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Torus graphs and simplicial posets

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Abstract

For several important classes of manifolds acted on by the torus, the information about the action can be encoded combinatorially by a regular n-valent graph with vector labels on its edges, which we refer to as the *torus graph*. By analogy with the *GKM-graphs*, we introduce the notion of *equivariant cohomology* of a torus graph, and show that it is isomorphic to the *face ring* of the associated *simplicial poset*. This extends a series of previous results on the equivariant cohomology of torus manifolds. As a primary combinatorial application, we show that a simplicial poset is *Cohen–Macaulay* if its face ring is Cohen–Macaulay. This completes the algebraic characterisation of Cohen–Macaulay posets initiated by Stanley. We also study *blow-ups* of torus graphs and manifolds from both the algebraic and the topological points of view.

Keywords: Torus graphs; Simplicial posets; Cohen–Macaulay posets; Torus manifolds; GKM-graphs; Equivariant cohomology; Blow-ups

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1. Introduction

The study of torus actions on manifolds is renowned for its close connections with combinatorics and combinatorial geometry. Two classes of actions are typical here; namely, the (smooth, compact) *algebraic toric varieties* and *Hamiltonian* torus actions on *symplectic manifolds*. Both are very special cases of a torus action $T^k \times M^{2n} \to M^{2n}$ on an even-dimensional manifold; and the relation to combinatorics comes from the study of the orbit poset and the orbit quotient. In the former case the notion of *fan*, which encodes both combinatorial and geometric data, allows one to completely translate algebraic geometry into combinatorics; in the latter case important information about the Hamiltonian action is contained in the *moment polytope*.

During the last two decades both examples have developed into several other classes of manifolds with torus action, mostly of purely topological nature. These manifolds are neither algebraic varieties nor symplectic manifolds in general, thereby enjoying a larger flexibility for topological or combinatorial applications, but still possess most of the important topological properties of their algebraic or symplectic predecessors. The study of toric varieties from a topological viewpoint led to the appearance of (*quasi*)*toric manifolds* [6], *multifans* [16] and *torus manifolds* [12]. The latter carry an effective half-dimensional torus action $T^n \times M^{2n} \to M^{2n}$ whose fixed point set is non-empty.

The concept of a *GKM-manifold* is closely related to Hamiltonian torus actions. According to [9], a compact $2n$-dimensional manifold M with an effective torus action $T^k \times M \to M$ ($k \leqslant n$) is called a GKM-manifold if the fixed point set is finite, M possesses an invariant almost complex structure, and the weights of the tangential T^k-representations at the fixed points are pairwise linearly independent. These manifolds are named after Goresky, Kottwitz and MacPherson, who studied them in [7]. They showed that the "one-skeleton" of such a manifold M, that is, the set of points fixed by at least a codimension-one subgroup of T^k, has the structure of a "labelled" graph (Γ, α), and that the most important topological information about M, such as its Betti numbers or equivariant cohomology ring, can be read directly from this graph. These graphs have since become known as *GKM-graphs* (or *moment graphs*); and their study has been of independent combinatorial interest since the appearance of Guillemin and Zara's paper [9]. The idea of associating a labelled graph to a manifold with a circle action also featured in Musin's work [20].

Both GKM- and torus manifolds have become objects of study in the emerging field of *toric topology*, and linking these important classes of torus actions together has been one of our aims here. Our concept of a *torus graph*, motivated by that of a GKM-graph, allows us to translate the important topological properties of torus manifolds into the language of combinatorics, like in the case of GKM-manifolds. Therefore, the study of torus graphs becomes our primary objective. A torus graph is a finite n-valent graph Γ (without loops, but with multiple edges allowed) with an *axial function* on the set $E(\Gamma)$ of oriented edges taking values in $\mathrm{Hom}(T^n, S^1) = H^2(BT^n)$ and satisfying certain compatibility conditions. These conditions (described in Section 3) are similar to those for GKM-graphs, but not exactly the same. For the graphs coming from torus manifolds the values of the axial function coincide with the weights of the tangential T^n-representations at the fixed points.

The notion of equivariant cohomology of a torus graph introduced in Section 3 is same as that of a GKM-graph given in [9] and [10]. However, unlike the case of GKM-graphs, we have been able to completely describe the equivariant cohomology ring of a torus graph in terms of generators and relations, by applying the methods of our previous work [18] to the associated *simplicial poset*. Simplicial posets have already shown their importance in the topological study of torus actions (see e.g. [18]); they also feature prominently in this paper. In Section 5 we associate a simplicial poset $\mathcal{P}(\Gamma)$ to an arbitrary torus graph Γ; our main result there (Theorem 5.5) establishes an isomorphism between the equivariant cohomology of Γ and the *face ring* of $\mathcal{P}(\Gamma)$. This theorem continues the series of results identifying the equivariant cohomology of a smooth toric variety, a (quasi)toric manifold [6], and a torus manifold [18, Theorem 7.5] with the face ring of the appropriate polytope, simplicial complex, or simplicial poset.

Despite the concepts of GKM- and torus graphs diverge in general, they have an important subclass of *n-independent* GKM-graphs in their intersection. Therefore, our methods and results about torus graphs are fully applicable to this subclass of GKM-graphs, which may be considered as a partial answer to some questions about GKM-graphs posed in the introduction of [10].

Apart from topological applications to the study of torus action, the concept of a torus graph and the associated simplicial poset appears to be of considerable interest for combinatorial commutative algebra. Since the appearance of Stanley's book [23] the face ring (or the *Stanley–Reisner ring*) of simplicial complex has become one of the most important media of applications of commutative-algebraic methods to combinatorics. The notion of a face ring has been later extended to simplicial posets in [22] (see also [18]). Our primary combinatorial application is the proof of equivalence of the Cohen–Macaulay properties for simplicial posets and their face rings. A poset is said to be *Cohen–Macaulay* if the face ring of its order complex is a Cohen–Macaulay ring. However, in the case of a simplicial poset $\mathcal{P}$ the face ring is defined for the poset $\mathcal{P}$ itself, not only for its order complex. Therefore, a natural question of whether the Cohen–Macaulay property can be read directly from the face ring of $\mathcal{P}$ arises. In [22] Stanley proved that the face ring of a Cohen–Macaulay simplicial poset is Cohen–Macaulay. In Theorem 6.9 we prove the converse. To do that one has to show that if the face ring of $\mathcal{P}$ is Cohen–Macaulay, then the face ring of the order complex of $\mathcal{P}$ is also Cohen–Macaulay. The passage to the order complex is known to topologists as the *barycentric subdivision*, and our proof proceeds inductively by decomposing the barycentric subdivision into a sequence of elementary *stellar subdivisions* and showing that the Cohen–Macaulay property is preserved at each step. Stellar subdivisions of simplicial posets are related to *blow-ups* of torus manifolds and torus graphs; we further explore this link in Section 8 by studying the behaviour of equivariant cohomology under these operations.

In Section 7 we give a partial answer to the question of characterising simplicial posets arising from torus graphs. We also discuss related notions of *orientation* and *orientability* of a torus graph. Here lies yet another distinction between the GKM- and torus graphs; all GKM-graphs are orientable by their definition.

In the last section we deduce certain combinatorial identities for the number of faces of simplicial posets and torus graphs, which may be regarded as a yet another generalisation to the *Dehn–Sommerville equations* for simple polytopes, sphere triangulations, Eulerian posets, etc.

2. Torus manifolds and equivariant cohomology

A *torus manifold* [12] is a $2n$-dimensional compact smooth manifold M with an effective (or faithful) action of an n-dimensional torus T whose fixed point set is non-empty. This fixed point set M^T is easily seen to consist of finite number of isolated points. A *characteristic submanifold* of M is a codimension-two connected component of the set fixed pointwise by a circle subgroup of T. An *omniorientation* [4] of M consists of a choice of orientation for M and for each characteristic submanifold. A choice of omniorientation allows us to regard the normal bundles to characteristic submanifolds as complex line bundles, and is particularly useful for studying the equivariant cohomology, characteristic classes, stably almost complex structures on torus manifolds, etc. Sometimes fixing an omniorientation is required in the definition of a torus manifold.

All the cohomology in this paper is taken with $\mathbb{Z}$ coefficients, unless otherwise specified.

Let $ET \to BT$ be the universal T-bundle, with T acting on ET freely from the right. Let $ET \times_T M$ be the orbit space of the T-action on $ET \times M$ defined by $(u,x) \to (ug^{-1}, gx)$ for $(u,x) \in ET \times M$ and $g \in T$. The projection onto the first factor gives rise to a fibration

$$M \to ET \times_T M \to BT \tag{2.1}$$

and the equivariant cohomology of M is defined as the ordinary cohomology of the total space of this fibration:

$$H_T^*(M) := H^*(ET \times_T M).$$

Assume that $H^{\mathrm{odd}}(M) = 0$. According to Lemma 2.1 of [18], this is equivalent to $H_T^*(M)$ being isomorphic to $H^*(M) \otimes H^*(BT)$ as an $H^*(BT)$-module. Therefore, $H_T^*(M)$ is a free $H^*(BT)$-module, and the Serre spectral sequence of the fibration (2.1) collapses. (This condition is referred to as the *equivariant formality* of M in [9, §1.1], although it is different from the notion of formality, either plain or equivariant, adopted in the rational homotopy theory.) Under such an assumption, the localisation theorem [14] implies that the restriction homomorphism

$$i^* : H_T^*(M) \to H_T^*\big(M^T\big) \tag{2.2}$$

is injective, where $i : M^T \to M$ is the inclusion. The image of $H_T^*(M)$ in $H_T^*(M^T)$ can be identified in the same way as it was done by Goresky, Kottwitz and MacPherson [7] for their class of manifolds (now known as the *GKM-manifolds*). We briefly describe their result here. Let Σ_M denote the set of 2-dimensional submanifolds of M each of which is fixed pointwise by a codimension one subtorus of T. Then every $S \in \Sigma_M$ is diffeomorphic to a sphere, contains exactly two T-fixed points, and is a connected component of the intersection of some $n-1$ characteristic submanifolds. Denote by T_S the isotropy subgroup of S. We have a canonical identification

$$H_T^*\big(M^T\big) = \mathrm{Map}\big(M^T, H^*(BT)\big),$$

and for each $p \in M^T$ there are exactly n spheres in Σ_M containing p.

Theorem 2.1. ([7], [8, §11.8], [11, Theorem 3.1]) *Suppose $H_T^*(M)$ is a free $H^*(BT)$-module. Then $f \in \mathrm{Map}(M^T, H^*(BT))$ belongs to the image of the map i^* from* (2.2) *if and only if*

$$r_S\big(f(p)\big) = r_S\big(f(q)\big) \quad \text{for every 2-sphere } S \text{ in } \Sigma_M$$

where r_S denotes the restriction $H^*(BT) \to H^*(BT_S)$ and p, q are the T-fixed points in S.

In [8] this result is stated for GKM-manifolds and with coefficients in a field of zero characteristic, but it also holds for torus manifolds with integer coefficients as stated above (see [18] and Example 3.3 below). The proof of [8] relies on a result of Chang and Skjelbred [5] and the localisation theorem [14]. In [11] the theorem is proved with integer coefficients in a much more general context of G-equivariant cohomology theories, under some additional assumptions.

The tangential T-representation $\tau_p M$ at $p \in M^T$ decomposes into a direct sum of irreducible real two-dimensional T-representations. An omniorientation on M determines orientations on the corresponding two-dimensional T-representation spaces, so that we may think of them as complex one-dimensional T-representations. Therefore, we have

$$\tau_p M = \bigoplus_{i=1}^{n} V(w_{p,i}) \tag{2.3}$$

where $V(w_{p,i})$ denotes a complex one-dimensional T-representations with weight $w_{p,i}$. The set of complex one-dimensional T-representations bijectively corresponds to $H^2(BT)$. Through this bijection, we may think of an element of $H^2(BT)$ as a weight of the corresponding T-representation.

