Applications of the operator $H(\alpha, \beta)$ to the Humbert double hypergeometric functions

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**Abstract**

By making use of some techniques based upon certain new inverse pairs of symbolic operators, the authors investigate several decomposition formulas associated with Humbert hypergeometric functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$, and $\Xi_2$. These operational representations are constructed and applied in order to derive the corresponding decomposition formulas. With the help of these inverse pairs of symbolic operators, as many as 34 decomposition formulas are found. Euler type integrals connected with Humbert functions are also presented.

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1. Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [1, p. 47]; see also the recent works [2–6] and the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [7, 8]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [9–11]. Especially, many problems in gas dynamics lead to those of degenerate second-order partial differential equations, which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [12, 13]. It is noted that Riemann’s functions and fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables [2–4].

In the investigation of the boundary value problems for these partial differential equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of the Gauss and Humbert types. We recall the Humbert functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$, and $\Xi_2$ in two variables defined by (see [14, p. 126])

$$
\Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(y)_m}{(\gamma)_{m+n}m!n!} x^m y^n \quad (|x| < 1, \ |y| < \infty),
$$

(1.1)
We introduce the following multivariable symbolic operators:

\( \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}m!n!} x^m y^n \) \quad (|x| < \infty, |y| < \infty), \hspace{1cm} (1.2)

\( \Phi_3(\beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}m!n!} x^m y^n \) \quad (|x| < \infty, |y| < \infty), \hspace{1cm} (1.3)

\( \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma_1)_m(\gamma_2)_nm!n!} x^m y^n \) \quad (|x| < 1, |y| < \infty), \hspace{1cm} (1.4)

\( \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma_1)_m(\gamma_2)_nm!n!} x^m y^n \) \quad (|x| < \infty, |y| < \infty), \hspace{1cm} (1.5)

\( \Xi_1(\alpha_1, \alpha_2, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta)_m}{(\gamma)_{m+n}m!n!} x^m y^n \) \quad (|x| < 1, |y| < \infty), \hspace{1cm} (1.6)

\( \Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}m!n!} x^m y^n \) \quad (|x| < 1, |y| < \infty), \hspace{1cm} (1.7)

where \((\alpha)_m = \Gamma(\alpha + m)/\Gamma(\alpha)\) is the Pochhammer symbol. For various multivariable hypergeometric functions including the Lauricella multivariable functions \( F_{\nu}^{(r)} \), \( F_{\nu}^{(h)} \), \( F_{\nu}^{(l)} \) and \( F_{\nu}^{(t)} \), Hasanov and Srivastava \([15,16]\) presented a number of decomposition formulas in terms of such hypergeometric functions as the Gauss and Appell functions. The main object of this sequel to the works of Hasanov and Srivastava \([15,16]\) is to show how some rather elementary techniques based upon certain inverse pairs of symbolic operators would lead us easily to several decomposition formulas associated with the Humbert hypergeometric functions \( \Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1 \) and \( \Xi_2 \). Over six decades ago, Burchall and Chaundy \([17,18]\) and Chaundy \([19]\) systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

\( \nabla_{xy}(h) := \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(h)_k k!} \), \hspace{1cm} (1.8)

\( \Delta_{xy}(h) := \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(1 - h - \delta_1 - \delta_2)_k k!} \)

\( = \sum_{k=0}^{\infty} \frac{(-1)^k(\delta_1)_k(\delta_2)_k}{(h + k - 1)_k(h + \delta_1)_k(h + \delta_2)_k k!} \), \hspace{1cm} (1.9)

and

\( \nabla_{xy}(h)\Delta_{xy}(g) := \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)\Gamma(\delta_1 + g)\Gamma(\delta_2 + g)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)\Gamma(g)\Gamma(\delta_1 + \delta_2 + g)} = \sum_{k=0}^{\infty} \frac{(g - h)_k(\delta_1)_k(\delta_2)_k}{(g + k - 1)_k(g + \delta_1)_k(g + \delta_2)_k k!} \)

