We evaluate some determinants related to Brioschi’s extension of the classical double alternant of Cauchy, and Scott’s extension of Brioschi’s.

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1. Introduction

We first recall Cauchy’s double alternant [2,4–6] of 1841.

Theorem 1

\[
\begin{vmatrix}
\frac{1}{x_1 - y_1} & \cdots & \frac{1}{x_1 - y_n} \\
\vdots & \ddots & \vdots \\
\frac{1}{x_n - y_1} & \cdots & \frac{1}{x_n - y_n}
\end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i)(y_i - y_j) \prod_{1 \leq i, j \leq n} (x_i - y_j).
\]

In 1857 Brioschi [1] (see also [7]) found a related determinant, which we will call Brioschi’s double alternant.
Theorem 2
\[
\begin{vmatrix}
\frac{1}{x_1-a_1} & \frac{1}{(x_1-a_1)^2} & \cdots & \frac{1}{(x_1-a_1)^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_{2n}-a_1} & \frac{1}{(x_{2n}-a_1)^2} & \cdots & \frac{1}{(x_{2n}-a_1)^2}
\end{vmatrix}
= (-1)^n \frac{\prod_{1 \leq i < j \leq 2n} (x_j - x_i) \prod_{1 \leq i < j \leq n} (a_i - a_j)^4}{\prod_{1 \leq j \leq n} (x_i - a_j)^2}.
\]

He does not give a proof, although he says it is easy: “si ha facilmente che il determinante” etc. Here is a generalization.

Theorem 3
\[
\begin{vmatrix}
\frac{1}{x_1-a_1} & \frac{1}{(x_1-a_1)(x_1-b_1)} & \cdots & \frac{1}{(x_1-a_n)(x_1-b_n)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_{2n}-a_1} & \frac{1}{(x_{2n}-a_1)(x_{2n}-b_1)} & \cdots & \frac{1}{(x_{2n}-a_n)(x_{2n}-b_n)}
\end{vmatrix}
= (-1)^n \frac{\prod_{1 \leq i < j \leq 2n} (x_j - x_i) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{\prod_{1 \leq j \leq n} (x_i - a_j)(x_i - b_j)}.
\]

This has a simple proof, and perhaps the easiest way to obtain Theorem 2 is as a corollary. Changing n to 2n in Theorem 1 and then renaming \(y_{2k-1}\) as \(a_k\) and \(y_{2k}\) as \(b_k\) for \(1 \leq k \leq n\) gives

\[
\begin{vmatrix}
\frac{1}{x_1-a_1} & \frac{1}{x_1-b_1} & \cdots & \frac{1}{x_1-a_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_{2n}-a_1} & \frac{1}{x_{2n}-b_1} & \cdots & \frac{1}{x_{2n}-a_n}
\end{vmatrix}
= \frac{\prod_{1 \leq i < j \leq 2n} (x_j - x_i) \prod_{k=1}^n (a_k - b_k) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{\prod_{1 \leq j \leq n} (x_i - a_j)(x_i - b_j)}.
\]

Next, subtract each odd column from the corresponding even column to get

\[
\begin{vmatrix}
\frac{1}{x_1-a_1} & \frac{b_1-a_1}{(x_1-a_1)(x_1-b_1)} & \cdots & \frac{b_n-a_n}{(x_1-a_n)(x_1-b_n)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_{2n}-a_1} & \frac{b_1-a_1}{(x_{2n}-a_1)(x_{2n}-b_1)} & \cdots & \frac{b_n-a_n}{(x_{2n}-a_n)(x_{2n}-b_n)}
\end{vmatrix}
= \frac{\prod_{1 \leq i < j \leq 2n} (x_j - x_i) \prod_{k=1}^n (a_k - b_k) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{\prod_{1 \leq j \leq n} (x_i - a_j)(x_i - b_j)}.
\]

Dividing out all the factors \(a_j - b_j\) we get Theorem 3. One can get rid of the factor \((-1)^n\) by changing

\[\prod_{1 \leq i < j \leq 2n} (x_j - x_i)\to \prod_{1 \leq i < j \leq 2n} (x_i - x_j),\]

and we have done so below in (1.3), but it appears in Brioschi’s original formulation, and we retain it to fit Theorem 7 below.
Theorem 3 is easily extended to the following result.

**Theorem 4**

\[
\begin{array}{cccc}
1 & c_1x_1 - y_1 & \cdots & 1 \\
& (x_1 - a_1)(x_1 - b_1) & \cdots & (x_1 - a_n)(x_1 - b_n) \\
\vdots & \vdots & \ddots & \vdots \\
1 & c_nx_n - y_n & \cdots & 1 \\
& (x_n - a_1)(x_n - b_1) & \cdots & (x_n - a_n)(x_n - b_n) \\
\end{array}
\]

\[
= \prod_{j=1}^{n} (c_j b_j - y_j) \prod_{1 \leq i < j \leq 2n} (x_i - x_j) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j) \\
= \prod_{1 \leq i < j \leq n} (x_i - a_j)(x_i - b_j). \\
\]

To see this, subtract \( c_j x_j - y_j \) times the \( j \)th odd column from the \( j \)th even column. We have

\[
\frac{c_j x_j - y_j}{(x_i - a_j)(x_i - b_j)} = \frac{c_j b_j - y_j}{(x_i - a_j)(x_i - b_j)},
\]

and if we take the common factor \( c_j b_j - y_j \) out of the \( j \)th even column for each \( j \), what remains is exactly the left side of Theorem 3.