3. Torus graphs

In their study of GKM-manifolds, Guillemin and Zara [9] introduced a combinatorial object called a *GKM-graph* and defined a notion of (equivariant) cohomology for such graphs accordingly. In this section we shall see that a similar idea works for torus manifolds with a little modification.

Let Γ be a connected regular n-valent graph, $V(\Gamma)$ the set of vertices of Γ, and $E(\Gamma)$ the set of oriented edges of Γ (so that each edge of Γ enters $E(\Gamma)$ with two possible orientations). We denote by $i(e)$ and $t(e)$ the initial and terminal points of $e \in E(\Gamma)$ respectively, and by $\bar{e}$ the edge e with the orientation reversed. For $p \in V(\Gamma)$ we set

$$E(\Gamma)_p := \big\{ e \in E(\Gamma) \;\big|\; i(e) = p \big\}.$$

A collection $\theta = \{\theta_e\}$ of bijections

$$\theta_e : E(\Gamma)_{i(e)} \to E(\Gamma)_{t(e)}, \quad e \in E(\Gamma),$$

is called a *connection* on Γ if

(a) $\theta_{\bar{e}}$ is the inverse of θ_e;
(b) $\theta_e(e) = \bar{e}$.

An n-valent graph Γ admits $((n-1)!)^g$ different connections, where g is the number of (non-oriented) edges in Γ. Slightly modifying the original definition of Guillemin and Zara [9], we call a map

$$\alpha : E(\Gamma) \to \mathrm{Hom}\big(T, S^1\big) = H^2(BT)$$

an *axial function* (associated with the connection θ) if it satisfies the following three conditions:

(a) $\alpha(\bar{e}) = \pm\alpha(e)$;
(b) elements of $\alpha(E(\Gamma)_p)$ are pairwise linearly independent (2-independent) for each $p \in V(\Gamma)$;
(c) $\alpha(\theta_e(e')) \equiv \alpha(e') \bmod \alpha(e)$ for any $e \in E(\Gamma)$ and $e' \in E(\Gamma)_{i(e)}$.

We also denote $T_e := \ker \alpha(e)$, the codimension-one subtorus in T determined by α and e. Then we may reformulate the condition (c) above by requiring that the restrictions of $\alpha(\theta_e(e'))$ and $\alpha(e')$ to $H^*(BT_e)$ coincide.

Remark. Guillemin and Zara required $\alpha(\bar{e}) = -\alpha(e)$ in (a) above. The connection θ which satisfies condition (c) above is unique if elements of $\alpha(E(\Gamma)_p)$ are 3-independent [9].

Definition. We call α a *torus axial function* if it is n-independent, that is, $\alpha(E(\Gamma)_p)$ is a basis of $H^2(BT)$ for each $p \in V(\Gamma)$. The triple (Γ, α, θ) is then called a *torus graph*. Since the connection θ is uniquely determined by the second remark above, we may suppress it in the notation. In what follows we only consider torus axial functions.

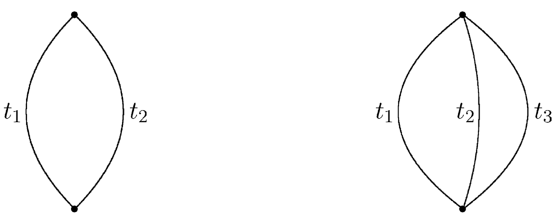
Remark. In comparison with the GKM-graphs, the definition of torus graphs weakens the assumption (a) on an axial function (by only requiring $\alpha(\bar{e}) = \pm\alpha(e)$ instead of $\alpha(\bar{e}) = -\alpha(e)$), but strengthens (b) (by requiring α to be n-independent instead of just 2-independent). Although the n-independence assumption is usually too strict for GKM-graphs (and leaves out some important examples), weakening the other assumption balances this, as is shown in our next examples.

Example 3.1. Let M be a torus manifold. Define a regular n-valent graph Γ_M whose vertex set is M^T and whose edges correspond to 2-spheres from Σ_M. The summands in (2.3) correspond to the oriented edges of Γ_M having p as the initial point. We assign $w_{p,i}$ to the oriented edge corresponding to $V(w_{p,i})$. This gives a function

$$\alpha_M : E(\Gamma_M) \to H^2(BT).$$

The normal bundle of the 2-sphere corresponding to an oriented edge in $E(\Gamma)$ decomposes into a Whitney sum of complex T-line bundles. This decomposition defines a connection θ_M in Γ_M. It is not difficult to see that α_M satisfies the three conditions from the definition of torus axial function.

Example 3.2. Two simple examples of torus graphs Γ are shown on Fig. 1. The first is 2-valent and the second is 3-valent. The axial function α assigns the basis elements $t_1, t_2 \in H^2(BT^2)$ (respectively $t_1, t_2, t_3 \in H^2(BT^3)$) to the two (respectively three) edges of Γ, regardless of the orientation. These torus graphs are not GKM-graphs, as the condition $\alpha(\bar{e}) = -\alpha(e)$ is not satisfied. Both come from torus manifolds, S^4 and S^6 respectively, where the torus action is obtained by suspending the standard coordinatewise torus actions on S^3 and S^5 (see [18, Example 3.2]).



(a) $n = 2$. (b) $n = 3$.

Fig. 1. Torus graphs.

Definition. The *equivariant cohomology* $H_T^*(\Gamma)$ of a torus graph Γ is a set of maps

$$f : V(\Gamma) \to H^*(BT)$$

such that for every $e \in E(\Gamma)$ the restrictions of $f(i(e))$ and $f(t(e))$ to $H^*(BT_e)$ coincide. Since $H^*(BT)$ is a ring, the vertex-wise multiplication endows the function space $H^*(BT)^{V(\Gamma)}$ with a ring structure. Its subspace $H_T^*(\Gamma)$ also becomes a ring because the restriction map $H^*(BT) \to H^*(BT_e)$ is multiplicative. Moreover, $H_T^*(\Gamma)$ is an algebra over $H^*(BT)$.

Example 3.3. If $H_T^*(M)$ is a free $H^*(BT)$-module (which happens if $H^{\mathrm{odd}}(M) = 0$) and $\Gamma_M := (\Gamma_M, \alpha_M, \theta_M)$ is the associated torus graph, then there is a ring isomorphism $H_T^*(\Gamma_M) \cong H_T^*(M)$. This follows either from Theorem 2.1 or from the explicit calculation of the two rings, given in Theorem 5.5 below and [18, Corollary 7.6].

4. Faces and Thom classes

Definition. Let (Γ, α, θ) be a torus graph, and Γ' a connected regular k-valent subgraph of Γ, where $0 \leqslant k \leqslant n$. We say that $(\Gamma', \alpha|E(\Gamma'))$ is a *k-dimensional face* of Γ if Γ' is invariant under the connection θ. We refer to $(n-1)$-dimensional faces as *facets*.

An intersection of faces is a union of faces. We define the *Thom class* of a k-face $F = (\Gamma', \alpha|E(\Gamma'))$ as a map $\tau_F : V(\Gamma) \to H^{2(n-k)}(BT)$ where

$$\tau_F(p) := \begin{cases} \prod_{i(e)=p,\, e \notin \Gamma'} \alpha(e) & \text{if } p \in V(\Gamma'), \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

Lemma 4.1. *τ_F is an element of $H_T^*(\Gamma)$.*

Proof. Let $e \in E(\Gamma)$. If neither vertex of e is contained in F, then the values of τ_F on both vertices of e are zero. If only one vertex of e, say $i(e)$, is contained in F, then $\tau_F(t(e)) = 0$, while $\tau_F(i(e)) = 0 \bmod \alpha(e)$, so that the restriction of $\tau_F(i(e))$ to $H^*(BT_e)$ is also zero. Finally, assume that the whole e is contained in F. Let e' be an edge such that $i(e') = i(e)$ and $e' \notin F$, so that $\alpha(e')$ is a factor in $\tau_F(i(e))$. Since F is invariant under the connection, we have $\theta_e(e') \notin F$. Therefore, $\alpha(\theta_e(e'))$ is one of the factors in $\tau_F(t(e))$. Now we have $\alpha(\theta_e(e')) \equiv \alpha(e') \bmod \alpha(e)$ by the definition of axial function. The same relation holds for every other factor in $\tau_F(i(e))$, whence the restrictions of $\tau_F(i(e))$ and $\tau_F(t(e))$ to $H^*(BT_e)$ coincide. □

Lemma 4.2. *If Γ is a torus graph, then there is a unique k-face of Γ containing any given k elements in $E(\Gamma)_p$.*

Proof. Let W be the k-dimensional subspace of $H^2(BT)$ spanned by the images of the given k edges in $E(\Gamma)_p$ under the axial function α. Take any element e from $E(\Gamma)_p$. Through the connection θ_e the given k edges in $E(\Gamma)_p$ map to some k edges in $E(\Gamma)_{t(e)}$. The α-images of these k edges in $E(\Gamma)_{t(e)}$ span the same subspace W in $H^2(BT)$. Proceeding in the same way, we translate the given k edges in $E(\Gamma)_p$ along the edges related to $E(\Gamma)_p$ via the connection. The uniqueness of the connection guarantees that the resulting graph is regular and k-valent. □

The intersection of two faces G and H of Γ is a finite set of faces. Denote by $G \vee H$ a minimal face containing both G and H. In general such a least upper bound may fail to exist or be non-unique; however it exists and is unique provided that the intersection $G \cap H$ is non-empty.

Lemma 4.3. *For any two faces G and H of Γ the corresponding Thom classes satisfy the relation*

$$\tau_G \tau_H = \tau_{G\vee H} \cdot \sum_{E \in G\cap H} \tau_E, \tag{4.2}$$

where we formally set $\tau_\Gamma = 1$ and $\tau_\emptyset = 0$, and the sum in the right-hand side is taken over connected components E of $G \cap H$.

Proof. Although the proof is the same as that of [18, Lemma 6.3], we include it here for reader's convenience. Take $p \in V(\Gamma)$. For a face F such that $p \in F$, we set

$$N_p(F) := \{ e \in E(\Gamma)\colon\ i(e) = p,\ e \notin F \},$$

which may be thought of as the set of directions normal to F at p. Then the identity (4.1) can be written as

$$\tau_F(p) = \prod_{e \in N_p(F)} \alpha(e) \tag{4.3}$$

where the right-hand side is understood to be 1 if $N_p(F) = \emptyset$ and 0 if $p \notin F$. If $p \notin G \cap H$, then $p \notin E$ for any connected component E of $G \cap H$ and either $p \notin G$ or $p \notin H$. Therefore, both sides of (4.2) take zero value on p. If $p \in G \cap H$, then

$$N_p(G) \cup N_p(H) = N_p(G \vee H) \cup N_p(E)$$

where E is the connected component of $G \cap H$ containing p, and $p \notin E'$ for any other connected component $E' \in G \cap H$. This together with (4.3) shows that both sides of (4.2) take the same value on p. □

Lemma 4.4. *The Thom classes τ_F corresponding to all proper faces of Γ constitute a set of ring generators for $H_T^*(\Gamma)$.*

Proof. Again, the proof is very similar to that of [18, Proposition 7.4]. Let $\eta \in H_T^{>0}(\Gamma)$ be a non-zero element. Set

$$Z(\eta) := \{ p \in V(\Gamma)\colon\ \eta(p) = 0 \}.$$

Take $p \in V(\Gamma)$ such that $p \notin Z(\eta)$. Then $\eta(p) \in H^*(BT)$ is non-zero and we can express it as a polynomial in $\{\alpha(e)\colon\ e \in E(\Gamma)_p\}$, which is a basis of $H^2(BT)$. Let

$$\prod_{e \in E(\Gamma)_p} \alpha(e)^{n_e}, \quad n_e \geqslant 0, \tag{4.4}$$

be a monomial entering $\eta(p)$ with a non-zero coefficient. Let F be the face spanned by the edges e with $n_e = 0$. Denote by $I(F)$ the ideal in $H^*(BT)$ generated by all elements $\alpha(e)$ with $e \in F$. Then $\eta(p) \notin I(F)$ since $\eta(p)$ contains monomial (4.4). Suppose $\eta(q) \in I(F)$ for some other vertex $q \in F$. Then $\eta(s) \in I(F)$ for any vertex $s \in F$ joined to q by an edge $f \subseteq F$ because $\eta(q) - \eta(s)$ is divisible by $\alpha(f)$ by the definition of axial function. Since F is a connected subgraph, $\eta(q) \in I(F)$ for any vertex $q \in F$, in contradiction with $\eta(p) \notin I(F)$. Hence, $\eta(q) \notin I(F)$, in particular $\eta(q) \neq 0$, for every vertex $q \in F$.