\( = \sum_{k=0}^{\infty} \frac{(h - g)_k(\delta_1)_k(\delta_2)_k}{(h)_k(1 - g - \delta_1 - \delta_2)_k k!} \left( \delta_1 := x \frac{\partial}{\partial x}; \delta_2 := y \frac{\partial}{\partial y} \right) \). \hspace{1cm} (1.10)

We introduce the following multivariable symbolic operators:

\( H_{x_1,\ldots,x_l}(\alpha, \beta) := \frac{\Gamma(\beta)\Gamma(\alpha + \delta_1 + \cdots + \delta_l)}{\Gamma(\alpha)\Gamma(\beta + \delta_1 + \cdots + \delta_l)} = \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(\beta - \alpha)_{k_1+\cdots+k_l}(-\delta_1)_{k_1} \cdots (-\delta_l)_{k_l}}{(\beta)_{k_1+\cdots+k_l}k_1! \cdots k_l!} \), \hspace{1cm} (1.11)

and

\( \tilde{H}_{x_1,\ldots,x_l}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta + \delta_1 + \cdots + \delta_l)}{\Gamma(\beta)\Gamma(\alpha + \delta_1 + \cdots + \delta_l)} = \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(\beta - \alpha)_{k_1+\cdots+k_l}(-\delta_1)_{k_1} \cdots (-\delta_l)_{k_l}}{(1 - \alpha - \delta_1 - \cdots - \delta_l)_{k_1+\cdots+k_l}k_1! \cdots k_l!} \), \hspace{1cm} (1.12)

\[\delta_j := x_j \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, l; \quad l \in \mathbb{N} := \{1, 2, 3, \ldots\}.\]
2. A set of decomposition formulas for Humbert functions

By making a simple application of the symbolic operators defined by (1.8) and (1.9), we begin by presenting each of the following operator formulas (2.1)-(2.35):

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x,y}(\alpha, \beta) \Phi_1(\alpha, \beta; y; x, y), \]  
(2.1)

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x,y}(\alpha, \alpha) \Phi_1(\alpha, \beta; y; x, y), \]  
(2.2)

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x,y}(\alpha, \gamma) \Phi_1(\alpha, \beta; \gamma; x, y), \]  
(2.3)

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x,y}(\gamma, \gamma)(1 - \gamma)^{-\beta} \Phi_1(\alpha, \beta; y; x, y), \]  
(2.4)

\[ (1 - \gamma)^{-\beta} \Phi_1(\alpha, \beta; y; x, y) = H_{x,y}(\gamma, \gamma) \Phi_1(\alpha, \beta; \gamma; x, y), \]  
(2.5)

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x}(\beta, \gamma) \Phi_1(\alpha, \beta; \gamma; x, y), \]  
(2.6)

\[ \Phi_1(\alpha, \beta; y; x, y) = H_{x}(\beta, \beta) \Phi_1(\alpha, \beta; \gamma; x, y), \]  
(2.7)

\[ \Phi_2(\beta_1, \beta_2; y; x, y) = H_{x,y}(\beta_1, \beta_2) \Phi_2(\beta_1, \beta_2; \gamma; x, y), \]  
(2.8)

\[ \Phi_2(\beta_1, \beta_2; y; x, y) = H_{x}(\beta_1, \beta_2) \Phi_2(\beta_1, \beta_2; \gamma; x, y), \]  
(2.9)

\[ \Phi_2(\beta_1, \beta_2; y; x, y) = H_{x}(\beta_1, \beta_1) \Phi_2(\beta_1, \beta_2; \gamma; x, y), \]  
(2.10)

\[ \Phi_2(\beta_1, \beta_2; y; x, y) = H_{x}(\beta_1, \beta_1) \Phi_2(\beta_1, \beta_2; \gamma; x, y), \]  
(2.11)

\[ \Phi_2(\beta_1, \beta_2; y; x, y) = H_{x}(\beta_1, \beta_1) \Phi_2(\beta_1, \beta_2; \gamma; x, y), \]  
(2.12)

\[ \Phi_3(\beta; y; x, y) = H_{x}(\beta, \beta) \Phi_3(\beta; \gamma; x, y), \]  
(2.13)

\[ \Phi_3(\beta; y; x, y) = H_{x}(\beta, \beta) \Phi_3(\beta; \gamma; x, y), \]  
(2.14)

\[ \Phi_3(\beta; y; x, y) = H_{x}(\beta, \beta) \Phi_3(\beta; \gamma; x, y), \]  
(2.15)

