \in \mathcal{A}, \quad \text{where } \delta \in \mathrm{Der}(\mathcal{A}).$$

For each bounded derivation δ_b on $\mathfrak{A}$, $\delta + \delta_b \in \mathrm{Der}(\mathcal{A})$. Johnson's uniqueness of norm theorem yields

Proposition 2.3. *All norms $\|\cdot\|_\delta$, $\delta \in$* $\mathrm{Der}(\mathcal{A})$*, on a domain $\mathcal{A}$ are equivalent.*

Each $\phi \in \mathrm{Dif}(\mathcal{A})$ defines a transformation T_ϕ of $\mathrm{Der}(\mathcal{A})$ by the formula

$$T_\phi(\delta) = \delta_\phi = \phi^{-1}\delta\phi|\mathcal{A}, \quad \text{for } \delta \in \mathrm{Der}(\mathcal{A}).$$

Then $T_{\phi\psi} = T_\psi T_\phi$, so $T : \phi \to T_\phi$ is an antirepresentation of the group $\mathrm{Dif}(\mathcal{A})$ into the set of all transformations of $\mathrm{Der}(\mathcal{A})$. Denote by $\mathcal{Z}(\delta)$ the stabilizer of δ in $\mathrm{Dif}(\mathcal{A})$:

$$\mathcal{Z}(\delta) = \{\phi \in \mathrm{Dif}(\mathcal{A}):\ \delta = \delta_\phi\}$$

and by $B(\delta)$ the subgroup of $\mathrm{Dif}(\mathcal{A})$ of diffeomorphisms which define *bounded* shifts of δ:

$$B(\delta) = \{\phi \in \mathrm{Dif}(\mathcal{A}):\ \text{the derivation } \delta_\phi - \delta \text{ is bounded on } \mathcal{A} \text{ in } \|\cdot\|\}.$$

If $\psi \in B(\delta)$ then $B(\delta) = B(\delta_\psi)$. Denote by $\mathfrak{A}^*$ the dual space of $\mathfrak{A}$.

Proposition 2.4. *Let $\delta \in \mathrm{Der}(\mathcal{A})$ and $\phi \in \mathrm{Dif}(\mathcal{A})$. If there exists $\Delta \in \mathrm{Der}(\mathcal{A})$ such that, for each $A \in \mathcal{A}$ and $F \in \mathfrak{A}^*$, $F(\Delta_{\phi^n}(A)) \to F(\delta(A))$, as $n \to \infty$, then $\phi \in \mathcal{Z}(\delta)$.*

Proof. Define $F_{\phi^{-1}}$ by $F_{\phi^{-1}}(A) = F(\phi^{-1}(A))$, for $A \in \mathfrak{A}$. Then $F_{\phi^{-1}} \in \mathfrak{A}^*$, so, for $A \in \mathcal{A}$,

$$\begin{aligned} F\big(\Delta_{\phi^{n+1}}(A)\big) &= F_{\phi^{-1}}\big(\Delta_{\phi^n}\big(\phi(A)\big)\big) \to F_{\phi^{-1}}\big(\delta\big(\phi(A)\big)\big) \\ &= F\big(\phi^{-1}\big(\delta\big(\phi(A)\big)\big)\big) = F\big(\delta_\phi(A)\big). \end{aligned}$$

Since $F(\Delta_{\phi^{n+1}}(A)) \to F(\delta(A))$, we have $F(\delta_\phi(A)) = F(\delta(A))$. Thus $\delta_\phi(A) = \delta(A)$, so $\phi \in \mathcal{Z}(\delta)$. $\square$

3. Domains of C*-algebras containing $C(H)$

For $x, y \in H$, the rank one operator $x \otimes y$ on H acts by the formula

$$(x \otimes y)z = (z, x)y \quad \text{for } z \in H, \quad \text{and} \quad \|x \otimes y\| = \|x\|\|y\|.$$

Let F be an operator on H. For $u, v \in H$,

$$\begin{gathered} (x \otimes y)(u \otimes v) = (v, x)(u \otimes y), \qquad (x \otimes y)^* = y \otimes x, \\ x \otimes \lambda y = \lambda(x \otimes y) = \bar{\lambda}x \otimes y, \\ F(x \otimes y) = x \otimes Fy, \quad (x \otimes y)F = F^*x \otimes y, \quad \text{if } y \in D(F),\ x \in D\big(F^*\big). \end{gathered} \tag{3.1}$$

For an algebra of operators $\mathcal{A}$, denote by $\mathcal{F}(\mathcal{A})$ the subalgebra of all finite rank operators in $\mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be a domain of $\mathfrak{A}$ and $C(H) \subseteq \mathfrak{A} \subseteq B(H)$. For $\delta \in \mathrm{Der}(\mathcal{A})$, let a symmetric operator S be its minimal implementation. Then*

(i) *the set of all rank one operators in $\mathcal{A}$ consists of all $y \otimes x$ with $x, y \in D(S)$;*
(ii) $\mathcal{F}(\mathcal{A}) = \{\sum_{i=1}^n x_i \otimes y_i:\ x_i, y_i \in D(S)\} = \mathcal{F}(\mathcal{A}_S)$.

Proof. First let us show that the set

$$E_\delta = \{x \in H\colon\ x \otimes x \in \mathcal{A}\}$$

is a dense linear subspace of H and each rank one operator in $\mathcal{A}$ has form $y \otimes x$, for $x, y \in E_\delta$. For each $x \in H$, $\|x\| = 1$, the rank one projection $x \otimes x$ belongs to $\mathfrak{A}$. It follows from [9, Proposition 3.4.9] that, for every $\varepsilon > 0$, there is a projection $p_\varepsilon \in D(\delta) = \mathcal{A}$ such that $\|x \otimes x - p_\varepsilon\| < \varepsilon$. Hence there is $x_\varepsilon \in H$, $\|x_\varepsilon\| = 1$, such that $p_\varepsilon = x_\varepsilon \otimes x_\varepsilon$, so $x_\varepsilon \in E_\delta$. As $\|x \otimes x - x_\varepsilon \otimes x_\varepsilon\| < \varepsilon$, we have

$$\bigl\|x - (x, x_\varepsilon)x_\varepsilon\bigr\| = \bigl\|(x \otimes x - x_\varepsilon \otimes x_\varepsilon)x\bigr\| < \varepsilon.$$

Thus the set E_δ is dense in H.

For $x \in E_\delta$ and $\lambda \in \mathbb{C}$, $\lambda x \in E_\delta$. If $y \in E_\delta$ and $\alpha = (y, x) \neq 0$, then $(x \otimes x)(y \otimes y) = \alpha(y \otimes x) \in \mathcal{A}$. Hence $y \otimes x \in \mathcal{A}$ and $x \otimes y = (y \otimes x)^* \in \mathcal{A}$. Therefore $(x \pm y) \otimes (x \pm y) \in \mathcal{A}$, so $x \pm y \in E_\delta$.

Let $(y, x) = 0$. Since E_δ is dense in H, there is z in E_δ such that $u = z - (x + y)$ satisfies $\|u\| < \frac{1}{4}\min(\|x\|, \|y\|)$. Then $(z, x) \neq 0$ and $(z, y) \neq 0$. Hence $x + z \in E_\delta$, $y + z \in E_\delta$ and

$$\begin{aligned}
\bigl|(x+z, y+z)\bigr| &= \bigl|(2x + y + u, x + 2y + u)\bigr| \\
&= \bigl|2\|x\|^2 + 2\|y\|^2 + \|u\|^2 + (u, x) + 2(u, y) + 2(x, u) + (y, u)\bigr| \\
&\geqslant 2\|x\|^2 + 2\|y\|^2 + \|u\|^2 - 3\bigl(\|x\| + \|y\|\bigr)\|u\| > 0.
\end{aligned}$$

Therefore $(x + z) - (y + z) = x - y \in E_\delta$. Similarly, $x + y \in E_\delta$. Thus E_δ is a linear space.

If $x \otimes y \in \mathcal{A}$ then $y \otimes x \in \mathcal{A}$, so $(x \otimes y)(y \otimes x) = \|x\|^2(y \otimes y) \in \mathcal{A}$. Hence $y \in E_\delta$. Similarly, $x \in E_\delta$. Conversely, let $x, y \in E_\delta$. Then $x + y, x + iy \in E_\delta$, so, by (3.1),

$$y \otimes x = \frac{1}{2}\bigl[(x + y) \otimes (x + y) - x \otimes x - y \otimes y\bigr] + \frac{i}{2}\bigl[(x + iy) \otimes (x + iy) - x \otimes x - y \otimes y\bigr]$$

belongs to $\mathcal{A}$. Thus each rank one operator in $\mathcal{A}$ has form $y \otimes x$ for $x, y \in E_\delta$.

We shall prove now that $E_\delta = D(S)$. Let $x \in E_\delta$. By (1.1), for $y \in D(S)$, we have $(x \otimes x)y = (y, x)x \in D(S)$. Since $D(S)$ is dense in H, $x \in D(S)$. Thus $E_\delta \subseteq D(S)$.

For each $A \in \mathcal{A}$, $A(x \otimes x) = x \otimes Ax \in \mathcal{A}$. Hence $Ax \in E_\delta$, so E_δ is invariant for all operators from $\mathcal{A}$. Then the operator $T = S|_{E_\delta}$ is closable, densely defined and implements δ. Hence its closure $\overline{T}$ also implements δ. Since S is a minimal implementation of δ, $\overline{T} = S$, that is, E_δ is a core of S. Therefore, for each $y \in D(S)$, there are y_n in E_δ such that $y_n \to y$ and $Sy_n \to Sy$. Then

$$\begin{aligned}
\|y \otimes y - y_n \otimes y_n\| &\leqslant \|y \otimes y - y \otimes y_n\| + \|y \otimes y_n - y_n \otimes y_n\| \\
&= \|y\|\|y - y_n\| + \|y - y_n\|\|y_n\| \to 0,
\end{aligned} \tag{3.2}$$

so $y_n \otimes y_n \to y \otimes y$. For all $x, y \in E_\delta$, we have $x \otimes y \in \mathcal{A}$ and it follows from (1.1) that

$$\delta(x \otimes y) = i[S, x \otimes y] = i(x \otimes Sy - Sx \otimes y). \tag{3.3}$$

Using it, we obtain as in (3.2) that

$$\delta(y_n \otimes y_n) = i(y_n \otimes Sy_n - Sy_n \otimes y_n) \to i(y \otimes Sy - Sy \otimes y).$$

Since δ is a closed derivation, $y \otimes y \in D(\delta) = \mathcal{A}$. Hence $y \in E_\delta$, so $E_\delta = D(S)$. Part (i) is proved.