Although it is a trivial extension, Theorem 4 has at least one advantage over Theorem 3: it can be converted to a corresponding trigonometric theorem. If we change \( x_i \) to \( e^{2x_i} \) and \( y_j \) to \( e^{2y_j} \), then \( x_i - y_j \) becomes

\[
e^{2x_i} - e^{2y_j} = e^{x_i+y_j} \left( e^{x_i-y_j} - e^{y_j-x_i} \right) = 2e^{x_i+y_j} \sinh(x_i - y_j).
\]

If we do this throughout Theorem 1, one can check that the exponential functions and the factors of 2 all cancel, leaving

\[
\frac{1}{\sin(x_1-y_1)} \cdots \frac{1}{\sin(x_1-y_n)} \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
\frac{1}{\sin(x_n-y_1)} \cdots \frac{1}{\sin(x_n-y_n)}
\]

\[
= \prod_{1 \leq i < j \leq n} \sin(x_i - x_j) \sin(y_i - y_j) \\
= \prod_{1 \leq i < j \leq n} \sin(x_i - y_j). \\
\] (1.2)

We will call this passage from results like Theorem 1 to results like (1.2) *trigonometric conversion*. If we set every \( c_j = 1 \) in Theorem 4, a trigonometric conversion then gives

\[
\begin{array}{cccc}
1 & \sin(x_1 - y_1) & \cdots & 1 \\
& \sin(x_1-a_1) \sin(x_1-b_1) & \cdots & \sin(x_1-a_n) \sin(x_1-b_n) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \sin(x_n - y_1) & \cdots & 1 \\
& \sin(x_n-a_1) \sin(x_n-b_1) & \cdots & \sin(x_n-a_n) \sin(x_n-b_n) \\
\end{array}
\]

\[
= \prod_{j=1}^{n} \sin(b_j - y_j) \prod_{1 \leq i < j \leq 2n} \sin(x_i - x_j) \\
= \prod_{1 \leq i < j \leq n} \sin(x_i - a_j) \sin(x_i - b_j) \\
\times \prod_{1 \leq i < j \leq n} \sin(a_i - a_j) \sin(a_i - b_j) \sin(b_i - a_j) \sin(b_i - b_j). \\
\] (1.3)
The trigonometric conversion of Theorem 3 is not as nice; the exponential functions do not all cancel because of the different degrees of the columns.

Before turning to generalizations, we prove in the next section several theorems related to Theorem 3.

2. Some determinants in the Brioschi family

We denote the left side of Theorem 3 by $B_n(x_1, \ldots, x_{2n}; a_1, \ldots, a_n; b_1, \ldots, b_n)$, or $B_n$ for short. Another way to evaluate $B_n$ is by locating it in a family of four determinants. We define the following determinant with $2n + 1$ rows and a special form of the last column.

Definition 1

\[
C_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n; u, v) = \begin{vmatrix}
1 & \frac{1}{(x_i - a_j)(x_i - u)} & \frac{1}{(x_i - a_j)(x_i - b_j)} & \frac{1}{(x_i - u)(x_i - v)} \\
\end{vmatrix}.
\]

If we subtract each odd column except the last from the even column to its right, then by (2.1) the $i$th entry in the $j$th even column becomes

\[
\frac{1}{(x_i - a_j)(x_i - u)} - \frac{1}{(x_i - a_j)(x_i - v)} = \frac{b_j - u}{(x_i - a_j)(x_i - b_j)(x_i - u)}.
\]

The $j$th odd column now has the factor $a_j - u$, the $j$th even column the factor $b_j - u$, and the $i$th row the factor $1/(x_i - u)$. It follows that

\[
C_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n; u, v)
= \frac{\prod_{j=1}^{n} (a_j - u)(b_j - u)}{\prod_{i=1}^{2n+1}(x_i - u)} D_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n; v), \tag{2.2}
\]

with the following definition.

Definition 2

\[
D_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n; v) = \begin{vmatrix}
1 & \frac{1}{(x_i - a_j)(x_i - v)} & \frac{1}{(x_i - a_j)(x_i - b_j)} & \frac{1}{x_i - v} \\
\end{vmatrix}.
\]

If we subtract each odd column of $D_n$ except the last from the even column to its right, then by (2.1) the $i$th entry in the $j$th even column becomes

\[
\frac{b_j - v}{(x_i - a_j)(x_i - b_j)(x_i - v)}.
\]