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = H_{x,y}(\alpha, \beta) \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.16)

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = H_{x}(\beta, \beta) \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.17)

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = H_{x}(\beta, \beta) \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.18)

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = H_{x}(\beta, \beta) \Phi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.19)

\[ (1 - \gamma)^{-\beta} \Phi_1(\alpha, \beta; y; x, y) = H_{x}(\beta, \beta) \Phi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.20)

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = H_{x}(\beta, \beta) \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y), \]  
(2.21)

\[ \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = H_{x,y}(\alpha, \gamma_1, \gamma_2; x, y), \]  
(2.22)

\[ \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = H_{x,y}(\alpha, \gamma_1, \gamma_2; x, y), \]  
(2.23)

\[ \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = H_{x,\epsilon}(\alpha, \gamma_1, \gamma_2; x, y), \]  
(2.24)

\[ \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = H_{x,\epsilon}(\alpha, \gamma_1, \gamma_2; x, y), \]  
(2.25)

\[ \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = H_{x,\epsilon}(\alpha, \gamma_1, \gamma_2; x, y), \]  
(2.26)

\[ \Xi_1(\alpha, \alpha_2, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha, \alpha_2) \Xi_1(\alpha, \alpha_2; \gamma; x, y), \]  
(2.27)

\[ \Xi_1(\alpha, \alpha_2, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha, \alpha_2) \Xi_1(\alpha, \alpha_2; \gamma; x, y), \]  
(2.28)

\[ \Xi_1(\alpha, \alpha_2, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha, \alpha_2) \Xi_1(\alpha, \alpha_2; \gamma; x, y), \]  
(2.29)

\[ \Xi_1(\alpha, \alpha_2, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha, \alpha_2) \Xi_1(\alpha, \alpha_2; \gamma; x, y), \]  
(2.30)

\[ \Xi_2(\alpha, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha) \Xi_2(\alpha, \beta; \gamma; x, y), \]  
(2.31)

\[ \Xi_2(\alpha, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha) \Xi_2(\alpha, \beta; \gamma; x, y), \]  
(2.32)

\[ \Xi_2(\alpha, \beta; \gamma; x, y) = H_{x,\epsilon}(\alpha) \Xi_2(\alpha, \beta; \gamma; x, y), \]  
(2.33)
\[ \Xi_2(\alpha, \beta; \gamma; x, y) = \frac{H_{x}(\varepsilon_2, \beta)}{H_{y}(\varepsilon_1, \gamma)} \Xi_2(\alpha, \varepsilon_2; \gamma; x, y), \] 
\[ \Xi_2(\alpha, \beta; \gamma; x, y) = H_{x}(\varepsilon, \gamma) \Xi_2(\alpha, \beta; \varepsilon; x, y). \] 

In view of the known Mellin–Barnes contour integral representations (we introduce a recent book [20] dealing with this topic comprehensively) for the Humbert functions \( \Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1 \) and \( \Xi_2 \), it is not difficult to give alternative proofs of the operator identities (2.1)–(2.35) above by using the Mellin and the inverse Mellin transformations (see, for example, [21]). The details involved in these alternative derivations of the operator identities (2.1)–(2.35) are omitted here. By virtue of the derivative formulas for the Humbert functions, and also some standard properties of hypergeometric functions, we find each of the following decomposition formulas for the Humbert functions \( \Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1 \) and \( \Xi_2 \) in two variables:

