ELSEVIER
Linear Algebra and its Applications 295 (1999) 3–6
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On a product of positive semidefinite matrices

A.R. Meenakshi [a], C. Rajian [b,*,1]

[a] Department of Mathematics, Annamalai University, Annamalai Nagar 608 002, Tamil Nadu, India
[b] Department of Mathematics, S.T. Hindu College, Nagercoil 629 002, Tamil Nadu, India

Received 22 February 1994; accepted 16 December 1998

Submitted by R.E. Hartwig

Abstract

Necessary and sufficient conditions are given for the product of two positive semidefinite (psd) matrices to be EP. As a consequence, it is shown that the product of two psd matrices is psd if and only if the product is normal. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

In [2], Hartwig and Spindelböck have raised the following question: If A and B are two positive semidefinite (psd) matrices satisfying the condition $[AA^{\dagger}, BB^{\dagger}] = 0$ when is AB EP? In this note we give a set of conditions, each of which is necessary and sufficient for a product of two psd matrices to be psd. Further, it is shown that for the given psd matrices A and B, AB is psd if and only if AB is normal. Also, it is proved that AB is psd if and only if AB is bi-dagger and star-dagger.

All matrices considered in this paper are square matrices with complex entries. We begin with a few basic definitions and notations. For any matrix A, its range space, null space, row space and its rank are denoted by $R(A)$, $N(A)$, $RS(A)$ and $r(A)$, respectively. A is said to be psd if there exists a matrix P such that $PP^* = A$. If A and its conjugate transpose A^* have the same range space,

* Corresponding author.
1 Supported by University Grants Commission, New Delhi, India.

0024-3795/99/$ – see front matter
PII: S 0 0 2 4 - 3 7 9 5 (9 9) 0 0 0 1 4 - 2

then A is called *EP*. A matrix X is called an $\{1,2\}$ *inverse* of A if $AXA = A$ and $XAX = X$. The unique commuting $\{1,2\}$ inverse of A, if it exists, is called the *group inverse* of A and is denoted by $A^{\#}$. The Moore–Penrose inverse (denoted as $A^{\dagger}$) is the unique solution to the equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. We shall assume familiarity with the basic theory of these inverses as given in [4]. For any two matrices A and B, $[A,B] = AB - BA$ is the *commutator* of A and B. A matrix A is called bi-EP if $[AA^{\dagger}, A^{\dagger}A] = 0$, bi-dagger if $(A^2)^{\dagger} = (A^{\dagger})^2$ and star-dagger if $[A^{\dagger}, A^*] = 0$. The *parallel sum* of two psd matrices A and B is defined to be $A(A+B)^{\dagger}B$ and is denoted by $A:B$. The *ordering* $A \geqslant B$ means that $A - B$ is psd.

The following characterization of the reverse order law (see [4], p. 182; [6], p. 68) is used in proving our main theorem.

Theorem 1. *For any two square matrices P and Q the following are equivalent*:

(1) P^*PQQ^* *is EP*.

(2) $R(P^*PQ) \subseteq R(Q)$ *and* $R(QQ^*P^*) \subseteq R(P^*)$.

(3) $(PQ)^{\dagger} = Q^{\dagger}P^{\dagger}$.

(4) $P^*PQQ^{\dagger}$ *and* $P^{\dagger}PQQ^*$ *are Hermitian*.

2. Main results

We now establish equivalent conditions for the product of two psd matrices to be EP in the following theorem.

Theorem 2. *Let A and B be two psd matrices. Then the following are equivalent*:

(1) AB *is EP*

(2) $R(AB) \subseteq R(B)$ *and* $RS(AB) \subseteq RS(A)$.

(3) $(A^{1/2}B^{1/2})^{\dagger} = B^{1/2^{\dagger}}A^{1/2^{\dagger}}$.

(4) $ABB^{\dagger}$ *and* $BAA^{\dagger}$ *are Hermitian*.

(5) (i) $[AA^{\dagger}, BB^{\dagger}] = 0$, (ii) $[A, AA^{\dagger}BB^{\dagger}] = 0$ *and* (iii) $[B, BB^{\dagger}AA^{\dagger}] = 0$.

(6) $[AA^{\dagger}, BB^{\dagger}] = 0$ *and* $[AB, B^{\dagger}A^{\dagger}] = 0$.

(7) $(AB)^{\#} = B^{\dagger}A^{\dagger}$.

(8) $AB[(A:B)(A:B)^{\dagger}] = AB$ *and* $[(A:B)(A:B)^{\dagger}]AB = AB$.