Clearly, all operators $\sum_{i=1}^n x_i \otimes y_i$, with $x_i,\ y_i \in D(S)$, belong to $\mathcal{F}(\mathcal{A})$. Conversely, each $A \in \mathcal{F}(\mathcal{A})$ has form $A = \sum_{i=1}^n x_i \otimes y_i$, where all x_i are linearly independent and all y_i are linearly independent. Since S implements δ and $D(S)$ is dense in H,

$$Az = \sum_{i=1}^n (z, x_i) y_i \in D(S) \quad \text{for all } z \in D(S).$$

Hence all $y_i \in D(S)$. As $A^* = \sum_{i=1}^n y_i \otimes x_i \in \mathcal{F}(\mathcal{A})$, all $x_i \in D(S)$. Thus $\mathcal{F}(\mathcal{A}) = \{\sum_{i=1}^n x_i \otimes y_i\colon x_i, y_i \in D(S)\}$. From this and from [6, Lemma 3.1] it follows that $\mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{A}_S)$. $\square$

For $\delta \in \mathrm{Der}(\mathcal{A})$, denote by $\mathcal{F}(\mathcal{A}, \delta)$ the closure of $\mathcal{F}(\mathcal{A})$ in $\|\cdot\|_\delta$. Recall that $\mathcal{F}_S$ is the closure of $\mathcal{F}(\mathcal{A}_S)$ in $\|\cdot\|_{\delta_S}$.

Corollary 3.2. *Let $\mathcal{A}$ be a domain in $\mathfrak{A}$, $C(H) \subseteq \mathfrak{A} \subseteq B(H)$. Let $\delta, \sigma \in \mathrm{Der}(\mathcal{A})$ and let S, T be, respectively, their minimal symmetric implementations. Then*

(i) *$\mathcal{F}(\mathcal{A}, \delta)$ is an ideal of $\mathcal{A}$ isometrically isomorphic to the algebra $(\mathcal{F}_S, \|\cdot\|_{\delta_S})$.*
(ii) *The algebras $\mathcal{F}(\mathcal{A}, \delta)$ and $\mathcal{F}(\mathcal{A}, \sigma)$ coincide and $D(S) = D(T)$.*

Proof. Since S implements δ, it follows from (1.1) that $\mathcal{A} = D(\delta) \subseteq \mathcal{A}_S$ and $\delta = \delta_S|D(\delta)$. Hence the norms $\|\cdot\|_{\delta_S}$ and $\|\cdot\|_\delta$ coincide on $\mathcal{A}$, so it follows from Lemma 3.1(ii) that $\mathcal{F}(\mathcal{A}, \delta)$ and $\mathcal{F}_S$ are isometrically isomorphic. As $\mathcal{F}_S$ is an ideal of $\mathcal{A}_S$ (see [6]), $\mathcal{F}(\mathcal{A}, \delta)$ is an ideal of $\mathcal{A}$.

By Proposition 2.3, the norms $\|\cdot\|_\delta$ and $\|\cdot\|_\sigma$ on $\mathcal{A}$ are equivalent, so the algebras $\mathcal{F}(\mathcal{A}, \delta)$ and $\mathcal{F}(\mathcal{A}, \sigma)$ coincide. Since $\mathcal{A} = D(\delta) = D(\sigma)$, we have from Lemma 3.1(i) that $D(S) = D(T)$. $\square$

Let S, T be minimal symmetric implementations of $\delta, \sigma \in \mathrm{Der}(\mathcal{A})$. It follows from Proposition 2.3 and Corollary 3.2 that the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ coincide and the norms $\|\cdot\|_{\delta_S}$ and $\|\cdot\|_{\delta_T}$ on them are equivalent. It was shown in [7, Theorem 4.4] that these norms are equal if and only if $S - t\mathbf{1} = \pm UTU^*$ for some $t \in \mathbb{R}$ and a unitary operator U. Below we consider the general case and obtain some necessary conditions that S and T satisfy.

Theorem 3.3. *Let $\mathcal{A}$ be a domain in $\mathfrak{A}$, $C(H) \subseteq \mathfrak{A} \subseteq B(H)$. Let symmetric operators S and T be minimal implementations of $\delta, \sigma \in \mathrm{Der}(\mathcal{A})$, respectively. Then*

(i) *S and T are either both selfadjoint or both non-selfadjoint;*
(ii) *there exist bounded invertible operators M from $(S - i\mathbf{1})D(S)$ onto $(T - i\mathbf{1})D(T)$ and N from $(S + i\mathbf{1})D(S)$ onto $(T + i\mathbf{1})D(T)$ such that*

$$T - i\mathbf{1} = M(S - i\mathbf{1}) \quad \text{and} \quad T + i\mathbf{1} = N(S + i\mathbf{1});$$

(iii) *the derivation $\sigma - \delta$ is bounded if and only if $T = S + R$ where R is selfadjoint and bounded.*

Proof. It was shown in [6] that the algebra $(\mathcal{F}_S, \|\cdot\|_{\delta_S})$ has a bounded approximate identity if and only if S is selfadjoint. By Corollary 3.2, $\mathcal{F}_S = \mathcal{F}_T$ and the norms $\|\cdot\|_{\delta_S}$ and $\|\cdot\|_{\delta_T}$ on them are equivalent. This yields (i).

By Corollary 3.2(ii), $D(S) = D(T)$. Fix $x \in D(S)$ with $\|x\| = 1$. By Proposition 2.3, there is $C > 0$ such that, for all $y \in D(S)$,

$$\|x \otimes y\|_\sigma = \|x \otimes y\| + \big\|\sigma(x \otimes y)\big\| \leqslant C\|x \otimes y\|_\delta = C\|x \otimes y\| + C\big\|\delta(x \otimes y)\big\|.$$

The operators $T - i\mathbf{1}$ and $S - i\mathbf{1}$ implement σ and δ. As $\|x \otimes y\| = \|x\|\|y\|$, we have from (3.3)

$$\begin{aligned} &\|x\|\|y\| + \big\|x \otimes (T - i\mathbf{1})y - \big((T + i\mathbf{1})x\big) \otimes y\big\| \\ &\quad \leqslant C\big(\|x\|\|y\| + \big\|x \otimes (S - i\mathbf{1})y - \big((S + i\mathbf{1})x\big) \otimes y\big\|\big). \end{aligned}$$

Therefore

$$\begin{aligned} \big\|(T - i\mathbf{1})y\big\| = \big\|x \otimes (T - i\mathbf{1})y\big\| &\leqslant \big\|x \otimes (T - i\mathbf{1})y - \big((T + i\mathbf{1})x\big) \otimes y\big\| + \big\|\big((T + i\mathbf{1})x\big) \otimes y\big\| \\ &\leqslant C\big(\|x\|\|y\| + \big\|x \otimes (S - i\mathbf{1})y - \big((S + i\mathbf{1})x\big) \otimes y\big\|\big) + \big\|(T + i\mathbf{1})x\big\|\|y\| \\ &\leqslant C\|y\| + C\big\|(S - i\mathbf{1})y\big\| + \big\|(S + i\mathbf{1})x\big\|\|y\| + \big\|(T + i\mathbf{1})x\big\|\|y\| \\ &\leqslant K\|y\| + C\big\|(S - i\mathbf{1})y\big\|. \end{aligned}$$

Since S is symmetric, $\|(S - i1)y\|^2 = \|Sy\|^2 + \|y\|^2$. Hence

$$\big\|(T - i\mathbf{1})y\big\| \leqslant (K + C)\big\|(S - i\mathbf{1})y\big\| \quad \text{for } y \in D(S). \tag{3.4}$$

It is well known that $(S \pm i\mathbf{1})D(S)$ are closed subspaces of H and $\operatorname{Ker}(S \pm i\mathbf{1}) = \{0\}$. Define an operator M from $(S - i\mathbf{1})D(S)$ into $(T - i\mathbf{1})D(T)$ by $M(S - i\mathbf{1})y = (T - i\mathbf{1})y$, for $y \in D(S)$. By (3.4), M is bounded. Similarly, the operator R from $(T - i\mathbf{1})D(T)$ into $(S - i\mathbf{1})D(S)$ defined by $R(T - i\mathbf{1})y = (S - i\mathbf{1})y$, for $y \in D(T)$, is bounded. Hence $R = M^{-1}$.

Similarly, there is a bounded invertible operator N from $(S + i\mathbf{1})D(S)$ on $(T + i\mathbf{1})D(T)$ such that $T + i\mathbf{1} = N(S + i\mathbf{1})$. Part (ii) is proved.

For $R = R^* \in B(H)$, the *-derivation $\delta_R(A) = i[R, A]$, $A \in \mathfrak{A}$, is bounded. Hence $\delta + \delta_R \in \operatorname{Der}(\mathcal{A})$ and $S + R$ is its minimal implementation.