\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\varepsilon-i)_{i+j}(\beta)_i}{(\gamma)_{i+j} i!} x^i y^j \Phi_1(\varepsilon + i + j; \beta + i; \gamma + i + j; x, y), \] 
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\alpha - \varepsilon)_{i+j}(\beta)_i}{(\gamma)_{i+j} i!} x^i y^j \Phi_1(\varepsilon, \beta + i; \gamma + i + j; x, y), \] 
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\gamma - \varepsilon)_{i+j}(\beta)_{i+j}}{(\gamma)_{i+j} i!} x^i y^j \Phi_1(\alpha + i + j; \beta + i; \varepsilon + i + j; x, y), \] 
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = e^{\gamma}(1 - x)^{-\beta} \Phi_1 \left( \gamma - \alpha; \beta; x; \frac{x}{x-1}, -y \right), \] 
\[ (1 - x)^{-\beta} e^{\gamma} = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}((\beta - \varepsilon)_i)}{(\gamma)_{i+j} i!} x^i y^j \Phi_1(\alpha + i; \varepsilon + i; \gamma + i; x, y), \] 
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta - \varepsilon)_i}{(\gamma)_i i!} x^i y^j \Phi_1(\alpha + i, \varepsilon; \gamma + i; x, y), \] 
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\gamma - \varepsilon)_{i+j}(\beta_1)_i(\beta_2)_i}{(\gamma)_{i+j} i!} x^i y^j \Phi_2(\beta_1 + i, \beta_2 + j; \varepsilon + i + j; x, y), \] 
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\beta_1 - \varepsilon)_i}{(\gamma)_i i!} x^i y^j \Phi_2(\beta_1, \beta_2; \gamma + i; x, y), \] 
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\varepsilon_1 - \beta_1)_i(\beta_2)_{i+j}}{(\gamma)_{i+j} i!} x^i y^j \Phi_2(\varepsilon_1 + i, \beta_2; \gamma + i + j; x, y), \] 
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\beta_1 - \varepsilon_1)(\beta_2 - \varepsilon_2)}{(\gamma)_{i+j} i!} x^i y^j \Phi_2(\varepsilon_1, \varepsilon_2; \gamma + i + j; x, y), \] 
\[ \Phi_3(\beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\gamma - \beta)_i}{(\gamma)_i i!} x^i y^j \Phi_3(\varepsilon + i + j; \gamma + i; x, y), \] 
\[ \Phi_3(\beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\beta - \varepsilon)_i}{(\gamma)_i i!} x^i y^j \Phi_3(\varepsilon, \gamma + i; x, y), \] 
\[ \Phi_3(\beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-i)_{i+j}(\gamma - \varepsilon)_{i+j}(\beta)_i}{(\gamma)_{i+j} i!} x^i y^j \Phi_3(\beta + i; \varepsilon + i + j; x, y). \]
\[
\psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha - \epsilon)_{i+j}(\beta)_i}{(\gamma_1)_i(i!)^j} x^i y^j \psi_1(\epsilon + i + j, \beta + i; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.51)

\[
\psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha)_{i}(\epsilon - \beta)_i}{(\gamma_1)_i(i!)} x^i \psi_1(\alpha + i, \epsilon + i; \gamma_1 + i, \gamma_2; x, y),
\]

(2.52)

\[
\psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{i=0}^{\infty} \frac{\alpha_i(\beta - \epsilon)_i}{(\gamma_1)_i(i!)} x^i \psi_1(\alpha + i, \epsilon; \gamma_1 + i, \gamma_2; x, y),
\]

(2.53)

\[
\psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = (1 - x)^{-\alpha} \psi_1\left(\alpha, \gamma_1 - \beta; \gamma_1, \gamma_2; \frac{x}{x - 1}, \frac{y}{1 - x}\right),
\]

(2.54)

\[
(1 - x)^{-\alpha} F_1(\alpha, \gamma_2; \frac{y}{1 - x}) = \sum_{i=0}^{\infty} \frac{\alpha_i(\gamma_1 - \beta)_i}{(\gamma_1)_i(i!)} x^i \psi_1(\alpha + i, \beta; \gamma_1 + i, \gamma_2; x, y),
\]