(9) $2AB(AA^{\dagger}:BB^{\dagger}) = AB$ *and* $2(AA^{\dagger}:BB^{\dagger})AB = AB$.

(10) $A \geqslant AA^{\dagger}BAA^{\dagger}$ *and* $B \geqslant BB^{\dagger}ABB^{\dagger}$.

(11) AB *is bi-dagger*

(12) AB *is bi-EP*.

(13) $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

(14) $A^2BB^{\dagger}$ *and* $B^2AA^{\dagger}$ *are Hermitian*.

Proof. Since A and B are psd matrices, we have by Corollary 2.3 of [3], that $r(AB) = r((AB)^2)$. Hence $(AB)^{\#}$ exists (see [4], p. 162). Further, if A is EP then

$A^{\dagger}$ is a polynomial in A (see [4], p. 173) and hence $AX = XA$ implies that $A^{\dagger}X = XA^{\dagger}$; in particular, $AA^{\dagger} = A^{\dagger}A$. The equivalence of (1)–(4) follow from Theorem 1 on replacing P by $A^{1/2} \geqslant 0$ and Q by $B^{1/2} \geqslant 0$.

(4) $\Rightarrow$ (5). If (4) holds, then $ABB^{\dagger} = BB^{\dagger}A$ and $BAA^{\dagger} = AA^{\dagger}B$. Moreover, since A and B are Hermitian, they are EP and hence $A^{\dagger}BB^{\dagger} = BB^{\dagger}A^{\dagger}$ and $B^{\dagger}AA^{\dagger} = AA^{\dagger}B^{\dagger}$. Now $AA^{\dagger}BB^{\dagger} = BAA^{\dagger}B^{\dagger} = BB^{\dagger}AA^{\dagger} \Rightarrow [AA^{\dagger}, BB^{\dagger}] = 0$. Furthermore, $ABB^{\dagger} = BB^{\dagger}A$ and $[AA^{\dagger}, BB^{\dagger}] = 0$ imply that $AAA^{\dagger}BB^{\dagger} = ABB^{\dagger} = BB^{\dagger}A = BB^{\dagger}AA^{\dagger}A = AA^{\dagger}BB^{\dagger}A$. Hence $[A, AA^{\dagger}BB^{\dagger}] = 0$. Similarly, $[B, AA^{\dagger}BB^{\dagger}] = 0$ holds.

(5) $\Rightarrow$ (4). If (5) holds, then $[AA^{\dagger}, BB^{\dagger}] = 0$ and $[A, AA^{\dagger}BB^{\dagger}] = 0$ implying $ABB^{\dagger} = AA^{\dagger}ABB^{\dagger} = A^{\dagger}ABB^{\dagger}A = BB^{\dagger}AA^{\dagger}A = BB^{\dagger}A$. This shows that $ABB^{\dagger}$ is Hermitian. Similarly, $[AA^{\dagger}, BB^{\dagger}] = 0$ and $[B, BB^{\dagger}AA^{\dagger}] = 0$ imply that $BAA^{\dagger}$ is Hermitian.

Next, we prove the equivalence of (4), (6) and (7).

(4) $\Rightarrow$ (6). If (4) holds, then $[AA^{\dagger}, BB^{\dagger}] = 0$. (See the derivation of (4) $\Rightarrow$ (5).) Now $ABB^{\dagger}A^{\dagger} = BB^{\dagger}AA^{\dagger} = B^{\dagger}BAA^{\dagger} = B^{\dagger}A^{\dagger}AB$ and so $[AB, B^{\dagger}A^{\dagger}] = 0$.

(6) $\Rightarrow$ (7). If (6) holds, then $[AA^{\dagger}, BB^{\dagger}] = 0$ shows that $B^{\dagger}A^{\dagger}$ is an $\{1, 2\}$ inverse of AB (see [2], p. 245). It is also a commuting $\{1, 2\}$ inverse of AB and so $(AB)^{\#} = B^{\dagger}A^{\dagger}$.

(7) $\Rightarrow$ (4). Since $(AB)^{\#}$ exists, we have $R(AB) = R((AB)^{\#})$ and $N(AB) = N((AB)^{\#})$ (see [4], p. 162). If (7) holds, then $R(AB) = R(B^{\dagger}A^{\dagger}) \subseteq R(B^{\dagger}) = R(B)$ and hence $BB^{\dagger}AB = AB$ (see [4], p. 55). Post-multiplication by $B^{\dagger}$ yields $BB^{\dagger}ABB^{\dagger} = ABB^{\dagger}$, showing that $ABB^{\dagger}$ is Hermitian. Similarly, one can show that $BAA^{\dagger}$ is Hermitian.