We can take $b_j - v$ out of the $j$th even column and $1/(x_i - v)$ out of the $i$th row. This gives

\[
D_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n; v)
= \frac{\prod_{j=1}^{n} (b_j - v)}{\prod_{i=1}^{2n+1}(x_i - v)} A_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n), \tag{2.3}
\]

where $A_n$ is a bordered generalized Brioschi determinant. More precisely, $A_n$ is the same as $B_n$ except with an extra row of the same type and an extra column of 1’s at the far right.
It is not difficult to relate $A_n$ to $B_n$. We subtract the last row from all the others and then expand on the last column. In the odd columns we have

$$
\frac{1}{x_i - a_j} - \frac{1}{x_{2n+1} - a_j} = \frac{x_{2n+1} - x_i}{(x_i - a_j)(x_{2n+1} - a_j)},
$$

and in the even columns

$$
\frac{1}{(x_i - a_j)(x_j - b_j)} - \frac{1}{(x_{2n+1} - a_j)(x_{2n+1} - b_j)} = \frac{x_{2n+1} - x_i}{(x_i - a_j)(x_j - b_j)},
$$

We can take $x_{2n+1} - x_i$ out of the $i$th row for $1 \leq i \leq 2n$, and $1/(x_{2n+1} - a_j)$ out of the $j$th odd column, and $1/(x_{2n+1} - a_j)(x_{2n+1} - b_j)$ out of the $j$th even column. Next, subtract the $j$th odd column from the $j$th even column. We have

$$
x_{2n+1} + x_i - a_j - b_j = \frac{1}{x_i - a_j} = \frac{x_{2n+1} - a_j}{(x_i - a_j)(x_j - b_j)},
$$

and we can pull out the factor $x_{2n+1} - a_j$ and cancel it with a factor that came out earlier. The remaining determinant is exactly $B_n$, so

$$
A_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n) = \frac{\prod_{i=1}^{2n} (x_i - x_j)}{\prod_{j=1}^{2n} (x_j - a_j)(x_j - b_j)} B_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n). \tag{2.4}
$$

Hence we have several corollaries of Theorem 3. From (2.4) we get

**Corollary 1**

$$
\begin{vmatrix}
\frac{1}{x_1 - a_1} & \frac{1}{(x_1 - a_1)(x_1 - b_1)} & \cdots & \frac{1}{x_1 - a_n} & \frac{1}{(x_1 - a_n)(x_1 - b_n)} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{x_{2n+1} - a_1} & \frac{1}{(x_{2n+1} - a_1)(x_{2n+1} - b_1)} & \cdots & \frac{1}{x_{2n+1} - a_n} & \frac{1}{(x_{2n+1} - a_n)(x_{2n+1} - b_n)} & 1
\end{vmatrix}
= (-1)^n \frac{\prod_{1 \leq i < j \leq 2n+1} (x_j - x_i) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{\prod_{1 \leq j \leq n} (x_i - a_j)(x_i - b_j)}. 
$$

Then (2.3) gives us

**Corollary 2**

$$
\frac{1}{(x_i - a_j)(x_i - b_j)} \frac{1}{x_i - v} = (-1)^n \prod_{1 \leq i < j \leq 2n+1} (x_j - x_i)
\times \frac{\prod_{j=1}^{2n+1} (b_j - v) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{\prod_{i=1}^{2n+1} (x_i - v) \prod_{1 \leq j \leq n} (x_i - a_j)(x_i - b_j)}. 
$$

Finally, (2.2) implies
Corollary 3

\[
\begin{vmatrix}
\frac{1}{(x_i - a_j)(x_i - u)} & \frac{1}{(x_i - a_j)(x_i - b_j)} & \frac{1}{(x_i - u)(x_i - v)}
\end{vmatrix} = (-1)^n \prod_{1 \leq i < j \leq 2n+1} (x_j - x_i) \\
\times \prod_{j=1}^n (a_j - u)(b_j - v) \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j) \\
\prod_{i=1}^{2n-1} (x_i - u)(x_i - v) \prod_{1 \leq i < j \leq 2n+1} (x_i - a_j)(x_i - b_j)
\]

If we want to prove Theorem 3 by this argument, we can close the circle by relating \( B_n \) to \( C_{n-1} \).

Subtracting the last row of \( B_n \) from all the others, as before we have

\[
\begin{align*}
\frac{1}{x_i - a_j} - \frac{1}{x_{2n} - a_j} &= \frac{x_{2n} - x_i}{(x_i - a_j)(x_{2n} - a_j)} \\
\frac{1}{(x_i - a_j)(x_i - b_j)} - \frac{1}{(x_{2n} - a_j)(x_{2n} - b_j)} &= \frac{(x_{2n} - x_i)(x_{2n} + x_i - a_j - b_j)}{(x_i - a_j)(x_i - b_j)(x_{2n} - a_j)(x_{2n} - b_j)}
\end{align*}
\]

in the odd columns and

\[
\begin{align*}
\frac{1}{(x_i - a_j)(x_i - b_j)} - \frac{1}{(x_{2n} - a_j)(x_{2n} - b_j)} &= \frac{x_{2n} - a_j}{(x_i - a_j)(x_{2n} - b_j)}
\end{align*}
\]