Conversely, let $\sigma - \delta$ be bounded. As $D(S) = D(T)$, the operator $R = T - S$ is symmetric on $D(S)$. There is $C > 0$ such that $\|\sigma(A) - \delta(A)\| \leqslant C\|A\|$ for all $A \in \mathcal{A}$. Hence, for all $x, y \in D(S)$, we have from (3.3) that $x \otimes y \in \mathcal{A}$ and

$$\big\|\sigma(x \otimes y) - \delta(x \otimes y)\big\| = \big\|i[T - S, x \otimes y]\big\| = \|x \otimes Ry - Rx \otimes y\| \leqslant C\|x \otimes y\| = C\|x\|\|y\|.$$

Fix x with $\|x\| = 1$. Then

$$\|Ry\| = \|x \otimes Ry\| \leqslant \|x \otimes Ry - Rx \otimes y\| + \|Rx \otimes y\| \leqslant C\|x\|\|y\| + \|Rx\|\|y\|.$$

Hence R is bounded on $D(S)$, so it extends to a selfadjoint bounded operator. $\square$

4. Diffeomorphisms of C*-algebras containing $C(H)$

Each *-automorphism ϕ of $C(H)$ is implemented by a unitary operator U: $\phi(B) = UBU^*$, for $B \in C(H)$ (see [8]). This is also true for all *-automorphisms ϕ of C*-subalgebras $\mathfrak{A}$ of $B(H)$ containing $C(H)$. Indeed, for $x, y, \in H$, set $R = \phi(x \otimes y)$. By (3.1), for all $z, u \in H$,

$$R^*z \otimes Ru = R(z \otimes u)R = \phi\big((x \otimes y)\phi^{-1}(z \otimes u)(x \otimes y)\big) = \big(\phi^{-1}(z \otimes u)y, x\big)R.$$

Hence R is a rank one operator, so ϕ and ϕ^{-1} map finite rank operators into finite rank operators. Thus $\phi(C(H)) = C(H)$ and there is a unitary U such that $\phi(B) = UBU^*$, for $B \in C(H)$. For $A \in \mathfrak{A}$ and all $x, y \in H$,

$$\begin{aligned} Ux \otimes UAy &= U(x \otimes Ay)U^* = \phi\big(A(x \otimes y)\big) = \phi(A)\phi(x \otimes y) \\ &= \phi(A)U(x \otimes y)U^* = \phi(A)(Ux \otimes Uy) = Ux \otimes \phi(A)Uy. \end{aligned}$$

Hence $\phi(A)Uy = UAy$ for all $y \in H$, so $\phi(A) = UAU^*$ for all $A \in \mathfrak{A}$.

Recall that we denote by $\mathcal{U}_S$ the group of all unitary operators in the algebra $\mathcal{A}_S$:

$$\mathcal{U}_S = \big\{U \in B(H)\colon\ U \text{ is unitary},\ UD(S) = D(S) \text{ and } [S, U]|_{D(S)} \text{ is bounded}\big\}$$

and set

$$\mathcal{Z}_S = \big\{U \in \mathcal{U}_S\colon\ \delta_S(U) = \lambda U \text{ for some } \lambda \in \mathbb{C}\big\}.$$

Theorem 4.1. *Let $\mathcal{A}$ be a domain in $\mathfrak{A}$, $C(H) \subseteq \mathfrak{A} \subseteq B(H)$, and let $\phi \in \operatorname{Dif}(\mathcal{A})$. Let, as above, a unitary $U \in B(H)$ implements ϕ: $\phi(A) = UAU^*$ for all $A \in \mathfrak{A}$. Then*

- (i) *if a symmetric operator S is a minimal implementation of $\delta \in \operatorname{Der}(\mathcal{A})$, then $UD(S) = D(S)$ and U^*SU is a minimal implementation of the *-derivation δ_ϕ;*
- (ii) *$\phi \in B(\delta)$ if and only if $U \in \mathcal{U}_S$;*
- (iii) *$\phi \in \mathcal{Z}(\delta)$ if and only if $U \in \mathcal{Z}_S$.*

Proof. By Lemma 3.1, $x \otimes x \in \mathcal{A}$ for $x \in D(S)$. Hence

$$\phi(x \otimes x) = U(x \otimes x)U^* = Ux \otimes Ux \in \mathcal{A},$$

so $Ux \in D(S)$. Thus $UD(S) \subseteq D(S)$. Since $\phi^{-1} \in \operatorname{Dif}(\mathcal{A})$ and implemented by U^*, we have $U^*D(S) \subseteq D(S)$. Therefore $UD(S) = D(S)$.

Let T implement δ. For all $A \in \mathcal{A}$, we have $UAU^* \in \mathcal{A}$, so $UAU^*D(T) \subseteq D(T)$. By (1.1),

$$\begin{aligned} \delta_\phi(A)|_{U^*D(T)} &= \phi^{-1}\big(\delta\big(\phi(A)\big)\big)\big|_{U^*D(T)} = U^*\delta\big(UAU^*\big)U\big|_{U^*D(T)} \\ &= U^*\delta\big(UAU^*\big)\big|_{D(T)} = U^*i\big[T, UAU^*\big]\big|_{D(T)} = i\big[U^*TU, A\big]\big|_{U^*D(T)}. \end{aligned} \tag{4.1}$$

Thus U^*TU implements δ_ϕ. Similarly, if R implements δ_ϕ, URU^* implements δ. Hence $T \to U^*TU$ is a one-to-one correspondence between the sets of implementations of δ and δ_ϕ. Since S is a minimal implementation of δ, U^*SU is a minimal implementation of δ_ϕ. Part (i) is proved.

It follows from (i) and from Theorem 3.3(iii) that $\delta_\phi - \delta$ is a bounded derivation if and only if $K = U^*SU - S$ is a bounded operator on $D(S)$. Hence $\phi \in B(\delta)$, if and only if the operator $[S, U] = UK$ is bounded on $D(S)$. Since $UD(S) = D(S)$, we have that $\phi \in B(\delta)$ if and only if $U \in \mathcal{U}_S$. Part (ii) is proved.

Let $\phi \in \mathcal{Z}(\delta)$. Then $U \in \mathcal{U}_S$ and, by (i), U^*SU is a minimal implementation of δ_ϕ and $D(U^*SU) = D(S)$. Since $\delta_\phi = \delta$ and S is a minimal implementation of δ, there is $\lambda \in \mathbb{C}$ such that $U^*SU = S + \lambda\mathbf{1}|_{D(S)}$. Hence $\delta_S(U) = i\lambda U$, so $U \in \mathcal{Z}_S$. Conversely, if $U \in \mathcal{Z}_S$ then $U^*SU = S + \lambda\mathbf{1}|_{D(S)}$. As U^*SU is a minimal implementation of δ_ϕ, we have $\delta_\phi = \delta$. $\square$

Let $\mathcal{A}$ be a domain in $\mathfrak{A}$, $C(H) \subseteq \mathfrak{A} \subseteq B(H)$. It follows from Theorem 4.1 that one can identify (modulo scalars from the unit circle) the group $\mathrm{Dif}(\mathcal{A})$ with the group of all unitary operators U on H whose action $A \to UAU^*$ preserve $\mathcal{A}$. For $\delta \in \mathrm{Der}(\mathcal{A})$, we will also identify the subgroups $B(\delta)$ and $\mathcal{Z}(\delta)$ with the corresponding subgroups of unitary operators. By Theorem 4.1, if S is a minimal implementation of δ then

$$\begin{aligned} B(\delta) &= \{U \in \mathcal{U}_S\colon\ U\mathcal{A}U^* = \mathcal{A}\} = \mathrm{Dif}(\mathcal{A}) \cap \mathcal{U}_S, \\ \mathcal{Z}(\delta) &= \{U \in \mathcal{Z}_S\colon\ U\mathcal{A}U^* = \mathcal{A}\} = \mathrm{Dif}(\mathcal{A}) \cap \mathcal{Z}_S. \end{aligned} \tag{4.2}$$

Proposition 4.2. *Let $\mathcal{A}$ be a domain in $\mathfrak{A}$, $C(H) \subseteq \mathfrak{A} \subseteq B(H)$, and let S be a minimal symmetric implementation of $\delta \in \mathrm{Der}(\mathcal{A})$. Then*

(i) *$B(\delta)$ is closed in $(\mathcal{A}_S, \|\cdot\|_{\delta_S})$ and $\mathcal{Z}(\delta)$ is closed in $(B(H), \|\cdot\|)$;*
(ii) *if $\mathcal{A}$ is an ideal of $\mathcal{A}_S$, then $B(\delta) = \mathcal{U}_S$ and $\mathcal{Z}(\delta) = \mathcal{Z}_S$.*

Proof. Let a sequence $\{U_n\}$ of unitaries in $B(\delta)$ converge to U in $(\mathcal{A}_S, \|\cdot\|_{\delta_S})$. Then $\|U - U_n\| \to 0$ and $\|\delta_S(U) - \delta_S(U_n)\| \to 0$. Hence U is unitary. For each $A \in \mathcal{A}$, we have $U_nAU_n^* \in \mathcal{A}$, $UAU^* \in \mathcal{A}_S$ and $\|UAU^* - U_nAU_n^*\| \to 0$. Hence

$$\begin{aligned} &\|\delta_S(UAU^*) - \delta(U_nAU_n^*)\| \\ &\quad = \|\delta_S(U)AU^* + U\delta(A)U^* + UA\delta_S(U^*) - \delta_S(U_n)AU_n^* - U_n\delta(A)U_n^* - U_nA\delta_S(U_n^*)\| \\ &\quad \leqslant \|\delta_S(U)AU^* - \delta_S(U_n)AU_n^*\| + \|U\delta(A)U^* - U_n\delta(A)U_n^*\| \\ &\qquad + \|UA\delta_S(U^*) - U_nA\delta_S(U_n^*)\| \to 0. \end{aligned}$$

Since δ is closed, $UAU^* \in \mathcal{A}$. Thus $U \in \mathrm{Dif}(\mathcal{A})$. As $U \in \mathcal{U}_S$, it follows from (4.2) that $U \in B(\delta)$.

Let $U_n \in \mathcal{Z}(\delta)$, $\delta_S(U_n) = \lambda_nU_n$, and let $U \in B(H)$ and $\|U - U_n\| \to 0$. If $\lambda_n \to \infty$, then $U_n/\lambda_n \to 0$ and $\delta_S(U_n/\lambda_n) = U_n \to U$. Since δ_S is a closed derivation, $U = 0$. This contradiction shows that $\{\lambda_n\}$ is bounded. Choose a subsequence converging to some λ and denote it also by $\{\lambda_n\}$. Then $\delta_S(U_n) = \lambda_nU_n \to \lambda U$. Since δ_S is a closed derivation, $U \in \mathcal{U}_S$ and $\delta_S(U) = \lambda U$. Hence U_n converge to U in $\|\cdot\|_{\delta_S}$ and, as above, $U \in \mathrm{Dif}(\mathcal{A})$. By (4.2), $U \in \mathcal{Z}(\delta)$. Part (i) is proved.