On the other hand, it follows from (4.1) that monomial (4.4) can be written as $u_F \tau_F(p)$ where u_F is a product of some Thom classes corresponding to faces containing F. Set $\eta' := \eta - u_F \tau_F \in H_T^*(\Gamma)$. Since $\tau_F(q) = 0$ for $q \notin F$, we have $\eta'(q) = \eta(q)$ for all such q. At the same time, $\eta(q) \neq 0$ for every vertex $q \in F$ by the argument from the previous paragraph. It follows that $Z(\eta') \supseteq Z(\eta)$. However, the number of monomials in $\eta'(p)$ is less than that in $\eta(p)$. Therefore, subtracting from η monomials in Thom classes we can eventually achieve an element λ such that $Z(\lambda)$ contains $Z(\eta)$ as a proper subset. Repeating this procedure, we end up at an element whose value on every vertex is zero. $\square$

In order to finish our description of the equivariant cohomology of torus graphs we need a combinatorial diversion to the concepts of simplicial posets and face rings.

5. Simplicial posets

We start by briefly reviewing simplicial posets and related algebraic notions. Then we prove our main result here, Theorem 5.5, which effectively describes the equivariant cohomology of torus graphs. The discussion of simplicial posets continues in the next section, where we concentrate on the Cohen–Macaulay property.

A poset $\mathcal{P}$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $\sigma \in \mathcal{P}$ the lower segment $[\hat{0}, \sigma]$ is a boolean lattice (the face poset of a simplex). We assume all our posets to be finite. An example of a simplicial poset is provided by the face poset of a simplicial complex, but there are many simplicial posets that do not arise in this way. We identify a simplicial complex with its face poset, thereby regarding simplicial complexes as particular cases of simplicial posets.

To each $\sigma \in \mathcal{P}$ we assign a geometric simplex whose face poset is $[\hat{0}, \sigma]$, and glue these geometrical simplices together according to the order relation in $\mathcal{P}$. We get a cell complex in which the closure of each cell can be identified with a simplex preserving the face structure and all the attaching maps are inclusions. We call it a *simplicial cell complex* and denote its underlying space by $|\mathcal{P}|$. In what follows we shall not distinguish between simplicial posets and simplicial cell complexes and refer to elements $\sigma \in \mathcal{P}$ as *simplices*.

Let $\mathcal{P}$ be a simplicial poset. The *rank function* on $\mathcal{P}$ is defined by setting $\operatorname{rk} \sigma = k$ for $\sigma \in \mathcal{P}$ if $[\hat{0}, \sigma]$ is the face poset of a $(k-1)$-dimensional simplex. The rank of $\mathcal{P}$ is the maximum of ranks of its elements. Let $\mathbf{k}$ be a commutative ring with unit. Introduce the graded polynomial ring $\mathbf{k}[v_\sigma\colon \sigma \in \mathcal{P} \setminus \hat{0}]$ with $\deg v_\sigma = 2 \operatorname{rk} \sigma$. We also write formally $v_{\hat{0}} = 1$. For any two simplices $\sigma, \tau \in \mathcal{P}$ denote by $\sigma \vee \tau$ the set of their least common upper bounds (*joins*), and by $\sigma \wedge \tau$ the set of their greatest common lower bounds (*meets*). Since $\mathcal{P}$ is a simplicial poset, $\sigma \wedge \tau$ consists of a single simplex whenever $\sigma \vee \tau$ is non-empty. There is the following simple characterisation of the subclass of simplicial complexes in simplicial posets.

Proposition 5.1. *$\mathcal{P}$ is a simplicial complex if and only if for any two elements $\sigma, \tau \in \mathcal{P}$ the set $\sigma \vee \tau$ is either empty or consists of a single simplex.*

Proof. Let $V(\mathcal{P}) = \{v_1, \dots, v_m\}$ be the set of vertices (rank one elements) of $\mathcal{P}$. Introduce a simplicial complex K on the vertex set $V(\mathcal{P})$ whose simplices are those subsets $\{v_{i_1}, \dots, v_{i_k}\}$ for which there is an element $\sigma \in \mathcal{P}$ with such vertex set. There is an obvious surjective order preserving map $\mathcal{P} \to K$ assigning to an element of $\mathcal{P}$ its vertex set. The injectivity of this map follows from the additional assumption on the joins. Therefore, $\mathcal{P}$ is (the face poset of) K. The other direction is obvious. □

Definition. [22] The *face ring* of a simplicial poset $\mathcal{P}$ is the quotient

$$\mathbf{k}[\mathcal{P}] := \mathbf{k}[v_\sigma\colon \sigma \in \mathcal{P} \setminus \hat{0}]/\mathcal{I}_\mathcal{P},$$

where $\mathcal{I}_\mathcal{P}$ is the ideal generated by the elements

$$v_\sigma v_\tau - v_{\sigma\wedge\tau} \cdot \sum_{\eta\in\sigma\vee\tau} v_\eta. \tag{5.1}$$

The sum over the empty set is assumed to be zero, so we have $v_\sigma v_\tau = 0$ if $\sigma \vee \tau = \emptyset$.

Remark. The definition above extends the notion of the face ring of a simplicial complex (also known as the *Stanley–Reisner ring*) to simplicial posets. In the case when $\mathcal{P}$ is a simplicial complex we may rewrite (5.1) as $v_\sigma v_\tau - v_{\sigma\wedge\tau} v_{\sigma\vee\tau}$ (because $\sigma \vee \tau$ is either empty or consists of a single simplex), and use the latter relation to express every element $\sigma \in \mathcal{P}$ as

$$v_\sigma = \prod_{v_i \in V(\sigma)} v_i,$$

where $V(\sigma)$ is the vertex set of σ. The relations between v_i's coming from (5.1) can now be written as

$$v_{i_1} \cdots v_{i_k} = 0 \quad \text{if } \{v_{i_1}, \dots, v_{i_k}\} \text{ does not span a simplex of } \mathcal{P}. \tag{5.2}$$

The face ring $\mathbf{k}[\mathcal{P}]$ is isomorphic to the quotient of the polynomial ring $\mathbf{k}[v_1, \dots, v_m]$ by (5.2), where $V(\mathcal{P}) = \{v_1, \dots, v_m\}$ and $\deg v_i = 2$. This is the standard form of the face ring of a simplicial complex [23].

We briefly remind several algebraic constructions from [18].

Lemma 5.2. [18, Lemma 5.4] *Every element of* $\mathbf{k}[\mathcal{P}]$ *can be uniquely written as a linear combination of monomials* $v_{\tau_1}^{\alpha_1} v_{\tau_2}^{\alpha_2} \cdots v_{\tau_k}^{\alpha_k}$ *corresponding to chains of fully ordered elements* $\tau_1 < \tau_2 < \cdots < \tau_k$ *of* $\mathcal{P}$.

In other words, the monomials $v_{\tau_1}^{\alpha_1} v_{\tau_2}^{\alpha_2} \cdots v_{\tau_k}^{\alpha_k}$ with $\tau_1 < \tau_2 < \cdots < \tau_k$ constitute an additive basis of $\mathbf{k}[\mathcal{P}]$. We refer to the expansion of an element $x \in \mathbf{k}[\mathcal{P}]$ in terms of this basis as the *chain decomposition* of x. To achieve a chain decomposition we inductively use *straightening relation* (5.1), which allows us to express the product of two unordered elements via the products of ordered elements. This can be restated by saying that the face ring is an example of an *algebra with straightening law* (see discussion in [22, p. 323]).

Given an element $\sigma \in \mathcal{P}$, define the *restriction map* s_σ as

$$s_\sigma : \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}]/(v_\tau \colon \tau \nleqslant \sigma).$$

Its codomain is isomorphic to a polynomial ring on $\operatorname{rk}\sigma$ generators. The following is a key algebraic statement, which has several geometric interpretations.

Lemma 5.3. [18, Lemma 5.6] *The sum* $s = \bigoplus_{\sigma \in \mathcal{P}} s_\sigma$ *of the restrictions to all elements of* $\mathcal{P}$,

$$s : \mathbf{k}[\mathcal{P}] \to \bigoplus_{\sigma \in \mathcal{P}} \mathbf{k}[\mathcal{P}]/(v_\tau \colon \tau \nleqslant \sigma),$$

is a monomorphism.

It is clear that to get a monomorphism it is enough to take the sum of restrictions to the maximal elements only. The proof of the above lemma uses the chain decomposition.

Now let Γ be a torus graph. By Lemma 4.2, the faces of Γ form a simplicial poset of rank n with respect to the reversed inclusion relation. We denote this simplicial poset by $\mathcal{P}(\Gamma)$. In Section 8 we shall discuss which simplicial posets may arise in this way. We prefer to stick to the original face-inclusions notation while dealing with torus graphs; then the face ring $\mathbf{k}[\mathcal{P}(\Gamma)]$ is a quotient of the polynomial ring on generators v_F, where F is a proper non-empty face of Γ, and $\deg v_F = 2(n - \dim F)$. We set formally $v_\emptyset = 0$ and $v_\Gamma = 1$; then the defining relation for the face ring is the same as (4.2).

Example 5.4. 1. Let Γ be the torus graph shown on Fig. 1(a), see Example 3.2. Denote its two edges by e and g, and the two vertices by p and q. The simplicial cell complex $\mathcal{P}(\Gamma)$ is obtained by gluing two segments along their boundaries. (It looks the same as Γ itself, but this is a mere coincidence, see the second example below.) The face ring $\mathbf{k}[\mathcal{P}(\Gamma)]$ is the quotient of the graded polynomial ring

$$\mathbf{k}[v_e, v_g, v_p, v_q], \qquad \deg v_e = \deg v_g = 2, \qquad \deg v_p = \deg v_q = 4$$

by the two relations

$$v_e v_g = v_p + v_q, \qquad v_p v_q = 0.$$

2. Now let Γ be the torus graph shown on Fig. 1(b). Denote its vertices by p and q, the edges by e, g, h, and the 2-faces by E, G, H so that e is opposite to E, etc. The simplicial cell complex $\mathcal{P}(\Gamma)$ is obtained by gluing two triangles along their boundaries. The face ring $\mathbf{k}[\mathcal{P}(\Gamma)]$ is isomorphic to the quotient of the graded polynomial ring

$$\mathbf{k}[v_E, v_G, v_H, v_p, v_q], \qquad \deg v_E = \deg v_G = \deg v_H = 2, \qquad \deg v_p = \deg v_q = 6$$

by the two relations

$$v_E v_G v_H = v_p + v_q, \qquad v_p v_q = 0.$$

(The generators corresponding to the edges can be excluded because of the relations $v_e = v_G v_H$, etc.)