in the even columns. We can take \( x_{2n} - x_i \) out of the \( i \)th row for \( 1 \leq i \leq 2n - 1 \), and \( 1/(x_{2n} - a_j) \) out of the \( j \)th odd column, and \( 1/(x_{2n} - a_j)(x_{2n} - b_j) \) out of the \( j \)th even column. Now subtract the \( j \)th odd column from the \( j \)th even column. We have

\[
\begin{align*}
x_{2n} + x_i - a_j - b_j &= \frac{1}{(x_i - a_j)(x_{2n} - a_j)} = \frac{x_{2n} - a_j}{(x_i - a_j)(x_{2n} - b_j)}
\end{align*}
\]

and we can pull out the factor \( x_{2n} - a_j \) and cancel it with a factor that came out earlier. This gives

\[
B_n = \frac{\prod_{j=1}^{n-1} (x_{2n} - x_j)}{\prod_{j=1}^n (x_{2n} - a_j)(x_{2n} - b_j)}
\]
times the same determinant with the last row replaced by \((1 \, 0 \, 1 \, 0 \, \cdots \, 1 \, 0)\). Then subtract the last odd column from all the other odd columns. We have

\[
\begin{align*}
\frac{1}{x_i - a_j} - \frac{1}{x_i - a_n} &= \frac{a_j - a_n}{(x_i - a_j)(x_i - a_n)}
\end{align*}
\]

and \( a_j - a_n \) comes out of the \( j \)th odd column. The last row is now \((0 \, 0 \, 0 \, \cdots \, 0 \, 1)\). Expanding on it, we get

\[
\begin{align*}
B_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n) &= -\frac{\prod_{j=1}^{n-1} (x_{2n} - x_j) \prod_{j=1}^{n-1} (a_j - a_n)}{\prod_{j=1}^n (x_{2n} - a_j)(x_{2n} - b_j)} \\
&\times C_{n-1}(x_1, \ldots, x_{2n-1}; a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n-1}; a_n, b_n).
\end{align*}
\]

(2.5)

Combining this with (2.2), (2.3), and (2.4), we get a functional equation for \( B_n \) alone, namely

\[
\begin{align*}
B_n(x_1, \ldots, x_{2n+1}; a_1, \ldots, a_n; b_1, \ldots, b_n) &= -(x_{2n} - x_{2n-1}) \\
&\times \frac{\prod_{j=1}^{2n-2} (x_{2n} - x_j)(x_{2n-1} - x_i) \prod_{j=1}^{n-1} (a_j - a_n)(a_j - b_n)(b_j - a_n)(b_j - b_n)}{\prod_{j=1}^n (x_{2n} - a_j)(x_{2n} - b_j)(x_{2n-1} - a_j)(x_{2n-1} - b_j) \prod_{j=1}^{2n-2} (x_i - a_n)(x_i - b_n)} \\
&\times B_{n-1}(x_1, \ldots, x_{2n-2}; a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n-1}).
\end{align*}
\]

(2.6)

Iterating this down to

\[
B_1(x_1, x_2; a_1; b_1) = -\frac{x_2 - x_1}{(x_1 - a_1)(x_1 - b_1)(x_2 - a_1)(x_2 - b_1)}
\]
or \( B_0(-) = 1 \) we get Theorem 3.
One can also add more columns to Theorem 3. For example, for $1 \leq i \leq 3n$ and $1 \leq j \leq n$ we have

$$\begin{vmatrix}
\frac{1}{x_i - a_j} & \frac{1}{x_i - a_j} \\
\frac{1}{x_i - a_j} & \frac{1}{x_i - a_j}
\end{vmatrix} = \frac{\prod_{1 \leq i < j \leq 3n} (x_j - x_i)}{\prod_{1 \leq i \leq 3n} (x_i - a_j)(x_i - b_j)(x_i - c_j)}$$

$$\times \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(a_i - c_j)(b_i - a_j)(b_i - b_j)(b_i - c_j)(c_i - a_j)(c_i - b_j)(c_i - c_j)$$

(2.7)

In the next section we will prove a generalization of this, and some other more general theorems.

3. An extension of Scott’s double alternant

Here is another natural way to add a third set of columns. For $1 \leq i \leq 3n$ and $1 \leq j \leq n$ we have the following evaluation.

**Theorem 5**

$$\begin{vmatrix}
\frac{1}{x_i - a_j} & \frac{1}{x_i - a_j} & \frac{1}{x_i - a_j} \\
\frac{1}{x_i - a_j} & \frac{1}{x_i - a_j} & \frac{1}{x_i - a_j}
\end{vmatrix} = (-1)^n \frac{\prod_{1 \leq i < j \leq 3n} (x_j - x_i)}{\prod_{1 \leq i \leq 3n} (x_i - a_j)(x_i - b_j)(x_i - c_j)}$$

$$\times \prod_{1 \leq i < j \leq n} (a_i - a_j)(a_i - b_j)(a_i - c_j)(b_i - a_j)(b_i - b_j)(b_i - c_j)(c_i - a_j)(c_i - b_j)(c_i - c_j)$$