(2.55)

\[
\psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha)_{i}(\gamma_2 - \epsilon)_i}{(\gamma_2)_i(i!)^j} y^j \psi_1(\alpha + i, \epsilon + i; x, y),
\]

(2.56)

\[
\psi_1(\alpha, \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(\alpha - \epsilon)_{i+j}}{(\gamma_1)_i(i!)} x^i \psi_2(\epsilon; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.57)

\[
\psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(\alpha - \epsilon)_{i+j}}{(\gamma_1)_i(i!)} x^i \psi_2(\alpha + i; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.58)

\[
\psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha)_{i+j}(\gamma_1 - \epsilon)_i}{(\gamma_1)_i(i!)} x^i \psi_2(\alpha + i; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.59)

\[
\psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha)_{i+j}(\gamma_2 - \epsilon)_i}{(\gamma_2)_i(i!)} y^j \psi_2(\alpha + i; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.60)

\[
\psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\alpha)_{i+j}(\gamma_1 - \epsilon)_i(\gamma_2 - \epsilon)_j}{(\gamma_1)_i(i!)} x^i y^j \psi_2(\alpha + i; \gamma_1 + i, \gamma_2 + j; x, y),
\]

(2.61)

\[
\Xi_1(\alpha_1, \alpha_2, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\epsilon - \alpha_1)(\epsilon - \alpha_2)}{(\gamma)_{i+j}(i!)^j} x^i y^j \Xi_1(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.62)

\[
\Xi_1(\alpha_1, \alpha_2, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{\beta_i(\alpha_1 - \epsilon)(\alpha_2 - \epsilon)}{(\gamma)_{i+j}(i!)^j} x^i y^j \Xi_1(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.63)

\[
\Xi_1(\alpha_1, \alpha_2, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}i(\alpha_1)(\alpha_2)}{(\gamma)_{i+j}(i!)^j} x^i y^j \Xi_1(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.64)

\[
\Xi_1(\alpha_1, \alpha_2, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\epsilon - \alpha_1)(\epsilon - \alpha_2)}{(\gamma)_{i+j}(i!)^j} x^i y^j \Xi_1(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.65)

\[
\Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-1)^{i}(\epsilon - \alpha_1)(\beta)_i}{(\gamma)_{i!}} x^i \Xi_2(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.66)

\[
\Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\alpha - \epsilon)_i(\beta)_i}{(\gamma)_{i!}} x^i \Xi_2(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.67)

\[
\Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(-1)^{i}(\alpha)(\beta)_i}{(\gamma)_{i!}} x^i \Xi_2(\epsilon + i, \beta + i; \gamma + i; x, y),
\]

(2.68)
\[ \Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta - \varepsilon_2)_i}{(\gamma)_i i!} x^i y^i \Xi_2(\alpha + i, \varepsilon_2; \gamma + i; x, y). \]  
\[ (2.69) \]

\[ \Xi_2(\alpha, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(\gamma - \varepsilon)_{i+j}(\alpha)_i(\beta)_j}{(\gamma)_{i+j}(\varepsilon)_{i+j} i! j!} x^i y^j \Xi_2(\alpha + i, \beta + j; \varepsilon + i + j; x, y). \]  
\[ (2.70) \]

Our operational derivations of the decomposition formulas (2.15)–(2.26) would indeed run parallel to those presented in the earlier works, which we have already cited in the preceding sections. In addition to the various operator expressions, we also make use of the following operator identities [22, p. 93]:

\[ (\delta + \alpha)_n [f(\xi)] = \xi^{1-a} \frac{d^n}{d\xi^n} [\xi^{a+n-1} f(\xi)] \]  
\[ (2.71) \]

and

\[ (-\delta)_n [f(\xi)] = (-1)^n \xi^{1-a} \frac{d^n}{d\xi^n} [f(\xi)], \quad (\delta := \xi \frac{d}{d\xi}; \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \]

for every analytic function \( f \). Many other analogous decomposition formulas can similarly be derived for the Humbert functions \( \Phi_1, \Phi_2, \Psi_1, \Psi_2, \Xi_1 \) and \( \Xi_2 \), but with various different parametric constraints.