By the definition of the equivariant cohomology of torus graph, it comes together with a monomorphism into the sum of polynomial rings:

$$r : H_T^*(\Gamma) \to \bigoplus_{V(\Gamma)} H^*(BT),$$

whose analogy with the algebraic restriction map s from Lemma 5.3 now becomes clear. The latter can now be written as

$$s : \mathbf{k}\big[\mathcal{P}(\Gamma)\big] \to \bigoplus_{p \in V(\Gamma)} \mathbf{k}\big[\mathcal{P}(\Gamma)\big]/(v_F\colon\ F \not\ni p).$$

Theorem 5.5. *$H_T^*(\Gamma)$ is isomorphic to the face ring $\mathbb{Z}[\mathcal{P}(\Gamma)]$. In other words, $H_T^*(\Gamma)$ is isomorphic to the quotient of the graded polynomial ring generated by the Thom classes τ_F modulo relations* (4.2).

Proof. We start by constructing a map

$$\mathbb{Z}[v_F\colon\ F \text{ a face}] \to H_T^*(\Gamma)$$

that sends v_F to τ_F. By Lemma 4.3, it factors through a map $\varphi : \mathbb{Z}[\mathcal{P}(\Gamma)] \to H_T^*(\Gamma)$. This map is surjective by Lemma 4.4. Finally, φ is injective, because we have a decomposition $s = r \circ \varphi$, and s is injective by Lemma 5.3. □

6. Cohen–Macaulay rings, complexes, and posets

A simplicial complex K is called *Cohen–Macaulay* (over $\mathbf{k}$) if its face ring $\mathbf{k}[K]$ is a Cohen–Macaulay ring (see, e.g., [23]). The Cohen–Macaulay property has several topological and algebraic interpretations (some of which we list below) and is very important for both topological and combinatorial applications of the face rings.

We shall not give a definition of a Cohen–Macaulay ring in the general case; instead we state a proposition characterising Cohen–Macaulay face rings of simplicial complexes.

Proposition 6.1. (See [23, Chapter II] or [2, Chapter 3].) *Assume K is an $(n-1)$-dimensional simplicial complex and $\mathbf{k}$ is an infinite field. Then $\mathbf{k}[K]$ is a Cohen–Macaulay ring if and only if there exists a sequence $\theta_1, \dots, \theta_n$ of linear* (*i.e., degree-two*) *elements of $\mathbf{k}[K]$ satisfying one of the two following equivalent conditions*:

(a) *θ_i is not a zero divisor in $\mathbf{k}[K]/(\theta_1, \dots, \theta_{i-1})$ for $i = 1, \dots, n$;*
(b) *$\theta_1, \dots, \theta_n$ are algebraically independent and $\mathbf{k}[K]$ is a free finitely generated module over its polynomial subring $\mathbf{k}[\theta_1, \dots, \theta_n]$.*

A sequence satisfying the first condition above is called *regular*. A sequence $\theta_1, \dots, \theta_n$ of algebraically independent linear elements for which $\mathbf{k}[K]$ is a finitely generated module over $\mathbf{k}[\theta_1, \dots, \theta_n]$ (i.e., $\mathbf{k}[K]/(\theta_1, \dots, \theta_n)$ is a finite-dimensional $\mathbf{k}$-vector space) is called an *lsop* (linear systems of parameters). Remember that $\dim K = n - 1$, so that an lsop in $\mathbf{k}[K]$ must have length n. An lsop always exists over an infinite field (see [1, Theorem 1.5.17]); the existence of an lsop over a finite field or $\mathbb{Z}$ is usually a subtle issue. Thus, we may reformulate Proposition 6.1 by saying that K is Cohen–Macaulay (over an infinite $\mathbf{k}$) if and only if $\mathbf{k}[K]$ admits a regular lsop. If $\mathbf{k}[K]$ is Cohen–Macaulay, then every lsop is regular [1, Theorem 2.1.2].

Linear systems of parameters in face rings may be detected by means of the following result.

Proposition 6.2. (See [23, Lemma III.2.4].) *A sequence of linear elements $\theta_1, \dots, \theta_n$ of $\mathbf{k}[K]$ is an lsop if and only if for every simplex $\sigma \in K$ the images $s_\sigma(\theta_1), \dots, s_\sigma(\theta_n)$ under the restriction map*

$$s_\sigma : \mathbf{k}[K] \to \mathbf{k}[K]/(v_i\colon\ i \notin \sigma)$$

generate the positive degree part of the polynomial ring $\mathbf{k}[K]/(v_i\colon\ i \notin \sigma)$.

A theorem due to Reisner gives a purely topological characterisation of Cohen–Macaulay complexes. We shall use the following version of Reisner's theorem, which is due to Munkres.

Theorem 6.3. [19, Corollary 3.4] *A simplicial complex K is Cohen–Macaulay if and only if the space $X = |K|$ satisfies*

$$\widetilde{H}^i(X) = 0 = \widetilde{H}^i(X, X \setminus p)$$

for any $p \in X$ and $i < \dim X$.

Now let $\mathcal{P}$ be a simplicial poset. Its *barycentric subdivision* $\mathcal{P}'$ is the order complex $\Delta(\overline{\mathcal{P}})$ of the poset $\overline{\mathcal{P}} = \mathcal{P} \setminus \hat{0}$. By the definition, $\mathcal{P}'$ is a genuine simplicial complex. Its geometric realisation can be obtained from the simplicial cell complex $\mathcal{P}$ by barycentrically subdividing each of its simplices.

Following Stanley [22], we call a simplicial poset $\mathcal{P}$ *Cohen–Macaulay* (over $\mathbf{k}$) if $\mathcal{P}'$ is a Cohen–Macaulay simplicial complex, that is, if the face ring $\mathbf{k}[\mathcal{P}']$ is Cohen–Macaulay. A question arises whether the Cohen–Macaulay property can be read directly from its face ring $\mathbf{k}[\mathcal{P}]$. By a result of Stanley [22, Corollary 3.7], the face ring of a Cohen–Macaulay simplicial poset is Cohen–Macaulay. The rest of this section is dedicated to the proof of the converse of this statement (see Theorem 6.9).

We call a simplicial subdivision of $\mathcal{P}$ *regular* if it is a genuine simplicial complex. For example, the barycentric subdivision is a regular subdivision. The following characterisation of Cohen–Macaulay simplicial posets follows from Theorem 6.3.

Corollary 6.4. *The following conditions are equivalent for $\mathcal{P}$ to be a Cohen–Macaulay poset*:

(a) *the barycentric subdivision of $\mathcal{P}$ is a Cohen–Macaulay complex*;
(b) *any regular subdivision of $\mathcal{P}$ is a Cohen–Macaulay complex*;
(c) *a regular subdivision of $\mathcal{P}$ is a Cohen–Macaulay complex.*

As a further corollary we obtain that Theorem 6.3 itself holds for arbitrary simplicial poset, i.e., the Cohen–Macaulay property for simplicial cell complexes is also of purely topological nature. All algebraic results from the beginning of this section also directly generalise to simplicial posets (for Proposition 6.2 see [3, Theorem 5.4]).

The barycentric subdivision $\mathcal{P}'$ can be obtained as the result of a sequence of *stellar subdivisions* of $\mathcal{P}$. Fix a $(k-1)$-dimensional simplex $\sigma \in \mathcal{P}$. The *star* of σ, its *boundary*, and *link* are the following subposets:

$$\begin{aligned} \operatorname{st}_{\mathcal{P}} \sigma &= \{\tau \in \mathcal{P}\colon\ \sigma \vee \tau \neq \emptyset\}, \\ \partial \operatorname{st}_{\mathcal{P}} \sigma &= \{\tau \in \mathcal{P}\colon\ \sigma \nleq \tau,\ \sigma \vee \tau \neq \emptyset\}, \\ \operatorname{lk}_{\mathcal{P}} \sigma &= \{\tau \in \mathcal{P}\colon\ \tau \wedge \sigma = \hat{0},\ \sigma \vee \tau \neq \emptyset\}. \end{aligned}$$

These correspond to the usual notions of star (or combinatorial neighbourhood) of a simplex in a triangulation, its boundary, and link.

Remark. The star of a simplex can be thought of as its "closed combinatorial neighbourhood." If $\mathcal{P}$ is (the poset of faces of) a simplicial complex, then the poset $\operatorname{lk}\sigma$ is isomorphic to the upper interval

$$\mathcal{P}_{>\sigma} = \{\rho \in \mathcal{P}\colon\ \rho > \sigma\},$$

and $\operatorname{st}\sigma = \sigma * \operatorname{lk}\sigma$ (here $*$ denotes the *join* of simplicial complexes). However this is not the case in general, see Example 6.7 below.

Definition. Let $\mathcal{P}$ be a simplicial poset and $\sigma \in \mathcal{P}$ a simplex. Assume first that $\mathcal{P}$ is a simplicial complex. Then the *stellar subdivision* of $\mathcal{P}$ at σ is a simplicial complex $\widetilde{\mathcal{P}}$ obtained by removing from $\mathcal{P}$ the star of σ and adding the cone over the boundary of the star:

$$\widetilde{\mathcal{P}} = (\mathcal{P} \setminus \operatorname{st}_{\mathcal{P}} \sigma) \cup \operatorname{cone}(\partial \operatorname{st}_{\mathcal{P}} \sigma). \tag{6.1}$$

Therefore, if v is the new vertex of $\widetilde{\mathcal{P}}$, then we have $\operatorname{lk}_{\widetilde{\mathcal{P}}} v = \partial \operatorname{st}_{\mathcal{P}} \sigma$ and $|\operatorname{st}_{\mathcal{P}} \sigma| \cong |\operatorname{st}_{\widetilde{\mathcal{P}}} v|$. Now, if $\mathcal{P}$ is an arbitrary simplicial poset, then its stellar subdivision $\widetilde{\mathcal{P}}$ at σ is obtained by stellarly subdividing each simplex containing σ. The term "subdvision" is justified by the fact that $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are homeomorphic as topological spaces.

Proposition 6.5. *The barycentric subdivision $\mathcal{P}'$ can be obtained as a sequence of stellar subdivisions of $\mathcal{P}$. Moreover, each stellar subdivision in the sequence is taken at a simplex whose star is a simplicial complex.*

Proof. Assume $\dim \mathcal{P} = n-1$. We start by taking stellar subdivisions of all $(n-1)$-dimensional simplices. Denote the resulting complex by $\mathcal{P}_1$. Then we take stellar subdivisions of $\mathcal{P}_1$ at all $(n-2)$-dimensional simplices corresponding to $(n-2)$-simplices of $\mathcal{P}$, and denote the result by $\mathcal{P}_2$. Proceeding this way, at the end we get a simplicial poset $\mathcal{P}_{n-1}$, which is obtained by stellar subdivisions of $\mathcal{P}_{n-2}$ at all 1-simplices corresponding to the edges of $\mathcal{P}$. Then $\mathcal{P}_{n-1}$ is $\mathcal{P}'$. To prove the second statement, assume that $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are the two subsequent complexes in the sequence, and $\widetilde{\mathcal{R}}$ is obtained from $\mathcal{R}$ by a stellar subdivision at σ. Then $\operatorname{st}_{\mathcal{R}} \sigma$ is isomorphic to $\sigma * (\mathcal{P}_{>\sigma})'$ and thereby is a simplicial complex. □

We proceed with two lemmas necessary to prove our main result.