This suggests a further generalization, which is our main theorem. We consider a determinant of order $k_1 + \cdots + k_n$, where the first $k_1$ columns look like

$$\begin{vmatrix}
\frac{1}{x_i - a_{j_1}} & \frac{1}{x_i - a_{j_1}} & \cdots & \frac{1}{x_i - a_{j_1}} \\
\frac{1}{x_i - a_{j_1}} & \frac{1}{x_i - a_{j_1}} & \cdots & \frac{1}{x_i - a_{j_1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{x_i - a_{j_1}} & \frac{1}{x_i - a_{j_1}} & \cdots & \frac{1}{x_i - a_{j_1}}
\end{vmatrix}$$

for some positive integer $k_1$, the next $k_2$ columns look like

$$\begin{vmatrix}
\frac{1}{x_i - a_{j_2}} & \frac{1}{x_i - a_{j_2}} & \cdots & \frac{1}{x_i - a_{j_2}} \\
\frac{1}{x_i - a_{j_2}} & \frac{1}{x_i - a_{j_2}} & \cdots & \frac{1}{x_i - a_{j_2}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{x_i - a_{j_2}} & \frac{1}{x_i - a_{j_2}} & \cdots & \frac{1}{x_i - a_{j_2}}
\end{vmatrix}$$

for some positive integer $k_2$, and so on through $n$ sets of columns, the last set looking like

$$\begin{vmatrix}
\frac{1}{x_i - a_{j_n}} & \frac{1}{x_i - a_{j_n}} & \cdots & \frac{1}{x_i - a_{j_n}} \\
\frac{1}{x_i - a_{j_n}} & \frac{1}{x_i - a_{j_n}} & \cdots & \frac{1}{x_i - a_{j_n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{x_i - a_{j_n}} & \frac{1}{x_i - a_{j_n}} & \cdots & \frac{1}{x_i - a_{j_n}}
\end{vmatrix}$$

We will use the phrase “first columns” for the columns of the form $1/(x_i - a_{j_1})$, and similarly for “second columns” and so on. It is convenient to set the total number of rows and columns $k_1 + k_2 + \cdots + k_n$ equal to $\omega$. We denote the determinant by $S_{k_1,\ldots,k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n)$, or $S$ for short.

**Theorem 6.** If $\omega = k_1 + \cdots + k_n$, then

$$S_{k_1,\ldots,k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{\binom{k_1}{2} + \cdots + \binom{k_n}{2}} \prod_{1 \leq i < j \leq \omega} (x_j - x_i) \prod_{1 \leq i \leq n} \prod_{r=1}^{k_i} \prod_{s=1}^{k_j} (a_{ir} - a_{js})$$

$$\prod_{i=1}^{\omega} \prod_{j=1}^{\omega} \prod_{r=1}^{k_i} (x_i - a_{ir})$$

The simplest way to prove this is probably to imitate the proof of Theorem 3 in Section 1. Change $n$ to $\omega$ in Theorem 1 and set $y_1, \ldots, y_{k_1}$ equal to $a_{11}, \ldots, a_{1k_1}$ respectively; $y_{k_1+1}, \ldots, y_{k_1+k_2}$ equal to $a_{21}, \ldots, a_{2k_2}$ respectively; and so on, finally setting the last $k_n$ $y$’s equal to $a_{nk_1}, \ldots, a_{nk_n}$ respectively. With this scheme, $y_i - y_j$ with $i < j$ in Theorem 1 translates to $a_{ir} - a_{js}$ either with $i < j$ or with $i = j$ and $r < s$, and Theorem 1 becomes the following.
Lemma 1

\[
\begin{vmatrix}
\frac{1}{x_1-a_{11}} & \cdots & \frac{1}{x_1-a_{1k_1}} & \cdots & \frac{1}{x_1-a_{1n}} & \cdots & \frac{1}{x_1-a_{nk_1}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{1}{x_{kn}-a_{kn}} & \cdots & \frac{1}{x_{kn}-a_{kn}} \\
\end{vmatrix}
\]

\[
= \prod_{1 \leq i < j \leq n} (x_j - x_i) \prod_{j=1}^{n} \prod_{1 \leq r < s \leq k_j} (a_{jr} - a_{js}) \prod_{1 \leq i < j \leq n} \prod_{s=1}^{k_j} (a_{ir} - a_{js}) \prod_{s=1}^{k_j} (x_i - a_{js}) \prod_{r=1}^{k_j} (x_i - a_{i_r}).
\]

In this determinant, subtract each first column from all the later columns in the same family (if any). We have

\[
\frac{1}{x_i - a_{js}} - \frac{1}{x_i - a_{j1}} = \frac{a_{js} - a_{j1}}{(x_i - a_{j1})(x_i - a_{js})}
\]

and we can pull out all the factors \(a_{js} - a_{j1}\). Next, subtract each of the new second columns from all the later columns in the same family, where we have

\[
\frac{1}{(x_i - a_{j1})(x_i - a_{js})} - \frac{1}{(x_i - a_{j1})(x_i - a_{j2})} = \frac{a_{js} - a_{j2}}{(x_i - a_{j1})(x_i - a_{j2})(x_i - a_{js})}.
\]

Pull out all the factors \(a_{js} - a_{j2}\) and continue in this way through all the columns. This transforms the determinant in Lemma 1 into the one in Theorem 6, multiplied by the factors

\[
\prod_{j=1}^{n} \prod_{1 \leq r < s \leq k_j} (a_{js} - a_{j_r}).
\]

Dividing these out gives the sign factor in Theorem 6 and completes the proof. It is also possible to give a functional equation proof as in Section 2.