### 3. An exemplifying proof of the decomposition formulas for Humbert functions

The various decomposition formulas for the Humbert functions \( \Phi_1, \Phi_2, \Psi_1, \Psi_2, \Xi_1 \) and \( \Xi_2 \) in two variables (which are presented here and in other places in the previously cited literature) can be proven in a fairly simple manner by suitably applying superposition of the inverse pairs of symbolic operators introduced in Section 1. As an example, we outline a proof of the decomposition formula (2.36). For the two variable Humbert function \( \Phi_1 \), it is seen from (2.1) that

\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \sum_{i,j=0}^{\infty} \frac{(\varepsilon - \alpha)_{i+j}(-\delta_1)_i(-\delta_2)_j}{(\varepsilon)_{i+j} i! j!} \Phi_1(\varepsilon, \beta; \gamma; x, y) \]  
\[ (3.1) \]

\[ \left( \delta_1 := \frac{x}{\partial_x}; \ \delta_2 := \frac{y}{\partial_y} \right). \]

Furthermore, by a straightforward computation, we have

\[ (-\delta_1)_i \Phi_1(\varepsilon, \beta; \gamma; x, y) = (-1)^i \frac{x^i(\varepsilon)_i(\beta)_i}{(\gamma)_i} \Phi_1(\varepsilon + i, \beta + i; \gamma + i; x, y) \]  
\[ (3.2) \]

and

\[ (-\delta_2)_i (-\delta_1)_i \Phi_1(\varepsilon, \beta; \gamma; x, y) = (-1)^{i+j} \frac{(\varepsilon)_{i+j}(\beta)_i}{(\gamma)_{i+j}} x^i y^j \Phi_1(\varepsilon + i + j, \beta + i; \gamma + i + j; x, y). \]  
\[ (3.3) \]

Upon substituting from (3.3) into (3.1), we finally arrive at the decomposition formula (2.36).

### 4. Integral representations via decomposition formulas

Here we observe that several integral representations of the Eulerian type can also be deduced from the corresponding decomposition formulas in Section 2. For example, by making use of the known integral representations:

\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 e^{\xi(1-x\xi)} (1-\xi)^{-\gamma-\alpha} d\xi, \]  
\[ (4.1) \]

\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta_1) \Gamma(\beta_2) \Gamma(\gamma - \beta_1 - \beta_2)} \int_0^1 \int_0^1 e^{\xi(1-x\xi)(\gamma - \beta_1 - \beta_2)} (1-\xi)^{-\gamma - \beta_1 - \beta_2 - 1} d\xi d\eta, \]  
\[ (4.2) \]

\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \frac{\Gamma(\gamma_1) \Gamma(\gamma_2)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma_1 - \alpha) \Gamma(\gamma_2 - \beta)} \int_0^1 \int_0^1 e^{\xi(1-x\xi)(\gamma_1 - \alpha)(\gamma_2 - \beta)} (1-\xi)^{-\gamma_1 - \alpha - 1} (1-x\xi)^{-\gamma_2 - \beta - 1} d\xi d\eta, \]  
\[ (4.3) \]
\[ Z_1(\alpha, \alpha_2, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha_2)\Gamma(\gamma - \alpha - \alpha_2)} \times \int_0^1 \int_0^1 e^{(1-\xi)\eta x_1 - 1} (1 - \xi)^{y-a_1-1}(1 - \eta)^{y-a_2-1}(1 - x_1^2)^{\beta_1} d\xi d\eta, \tag{4.4} \]
\[ Z_2(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \xi^{\alpha-1}(1 - \xi)^{\gamma-\alpha-1}(1 - x_1^2)^{\beta} F_1[y - \alpha; (1 - \xi) y] d\xi, \tag{4.5} \]

we find each of the following integral representations (4.6)–(4.20):