Lemma 6.6. *Let $\mathcal{P}$ be a $(n-1)$-dimensional simplicial poset with the vertex set $V(\mathcal{P})=\{v_1,\dots,v_m\}$, and assume that the first k vertices $v_1,\dots,v_k$ span a face σ. Assume further that $\mathrm{st}_{\mathcal{P}}\,\sigma$ is a simplicial complex, and consider the stellar subdivision $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ at σ. Let v denote the degree-two generator of $\mathbf{k}[\widetilde{\mathcal{P}}]$ corresponding to the added vertex. Then there exists a unique map $\beta:\mathbf{k}[\mathcal{P}]\to\mathbf{k}[\widetilde{\mathcal{P}}]$ such that*

$$\begin{aligned} v_\tau &\mapsto v_\tau && \text{if } \tau\notin \mathrm{st}_{\mathcal{P}}\,\sigma,\\ v_i &\mapsto v+v_i, && i=1,\dots,k,\\ v_i &\mapsto v_i, && i=k+1,\dots,m \end{aligned}$$

(*we use the same notation for the vertices and the corresponding degree-two generators of the face ring*). *Moreover, β is a monomorphism, and if $\theta_1,\dots,\theta_n$ is an lsop in $\mathbf{k}[\mathcal{P}]$, then $\beta(\theta_1),\dots,\beta(\theta_n)$ is an lsop in $\mathbf{k}[\widetilde{\mathcal{P}}]$.*

Proof. In order to define the map β completely we have to specify the images of v_τ for $\tau\in\mathrm{st}_{\mathcal{P}}\,\sigma$. Choose such a v_τ and let $V(\tau)=\{v_{i_1},\dots,v_{i_\ell}\}$ be its vertex set. Then we have the following identity in $\mathbf{k}[\mathcal{P}]=\mathbf{k}[v_\tau\colon\ \tau\in\mathcal{P}\setminus\hat{0}]/\mathcal{I}_{\mathcal{P}}$:

$$v_{i_1}\cdots v_{i_\ell}=v_\tau+\sum_{\eta\colon\ V(\eta)=V(\tau),\,\eta\neq\tau} v_\eta. \tag{6.2}$$

For every v_η in the latter sum we have $\eta\notin\mathrm{st}_{\mathcal{P}}\,\sigma$ since $\mathrm{st}_{\mathcal{P}}\,\sigma$ is a simplicial complex (see Proposition 5.1). Since β is already defined on the product in the left-hand side and on the sum in the right-hand side, this uniquely determines $\beta(v_\tau)$. Therefore, the map β is defined on all monomials described in Lemma 5.2, and we may construct a map of $\mathbf{k}$-modules $\beta:\mathbf{k}[\mathcal{P}]\to\mathbf{k}[\widetilde{\mathcal{P}}]$ using the chain decomposition.

Next we have to check that β is a ring homomorphism. Consider the projection

$$p:\mathbf{k}[\mathcal{P}]\to\mathbf{k}[\mathcal{P}]/(v_\tau\colon\ \tau\notin\mathrm{st}_{\mathcal{P}}\,\sigma)=\mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]$$

and denote its kernel by R. Similarly, denote $\widetilde{R}=\ker(\tilde{p}:\mathbf{k}[\widetilde{\mathcal{P}}]\to\mathbf{k}[\mathrm{st}_{\widetilde{\mathcal{P}}}\,v])$. The ideal R has a $\mathbf{k}$-basis consisting of monomials $v_{\tau_1}^{\alpha_1}v_{\tau_2}^{\alpha_2}\cdots v_{\tau_k}^{\alpha_k}$ with $\tau_1<\tau_2<\dots<\tau_k$, $\alpha_i>0$ for $1\leqslant i\leqslant k$ and $\tau_k\notin\mathrm{st}_{\mathcal{P}}\,\sigma$. Since the simplicial cell complexes $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ do not differ on the complement to $\mathrm{st}_{\mathcal{P}}\,\sigma$ and $\mathrm{st}_{\widetilde{\mathcal{P}}}\,v$ respectively, the map β restricts to the identity isomorphism $R\to\widetilde{R}$.

The map β induces an additive map $\mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]\to\mathbf{k}[\mathrm{st}_{\widetilde{\mathcal{P}}}\,v]$, and our next observation will be that the latter is a ring homomorphism. Since $\mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]$ is generated in degree two, we need to check that β vanishes on monomials (5.2), that is, that $\beta(\mathcal{I}_{\mathrm{st}_{\mathcal{P}}\sigma})\subset\mathcal{I}_{\mathrm{st}_{\widetilde{\mathcal{P}}}v}$. We may assume that $\{v_{i_1},\dots,v_{i_\ell}\}$ is a minimal non-simplex of $\mathrm{st}_{\mathcal{P}}\,\sigma$, that is, every its proper subset is a simplex. Then we have $\{v_{i_1},\dots,v_{i_\ell}\}\cap V(\sigma)=\emptyset$ by the definition of the star. Therefore, $\beta(v_{i_1}\cdots v_{i_\ell})=v_{i_1}\cdots v_{i_\ell}$, which belongs to $\mathcal{I}_{\mathrm{st}_{\widetilde{\mathcal{P}}}v}$.

Now we have the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "\cong"] & \mathbf{k}[\mathcal{P}] \arrow[r, "p"] \arrow[d, "\beta"] & \mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{R} \arrow[r] & \mathbf{k}[\widetilde{\mathcal{P}}] \arrow[r, "\tilde{p}"] & \mathbf{k}[\mathrm{st}_{\widetilde{\mathcal{P}}}\,v] \arrow[r] & 0, \end{tikzcd} \tag{6.3}$$

in which the left and right vertical arrows are ring homomorphisms. We need to check that $\beta(x_1x_2) = \beta(x_1)\beta(x_2)$ for every $x_1, x_2 \in \mathbf{k}[\mathcal{P}]$. Since $\mathbf{k}[\mathcal{P}] = R \oplus \mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]$ as $\mathbf{k}$-modules, we may write $x_i = r_i + s_i$ with $r_i \in R$, $s_i \in \mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]$, $i = 1, 2$. For every $s \in \mathbf{k}[\mathrm{st}_{\mathcal{P}}\,\sigma]$ we have $\beta(s) = s + vx$ for some $x \in \mathbf{k}[\mathrm{st}_{\widetilde{\mathcal{P}}}\,v]$, and $rv = 0$ in $\mathbf{k}[\widetilde{\mathcal{P}}]$ for every $r \in \widetilde{R}$. Note also that $rs \in R$ for every $r \in R$, $s \in S$, as R is an ideal. Then we have

$$\beta(x_1x_2) = \beta(r_1r_2 + r_1s_2 + r_2s_1 + s_1s_2) = r_1r_2 + r_1s_2 + r_2s_1 + \beta(s_1s_2),$$

and

$$\beta(x_1)\beta(x_2) = \bigl(r_1 + \beta(s_1)\bigr)\bigl(r_2 + \beta(s_2)\bigr) = r_1r_2 + r_1\beta(s_2) + r_2\beta(s_1) + \beta(s_1)\beta(s_2).$$

Since $r_1\beta(s_2) = r_1(s_2 + vx_2) = r_1s_2$, $r_2\beta(s_1) = r_2s_1$ and $\beta(s_1s_2) = \beta(s_1)\beta(s_2)$, we conclude that $\beta(x_1x_2) = \beta(x_1)\beta(x_2)$. Thus, β is a ring homomorphism.

The rest of the statement follows by considering the commutative diagram of restriction maps (see Lemma 5.3)

$$\begin{array}{ccc}
\mathbf{k}[\mathcal{P}] & \xrightarrow{\quad\beta\quad} & \mathbf{k}[\widetilde{\mathcal{P}}] \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
\bigoplus_{\zeta\in\mathcal{P}} \mathbf{k}[\mathcal{P}]/(v_\tau\colon\ \tau \nleqslant \zeta) & \xrightarrow{s(\beta)} & \bigoplus_{\zeta\in\widetilde{\mathcal{P}}} \mathbf{k}[\widetilde{\mathcal{P}}]/(v_\tau\colon\ \tau \nleqslant \zeta).
\end{array} \tag{6.4}$$

The map $s(\beta)$ sends each direct summand of its domain isomorphically to at least one summand of its codomain, and therefore, is a monomorphism (its exact form can be easily guessed from the definition of β). Thus, β is also a monomorphism. Finally, the statement about lsop follows from the above diagram and Proposition 6.2. □

Note that if we map v_i identically for $i = 1, \ldots, k$ in the lemma above, then the map $\mathbf{k}[\mathcal{P}] \to \mathbf{k}[\widetilde{\mathcal{P}}]$ would still exists, but fail to be a monomorphism.

Example 6.7. The assumption in Lemma 6.6 is not satisfied if we take $\mathcal{P}$ to be the simplicial cell complex obtained by identifying two 2-simplices along their boundaries and make a stellar subdivision at a 1-simplex (the star of a 1-simplex in $\mathcal{P}$ is the whole $\mathcal{P}$). However, in the process of barycentric subdivision of $\mathcal{P}$ we first make the stellar subdivisions at 2-dimensional simplices, and the stars of 1-simplices in the resulting complex are simplicial complexes. Note also that if $\mathrm{st}_{\mathcal{P}}\,\sigma$ is not a simplicial complex, then the map $\beta : \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}']$ is not determined by the conditions specified in Lemma 6.6. That is, the images of degree-two generators do not determine the images of generators of higher degree. However, it is still possible to define the map $\beta : \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}']$ for an arbitrary simplicial poset; see Section 8.

Lemma 6.8. *Assume that $\mathbf{k}[\mathcal{P}]$ is a Cohen–Macaulay ring, and $\widetilde{\mathcal{P}}$ a stellar subdivision of $\mathcal{P}$ at σ such that $\mathrm{st}_{\mathcal{P}}\,\sigma$ is a simplicial complex. Then*

(a) $\mathrm{st}_{\mathcal{P}}\,\sigma$ *is a Cohen–Macaulay complex*;
(b) $\mathbf{k}[\widetilde{\mathcal{P}}]$ *is a Cohen–Macaulay ring.*

Proof. (a) As $\operatorname{st}_{\mathcal{P}}\sigma = \sigma * \operatorname{lk}_{\mathcal{P}}\sigma$, it is enough to prove that $\operatorname{lk}_{\mathcal{P}}\sigma$ is Cohen–Macaulay. This can be done by showing that the simplicial homology of $\operatorname{lk}_{\mathcal{P}}\sigma$ is a direct summand in the local cohomology of $\mathbf{k}[\mathcal{P}]$, as in the proof of Hochster's theorem (see [23, Theorem II.4.1] or [1, Theorem 5.3.8]).

(b) (Compare proof of Lemma 9.2 of [18].) Choose an lsop $\theta_1, \dots, \theta_n \in \mathbf{k}[\mathcal{P}]$ (we can always assume that an lsop exists by passing to an infinite extension field, see [1, Theorem 2.1.10]) and denote $\tilde{\theta}_i = \beta(\theta_i)$. Applying the functors $\otimes_{\mathbf{k}[\theta_1,\dots,\theta_n]}\mathbf{k}$ and $\otimes_{\mathbf{k}[\tilde{\theta}_1,\dots,\tilde{\theta}_n]}\mathbf{k}$ to diagram (6.3), we get a map between the long exact sequences for Tor. Consider the following fragment (we denote $\operatorname{Tor}_\theta = \operatorname{Tor}_{\mathbf{k}[\theta_1,\dots,\theta_n]}$):

$$\begin{tikzcd}
\operatorname{Tor}_\theta^{-2}(\mathbf{k}[\operatorname{st}\sigma],\mathbf{k}) \arrow[r, "f"] \arrow[d] & \operatorname{Tor}_\theta^{-1}(R,\mathbf{k}) \arrow[r] \arrow[d, "\cong"] & \operatorname{Tor}_\theta^{-1}(\mathbf{k}[\mathcal{P}],\mathbf{k}) \arrow[r] \arrow[d] & \operatorname{Tor}_\theta^{-1}(\mathbf{k}[\operatorname{st}\sigma],\mathbf{k}) \arrow[d] \\
\operatorname{Tor}_{\tilde{\theta}}^{-2}(\mathbf{k}[\operatorname{st}v],\mathbf{k}) \arrow[r, "\tilde{f}"] & \operatorname{Tor}_{\tilde{\theta}}^{-1}(\widetilde{R},\mathbf{k}) \arrow[r] & \operatorname{Tor}_{\tilde{\theta}}^{-1}(\mathbf{k}[\widetilde{\mathcal{P}}],\mathbf{k}) \arrow[r] & \operatorname{Tor}_{\tilde{\theta}}^{-1}(\mathbf{k}[\operatorname{st}v],\mathbf{k}).
\end{tikzcd}$$