If $\mathcal{A}$ is an ideal of $\mathcal{A}_S$ then, for each U in $\mathcal{U}_S$, the map $A \to UAU^*$ preserves $\mathcal{A}$. Hence $\mathrm{Dif}(\mathcal{A}) \supseteq \mathcal{U}_S \supseteq \mathcal{Z}_S$ and (ii) follows from (4.2). $\square$

5. Structure of the group $\mathcal{Z}_S$

Let $U \in \mathcal{Z}_S$ and $\delta_S(U) = \lambda U$, for $\lambda \in \mathbb{C}$. Then $U^* \in \mathcal{U}_S$ and $\delta_S(U^*) = \delta_S(U)^* = \bar{\lambda} U^*$. As

$$0 = \delta_S(\mathbf{1}) = \delta_S(U^*U) = U^*\delta_S(U) + \delta_S(U^*)U = (\lambda + \bar{\lambda})U^*U = (\lambda + \bar{\lambda})\mathbf{1},$$

we have $\mathrm{Re}(\lambda) = 0$. For each $t \in \mathbb{R}$, set

$$\mathcal{Z}_S(t) = \{U \in \mathcal{Z}_S\colon\ \delta_S(U) = itU\} \quad \text{and} \quad \Gamma_S = \{t \in \mathbb{R}\colon\ \mathcal{Z}_S(t) \neq \emptyset\}.$$

Then $\mathcal{Z}_S(t)\mathcal{Z}_S(s) \subseteq \mathcal{Z}_S(t+s)$, for $t, s \in \Gamma_S$, so $U\mathcal{Z}_S(0) \subseteq \mathcal{Z}_S(t)$ and $U^*\mathcal{Z}_S(t) \subseteq \mathcal{Z}_S(0)$, for $U \in \mathcal{Z}_S(t)$. Hence

$$\begin{gathered}\mathcal{Z}_S(t) = U\mathcal{Z}_S(0) = \mathcal{Z}_S(0)U \quad \text{for each } U \in \mathcal{Z}_S(t),\\ \mathcal{Z}_S(-t) = \mathcal{Z}_S(t)^*, \qquad \mathcal{Z}_S(t+s) = \mathcal{Z}_S(t)\mathcal{Z}_S(s) \quad \text{and} \quad \mathcal{Z}_S = \bigcup_{t \in \Gamma_S} \mathcal{Z}_S(t).\end{gathered} \tag{5.1}$$

All sets $\mathcal{Z}_S(t)$ are norm closed and $\mathcal{Z}_S(0)$ is a selfadjoint normal subgroup of the group $\mathcal{Z}_S$; Γ_S is a subgroup of $\mathbb{R}$ by addition, isomorphic to the quotient group $\mathcal{Z}_S/\mathcal{Z}_S(0)$.

Denote by $\Lambda(S)$ and $\Lambda(S^*)$ the sets of all eigenvalues of operators S and S^* and by $H_\lambda(S)$ and $H_\lambda(S^*)$ the corresponding eigenspaces of S and S^*. For a selfadjoint S, let $E_S(\lambda)$ be the spectral resolution of the identity of S. Then

$$E_{S-t\mathbf{1}}(\lambda) = E_S(\lambda + t). \tag{5.2}$$

Let $t \in \mathbb{R} - \{0\}$. We say that a unitary operator U on H is an (S,t)-*shift* if

$$UD(S) = D(S) \quad \text{and} \quad UE_S(\lambda)U^* = E_S(\lambda + t) \quad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 5.1.

(i) *If $\mathcal{Z}_S(t) \neq \{0\}$ then the map $\lambda \to \lambda + t$ is an isomorphism of the sets* $\mathrm{Sp}(S)$, $\Lambda(S)$, $\mathrm{Sp}(S^*)$, $\Lambda(S^*)$. *For $U \in \mathcal{Z}_S(t)$ and all $\lambda \in \Lambda(S)$ and $\mu \in \Lambda(S^*)$,*

$$H_{\lambda+t}(S) = UH_\lambda(S) \quad \textit{and} \quad H_{\mu+t}(S^*) = UH_\mu(S^*).$$

(ii) *If S is selfadjoint then $U \in \mathcal{Z}_S(t)$, $t \neq 0$, if and only if U is an (S,t)-shift.*

Proof. We have $UD(S) = D(S)$, $U^*D(S) = D(S)$ and

$$(SU - US)|_{D(S)} = tU|_{D(S)} \quad \text{and} \quad (SU^* - U^*S)|_{D(S)} = -tU^*|_{D(S)}. \tag{5.3}$$

Hence

$$U^*(S - (\lambda + t)\mathbf{1})U|_{D(S)} = (S - \lambda\mathbf{1})|_{D(S)} \quad \text{for each } \lambda \in \mathbb{C}.$$

Therefore $\lambda \in \mathrm{Sp}(S)$ if and only if $\lambda + t \in \mathrm{Sp}(S)$. Hence $\lambda \to \lambda + t$ is an isomorphism of $\mathrm{Sp}(S)$.

By (5.3), for $x \in D(S)$ and $y \in D(S^*)$,

$$(Sx, Uy) = \big(U^*Sx, y\big) = \big(SU^*x, y\big) + \big(tU^*x, y\big) = \big(x, \big(US^* + tU\big)y\big).$$

Hence $Uy \in D(S^*)$ and $(S^*U - US^*)y = tUy$. Similarly, $U^*y \in D(S^*)$ and $(S^*U^* - U^*S^*)y = -tU^*y$. Therefore $UD(S^*) = D(S^*)$, $U^*D(S^*) = D(S^*)$ and

$$\big(S^*U - US^*\big)\big|_{D(S^*)} = tU|_{D(S^*)} \quad \text{and} \quad \big(S^*U^* - U^*S^*\big)\big|_{D(S^*)} = -tU^*|_{D(S^*)}. \tag{5.4}$$

Hence, as above, we have that $\lambda \to \lambda + t$ is an isomorphism of $\mathrm{Sp}(S^*)$.

For $\lambda \in \Lambda(S)$, we have from (5.3) that $UH_\lambda \subseteq H_{\lambda+t}$ and $U^*H_\lambda \subseteq H_{\lambda-t}$. Therefore

$$UH_\lambda \subseteq H_{\lambda+t} = UU^*H_{\lambda+t} \subseteq UH_\lambda.$$

Hence $UH_\lambda = H_{\lambda+t}$ and $\lambda \to \lambda + t$ is an isomorphism of $\Lambda(S)$. Using (5.4), we obtain that the same is true for $\Lambda(S^*)$. Part (i) is proved.

For any unitary U, the operator USU^* is selfadjoint,

$$D\big(USU^*\big) = UD(S) \quad \text{and} \quad E_{USU^*}(\lambda) = UE_S(\lambda)U^* \quad \text{for all } \lambda \in \mathbb{R}. \tag{5.5}$$

Let $U \in \mathcal{Z}_S(t)$. By (5.3), $UD(S) = U^*D(S) = D(S)$ and $USU^*|_{D(S)} = S - t\mathbf{1}|_{D(S)}$. Hence it follows from (5.2) that

$$E_{USU^*}(\lambda) = E_{S-t\mathbf{1}}(\lambda) = E_S(\lambda + t) \quad \text{for all } \lambda \in \mathbb{R}.$$

Taking into account (5.5), we have $UE_S(\lambda)U^* = E_S(\lambda + t)$. Thus U is an (S, t)-shift.

Conversely, let U be an (S, t)-shift. Then $UD(S) = D(S)$ and $UE_S(\lambda)U^* = E_S(\lambda + t)$ for all $\lambda \in \mathbb{R}$. Hence it follows from (5.2) and (5.5) that

$$E_{USU^*}(\lambda) = UE_S(\lambda)U^* = E_{S-t\mathbf{1}}(\lambda),$$

so $USU^*|_{D(S)} = S - t\mathbf{1}|_{D(S)}$. Thus

$$\delta_S(U)|_{D(S)} = i(SU - US)|_{D(S)} = itU|_{D(S)}.$$

Therefore $U \in \mathcal{Z}_S(t)$. $\square$

Theorem 5.1 has an especially simple form when S is *diagonal*, that is, $H = \bigoplus_{\lambda \in \Lambda(S)} H_\lambda$.

Corollary 5.2. *Let S be a diagonal selfadjoint operator. Then $t \in \Gamma_S$ if and only if $\lambda \to \lambda + t$ is an isomorphism of $\Lambda(S)$ and* $\dim H_\lambda = \dim H_{\lambda+t}$ *for all $\lambda \in \Lambda(S)$.*

From Corollary 5.2 it follows that, for any subgroup Γ of $\mathbb{R}$, there is a diagonal S with $\Gamma_S = \Gamma$.

If for each $t \in \Gamma_S$, there is $U_t \in \mathcal{Z}_S(t)$ such that $\mathbf{U} = \{U_t\colon t \in \Gamma_S\}$ is a group, then $\mathbf{U}$ is called a *resolving subgroup of* $\mathcal{Z}_S$. It is commutative and consists of unitary operators U_t, $t \in \Gamma_S$, satisfying

$$U_tD(S) = D(S) \quad \text{and} \quad (SU_t - U_tS)|_{D(S)} = tU_t|_{D(S)}.$$

This relation is called the *infinitesimal Weyl relation* for the group $\mathbf{U}$ and the operator S (see [4]). It follows from (5.1) that $\mathcal{Z}_S$ is the semi-direct product of $\mathbf{U}$ and the normal subgroup $\mathcal{Z}_S(0)$.