\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\gamma - \epsilon)} \int_0^1 e^{y_1 \xi^{\gamma-1}(1 - \xi)^{\gamma-\epsilon-1}(1 - x_1^2)^{\beta_1}} \Phi_1(\epsilon - \alpha, \beta; \epsilon; \frac{x_1}{x_1^2 - 1}, -y_1) d\xi, \tag{4.6} \]
\[ (\Re(\gamma) > \Re(\epsilon) > 0), \]
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\gamma - \epsilon)\Gamma(\epsilon - \alpha)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - x_1^2)^{\beta_1}} \Phi_1 \times \left( \alpha - \epsilon, \beta; \gamma - \epsilon; \frac{x(1 - \xi)}{1 - x_1^2}, y(1 - \xi) \right) d\xi d\eta, \tag{4.7} \]
\[ (\Re(\gamma) > \Re(\epsilon) > 0), \]
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \alpha)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - x_1^2)^{\beta_1}} \Phi_1 \times \left( \alpha - \epsilon, \beta; \gamma - \epsilon; \frac{x(1 - \xi)}{1 - x_1^2}, y(1 - \xi) \right) d\xi d\eta, \tag{4.8} \]
\[ (\Re(\gamma) > \Re(\epsilon) > 0), \]
\[ \Phi_1(\alpha, \beta; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \alpha)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - x_1^2)^{\beta_1}} \Phi_1 \times \left( \alpha - \epsilon, \beta; \gamma - \epsilon; \frac{x(1 - \xi)}{1 - x_1^2}, y(1 - \xi) \right) d\xi d\eta, \tag{4.9} \]
\[ (\Re(\gamma) > \Re(\epsilon) > 0), \]
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma - \beta_1 - \beta_2)} \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\beta_1-1}(1 - \eta)\beta_2-1} F_1(\gamma - \epsilon; \gamma - \epsilon; -x_1^2, y(1 - \xi)\eta) d\xi d\eta, \tag{4.10} \]
\[ (\Re(\epsilon) > \Re(\beta_1) > 0; \Re(\beta_2) > 0), \]
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \beta_1 - \beta_2)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - \eta)\beta_2-1} F_1(\epsilon - \beta_1; \epsilon; -x_1^2, y(1 - \xi)\eta) d\xi d\eta, \tag{4.11} \]
\[ (\Re(\gamma) - \Re(\beta_1) - \Re(\beta_2) > 0; \Re(\epsilon) > 0; \Re(\beta_2) > 0), \]
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \beta_1 - \beta_2)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - \eta)\beta_2-1} F_1(\epsilon - \beta_1; \epsilon; -x_1^2, y(1 - \xi)\eta) d\xi d\eta, \tag{4.12} \]
\[ (\Re(\gamma) - \Re(\beta_1) - \Re(\beta_2) > 0; \Re(\epsilon) > 0; \Re(\beta_2) > 0), \]
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \beta_1 - \beta_2)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - \eta)\beta_2-1} F_1(\epsilon - \beta_1; \epsilon; -x_1^2, y(1 - \xi)\eta) d\xi d\eta, \tag{4.13} \]
\[ (\Re(\gamma) - \Re(\beta_1) - \Re(\beta_2) > 0; \Re(\epsilon) > 0; \Re(\beta_2) > 0), \]
\[ \Phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon)\Gamma(\epsilon - \beta_1 - \beta_2)\Gamma(\gamma - \epsilon)} \times \int_0^1 \int_0^1 e^{y_1 \xi^\gamma(1 - \xi)^{\gamma-\epsilon-1}(1 - \eta)\beta_2-1} F_1(\epsilon - \beta_1; \epsilon; -x_1^2, y(1 - \xi)\eta) d\xi d\eta, \tag{4.14} \]
\[ (\Re(\gamma) - \Re(\beta_1) - \Re(\beta_2) > 0; \Re(\epsilon) > 0; \Re(\beta_2) > 0), \]
\[ \Psi_1(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \frac{\Gamma(\gamma_1)\Gamma(\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma_1 - \beta)} \int_0^1 \int_0^1 e^{\frac{y}{x\xi}} \xi^{\beta - 1} \eta^{\alpha - 1} (1 - \xi)^{\gamma_1 - \beta - 1} (1 - \eta)^{\epsilon - \alpha - 1} \]
\[ \times (1 - x\xi)^{-\alpha} F_1 \left[ \gamma_2 - \epsilon; \gamma_2; \frac{y}{x\xi - 1} \right] \, d\xi \, d\eta \]
(4.15)