Since $\mathbf{k}[\mathcal{P}]$ is Cohen–Macaulay, $\operatorname{Tor}_\theta^{-1}(\mathbf{k}[\mathcal{P}],\mathbf{k}) = 0$, therefore, f is onto. Then $\tilde{f}$ is also onto. Since $\operatorname{st}_{\mathcal{P}}\sigma$ (and $\operatorname{st}_{\widetilde{\mathcal{P}}} v$) is a simplicial complex and $|\operatorname{st}_{\mathcal{P}}\sigma| \cong |\operatorname{st}_{\widetilde{\mathcal{P}}} v|$, part (a) of this lemma and Theorem 6.3 imply that $\mathbf{k}[\operatorname{st} v]$ is Cohen–Macaulay. Therefore,

$$\operatorname{Tor}_{\tilde{\theta}}^{-1}\big(\mathbf{k}[\operatorname{st}v],\mathbf{k}\big) = 0.$$

Since $\tilde{f}$ is surjective, we also have $\operatorname{Tor}_{\tilde{\theta}}^{-1}(\mathbf{k}[\widetilde{\mathcal{P}}],\mathbf{k}) = 0$. Then $\mathbf{k}[\widetilde{\mathcal{P}}]$ is free as a module over $\mathbf{k}[\tilde{\theta}_1,\dots,\tilde{\theta}_n]$ (see [15, Lemma VII.6.2]) and thereby is Cohen–Macaulay. □

Theorem 6.9. *The face ring $\mathbf{k}[\mathcal{P}]$ of a simplicial poset $\mathcal{P}$ is Cohen–Macaulay if and only if $\mathcal{P}$ is Cohen–Macaulay.*

Proof. Assume $\mathbf{k}[\mathcal{P}]$ is Cohen–Macaulay. Since the barycentric subdivision $\mathcal{P}'$ is obtained by a sequence of stellar subdivisions, subsequent application of Lemma 6.8 gives that $\mathbf{k}[\mathcal{P}']$ is also Cohen–Macaulay. Hence, $\mathcal{P}$ is a Cohen–Macaulay poset. The converse statement is [22, Corollary 3.7]. □

7. Pseudomanifolds and orientations

The question of identifying the class of simplicial posets which arise as $\mathcal{P}(\Gamma)$ for some torus graph Γ might be a difficult one, although our next statement sheds some light on this problem. The following is a straightforward extension of the notion of *pseudomanifold* [23, Definition 0.3.15] to simplicial posets.

Definition. A simplicial poset $\mathcal{P}$ is called an $(n-1)$-*dimensional pseudomanifold* (without boundary) if

(a) for every element $\sigma \in \mathcal{P}$ there is an element τ of rank n such that $\sigma \leqslant \tau$ (in other words, $\mathcal{P}$ is *pure* $(n-1)$-*dimensional*);
(b) for every element $\sigma \in \mathcal{P}$ of rank $(n-1)$ there are exactly two elements τ of rank n such that $\sigma < \tau$;

(c) for every two elements τ and τ' of rank n there is a sequence $\tau = \tau_1, \tau_2, \ldots, \tau_k = \tau'$ of elements such that $\operatorname{rk}\tau_i = n$ and $\tau_i \wedge \tau_{i+1}$ contains an element of rank $(n-1)$ for $i = 1, \ldots, k-1$.

Examples of pseudomanifolds are provided by triangulations or simplicial cell decompositions of topological manifolds. However, not every pseudomanifold arises in this way, see Example 7.2 below.

Theorem 7.1.

(a) *Let Γ be a torus graph; then $\mathcal{P}(\Gamma)$ is a pseudomanifold, and the face ring $\mathbb{Z}(\mathcal{P})$ admits an lsop.*
(b) *Given an arbitrary pseudomanifold $\mathcal{P}$ and an lsop in $\mathbb{Z}(\mathcal{P})$, one can canonically construct a torus graph $\Gamma_{\mathcal{P}}$.*

Moreover, $\Gamma_{\mathcal{P}(\Gamma)} = \Gamma$.

Proof. (a) Vertices of $\mathcal{P}(\Gamma)$ correspond to $(n-1)$-faces of Γ. As every face of Γ contains a vertex and Γ is n-valent, $\mathcal{P}(\Gamma)$ is pure $(n-1)$-dimensional. Condition (b) from the definition of a pseudomanifold follows from the fact that every edge of Γ has exactly two vertices, while (c) follows from the connectivity of Γ. In order to find an lsop, we identify $\mathbb{Z}[\mathcal{P}(\Gamma)]$ with a subset of $H^*(BT)^{V(\Gamma)}$ (see Theorem 5.5) and consider the constant map $c : H^*(BT) \to H^*(BT)^{V(\Gamma)}$. It factors through a monomorphism $H^*(BT) \to \mathbb{Z}[\mathcal{P}(\Gamma)]$, and Proposition 6.2 guarantees that the c-image of a basis in $H^*(BT)$ is an lsop.

(b) Let $\mathcal{P}$ be a pseudomanifold of dimension $(n-1)$. Define a graph $\Gamma_{\mathcal{P}}$ whose vertices correspond to $(n-1)$-dimensional simplices $\sigma \in \mathcal{P}$, and in which the number of edges between two vertices σ and σ' equals the number of $(n-2)$-dimensional simplices in $\sigma \wedge \sigma'$. Then $\Gamma_{\mathcal{P}}$ is a connected n-valent graph, and we need to define an axial function. The following argument is similar to that of [17, §3], compare also a similar treatment of “edge vectors” in [21]. We can regard an lsop as a map $\lambda : H^*(BT) \to \mathbb{Z}[\mathcal{P}]$. As usual, assume that $\mathcal{P}$ has m vertices and let $v_1, \ldots, v_m$ be the corresponding degree-two generators of $\mathbb{Z}[\mathcal{P}]$. Then for $t \in H^2(BT)$ we can write

$$\lambda(t) = \sum_{i=1}^{m} \lambda_i(t) v_i,$$

where λ_i is a linear function on $H^2(BT)$, that is, an element of $H_2(BT)$. Let e be an oriented edge of Γ and $p = i(e)$ its initial vertex. This vertex corresponds to an $(n-1)$-simplex of $\mathcal{P}$, and we denote by $I(p) \subset \{1, \ldots, m\}$ the set of its vertices in $\mathcal{P}$; note that $|I(p)| = n$. Since λ is an lsop, the set $\{\lambda_i : i \in I(p)\}$ is a basis in $H_2(BT)$. Now we define the axial function $\alpha : E(\Gamma) \to H^2(BT)$ by requiring that its value on $E(\Gamma)_p$ is the dual basis to $\{\lambda_i : i \in I(p)\}$. In more detail, the edge e corresponds to an $(n-2)$-simplex of $\mathcal{P}$ and let $\ell \in I(p)$ be the unique vertex which is not in this $(n-2)$-simplex. Then we define $\alpha(e)$ by requiring that

$$\langle \alpha(e), \lambda_i \rangle = \delta_{i\ell}, \quad i \in I(p), \tag{7.1}$$

where $\delta_{i\ell}$ is the Kronecker delta. Now we have to check the three conditions from the definition of a torus axial function. Let $p' = t(e) = i(\bar{e})$. Note that the intersection of $I(p)$ and $I(p')$ consists of at least $(n-1)$ elements. If $I(p) = I(p')$ then Γ has only two vertices, like in Example 3.2, while $\mathcal{P}$ is obtained by gluing together two $(n-1)$-simplices along their boundaries, see Example 6.7. Otherwise, $|I(p) \cap I(p')| = n-1$ and we have $\ell \notin I(p')$. Let ℓ' be an element such that $\ell' \in I(p')$ but $\ell' \notin I(p)$. Then (7.1) guarantees that $\langle \alpha(e), \lambda_i \rangle = \langle \alpha(\bar{e}), \lambda_i \rangle = 0$ for $i \in I(p) \cap I(p')$. As we work with integral bases, this implies $\alpha(\bar{e}) = \pm\alpha(e)$. It also follows that $\alpha(E(\Gamma)_p \setminus e)$ and $\alpha(E(\Gamma)_{p'} \setminus \bar{e})$ give the same bases in the quotient space $H^2(BT)/\alpha(e)$. Identifying these bases, we obtain a connection $\theta_e : E(\Gamma)_p \to E(\Gamma)_{p'}$ satisfying $\alpha(\theta_e(e')) \equiv \alpha(e') \bmod \alpha(e)$ for any $e' \in E(\Gamma)_p$, as needed. The rest of the statement is straightforward. □

Note that the above theorem does not give a complete characterisation of simplicial posets of the form $\mathcal{P}(\Gamma)$, as it may happen that $\mathcal{P}(\Gamma_{\mathcal{P}}) \neq \mathcal{P}$. In fact, here is a counterexample.

Example 7.2. Let $\mathcal{P}$ be a triangulation of a 2-dimensional sphere different from the boundary of a simplex. Choose two vertices that are not joined by an edge. Let $\mathcal{P}'$ be the complex obtained by identifying these two vertices. Then $\mathcal{P}'$ is a pseudomanifold. If $\mathbb{Z}[\mathcal{P}]$ admits an lsop, so does $\mathbb{Z}[\mathcal{P}']$ (this easily follows from Proposition 6.2). However, $\mathcal{P}(\Gamma_{\mathcal{P}'}) \neq \mathcal{P}'$ (in fact, $\mathcal{P}(\Gamma_{\mathcal{P}'}) = \mathcal{P}$). It follows that $\mathcal{P}'$ does not arise from any torus graph.

Definition. We say that an assignment $o : V(\Gamma) \to \{\pm 1\}$ is an *orientation* on Γ if $o(i(e))\alpha(e) = -o(i(\bar{e}))\alpha(\bar{e})$ for every $e \in E(\Gamma)$.

Example 7.3. Let M be a torus manifold which admits a T-invariant almost complex structure. The almost complex structure induce orientations on M and its characteristic submanifolds (an omniorientation). The associated torus axial function α_M satisfies $\alpha_M(\bar{e}) = -\alpha_M(e)$ for any oriented edge e. In this case we can take $o(p) = 1$ for every $p \in V(\Gamma_M)$.

Proposition 7.4. *An omniorientation of a torus manifold M induces an orientation of the associated torus graph Γ_M.*

Proof. Given a vertex $p \in M^T = V(\Gamma_M)$ we set $o(p) = 1$ if the canonical orientation of the sum of complex one-dimensional representation spaces in the right-hand side of (2.3) coincides with the orientation of $\tau_p M$ induced by the orientation of M, and set $o(p) = -1$ otherwise. □

Example 7.5. Let Γ be a complete graph on four vertices p_1, p_2, p_3, p_4. Choose a basis $t_1, t_2, t_3 \in H^2(BT^3)$ and define an axial function by setting

$$\alpha(p_1p_2) = \alpha(p_3p_4) = t_1, \qquad \alpha(p_1p_3) = \alpha(p_2p_4) = t_2, \qquad \alpha(p_1p_4) = \alpha(p_2p_3) = t_3$$

and $\alpha(\bar{e}) = \alpha(e)$ for any oriented edge e. A direct check shows that this torus graph is not orientable. In fact, this graph is associated with the pseudomanifold (simplicial cell complex) shown on Fig. 2 via the construction of Theorem 7.1(b). This pseudomanifold $\mathcal{P}$ is homeomorphic to $\mathbb{R}P^2$ (the opposite outer edges are identified according to the arrows shown), the ring $\mathbb{Z}[\mathcal{P}]$ has three two-dimensional generators v_p, v_q, v_r, which constitute an lsop. Note that $\mathbb{R}P^2$ itself is non-orientable; in fact, this example is generalised by the following proposition.