Proposition 5.3. *If Γ_S has a minimal positive element μ, then $\Gamma_S = \{n\mu\colon n \in \mathbb{Z}\}$ and, for each $U \in \mathcal{Z}_S(\mu)$, $\mathbf{U} = \{U^n\colon n \in \mathbb{Z}\}$ is a resolving subgroup of $\mathcal{Z}_S$.*

Proof. We only need to show that $\Gamma_S = \{n\mu\colon n \in \mathbb{Z}\}$. If there is $\lambda \in \Gamma_S$ such that $\lambda \neq n\mu$, for all $n \in \mathbb{Z}$, then $m\mu < \lambda < (m+1)\mu$ for some $m \in \mathbb{Z}$. Then $\alpha = \lambda - m\mu \in \Gamma_S$ and $0 < \alpha < \mu$; a contradiction. □

Example 5.4.

(1) Let $\{e_n\}_{n\in\mathbb{Z}}$ be a basis in H and let $Se_n = ne_n$. Then $\mathcal{Z}_S(0)$ consists of all unitary diagonal operators, $\Gamma_S = \mathbb{Z}$, the unitary $U : e_n \to e_{n+1}$ belongs to $\mathcal{Z}_S(1)$ and $\{U^n\colon n \in \mathbb{Z}\}$ is a resolving subgroup of $\mathcal{Z}_S$.
(2) Let T be the operator of multiplication by t on $L_2(\mathbb{R})$. The subgroup $\mathcal{Z}_T(0)$ consists of all multiplication operators by functions $g \in L_\infty(\mathbb{R})$ such that $|g(t)| = 1$; $\Gamma_T = \mathbb{R}$, the shift operators $U_r : h(t) \to h(t - r)$ belong to $\mathcal{Z}_T(r)$ and $\{U_r\colon r \in \mathbb{R}\}$ is a resolving subgroup of $\mathcal{Z}_T$.

If $U \in \mathcal{Z}_S(t)$ and $V \in \mathcal{Z}_T(t)$ then $U \oplus V \in \mathcal{Z}_{S\oplus T}(t)$. Hence if $\{U_t\colon t \in \mathbb{R}\}$ and $\{V_t\colon t \in \mathbb{R}\}$ are resolving subgroups of $\mathcal{Z}_S$ and $\mathcal{Z}_T$, then $\{U_t \oplus V_t\colon t \in \mathbb{R}\}$ is a resolving subgroup of $\mathcal{Z}_{S\oplus T}$. We also have that $\Gamma_S \cap \Gamma_T \subseteq \Gamma_{S\oplus T}$. The group $\Gamma_{S\oplus T}$ is often larger than the groups Γ_S and Γ_T. Indeed, let S and T be diagonal operators on H and K, $\mathrm{Sp}(S) = \mathrm{Sp}(T) = \mathbb{Z}$ and

$$H = \bigoplus_{n\in\mathbb{Z}} H_n, \quad K = \bigoplus_{n\in\mathbb{Z}} K_n, \quad \dim H_0 = \dim K_k = 1, \quad \dim H_k = \dim K_0 = 2 \quad \text{for } k \neq 0.$$

Then $\Gamma_S = \Gamma_T = \{0\}$ and $\Gamma_{S\oplus T} = \mathbb{Z}$.

Assume that $\Gamma_S = \mathbb{R}$ and the resolving group $\mathbf{U} = \{U_t\colon t \in \mathbb{R}\}$ is strongly continuous: for all $x \in H$, $\|U_t x - x\| \to 0$ as $t \to 0$. Let a selfadjoint operator T be the generator of $\mathbf{U}$. Repeating the argument in [5, p. 497], we obtain that there is a linear manifold $\mathcal{D}$ in $D(S) \cap D(T)$ dense in H such that the operators S and T satisfy the *canonical commutation relation*:

$$(ST - TS)|_{\mathcal{D}} = \mathbf{1}|_{\mathcal{D}}.$$

For a symmetric operator S, define the **-commutant of* S by the formula

$$\mathcal{C}_S = \operatorname{Ker}\delta_S = \bigl\{A \in \mathcal{A}_S\colon \delta_S(A) = 0\bigr\}, \quad \text{so } \mathcal{Z}_S(0) = \mathcal{U}_S \cap \mathcal{C}_S. \tag{5.6}$$

For selfadjoint S, $\mathcal{C}_S$ is the set of all bounded operators commuting with all projections $E_S(\lambda)$, $\lambda \in \mathbb{R}$. If S is non-selfadjoint, $\mathcal{C}_S$ is often trivial: $\mathcal{C}_S = \mathbb{C}\mathbf{1}$, so $\mathcal{Z}_S(0) = \{z\mathbf{1}\colon |z| = 1\}$.

For some selfadjoint operators S, the algebras $\mathcal{A}_S$ and $\mathcal{C}_S$ coincide modulo compact operators. Let $H = \bigoplus_{i=-\infty}^{\infty} H_i$ and $S|_{H_i} = \lambda_i \mathbf{1}_{H_i}$ with all distinct λ_i. Set

$$d_S(k) = \Bigl(\inf_{i\in\mathbb{Z}} |\lambda_{i+k} - \lambda_i|\Bigr)^{-1} \quad \text{for } k \neq 0.$$

It was proved in [7] that $\mathcal{A}_S = \mathcal{C}_S + (\mathcal{A}_S \cap C(H))$ if all $\dim(H_i) < \infty$,

$$\lim_{|i|\to\infty} (\lambda_{i+1} - \lambda_i) = \infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}\setminus\{0\}} d_S(k) \text{ converges.} \tag{5.7}$$

In particular, (5.7) holds if $\lambda_i = \operatorname{sgn}(i)|i|^{1+\alpha}$ for any $\alpha > 0$.

For $\lambda \in \Lambda(S)$, let P_λ be the projection on the eigenspace H_λ of S. If $\lambda \neq \mu$, $P_\lambda P_\mu = 0$. Let $A \in C(H)$. Each sequence $\{x_{\lambda_n}\}$, $x_{\lambda_n} \in H_{\lambda_n}$ with $\|x_{\lambda_n}\| = 1$ and distinct λ_n, weakly converges to 0. Hence $\|Ax_{\lambda_n}\| \to 0$. This implies that $\Lambda_A = \{\lambda \in \Lambda(S)\colon\ AP_\lambda \neq 0\}$ is a finite or countable set: $\Lambda_A = \{\lambda_i\}$, and $\|AP_{\lambda_i}\| \to 0$ as $i \to \infty$. Thus the series $\sum_{\lambda_i\in\Lambda_A} P_{\lambda_i} A P_{\lambda_i}$ converges in norm, so

$$\rho : A \to \sum_{\lambda\in\Lambda(S)} P_\lambda A P_\lambda = \sum_{\lambda_i\in\Lambda_A} P_{\lambda_i} A P_{\lambda_i} \tag{5.8}$$

is a map from $C(H)$ into $C(H)$ and $\|\rho\| = 1$. Let $Q = \mathbf{1} - \sum_{\lambda\in\Lambda(S)} P_\lambda$ and set

$$\mathcal{D}_S = \bigl\{A \in C(H)\colon\ AQ = QA = 0 \text{ and } P_\lambda A = AP_\lambda \text{ for all } \lambda \in \Lambda(S)\bigr\}.$$

Then $\mathcal{D}_S$ is a C*-subalgebra of $C(H)$. For each $A \in C(H)$, $\rho(A) \in \mathcal{D}_S$ and, for each $A \in \mathcal{D}_S$,

$$A = \Biggl(Q + \sum_{\lambda\in\Lambda(S)} P_\lambda\Biggr) A = \rho(A), \quad \text{so} \quad \mathcal{D}_S = \bigl\{\rho(A)\colon\ A \in C(H)\bigr\}.$$

Lemma 5.5.

(i) *$\mathcal{C}_S$ is a W^*-algebra and*

$$\begin{aligned}\mathcal{C}_S &= \bigl\{A \in B(H)\colon\ AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S),\ [S, A]|_{D(S)} = 0\bigr\} \\ &= \bigl\{A \in B(H)\colon\ AD(S) \subseteq D(S),\ AD\bigl(S^*\bigr) \subseteq D\bigl(S^*\bigr),\ [S, A]|_{D(S)} = \bigl[S^*, A\bigr]\big|_{D(S^*)} = 0\bigr\}.\end{aligned}$$

(ii) *All $P_\lambda \in \mathcal{C}_S \cap \mathcal{C}'_S$, for $\lambda \in \Lambda(S)$, and $\mathcal{D}_S = \mathcal{C}_S \cap C(H)$.*

Proof. The first equality in (i) follows from (1.2) and (5.6). For $A \in \mathcal{C}_S$, $A^* \in \mathcal{A}_S$ and $\delta_S(A^*) = \delta_S(A)^* = 0$. Thus $A^* \in \mathcal{C}_S$, so $\mathcal{C}_S$ is a *-algebra. Denote by Π the last set in (i). Let $A \in \mathcal{C}_S$. For all $x \in D(S^*)$ and $y \in D(S)$,

$$(Ax, Sy) = \bigl(x, A^*Sy\bigr) = \bigl(x, SA^*y\bigr) = \bigl(AS^*x, y\bigr).$$

Hence $AD(S^*) \subseteq D(S^*)$ and $AS^*|_{D(S^*)} = S^*A|_{D(S^*)}$, so $[S^*, A]|_{D(S^*)} = 0$. Thus $\mathcal{C}_S \subseteq \Pi$.

Conversely, let $A \in \Pi$. Then, for all $x \in D(S^*)$ and $y \in D(S)$,

$$\bigl(S^*x, A^*y\bigr) = \bigl(AS^*x, y\bigr) = \bigl(S^*Ax, y\bigr) = \bigl(x, A^*Sy\bigr).$$

Hence $A^*y \in D(S^{**}) = D(S)$, so $A^*D(S) \subseteq D(S)$. Thus $\Pi \subseteq \mathcal{C}_S$, so $\Pi = \mathcal{C}_S$.