(2) $\Longleftrightarrow$ (8). If (2) holds, then $R(AB) \subseteq R(B)$ and $RS(AB) \subseteq RS(A) = R(A)$, it is also known that $R(AB) \subseteq R(A)$ and $RS(AB) \subseteq RS(B) = R(B)$. Thus, $R(AB), RS(AB) \subseteq R(A) \cap R(B) = R(A : B)$. This implies that $[(A : B)(A : B)^{\dagger}]AB = AB$ and $AB[(A : B)(A : B)^{\dagger}] = AB$. Each of the above steps is clearly reversible, and hence (8) $\Longleftrightarrow$ (2).

(2) $\Longleftrightarrow$ (9). Since $AA^{\dagger}$ and $BB^{\dagger}$ are orthogonal projections along $R(A)$ and $R(B)$, respectively, it follows that $2(AA^{\dagger} : BB^{\dagger})$ is the orthogonal projection along $R(AA^{\dagger}) \cap R(BB^{\dagger})$ (see [6], p. 189) and so $R[2(AA^{\dagger} : BB^{\dagger})] = R(A) \cap R(B)$. Hence the equivalence of (2) and (9) holds.

(4) $\Longleftrightarrow$ (10). Since A and B are psd and $AA^{\dagger}, BB^{\dagger}$ are idempotent Hermitian, by Corollary 2 of [1], we have $A \geqslant BB^{\dagger}ABB^{\dagger}$ if and only if $ABB^{\dagger} = BB^{\dagger}A$ and $B \geqslant AA^{\dagger}BAA^{\dagger}$ if and only if $BAA^{\dagger} = AA^{\dagger}B$. Hence we have the equivalence of (4) and (10).

By Corollary 3 of [2] and by the existence of $(AB)^{\#}$ it follows that AB is EP if and only if AB is bi-dagger; the latter holds if and only if AB is bi-EP. Hence we have the equivalence of (1), (11) and (12).

From the fact that A and B are psd, it is clear that (4) and (14) are equivalent. Also, the equivalence of (13) and (14) follows from Theorem 1. □

As an application of Theorem 2, we have the following:

Theorem 3. *Let A and B be two psd matrices. Then the following are equivalent*:
(1) AB *is normal*.
(2) $AB \geqslant 0$.
(3) (i) $[AA^{\dagger}, BB^{\dagger}] = 0$, (ii) $[AB, B^{\dagger}A^{\dagger}] = 0$ *and* (iii)$[B^{\dagger}A^{\dagger}, BA] = 0$.
(4) AB *is bi-dagger and star-dagger*.

Proof. (1) $\Longleftrightarrow$ (2). If (1) holds, then AB is normal. Since A and B are psd, the eigen values of AB are non-negative. Hence AB is psd. The implication of (1) from (2) is obvious.

(1) $\Rightarrow$ (3). If AB is normal, then $AB(AB)^{\dagger} = (AB)^{\dagger}AB$ and $(AB)^{*}(AB)^{\dagger} = (AB)^{\dagger}(AB)^{*} \Rightarrow BA(AB)^{\dagger} = (AB)^{\dagger}BA$.

Now the equivalence of (1) and (13) of Theorem 2 implies that $BAB^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}BA$ and therefore (3) (iii) holds. Proving that (3) (ii) holds is similar and (3) (i) is obvious.

(3) $\Rightarrow$ (1). From the equivalence of (6) and (1) in Theorem 2, (3) (i) and (3) (ii) imply that AB is EP. Further, (3) (iii) implies that AB is star-dagger; hence by Theorem 3 of [5] AB is normal.

The equivalence of (1) and (4) is obvious. □

Acknowledgement

The authors thank the referees for giving several helpful comments and suggestions to improve the presentation of paper.

References

[1] J.K. Baksalary, R. Kala, K. Klaczynski, The matrix inequality $M \geqslant B^{*}MB$, Linear Algebra Appl. 54 (1983) 77–86.
[2] R.E. Hartwig, K. Spindelböck, Matrices A for which A^{*} and $A^{\dagger}$ commute, Linear and Multilinear Algebra 14 (1984) 241–256.
[3] Y. Hong, R.A. Horn, The Jordan canonical forms of product of a Hermitian and a positive semidefinite matrix, Linear Algebra Appl. 147 (1991) 373–386.
[4] A. Ben Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, Wiley, New York, 1974.
[5] C.D. Meyer, Some remarks on EPr matrices and generalized inverses, Linear Algebra Appl. 3 (1970) 275–278.
[6] C.R. Rao, S.K. Mitra, Generalized Inverse of Matrices and Its Applications, Wiley, New York, 1971.