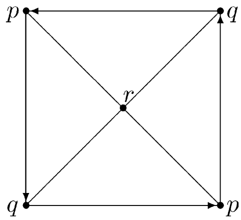


Fig. 2. Simplicial cell decomposition of $\mathbb{R}P^2$ with 3 vertices.

Proposition 7.6. *A torus graph Γ is orientable if and only if the associated pseudomanifold $\mathcal{P}(\Gamma)$ is orientable.*

Proof. Let $p \in V(\Gamma)$ and σ the corresponding $(n-1)$-simplex of $\mathcal{P}(\Gamma)$. There is a canonical one-to-one correspondence between $E(\Gamma)_p$ and the vertex set of σ, see the proof of Theorem 7.1(b). Choose a basis of $H^2(BT)$. Assume first that $\mathcal{P}(\Gamma)$ is oriented. Choose a "positive" (that is, compatible with the orientation) order of vertices of σ; this allows to regard $\alpha(E(\Gamma)_p)$ as a basis of $H^2(BT)$. We set $o(p) = 1$ if this is a positively oriented basis, and $o(p) = -1$ otherwise. This defines an orientation on Γ. To prove the opposite direction we just reverse this procedure. □

8. Blow-ups of torus manifolds and torus graphs

Here we relate the following three geometric constructions:

(a) blowing up a torus manifold at a facial submanifold [18, §9];
(b) cutting a face from a simple polytope or, more generally, blowing up a GKM graph [9, §2.2];
(c) stellar subdivision of a simplicial poset.

Let M be a torus manifold with the projection map $\pi : M \to Q$ onto the orbit space, and F a face of Q (details may be found in [12] or [18]; a reader less familiar with torus manifolds may assume M to be a smooth projective toric variety, in which case Q is a convex simple polytope). Replacing the facial submanifold $M_F = \pi^{-1}(F)$ of M by the complex projectivisation $P(\nu_F)$ of its normal bundle ν_F, we obtain a new torus manifold $\widetilde{M}$. The passage from M to $\widetilde{M}$ is called *blowing-up* of M at M_F. The orbit space $\widetilde{Q}$ of $\widetilde{M}$ is then obtained by "cutting off" the face F from Q. As explained in Example 3.1, the 1-skeleton of Q is a torus graph. The general construction of blow-up of a GKM-graph is described in [9, §2.2.1]; in particular, it applies to torus graphs and agrees with the topological picture for the graphs coming from manifolds. We briefly review their construction below, and illustrate it in a couple of examples.

Let F be a k-face of Γ (of codimension $n-k$). The *blow-up* of Γ at F, denoted $\widetilde{\Gamma}$, has vertex set $V(\widetilde{\Gamma}) = (V(\Gamma) \setminus V(F)) \cup V(F)^{n-k}$, that is, each vertex $p \in V(F)$ is replaced by $(n-k)$ vertices $\widetilde{p}_1, \dots, \widetilde{p}_{n-k}$. It is convenient to regard those points as chosen close to p on edges from $E_p(\Gamma) \setminus E_p(F)$, and we denote by p'_i the endpoint of the edge containing both p and $\widetilde{p}_i$, $i = 1, \dots, n-k$. (We also assume $\theta_{pq}(pp'_i) = qq'_i$ if p and q are joined by an edge in F.) Then we have four types of edges in $\widetilde{\Gamma}$, and the corresponding values of the axial function $\widetilde{\alpha} : E(\widetilde{\Gamma}) \to H^*(BT)$:

(a) $\widetilde{p}_i \widetilde{p}_j$ for every $p \in V(F)$; $\widetilde{\alpha}(\widetilde{p}_i \widetilde{p}_j) = \alpha(pp'_j) - \alpha(pp'_i)$;

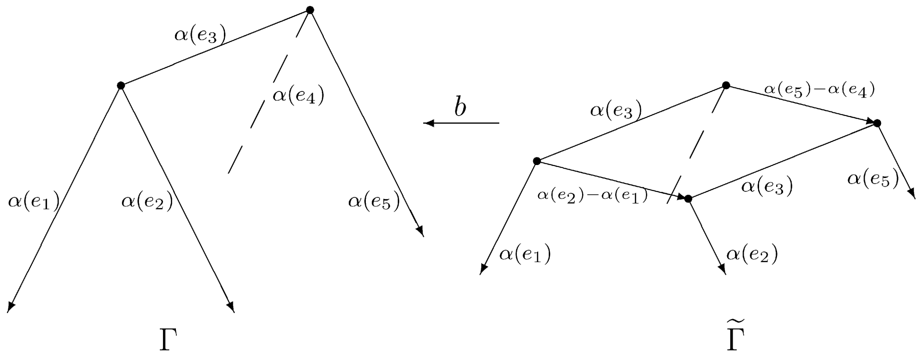


Fig. 3. Blow up at an edge.

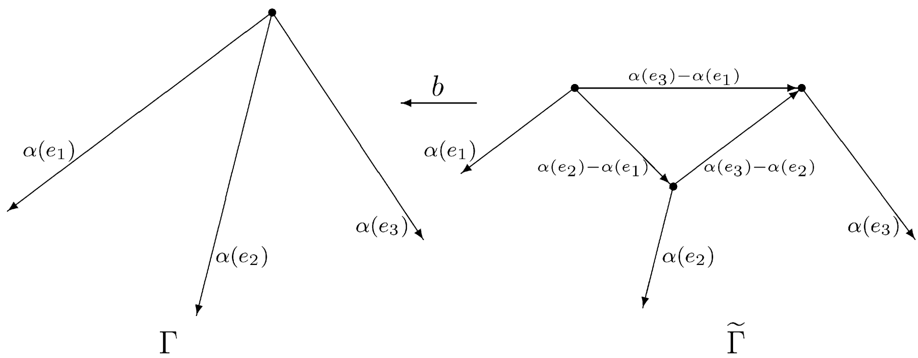


Fig. 4. Blow up at a vertex.

(b) $\widetilde{p}_i\widetilde{q}_i$ if p and q were joined by an edge in F; $\widetilde{\alpha}(\widetilde{p}_i\widetilde{q}_i) = \alpha(pq)$;
(c) $\widetilde{p}_i p_i'$ for every $p \in V(F)$; $\widetilde{\alpha}(\widetilde{p}_i p_i') = \alpha(pp_i')$;
(d) edges "coming from Γ," that is, $e \in E(\Gamma)$ such that $i(e) \notin V(F)$ and $t(e) \notin V(F)$; $\widetilde{\alpha}(e) = \alpha(e)$,

see Fig. 3 ($n = 3$, $k = 1$) and Fig. 4 ($n = 3$, $k = 0$).

There is a *blow-down map* $b : \widetilde{\Gamma} \to \Gamma$ preserving the face structure. The face $F \subset \Gamma$ is blown up to a new facet $\widetilde{F} \subset \widetilde{\Gamma}$ (unless F itself was a facet, in which case $\widetilde{\Gamma} = \Gamma$). For every face $H \subset \Gamma$ which is not contained in F, there is a unique face $\widetilde{H} \subset \widetilde{\Gamma}$ that is mapped onto H. The blow-down map induces an equivariant cohomology map $b^* : H_T^*(\Gamma) \to H_T^*(\widetilde{\Gamma})$. In fact, this map can be easily identified by the following commutative diagram

$$\begin{tikzcd} H_T^*(\Gamma) \arrow[r, "b^*"] \arrow[d, "r"'] & H_T^*(\widetilde{\Gamma}) \arrow[d, "\widetilde{r}"] \\ H^*(BT)^{V(\Gamma)} \arrow[r, "V(b)^*"] & H^*(BT)^{V(\widetilde{\Gamma})} \end{tikzcd} \tag{8.1}$$

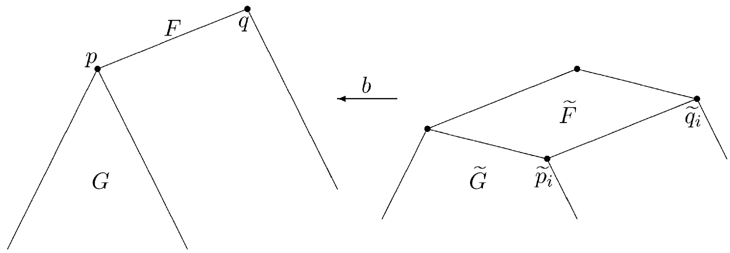


Fig. 5.

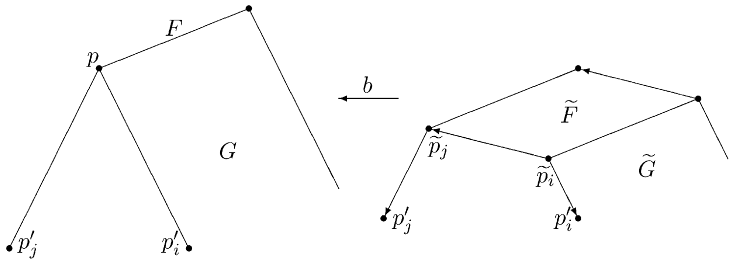


Fig. 6.

(compare (6.4)), where r and $\widetilde{r}$ are the monomorphisms from the definition of equivariant cohomology of a torus graph, and $V(b)^*$ is the map induced by the set map $V(b): V(\widetilde{\Gamma}) \to V(\Gamma)$. The next lemma describes the images of the two-dimensional generators $\tau_G \in H_T^*(\Gamma)$ corresponding to the facets $G \subset \Gamma$.

Lemma 8.1. *Given a facet $G \subset \Gamma$, we have $b^*(\tau_G) = \tau_{\widetilde{F}} + \tau_{\widetilde{G}}$ if $F \subset G$ and $b^*(\tau_G) = \tau_{\widetilde{G}}$ otherwise.*

Proof. We use (8.1) and check that the images of τ_G and $\tau_{\widetilde{F}} + \tau_{\widetilde{G}}$ (or $\tau_{\widetilde{G}}$) under the horizontal maps agree. Let $p \in V(\Gamma)$ be a vertex. If $p \notin F$, then $b^{-1}(p) = p$ and $r(\tau_G)(p) = \widetilde{r}(\tau_{\widetilde{G}})(p)$, $\widetilde{r}(\tau_{\widetilde{F}})(p) = 0$. Thus we may assume $p \in F$, and then we have $b(\widetilde{p}_i) = p$, $i = 1, \dots, n-k$.

First consider the case $F \not\subset G$. If $p \notin G$, then $r(\tau_G)(p) = \widetilde{r}(\tau_{\widetilde{G}})(\widetilde{p}_i) = 0$. Otherwise $p \in G \cap F$. Let e be the unique edge such that $e \in E_p(\Gamma)$ and $e \notin G$. Then $e = pq$ for some $q \in V(F)$ (because $F \not\subset G$). From (4.1) we obtain

$$r(\tau_G)(p) = \alpha(pq) = \widetilde{\alpha}(\widetilde{p}_i\widetilde{q}_i) = \widetilde{r}(\tau_{\widetilde{G}})(\widetilde{p}_i),$$

see Fig. 5. It follows that $V(b)^* r(\tau_G) = \widetilde{r}(\tau_{\widetilde{G}})$, and therefore, $b^*(\tau_G) = \tau_{\widetilde{G}}$.