Let $\mathcal{C}_S \ni A_\lambda \to A$ in the weak operator topology (wot). Then $A_\lambda^* \overset{\text{wot}}{\to} A^*$. As $S^{**} = S$, we have, for all $x \in D(S^*)$ and $y \in D(S)$,

$$\begin{aligned}\big(S^*x, Ay\big) &= \big(A^*S^*x, y\big) = \lim_\lambda \big(A_\lambda^* S^* x, y\big) = \lim_\lambda \big(S^* A_\lambda^* x, y\big)\\ &= \lim_\lambda \big(A_\lambda^* x, Sy\big) = \big(A^*x, Sy\big) = (x, ASy).\end{aligned}$$

Hence $Ay \in D(S^{**}) = D(S)$ and $SAy = ASy$. Thus $AD(S) \subseteq D(S)$ and $[S, A]|_{D(S)} = 0$. Similarly, $A^*D(S) \subseteq D(S)$. Therefore $A \in \mathcal{C}_S$, so $\mathcal{C}_S$ is a W*-algebra. Part (i) is proved.

Since $P_\lambda D(S) \subseteq H_\lambda \subseteq D(S)$ and

$$P_\lambda S|_{D(S)} = SP_\lambda|_{D(S)} = \lambda P_\lambda|_{D(S)}, \tag{5.9}$$

we have $\delta_S(P_\lambda) = 0$, so $P_\lambda \in \mathcal{C}_S$. Let $A \in \mathcal{C}_S$. For $x \in H_\lambda$, we have $SAx = ASx = \lambda Ax$, so $Ax \in H_\lambda$. Hence $P_\lambda A P_\lambda = AP_\lambda$. Since $\mathcal{C}_S$ is a *-algebra, $P_\lambda A = AP_\lambda$, so $P_\lambda \in \mathcal{C}_S \cap \mathcal{C}_S'$.

Let $A \in C(H)$. For each $\lambda \in \Lambda(S)$, $P_\lambda A P_\lambda \in C(H)$ and $P_\lambda A P_\lambda D(S) \subseteq H_\lambda \subseteq D(S)$. By (5.9),

$$\delta_S(P_\lambda A P_\lambda)|_{D(S)} = i(SP_\lambda A P_\lambda|_{D(S)} - P_\lambda A P_\lambda S|_{D(S)}) = 0.$$

Hence $P_\lambda A P_\lambda \in \mathcal{C}_S \cap C(H)$. As $\rho(A)$ is the norm limit of sums of the operators $P_\lambda A P_\lambda$, $\lambda \in \Lambda_A$, we have $\rho(A) \in \mathcal{C}_S \cap C(H)$. Thus $\mathcal{D}_S \subseteq \mathcal{C}_S \cap C(H)$.

Conversely, let $A = A^* \in \mathcal{C}_S \cap C(H)$. Then $A = \sum_i \alpha_i P_i$, where P_i are finite-dimensional mutually orthogonal projections from $\mathcal{C}_S \cap C(H)$ and $|\alpha_i| \to 0$. Each subspace $P_i H$ lies in $D(S)$ and the operator $S|_{P_i H}$ is selfadjoint. Hence $P_i H = \bigoplus_j K_{ij}$, where $S|_{K_{ij}} = \lambda_{ij} P_{K_{ij}}$. Therefore $\lambda_{ij} \in \Lambda(S)$. As $P_\lambda \in \mathcal{C}_S'$, each P_i commutes with all P_λ, so $P_{K_{ij}} = P_{\lambda_{ij}} P_i P_{\lambda_{ij}} \in \mathcal{D}_S$. Hence $P_i = \sum_j P_{K_{ij}} \in \mathcal{D}_S$. As $\mathcal{D}_S$ is norm closed, $A \in \mathcal{D}_S$. Thus $\mathcal{D}_S = \mathcal{C}_S \cap C(H)$. □

Recall that a closed subspace L of H (the projection Q on L) *reduces* a symmetric operator S if

$$QD(S) \subseteq D(S) \quad \text{and} \quad SQ|_{D(S)} = QS|_{D(S)}. \tag{5.10}$$

The operator S is called *simple* if it has no reducing subspaces where it induces a *selfadjoint operator*; it is called *irreducible* if it has no reducing subspaces.

Denote by $H^{(n)}$, $1 \leqslant n \leqslant \infty$, the orthogonal sum of n copies of H and by $S^{(n)}$ the orthogonal sum of n copies of S. Lemma 5.5(i) and (5.10) yield

Lemma 5.6.

(i) *A projection Q reduces a symmetric operator S if and only $Q \in \mathcal{C}_S$.*
(ii) *S is irreducible if and only if $\mathcal{C}_S = \mathbb{C}\mathbf{1}$.*
(iii) *If S is irreducible, $\mathcal{C}_{S^{(n)}}$ consists of all block-matrix bounded operators $(\lambda_{ij}\mathbf{1}_H)$ on $H^{(n)}$ with $\lambda_{ij} \in \mathbb{C}$.*

Let S^* be the adjoint of S. The deficiency subspaces $N_\pm(S) = \{x \in D(S^*)\colon\ S^*x = \pm ix\}$ of S are closed in H and $n_\pm(S) = \dim N_\pm(S)$ are called the deficiency indices of S. The operator S is *selfadjoint* if $n_-(S) = n_+(S) = 0$; it is *maximal symmetric* if either $n_-(S) = 0$ or $n_+(S) = 0$.

Recall that symmetric operators R on K and S on H are *isomorphic* if

$$UD(R) = D(S) \quad \text{and} \quad UR|_{D(R)} = SU|_{D(R)}, \tag{5.11}$$

for some unitary operator U from K on H. If S and R are isomorphic, $\mathcal{Z}_S = \mathcal{Z}_R$ and $\Gamma_S = \Gamma_R$.

The operator $T = i\frac{d}{dt}$ on $H = L_2(0, \infty)$ with

$$D(T) = \{h \in H\colon h \text{ are absolutely continuous, } h' \in H \text{ and } h(0) = 0\}$$

is simple and maximal symmetric with $n_-(T) = 1$, $n_+(T) = 0$. For each $r \in \mathbb{R}$, the multiplication operator

$$V_r h(t) = e^{-irt} h(t) \tag{5.12}$$

on H is unitary. It is easy to check that $V_r \in \mathcal{U}_T$ and $\delta_T(V_r) = irV_r$ for all $r \in \mathbb{R}$, so $V_r \in \mathcal{Z}_T(r)$.

It is well known (see [1]) that each simple maximal symmetric operator S is isomorphic

$$\text{either to } T^{(k)} \quad \text{if } n_-(S) = k,\ n_+(S) = 0; \quad \text{or to } -T^{(k)} \quad \text{if } n_-(S) = 0,\ n_+(S) = k. \tag{5.13}$$

Using Lemma 5.6, we have the following description of $\mathcal{Z}_S$ for simple maximal symmetric operators.

Theorem 5.7. *Let S be a simple maximal symmetric operator satisfying* (5.13)*, for some k, and let V_r, $r \in \mathbb{R}$, be the unitary operators defined in* (5.12)*. Then $\Gamma_S = \mathbb{R}$, $\{V(r)^{(k)}\colon r \in \mathbb{R}\}$ is a resolving subgroup of $\mathcal{Z}_S$ and $\mathcal{Z}_S(0)$ consists of all unitary block-matrix operators $(\lambda_{ij}\mathbf{1}_H)$ on $H^{(k)}$ with $\lambda_{ij} \in \mathbb{C}$.*

We consider now the following criteria for a symmetric operator to be irreducible.

Lemma 5.8. *Let S be a symmetric operator. Let $\{\lambda_n\}$ be eigenvalues of S^* with one-dimensional eigenspaces: $H_n = \mathbb{C}h_n$ and let the linear span of all h_n be dense in H. Suppose that all $h_n \notin D(S)$. If there are $\mu_n \in \mathbb{C}$ such that $h_n - \mu_n h_1 \in D(S)$, for all n, then S is irreducible.*

Proof. Let a projection Q belong to $\mathcal{C}_S$. It commutes with S^*, so $S^*Qh_n = QS^*h_n = \lambda_n Qh_n$. Since H_n are one-dimensional, $Qh_n = \alpha_n h_n$, where $\alpha_n = 0$ or 1. Since Q preserves $D(S)$,

$$Q(h_n - \mu_n h_1) = \alpha_n h_n - \mu_n \alpha_1 h_1 = \alpha_n(h_n - \mu_n h_1) + (\alpha_n - \alpha_1)\mu_n h_1 \in D(S).$$

Hence $(\alpha_n - \alpha_1)\mu_n h_1 \in D(S)$ for all n. Since all $\mu_n \neq 0$, all $\alpha_n = \alpha_1$. Thus Q is either $\mathbf{1}$ or $\mathbf{0}$. $\square$

We shall now consider an irreducible non-maximal symmetric operator with a resolving subgroup. The symmetric operator $S = i\frac{d}{dt}$ on $H = L_2(0, 2\pi)$ with

$$D(S) = \{h \in H\colon h \text{ is absolutely continuous, } h' \in H \text{ and } h(0) = h(2\pi) = 0\}$$

has $n_-(S) = n_+(S) = 1$ (see [1]). It is irreducible. Indeed, $S^* = i\frac{d}{dt}$ and

$$D(S^*) = \{h \in H\colon h \text{ is absolutely continuous and } h' \in H\}.$$

The functions $h_n(t) = e^{int}$, $-\infty < n < \infty$, form an orthonormal basis in H, $h_n \in D(S^*)$ and $h_n \notin D(S)$. Moreover, $S^* h_n = -nh_n$ and $h_n - h_0 \in D(S)$ for all n. Hence, by Lemma 5.8, S is irreducible. For each real r, the multiplication operator $U_r h(t) = e^{-irt}h(t)$ on H is unitary, $U_r \in \mathcal{U}_S$ and $\delta_S(U_r) = irU_r$. This yields

Proposition 5.9. *$\Gamma_S = \mathbb{R}$, $\mathcal{Z}_S(0) = \{z\mathbf{1}\colon |z| = 1\}$ and $\{U_r\colon r \in \mathbb{R}\}$ is a resolving subgroup of $\mathcal{Z}_S$.*

All selfadjoint extensions of the operator S above can be parametrized by $\omega \in [0, 2\pi)$ (see [1]):

$$S_\omega = i\frac{d}{dt},$$

$$D(S_\omega) = \{h \in H\colon h \text{ are absolutely continuous, } h' \in H \text{ and } h(2\pi) = e^{i\omega}h(0)\} \quad \text{and}$$

$$\mathrm{Sp}(S_\omega) = \left\{\lambda_n = n - \frac{\omega}{2\pi}\colon n \in \mathbb{Z}\right\}$$

with the eigenvectors $h_n(t) = e^{-i\lambda_n t}$. Each operator $S_\omega + \frac{\omega}{2\pi}\mathbf{1}$ is isomorphic to the diagonal selfadjoint operator in Example 5.4(1). Hence $\Gamma_{S_\omega} = \mathbb{Z}$, the groups $\mathcal{Z}_{S_\omega}$ have resolving subgroups and large subgroups $\mathcal{Z}_{S_\omega}(0)$.