The special case where all the \(k_j\) are equal to \(k\) is of interest.

Theorem 7

\[
S_{k,k,\ldots,k}(\bar{x}; \bar{a}_1, \ldots, \bar{a}_n) = (-1)^n(n_2) \frac{\prod_{1 \leq i < j \leq \min(k)} (x_j - x_i) \prod_{1 \leq i < j \leq n} (a_{i_r} - a_{js})}{\prod_{r=1}^{\min(k)} \prod_{i=1}^{n} \prod_{s=1}^{k} (x_i - a_{i_r})}.
\]

Theorem 1 is the case \(k = 1\), Theorem 3 the case \(k = 2\), and Theorem 5 the case \(k = 3\).

If every \(a_{ij}\) equals \(a_1\), every \(a_{2j}\) equals \(a_2\), and so on, Theorem 7 reduces to Scott’s generalization [9] of Brioschi’s double alternant; see also [8]. If \(a_{ij} = q^{i-1}a_i\) for every \(i\) and \(j\), then Theorem 6 reduces to Theorem 2 in [3] or Theorem 24 in [5].

We observe a consequence of the above argument. Let \(T_{k_1,\ldots,k_n}(\bar{x}; \bar{a}_1, \ldots, \bar{a}_n)\), or \(T\) for short, denote the determinant of order \(k_1 + \cdots + k_n\) where the \(\bar{a}_j\) columns have the form

\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
x_i - a_{j1} & (x_i - a_{j1})(x_i - a_{j2}) & (x_i - a_{j1})(x_i - a_{j3}) & \cdots & (x_i - a_{j1})(x_i - a_{jk}) \\
\end{vmatrix}
\]

for some positive integer \(k_j\), which is one greater than the number of columns of degree \(-2\). We encountered this determinant in the course of proving Theorem 6, so we have only to divide Lemma 1 by
\[
\prod_{j=1}^{n} \prod_{s=2}^{k_j} (a_{js} - a_{j1})
\]
to obtain the evaluation of \( T \).

**Theorem 8.** If \( \omega = k_1 + \cdots + k_n \), then

\[
T_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_1+\cdots+k_n-n} \prod_{1 \leq i < j \leq \omega} (x_j - x_i) \prod_{j=1}^{n} \prod_{2 \leq r < s \leq k_j} (a_{jr} - a_{js}) \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k_i} a_{ir} - a_{js} \prod_{i=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_j} (x_i - a_{jr}).
\]

This can be rewritten to make the sign factor look like Theorem 6.

\[
T_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{(k_1) + \cdots + (k_n)} \prod_{1 \leq i < j \leq \omega} (x_j - x_i) \prod_{j=1}^{n} \prod_{2 \leq r < s \leq k_j} (a_{jr} - a_{js}) \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k_i} a_{ir} - a_{js} \prod_{i=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_j} (x_i - a_{jr}).
\]

When every \( k_j = 3 \) we have (2.7).

**4. Further extensions**

It is easy to add numerators to Theorem 8, as in Theorem 4. We consider the determinant \( \tau_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \) where the \( \vec{a}_j \) columns have the form

\[
\frac{1}{x_i - a_{j1}} c_{j2} x_i - b_{j2} \frac{c_{j3} x_i - b_{j3}}{(x_i - a_{j1})(x_i - a_{j2})} \cdots \frac{c_{jk} x_i - b_{jk}}{(x_i - a_{j1})(x_i - a_{jk})}
\]

for each \( j, 1 \leq j \leq n \). Subtracting \( c_{jk} \) times the \( j \)th first column from the \( k \)th \( \vec{a}_j \) column, we have

\[
\frac{c_{jk} x_i - b_{jk}}{(x_i - a_{j1})(x_i - a_{jk})} - \frac{c_{jk}}{x_i - a_{j1}} = \frac{c_{jk} a_{jk} - b_{jk}}{(x_i - a_{j1})(x_i - a_{jk})}.
\]

If we take out \( c_{jk} a_{jk} - b_{jk} \) for each \( j \) and \( k \) then we are back to the determinant in Theorem 8. This shows the following result.