Now let $F \subset G$. In this case the unique edge e such that $e \in E_p(\Gamma)$ and $e \notin G$ is of type pp'_j, see Fig. 6. Using (4.1) we calculate

$$\begin{aligned} r(\tau_G)(p) &= \alpha(pp'_j), \\ \widetilde{r}(\tau_{\widetilde{G}})(\widetilde{p}_i) &= \widetilde{\alpha}(\widetilde{p}_i\widetilde{p}_j) = \alpha(pp'_j) - \alpha(pp'_i), \\ \widetilde{r}(\tau_{\widetilde{F}})(\widetilde{p}_i) &= \widetilde{\alpha}(\widetilde{p}_i p'_i) = \alpha(pp'_i). \end{aligned}$$

As in the previous case it follows that $V(b)^* r(\tau_G) = \widetilde{r}(\tau_{\widetilde{F}}) + \widetilde{r}(\tau_{\widetilde{G}})$, and therefore, $b^*(\tau_G) = \tau_{\widetilde{F}} + \tau_{\widetilde{G}}$. $\square$

Corollary 8.2. *After the identifications $H_T^*(\Gamma) \cong \mathbb{Z}[\mathcal{P}(\Gamma)]$ and $H_T^*(\widetilde{\Gamma}) \cong \mathbb{Z}[\mathcal{P}(\widetilde{\Gamma})]$, the equivariant cohomology map induced by the blow-down $b: \widetilde{\Gamma} \to \Gamma$ coincides with the map β from Lemma* 6.6*.*

Proof. Remember that the poset $\mathcal{P}(\Gamma)$ is formed by the faces of Γ with the reversed inclusion relation, and the isomorphism $H_T^*(\Gamma) \cong \mathbb{Z}[\mathcal{P}(\Gamma)]$ is established by identifying τ_H with v_H for all faces $H \subset \Gamma$. Let $\sigma \in \mathcal{P}(\Gamma)$ be the element corresponding to the face F. Then an element $\tau \in \mathcal{P}(\Gamma)$ satisfies $\tau \in \mathrm{st}_{\mathcal{P}}\,\sigma$ if and only if the corresponding face $H \subset \Gamma$ satisfies $F \cap H \neq \emptyset$. The degree-two generators v_i, $i = 1, \dots, m$, of $\mathbb{Z}[\mathcal{P}(\Gamma)]$ (or $\mathbb{Z}[\mathcal{P}(\widetilde{\Gamma})]$) correspond to the generators τ_G of $H_T^*(\Gamma)$ (or $\tau_{\widetilde{G}}$ of $H_T^*(\widetilde{\Gamma})$ respectively). Making the appropriate identifications, we see that the map from Lemma 6.6 is determined by the conditions

$$\begin{aligned} &\tau_H \mapsto \tau_H && \text{if } F \cap H = \emptyset, \\ &\tau_G \mapsto \tau_{\widetilde{F}} + \tau_{\widetilde{G}} && \text{if } F \subset G, \\ &\tau_G \mapsto \tau_{\widetilde{G}} && \text{if } F \not\subset G. \end{aligned}$$

The blow-down map b^* satisfies these conditions, whence the proof follows. □

Returning to torus manifolds, in [18, Lemma 9.2] we proved that $H^{\mathrm{odd}}(M) = 0$ implies $H^{\mathrm{odd}}(\widetilde{M}) = 0$. Also, $H^{\mathrm{odd}}(M) = 0$ implies that $\mathbb{Z}[\mathcal{P}]$ is Cohen–Macaulay [18, Theorem 7.7] (here $\mathcal{P}$ is the face poset of the orbit space Q). Now the above mentioned analogy between the proof of [18, Lemma 9.2] and the proof of Lemma 6.8 becomes even more transparent. Note that we have isomorphisms $H_T^*(M_F) \cong \mathbb{Z}[\mathrm{st}_{\mathcal{P}}\,\sigma]$ and, $H_T^*(\widetilde{M}_{\widetilde{F}}) \cong \mathbb{Z}[\mathrm{st}_{\widetilde{\mathcal{P}}}\,v]$. We are also ready to give the proof of the other direction of [18, Lemma 9.2], promised in the end of Section 9 of [18].

Lemma 8.3. $H^{\mathrm{odd}}(\widetilde{M}) = 0$ *if and only if* $H^{\mathrm{odd}}(M) = 0$.

Proof. Assume $H^{\mathrm{odd}}(\widetilde{M}) = 0$. Then $\mathbb{Z}[\widetilde{\mathcal{P}}]$ is Cohen–Macaulay by Theorem 7.7 of [18]. We claim that $\mathbb{Z}[\mathcal{P}]$ is also Cohen–Macaulay (i.e, the converse of Lemma 6.8 holds). Indeed, by Theorem 6.9, $\widetilde{\mathcal{P}}$ is a Cohen–Macaulay poset. Choose a simplicial complex $\mathcal{S}$ which is a common subdivision of $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ (for example, we can take $\mathcal{S}$ to be the barycentric subdivision of $\widetilde{\mathcal{P}}$). By Corollary 6.4, $\mathcal{S}$ is a Cohen–Macaulay complex, whence $\mathcal{P}$ is a Cohen–Macaulay poset. Applying Theorem 6.9 again we get that $\mathbb{Z}[\mathcal{P}]$ is Cohen–Macaulay. Then $H^{\mathrm{odd}}(M) = 0$ by Theorem 7.7 of [18]. The other direction of the lemma is already proven in [18]. □

9. Dehn–Sommerville equations

Let $\mathcal{P}$ be a simplicial poset of rank n (i.e., of dimension $n-1$). Let f_i denote the number of i-dimensional simplices in $\mathcal{P}$, $0 \leqslant i \leqslant n-1$. Since $\mathcal{P}$ has a unique initial element $\hat{0}$, we have $f_{-1} = 1$. The *h-vector* $\boldsymbol{h}(\mathcal{P}) = (h_0, \dots, h_n)$ of $\mathcal{P}$ is defined from the polynomial identity

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=0}^{n} f_{i-1}(t-1)^{n-i}. \tag{9.1}$$

Let $\mathcal{P}_{\geqslant\sigma} = \{\tau \in \mathcal{P}\colon\ \tau \geqslant \sigma\}$ be the subposet $\mathcal{P}$ with the induced rank function. For a simplex $\sigma \in \mathcal{P}$ we set

$$\chi(\mathcal{P}_{\geqslant\sigma}) := \sum_{\tau\geqslant\sigma} (-1)^{\operatorname{rk}\tau-1}. \tag{9.2}$$

Theorem 9.1.

$$\sum_{i=0}^{n} (h_{n-i} - h_i)t^i = \sum_{\sigma\in\mathcal{P}} \bigl(1 + (-1)^n \chi(\mathcal{P}_{\geqslant\sigma})\bigr)(t-1)^{n-\operatorname{rk}\sigma}.$$

In particular, the Dehn–Sommerville equations $h_i = h_{n-i}$ hold if $\chi(\mathcal{P}_{\geqslant\sigma}) = (-1)^{n-1}$ for every $\sigma \in \mathcal{P}$.

Proof. The argument below is essentially the same as that used by Hibi in [13, p. 91]. We have

$$\begin{aligned}
\sum_{i=0}^{n} h_i t^i &= t^n \sum_{i=0}^{n} h_i (1/t)^{n-i} = t^n \sum_{i=0}^{n} f_{i-1}\bigl((1-t)/t\bigr)^{n-i} \quad \text{by (9.1)} \\
&= \sum_{i=0}^{n} f_{i-1} t^i (1-t)^{n-i} = \sum_{\tau\in\mathcal{P}} t^{\operatorname{rk}\tau}(1-t)^{n-\operatorname{rk}\tau} \\
&= \sum_{\tau\in\mathcal{P}} \sum_{\sigma\leqslant\tau} (t-1)^{\operatorname{rk}\tau-\operatorname{rk}\sigma}(1-t)^{n-\operatorname{rk}\tau} \\
&= \sum_{\tau\in\mathcal{P}} \sum_{\sigma\leqslant\tau} (-1)^{n-\operatorname{rk}\tau}(t-1)^{n-\operatorname{rk}\sigma} \\
&= \sum_{\sigma\in\mathcal{P}} (t-1)^{n-\operatorname{rk}\sigma} \sum_{\tau\geqslant\sigma} (-1)^{n-\operatorname{rk}\tau} \\
&= \sum_{\sigma\in\mathcal{P}} (t-1)^{n-\operatorname{rk}\sigma}(-1)^{n-1}\chi(\mathcal{P}_{\geqslant\sigma}) \quad \text{by (9.2)}
\end{aligned} \tag{9.3}$$

where the fifth equality follows from the binomial expansion of the right-hand side of the identity $t^{\operatorname{rk}\tau} = ((t-1)+1)^{\operatorname{rk}\tau}$.

On the other hand, we have

$$\sum_{i=0}^{n} h_{n-i} t^i = \sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=0}^{n} f_{i-1}(t-1)^{n-i} = \sum_{\sigma\in\mathcal{P}} (t-1)^{n-\operatorname{rk}\sigma}. \tag{9.4}$$

Subtracting (9.3) from (9.4) we obtain the theorem. □

Corollary 9.2. [2] *If K is a triangulation of a closed $(n-1)$-manifold, then*

$$h_{n-i} - h_i = (-1)^i \binom{n}{i} \bigl(\chi(K) - \chi\bigl(S^{n-1}\bigr)\bigr).$$

Proof. Let $\mathcal{P}$ be the face poset of K with an added initial element (corresponding to the empty simplex). Then for any $\sigma \in \mathcal{P}$ we have

$$\chi(\mathcal{P}_{\geqslant\sigma}) = \sum_{\tau>\sigma}(-1)^{\operatorname{rk}\tau-1} + (-1)^{\operatorname{rk}\sigma-1} = (-1)^{\operatorname{rk}\sigma}\left(\sum_{\tau>\sigma}(-1)^{\operatorname{rk}\tau-\operatorname{rk}\sigma-1} - 1\right)$$
$$= (-1)^{\operatorname{rk}\sigma}\left(\sum_{\emptyset\neq\rho\in\operatorname{lk}_K\sigma}(-1)^{\dim\rho} - 1\right) = (-1)^{\operatorname{rk}\sigma}\big(\chi(\operatorname{lk}_K\sigma) - 1\big)$$

(since K a simplicial complex, the poset of non-empty faces of $\operatorname{lk}_K\sigma$ is isomorphic to $\mathcal{P}_{>\sigma}$ with shifted rank function). Now, because K is a triangulation of a closed $(n-1)$-manifold, the link of a non-empty simplex σ is a homology sphere of dimension $(n-\operatorname{rk}\sigma-1)$. Therefore, $\chi(\operatorname{lk}_K\sigma) = 1 + (-1)^{n-\operatorname{rk}\sigma-1}$ and $\chi(\mathcal{P}_{\geqslant\sigma}) = (-1)^{n-1}$ for $\sigma \neq \emptyset$. We also have $\operatorname{lk}_K\emptyset = K$. It follows from Theorem 9.1 that

$$\sum_{i=0}^{n}(h_{n-i}-h_i)t^i = \big(1+(-1)^n\big(\chi(K)-1\big)\big)(t-1)^n$$
$$= (-1)^n\big(\chi(K)-\chi\big(S^{n-1}\big)\big)(t-1)^n.$$

Comparing the coefficients of t^i of both sides, we obtain the corollary. □

For a face F of a torus graph Γ, we define its Euler number $\chi(F)$ by

$$\chi(F) := \sum_{H\subseteq F}(-1)^{\dim H} \tag{9.5}$$

where H is a face of F.

Corollary 9.3.

$$\sum_{i=0}^{n}(h_{n-i}-h_i)t^i = \sum_{F\subseteq\Gamma}\big(1-\chi(F)\big)(t-1)^{\dim F}.$$

In particular, the equations $h_i = h_{n-i}$ hold if $\chi(F) = 1$ for every face F of Γ.

Proof. We apply Theorem 9.1 to the simplicial poset $\mathcal{P}(\Gamma)$ associated with the graph Γ. Given a face F, denote by σ the corresponding element of $\mathcal{P}(\Gamma)$. Then $\operatorname{rk}\sigma = (n-\dim F)$ and

$$\chi(\mathcal{P}_{\geqslant\sigma}) = \sum_{\tau\geqslant\sigma}(-1)^{\operatorname{rk}\tau-1} = \sum_{H\subseteq F}(-1)^{n-\dim H-1} = (-1)^{n-1}\chi(F).$$

This together with Theorem 9.1 proves the corollary. □

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