6. Extension of domains

Recall (see Introduction) that, for each symmetric operator S, the algebra $\mathcal{F}_S$ and the algebra

$$\mathcal{J}_S = \{A \in \mathcal{A}_S \cap C(H)\colon \delta_S(A) \in C(H)\}$$

are domains of $C(H)$ and $\delta_S^{\min} = \delta_S|\mathcal{F}_S$ and $\delta_S^{\max} = \delta_S|\mathcal{J}_S$ are closed *-derivations of $C(H)$ with minimal implementation S. If S is selfadjoint then $\mathcal{J}_S = \mathcal{F}_S$. If $\mathcal{A}$ is a domain in $C(H)$, then $\mathcal{F}_S \subseteq \mathcal{A} \subseteq \mathcal{J}_S$ for some symmetric operator S on H. Indeed, let $\delta \in \mathrm{Der}(\mathcal{A})$ and let S be a minimal implementation of δ. Since $\delta_S^{\min}$ and $\delta_S^{\max}$ are the minimal and the maximal closed *-derivations of $C(H)$ with minimal implementation S, $\delta_S^{\min} \subseteq \delta \subseteq \delta_S^{\max}$. Thus $\mathcal{F}_S \subseteq \mathcal{A} \subseteq \mathcal{J}_S$.

In this section we construct a variety of domains of C*-algebras that contain $\mathcal{J}_S$. We will consider C*-subalgebras of $B(H)$ that *do not contain* $C(H)$. Let $\mathfrak{A}$ be such a C*-subalgebra of $B(H)$ and let $\mathcal{A}$ be a domain of $\mathfrak{A}$. Assume that there is a derivation in $\mathrm{Der}(\mathcal{A})$ implemented by S: $\mathcal{A} \subseteq \mathcal{A}_S$ and $\delta_S|\mathcal{A} \in \mathrm{Der}(\mathcal{A})$. It follows from [3, Corollary 1.8.4] that $\mathfrak{A} + C(H)$ is a C*-algebra. Since $\mathcal{J}_S$ is an ideal of $\mathcal{A}_S$, $\mathcal{A} + \mathcal{J}_S$ is a dense *-subalgebra of $\mathfrak{A} + C(H)$ and δ_S maps $\mathcal{A} + \mathcal{J}_S$ into $\mathfrak{A} + C(H)$, so $\delta_S|(\mathcal{A} + \mathcal{J}_S)$ is a *-derivation of $\mathfrak{A} + C(H)$. Below we consider some conditions for $\delta_S|(\mathcal{A} + \mathcal{J}_S)$ to be closed. These conditions will imply that $\mathcal{A} + \mathcal{J}_S$ is a domain of $\mathfrak{A} + C(H)$.

Let δ be the *maximal closed* *-derivation of $\mathfrak{A}$ implemented by S, that is,

$$G(\delta) = G(\delta_S) \cap (\mathfrak{A} \oplus \mathfrak{A}), \quad \text{where } G(\delta) = \{A \oplus \delta(A)\colon A \in D(\delta)\} \tag{6.1}$$

is the graph of δ. In particular, $\delta_S^{\max}$ is the maximal derivation of $C(H)$ implemented by S.

Theorem 6.1. *Let $\mathcal{A}$ be a domain of $\mathfrak{A}$, $C(H) \nsubseteq \mathfrak{A} \subseteq B(H)$, and let $\delta_S|\mathcal{A} \in \mathrm{Der}(\mathcal{A})$. Suppose that*

(i) *$\delta_S|\mathcal{A}$ is the maximal closed *-derivation of $\mathfrak{A}$ implemented by S, that is,* (6.1) *holds*;
(ii) *there exists a bounded linear map θ from $C(H)$ onto $\mathfrak{A} \cap C(H)$ such that*

$$\theta(A) = A \quad \textit{for all } A \in \mathfrak{A} \cap C(H), \tag{6.2}$$

and θ commutes with $\delta_S^{\max}$:

$$\theta(A) \in \mathcal{J}_S \quad \textit{and} \quad \delta_S^{\max}\big(\theta(A)\big) = \theta\big(\delta_S^{\max}(A)\big) \quad \textit{for all } A \in \mathcal{J}_S. \tag{6.3}$$

Then $\mathcal{B} = \mathcal{A} + \mathcal{J}_S$ is a domain of $\mathfrak{A} + C(H)$, $\delta_S|\mathcal{B} \in \mathrm{Der}(\mathcal{B})$ and S is its minimal implementation.

Proof. Set $\delta = \delta_S|\mathcal{B}$. Since S implements δ and $\mathcal{F}_S \subseteq \mathcal{J}_S \subseteq \mathcal{B}$, it is easy to see that S is a minimal implementation of δ. Thus we only need to prove that δ is closed.

By (6.2), $C(H) = (\mathfrak{A} \cap C(H)) \dotplus \mathrm{Ker}(\theta)$ and $\mathrm{Ker}(\theta)$ is a closed subspace of $C(H)$. Therefore

$$\mathfrak{A} + C(H) = \mathfrak{A} \dotplus \mathrm{Ker}(\theta) \tag{6.4}$$

is the direct sum of $\mathfrak{A}$ and $\mathrm{Ker}(\theta)$. Let $A \in \mathcal{J}_S$. Since $\theta(A) \in \mathfrak{A} \cap C(H)$, we have from (6.3) that

$$\theta(A) \in \mathfrak{A} \cap \mathcal{J}_S \quad \text{and} \quad \delta_S\big(\theta(A)\big) = \delta_S^{\max}\big(\theta(A)\big) = \theta\big(\delta_S^{\max}(A)\big) \in \mathfrak{A} \cap C(H). \tag{6.5}$$

Hence $\theta(A) \oplus \delta_S(\theta(A))$ belongs to $\mathfrak{A} \oplus \mathfrak{A}$ and to $G(\delta_S)$. Since $\delta_S|\mathcal{A}$ satisfies (6.1), we have $\theta(A) \in \mathcal{A}$. Therefore $A = \theta(A) + (A - \theta(A))$ and $A - \theta(A) \in \mathcal{J}_S \cap \mathrm{Ker}(\theta)$. Thus, by (6.4),

$$\mathcal{B} = \mathcal{A} + \mathcal{J}_S = \mathcal{A} \dotplus \big(\mathcal{J}_S \cap \mathrm{Ker}(\theta)\big).$$

Let $A_n \in \mathcal{A}$, $B_n \in \mathcal{J}_S \cap \mathrm{Ker}(\theta)$, $A, T \in \mathfrak{A}$ and $B, R \in \mathrm{Ker}(\theta)$, let $A_n + B_n \to A + B \in \mathfrak{A} + C(H)$ and let $\delta(A_n + B_n) \to T + R$. By (6.5), $\theta(\delta_S^{\max}(B_n)) = \delta_S^{\max}(\theta(B_n)) = 0$. Hence $\delta_S^{\max}(B_n) \in \mathrm{Ker}(\theta)$. Thus

$$\delta(A_n + B_n) = \delta_S(A_n) + \delta_S^{\max}(B_n) \to T + R, \quad \text{where } \delta_S(A_n) \in \mathfrak{A} \text{ and } \delta_S^{\max}(B_n) \in \mathrm{Ker}(\theta).$$

If a sequence in the direct sum of closed subspaces converges, the components of its elements also converge. Hence, by (6.4), $A_n \to A$, $B_n \to B$, $\delta_S(A_n) \to T$ and $\delta_S^{\max}(B_n) \to R$. As $\delta_S|\mathcal{A}$ is a closed derivation, we have $A \in \mathcal{A}$ and $\delta_S(A) = T$. As $\delta_S^{\max}$ is a closed derivation, $B \in \mathcal{J}_S \cap \mathrm{Ker}(\theta)$ and $\delta_S^{\max}(B) = R$. Thus $A + B \in \mathcal{B}$ and $\delta(A + B) = T + R$, so δ is a closed *-derivation. □

Corollary 6.2. *Let $\mathcal{A}$ be a domain of $\mathfrak{A} \subseteq B(H)$ and $\delta_S|\mathcal{A} \in \mathrm{Der}(\mathcal{A})$ satisfy* (6.1). *If $\mathfrak{A} \cap C(H) = \{0\}$, then $\mathcal{B} = \mathcal{A} + \mathcal{J}_S$ is a domain of $\mathfrak{A} + C(H)$, $\delta_S|\mathcal{B} \in \mathrm{Der}(\mathcal{B})$ and S is its minimal implementation.*

Let $\mathfrak{A}$ be a C*-subalgebra of $\mathcal{C}_S$. Then $\delta_S|\mathfrak{A} = 0$ and it satisfies (6.1). For $\lambda \in \Lambda(S)$, let P_λ be the projection on the eigenspace H_λ of S. By Lemma 5.5(ii), $P_\lambda \mathcal{C}_S P_\lambda \subseteq \mathcal{C}_S$. Assume also that

$$P_\lambda\big(\mathfrak{A} \cap C(H)\big)P_\lambda \subset \mathfrak{A} \quad \text{for all } \lambda \in \Lambda(S). \tag{6.6}$$

Then all $P_\lambda(\mathfrak{A} \cap C(H))P_\lambda$ are C*-algebras. Since $\delta_S(A) = 0$, for $A \in \mathcal{C}_S \cap C(H)$,

$$\mathcal{C}_S \cap C(H) \subseteq \mathcal{C}_S \cap \mathcal{J}_S. \tag{6.7}$$

Let $A \in C(H)$. Recall (see (5.8)) that $AP_\lambda \neq 0$ for a finite or countable subset $\Lambda_A = \{\lambda_i\}$ of $\Lambda(S)$,

$$\|AP_{\lambda_i}\| \to 0 \quad \text{and} \quad \rho(A) = \sum_{\lambda \in \Lambda(S)} P_\lambda A P_\lambda = \sum_{\lambda_i \in \Lambda_A} P_{\lambda_i} A P_{\lambda_i} \in \mathcal{C}_S \cap C(H) \subseteq \mathcal{C}_S \cap \mathcal{J}_S,$$

where the series converges in norm.