**Theorem 9.** If \( \omega = k_1 + \cdots + k_n \), then

\[
\tau_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_1+\cdots+k_n-n} \prod_{1 \leq i < j \leq \omega} (x_j - x_i) \prod_{j=1}^{n} \left( \prod_{k=2}^{k_j} (c_{jk} a_{jk} - b_{jk}) \prod_{2 \leq r < s \leq k_j} (a_{jr} - a_{js}) \right) \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k_i} a_{ir} - a_{js} \prod_{i=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_j} (x_i - a_{jr}).
\]

We can also obtain a variation of this. Let \( U_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \), or \( U \) for short, denote the determinant of order \( k_1 + \cdots + k_n \) where the \( \vec{a}_j \) columns have the form

\[
\frac{1}{x_i - a_{j1}} c_{j2} x_i - b_{j2} \frac{c_{j3} x_i - b_{j3}}{(x_i - a_{j1})(x_i - a_{j2})} \cdots \frac{c_{jk} x_i - b_{jk}}{(x_i - a_{j1})(x_i - a_{jk})}.
\]
If \( a_{jk} \neq a_{j(k-1)} \), we can write
\[
\frac{c_{jk}x_i - b_{jk}}{(x_i - a_{j(k-1)})(x_i - a_{jk})} = \frac{1}{a_{jk} - a_{j(k-1)}} \left( \frac{c_{jk}a_{jk} - b_{jk}}{x_i - a_{jk}} - \frac{c_{jk}a_{j(k-1)} - b_{jk}}{x_i - a_{j(k-1)}} \right).
\]
Therefore, adding \((c_{j2}a_{j1} - b_{j2})/(a_{j2} - a_{j1})\) times the first \( \vec{a}_j \) column to the second will allow us to pull out \((c_{j2}a_{j2} - b_{j2})/(a_{j2} - a_{j1})\) from the second \( \vec{a}_j \) column, leaving \( 1/(x_i - a_{j2}) \). Next add \((c_{j3}a_{j2} - b_{j3})/(a_{j3} - a_{j2})\) times the new second column to the third, so we can pull out \((c_{j3}a_{j3} - b_{j3})/(a_{j3} - a_{j2})\) and be left with \( 1/(x_i - a_{j3}) \), and so on. These operations reduce \( U \) to
\[
\prod_{j=1}^{n} \prod_{s=2}^{k_i} c_{js}a_{js} - b_{js} = \prod_{j=1}^{n} \prod_{s=2}^{k_i} a_{js} - a_{j(s-1)}
\]
times the determinant in Lemma 1. Hence we have the following result.

**Theorem 10.** If \( \omega = k_1 + \cdots + k_n \), then
\[
U_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_1+\cdots+k_n-n} \prod_{1 \leq i < j \leq \omega} (x_i - x_j) \times \prod_{i=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_i} a_{ir} - a_{js} \]
\[
\times \prod_{1 \leq i < j \leq \omega} (x_i - a_{jr}) \prod_{1 \leq r < s \leq k_j} (a_{jr} - a_{js}) \prod_{i \leq j \leq n} (a_{ir} - a_{js}).
\]
If we set all the \( c_{jk} = 1 \), trigonometric conversions of the last two results come out nicely after a tedious calculation. If \( \Theta_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \) denotes the determinant of order \( k_1 + \cdots + k_n \) where the \( \vec{a}_j \) columns have the form
\[
\begin{array}{cccc}
\sin(x_i - a_{j1}) & \sin(x_i - a_{j2}) & \sin(x_i - a_{j3}) & \cdots \\
\sin(x_i - a_{j1}) & \sin(x_i - a_{j2}) & \sin(x_i - a_{j3}) & \cdots \\
\end{array}
\]
and if \( \Phi_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \) denotes the determinant of order \( k_1 + \cdots + k_n \) where the \( \vec{a}_j \) columns have the form
\[
\begin{array}{cccc}
1 & \sin(x_i - b_{j1}) & \sin(x_i - b_{j2}) & \sin(x_i - b_{j3}) & \cdots \\
\sin(x_i - a_{j1}) & \sin(x_i - a_{j2}) & \sin(x_i - a_{j3}) & \cdots \\
\end{array}
\]
then we have the following evaluations.

**Corollary 4.** If \( \omega = k_1 + \cdots + k_n \), then
\[
\Theta_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_1+\cdots+k_n-n} \prod_{1 \leq i < j \leq n} (a_{ir} - a_{js}) \\
\prod_{1 \leq i < j \leq \omega} \sin(x_j - x_i) \prod_{j=1}^{n} \left( \prod_{k=2}^{k_j} \prod_{r=1}^{n} \sin(a_{jk} - b_{jk}) \prod_{2 \leq r < s \leq k_j} \sin(a_{jr} - a_{js}) \right) \\
\prod_{r=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_j} \sin(x_i - a_{jr})
\]
and
\[ \Phi_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_1 + \cdots + k_n - n} \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k_i} \prod_{s=1}^{k_j} \sin(a_{ir} - a_{js}) \]
\[ \prod_{1 \leq i < j \leq \omega} \sin(x_j - x_i) \prod_{i=1}^{n} \left( \prod_{j=1}^{b_i} \sin(a_{ik} - b_{jk}) \prod_{1 \leq r < s \leq k_j - 1} \sin(a_{jr} - a_{js}) \right) \]