\[ (\Re(\gamma_1) > \Re(\beta) > 0; \Re(\epsilon) > \Re(\alpha) > 0), \]

\[ \Xi_1(\alpha_1, \alpha_2, \beta; \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon_1)\Gamma(\epsilon_2)\Gamma(\gamma - \epsilon_1 - \epsilon_2)} \int_0^1 \int_0^1 e^{(1 - \epsilon)\eta\xi^{1 - \epsilon - 1}} \eta^{\gamma - \epsilon - \epsilon_1 - \epsilon_2 - 1} \xi^{\epsilon_1 - \epsilon - 1} \]
\[ \times (1 - \eta)^{\gamma - \epsilon - \epsilon_2 - 1} F(\alpha_1 - \epsilon_1, \alpha_2 - \epsilon_2; \beta; \gamma - \epsilon_1 - \epsilon_2; \frac{x(1 - \xi)(1 - \eta)}{1 - x\xi}), \]
\[ \times y(1 - \xi)(1 - \eta) \right] \, d\xi \, d\eta \]
(4.16)

\[ (\Re(\gamma - \epsilon_1 - \epsilon_2) > 0; \Re(\epsilon_1) > 0; \Re(\epsilon_2) > 0), \]

\[ \Xi_2(\alpha, \beta; \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\epsilon_1)\Gamma(\gamma - \epsilon_1 - 1)} \int_0^1 \xi^{\gamma - \gamma_1 - 1} (1 - \xi)^{\gamma - \epsilon - \epsilon_1 - 1} F(\alpha, \beta; \gamma - \epsilon_1 - 1; \frac{x(1 - \xi)}{1 - x\xi}), \]
\[ (\Re(\gamma) > \Re(\epsilon_1) > 0), \]

\[ \Xi_2(\alpha, \beta; \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\epsilon_1 - \alpha)\Gamma(\gamma - \epsilon_1 - 1)} \int_0^1 \int_0^1 \xi^{\gamma - \gamma_1 - 1} \eta^{\alpha - 1} \]
\[ \times (1 - \xi)^{\gamma - \gamma_1 - 1} (1 - \eta)^{\epsilon - 1 - \alpha - 1} (1 - x\xi)^{\beta - 1} F_1 \left( \gamma - \epsilon_1 - 1; \gamma - \epsilon_1 - 1; \frac{x(1 - \xi)}{1 - x\xi}, \frac{y(1 - \xi)}{1 - x\xi} \right) \, d\xi \, d\eta \]
(4.19)

\[ (\Re(\gamma) > \Re(\epsilon_1) > \Re(\alpha) > 0), \]

\[ \Xi_2(\alpha, \beta; \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \epsilon_1 - 1)\Gamma(\epsilon_1 - \beta)} \int_0^1 \int_0^1 \xi^{\gamma - \gamma_1 - 1} \eta^{\beta - 1} (1 - \xi)^{\gamma - \gamma_1 - 1} (1 - \eta)^{\epsilon_1 - 1 - \beta - 1} \]
\[ \times (1 - x\xi)^{-\beta} F_1 \left( \gamma - \epsilon_1 - 1; \gamma - \epsilon_1 - 1; \frac{x(1 - \xi)}{1 - x\xi}, \frac{y(1 - \xi)}{1 - x\xi} \right) \, d\xi \, d\eta \]
(4.20)

(\Re(\gamma) > \Re(\epsilon_1) > \Re(\beta) > 0).

We conclude this paper by remarking that mutually inverse operators (1.11) and (1.12) can be applied to other multivariate hypergeometric functions.

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References