We will now construct a bounded linear map θ from $C(H)$ onto $\mathfrak{A} \cap C(H)$ satisfying (6.2) and (6.3). We will use for this the well-known result that, for each C*-subalgebra $\mathfrak{B}$ of $C(H)$, there is a conditional expectation from $C(H)$ onto $\mathfrak{B}$. Thus, for each $\lambda \in \Lambda(S)$, there is a conditional expectation θ_λ from the algebra $P_\lambda C(H) P_\lambda$ which is isomorphic to $C(H_\lambda)$ onto the C*-subalgebra $P_\lambda(\mathfrak{A} \cap C(H))P_\lambda$ of $P_\lambda C(H) P_\lambda$. Set

$$\theta(A) = \sum_{\lambda \in \Lambda(S)} \theta_\lambda(P_\lambda A P_\lambda) = \sum_{\lambda_i \in \Lambda_A} \theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i}) \quad \text{for all } A \in C(H). \tag{6.8}$$

Since $\|\theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i})\| \leqslant \|AP_{\lambda_i}\| \to 0$ and since $\theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i})$ belong to $P_{\lambda_i} C(H) P_{\lambda_i}$ and, hence, mutually orthogonal, the series in (6.8) is norm convergent. Hence we have from (6.7) that

$$\theta(A) \in \mathfrak{A} \cap C(H) \subseteq \mathcal{C}_S \cap C(H) \subseteq \mathcal{C}_S \cap \mathcal{J}_S \quad \text{for all } A \in C(H), \tag{6.9}$$

so θ maps $C(H)$ into $\mathfrak{A} \cap C(H)$. Moreover, θ is linear and bounded, since

$$\big\|\theta(A)\big\| = \sup \big\|\theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i})\big\| \leqslant \sup \|P_{\lambda_i} A P_{\lambda_i}\| \leqslant \|A\|.$$

Since projections P_λ, $\lambda \in \Lambda(S)$, are mutually orthogonal, $P_\lambda \rho(A) P_\lambda = P_\lambda A P_\lambda$ (see (5.8)). Hence

$$\theta\big(\rho(A)\big) = \sum_{\lambda \in \Lambda(S)} \theta_\lambda\big(P_\lambda \rho(A) P_\lambda\big) = \sum_{\lambda_i \in \Lambda_A} \theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i}) = \theta(A).$$

Since $\theta_\lambda(P_\lambda A P_\lambda) \in P_\lambda(\mathfrak{A} \cap C(H))P_\lambda$, we have $P_\lambda \theta(A) P_\lambda = \theta_\lambda(P_\lambda A P_\lambda)$, so (see (5.8))

$$\rho\big(\theta(A)\big) = \sum_{\lambda \in \Lambda(S)} P_\lambda \theta(A) P_\lambda = \sum_{\lambda_i \in \Lambda_A} \theta_{\lambda_i}(P_{\lambda_i} A P_{\lambda_i}) = \theta(A).$$

Thus

$$\theta\big(\rho(A)\big) = \rho\big(\theta(A)\big) = \theta(A) \quad \text{for all } A \in C(H). \tag{6.10}$$

Let $A \in \mathfrak{A} \cap C(H)$. Then $A \in \mathcal{C}_S \cap C(H)$ and we have from Lemma 5.5(ii) that $A = \rho(B)$ for some $B \in C(H)$. Since $P_\lambda A P_\lambda \in P_\lambda(\mathfrak{A} \cap C(H))P_\lambda$ and θ_λ are conditional expectations, $\theta_\lambda(P_\lambda A P_\lambda) = P_\lambda A P_\lambda = P_\lambda \rho(B) P_\lambda = P_\lambda B P_\lambda$. Thus (6.2) holds, since

$$\theta(A) = \sum_{\lambda \in \Lambda(S)} \theta_\lambda(P_\lambda A P_\lambda) = \sum_{\lambda \in \Lambda(S)} P_\lambda B P_\lambda = \rho(A) = A.$$

Let now $A \in \mathcal{J}_S$. Since $P_\lambda H \subset D(S)$, for all $\lambda \in \Lambda(S)$, we have from (5.9)

$$P_\lambda \delta_S^{\max}(A) P_\lambda = P_\lambda \delta_S(A) P_\lambda = P_\lambda i[S, A] P_\lambda = i(P_\lambda S A P_\lambda - P_\lambda A S P_\lambda) = 0.$$

Hence

$$\rho\big(\delta_S^{\max}(A)\big) = \sum_{\lambda \in \Lambda(S)} P_\lambda \delta_S^{\max}(A) P_\lambda = 0.$$

As $\delta_S^{\max}(A) \in C(H)$, we have from (6.10) that

$$\theta\big(\delta_S^{\max}(A)\big) = \theta\big(\rho\big(\delta_S^{\max}(A)\big)\big) = 0.$$

By (6.9), $\theta(C(H)) \subseteq \mathcal{C}_S \cap \mathcal{J}_S$, so that $\delta_S^{\max}(\theta(A)) = 0$. Thus

$$\delta_S^{\max}\big(\theta(A)\big) = 0 = \theta\big(\delta_S^{\max}(A)\big) \quad \text{for all } A \in \mathcal{J}_S.$$

Therefore (6.3) holds and Theorem 6.1 yields

Theorem 6.3. *Let a C^*-subalgebra $\mathfrak{A}$ of $\mathcal{C}_S$ satisfy* (6.6). *Then $\mathcal{B} = \mathfrak{A} + \mathcal{J}_S$ is a domain of the C^*-algebra $\mathfrak{A} + C(H)$, $\delta_S|\mathcal{B}$ is a closed *-derivation of $\mathfrak{A} + C(H)$ with minimal implementation S.*

Finally, we consider derivations δ with $\mathcal{Z}(\delta) = \mathcal{Z}_S$, where S is a minimal implementation of δ.

Proposition 6.4.

(i) *Let T be a minimal symmetric implementation of $\delta \in \mathrm{Der}(\mathcal{F}_S)$. Then $B(\delta) = \mathcal{U}_T$ and $\mathcal{Z}(\delta) = \mathcal{Z}_T$.*
(ii) *$B(\delta_S^{\max}) = \mathcal{U}_S$ and $\mathcal{Z}(\delta_S^{\max}) = \mathcal{Z}_S$.*
(iii) *Let $\delta = \delta_S|(\mathcal{C}_S + \mathcal{J}_S)$. Then $\mathcal{Z}(\delta) = \mathcal{Z}_S$.*

Proof. As $\delta_S^{\min}, \delta \in \mathrm{Der}(\mathcal{F}_S)$, it follows from Corollary 3.2 that

$$\mathcal{F}_S = \mathcal{F}\big(\mathcal{F}_S, \delta_S^{\min}\big) = \mathcal{F}(\mathcal{F}_S, \delta) = \mathcal{F}_T.$$

Since $\mathcal{F}_T$ is an ideal of $\mathcal{A}_T$, Proposition 4.2(ii) yields (i).

As $\mathcal{J}_S$ is an ideal of $\mathcal{A}_S$, Proposition 4.2(ii) also yields (ii).

Let $U \in \mathcal{Z}_S(t)$ and $A \in \mathcal{C}_S$. Since $\mathcal{Z}_S(t) \subseteq \mathcal{A}_S$ and $\mathcal{C}_S \subseteq \mathcal{A}_S$, we have $UAU^* \in \mathcal{A}_S$. Further

$$\delta_S\big(UAU^*\big) = \delta_S(U)AU^* + U\delta_S(A)U^* + UA\delta_S\big(U^*\big) = itUAU^* + UA\big(-itU^*\big) = 0.$$

Hence $UAU^* \in \mathcal{C}_S$, so $U\mathcal{C}_S U^* = \mathcal{C}_S$. As $\mathcal{J}_S$ is an ideal of $\mathcal{A}_S$,

$$UD(\delta)U^* = U(\mathcal{C}_S + \mathcal{J}_S)U^* \subseteq D(\delta).$$

Thus $U \in \mathcal{Z}(\delta)$, so $\mathcal{Z}_S \subseteq \mathcal{Z}(\delta)$. Since always $\mathcal{Z}(\delta) \subseteq \mathcal{Z}_S$, we have $\mathcal{Z}(\delta) = \mathcal{Z}_S$. $\square$

Let S be a selfadjoint operator on $H = \bigoplus_{i=-\infty}^{\infty} H_i$ and let $S|_{H_i} = \lambda_i \mathbf{1}_{H_i}$ with all distinct λ_i. The group Γ_S is described in Corollary 5.2 and $\mathcal{C}_S$ consists of bounded operators commuting with all P_{λ_i}. By Theorem 6.3 and Proposition 6.4, $\delta = \delta_S|(\mathcal{C}_S + \mathcal{J}_S)$ is a closed *-derivation of the C*-algebra $\mathcal{C}_S + C(H)$ and $\mathcal{Z}(\delta) = \mathcal{Z}_S$.

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