We can also add numerators to Theorem 6. For example, we can add linear ones as in Theorems 9 and 10. Suppose the \( \vec{a}_j \) columns have the form
\[ \frac{1}{x_i - a_{j1}} \frac{c_{j2}x_i - b_{j2}}{(x_i - a_{j1})(x_i - a_{j2})} \cdots \frac{c_{jk}x_i - b_{jk}}{(x_i - a_{j1})(x_i - a_{j2}) \cdots (x_i - a_{jk})} \]
for \( 1 \leq j \leq n \). We denote this determinant by \( V_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \), or \( V \) for short. If we subtract \( c_{j2} \) times the first column from the second in each set of columns, we have
\[ \frac{c_{j2}x_i - b_{j2}}{(x_i - a_{j1})(x_i - a_{j2})} \frac{c_{j2}}{x_i - a_{j1}} = \frac{c_{j2}a_{j2} - b_{j2}}{(x_i - a_{j1})(x_i - a_{j2})} \]
and we can pull out the factor \( c_{j2}a_{j2} - b_{j2} \), reducing the numerator of the second column to 1. In general, after reducing the numerator of the \( (k - 1) \)st column to 1 we subtract it \( c_{jk} \) times from the \( k \)th column. We have
\[ \frac{c_{jk}x_i - b_{jk}}{(x_i - a_{j1}) \cdots (x_i - a_{jk})} \frac{c_{jk}}{(x_i - a_{j1}) \cdots (x_i - a_{jk})} = \frac{c_{jk}a_{jk} - b_{jk}}{(x_i - a_{j1}) \cdots (x_i - a_{jk})} \]
and \( c_{jk}a_{jk} - b_{jk} \) comes out of the \( k \)th column. This ultimately reduces \( V \) to the determinant \( S \) in Theorem 6, so we have the following result.

**Theorem 11.** If \( \omega = k_1 + \cdots + k_n \), then
\[ V_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{k_j + \cdots + k_s} \prod_{1 \leq i < j \leq \omega} \sin(x_j - x_i) \prod_{i=1}^{n} \left( \prod_{j=1}^{b_i} \sin(a_{ik} - b_{jk}) \prod_{1 \leq r < s \leq k_j - 1} \sin(a_{jr} - a_{js}) \right) \]

Another nice example is to give the \( \vec{a}_j \) columns the form
\[ \frac{1}{x_i - a_{j1}} \frac{c_{j1}x_i - b_{j1}}{(x_i - a_{j1})(x_i - a_{j2})} \cdots \frac{c_{j(s-1)}x_i - b_{j(s-1)}}{(x_i - a_{j1})(x_i - a_{j2}) \cdots (x_i - a_{js})} \]
for \( 1 \leq j \leq n \). We denote this determinant by \( W_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) \).
If we subtract \( c_{jk} \) times the \( k \)th column from the \( (k + 1) \)st column for each \( j \) and \( k \), working from right to left within each set of columns, we have
\[ \frac{(c_{j1}x_i - b_{j1})(c_{j2}x_i - b_{j2}) \cdots (c_{j(s-1)}x_i - b_{j(s-1)})}{(x_i - a_{j1})(x_i - a_{j2}) \cdots (x_i - a_{js})} \]
\[ - \frac{c_{j(s-1)}(c_{j1}x_i - b_{j1})(c_{j2}x_i - b_{j2}) \cdots (c_{j(s-2)}x_i - b_{j(s-2)})}{(x_i - a_{j1})(x_i - a_{j2}) \cdots (x_i - a_{j(s-1)})} \]
\[ = \frac{(c_{j(s-1)}a_{js} - b_{j(s-1)})(c_{j1}x_i - b_{j1})(c_{j2}x_i - b_{j2}) \cdots (c_{j(s-2)}x_i - b_{j(s-2)})}{(x_i - a_{j1})(x_i - a_{j2}) \cdots (x_i - a_{js})} \]
and we can pull out the first numerator factor. Continuing in this way we can reduce all the numerators to 1, and the factors that come out are
\[ \prod_{j=1}^{n} \prod_{1 \leq r < s \leq k_j} (c_{jr}a_{js} - b_{jr}). \]

Theorem 6 then implies the following result.

**Theorem 12.** If \( \omega = k_1 + \cdots + k_n \), then
\[
W_{k_1, \ldots, k_n}(\vec{x}; \vec{a}_1, \ldots, \vec{a}_n) = (-1)^{\binom{k_1}{2} + \cdots + \binom{k_n}{2}}
\times \frac{\prod_{1 \leq i < j \leq \omega} (x_j - x_i) \prod_{j=1}^{n} \prod_{1 \leq r < s \leq k_j} (c_{jr}a_{js} - b_{jr}) \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k_i} \prod_{s=1}^{k_j} (a_{ir} - a_{js})}{\prod_{i=1}^{\omega} \prod_{j=1}^{n} \prod_{r=1}^{k_j} (x_i - a_{jr})}.
\]

In principle, we can use any polynomials \( P_{jk}(x) \) of degree \( k \) or less in place of
\[
(c_{j1}x - b_{j1})(c_{j2}x - b_{j2}) \cdots (c_{jk}x - b_{jk})
\]
and eliminate the numerators by a partial fractions argument as in the proof of Theorem 10. If we set all the \( c_{jk} = 1 \), a trigonometric conversion of Theorem 12 succeeds after an unpleasant calculation.

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**References**


