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Symmetry analysis of an integrable Ito coupled system

Hassan A. Zedan [1]

Department of Mathematics, Faculty of Science, Kafr El-Sheikh University, Egypt

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ABSTRACT

In this paper, we study the invariance analysis, integrability properties and P-property of the Ito coupled nonlinear partial differential equations. We explore several new solutions for the Ito system through the Lie symmetry analysis. Moreover, this work has been devoted to study the integrability aspects of the Ito system through higher order symmetries. We are also investigating the existence of higher order symmetries for the Ito system. Interestingly our investigations reveal a rich variety of particular solutions, which have not been reported in the literature, for this model.

1. Introduction

For the past two decades, the Lie group method has been applied to solve a wide range of problems and to explore many physically interesting solutions of nonlinear phenomena [1–4]. In recent years several extensions and modifications of the classical Lie algorithm have been proposed in order to arrive at new solutions of partial differential equations (PDEs) [5].

Lie symmetry analysis is one of the most powerful methods to obtain particular solutions of differential equations [6]. It is based on the study of their invariance with respect to one-parameter Lie group of point transformations whose infinitesimal generators are represented as vector fields. Once the Lie groups that leave the differential equations invariant are known, we can construct an exact solution called a group invariant solution which is invariant under the transformation.

In this paper, we investigate the invariance analysis and the Painleve analysis to the following Nonlinear Ito coupled system

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2(v_{xxx} + 3uv_x + 3vu_x) - 12ww_x, \\ w_t &= w_{xxx} + 3uw_x. \end{aligned} \tag{1.1}$$

Let us consider a one-parameter Lie group of infinitesimal transformations of the form:

$$\begin{aligned} U &= u + \varepsilon\eta_1(x, t, u, v, w), \\ V &= v + \varepsilon\eta_2(x, t, u, v, w), \\ W &= w + \varepsilon\eta_3(x, t, u, v, w), \\ X &= x + \varepsilon\zeta_1(x, t, u, v, w), \\ T &= t + \varepsilon\zeta_2(x, t, u, v, w), \quad \varepsilon \ll 1. \end{aligned} \tag{1.2}$$

The functions $\eta_1, \eta_2, \eta_3, \zeta_1$ and ζ_2 are the infinitesimal of transformations for the variables u, v, w, x and t respectively. In order to find the infinitesimal we need to extend the group to calculate how derivative terms transform. The transformation

E-mail address: hassanzedan2003@yahoo.com.

[1] Current Address: Math. Dept., Faculty of Science, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

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(1.2), together with the transformations for the first, second, ..., derivatives, are called the first, second, ..., extensions. We denote the infinitesimal for u_t, v_t, w_t, u_x, v_x, w_x, u_y, v_y, w_y, u_{tt} by $\eta_{1t}^{(1)}, \eta_{2t}^{(1)}, \eta_{3t}^{(1)}, \eta_{1x}^{(1)}, \eta_{2x}^{(1)}, \eta_{3x}^{(1)}, \eta_{1y}^{(1)}, \eta_{2y}^{(1)}, \eta_{2y}^{(1)}, \eta_{1tt}^{(2)}$, respectively. Using these various extensions, the infinitesimal criterion for the invariance of (1.1) under group (1.2) admits an infinitesimal generator of the form:

$$\begin{aligned}
\eta_{1t}^{(1)} &= \eta_{2x}^{(1)},\\
\eta_{2t}^{(1)} &= -2(\eta_{2xxx}^{(3)} + 3\eta_1 v_x + 3\eta_{2x}^{(1)} u + 3\eta_2 u_x + 3\eta_{1x}^{(1)} v) - 12\eta_3 w_x - 12\eta_{3x}^{(1)} w,\\
\eta_{3t}^{(1)} &= \eta_{3xxx}^{(3)} + 3\eta_1 w_x + 3\eta_{3x}^{(1)} u
\end{aligned} \tag{1.3}$$

where $\eta_{1t}^{(1)}, \eta_{2t}^{(1)}, \eta_{3t}^{(1)}, \eta_{1x}^{(1)}, \eta_{2x}^{(1)}, \eta_{3x}^{(1)}, \eta_{1y}^{(1)}, \eta_{2y}^{(1)}, \eta_{2y}^{(1)}, \eta_{1tt}^{(2)}$ are extending infinitesimal of transformations given by [1–4]. The invariance of Eq. (1.1) under the infinitesimal transformations (1.2) leads to [9]:

$$\eta_1 = -2au, \qquad \eta_2 = -4av, \qquad \eta_3 = -3aw, \qquad \zeta_1 = ax + b, \qquad \zeta_2 = 3at + c \tag{1.4}$$

where a, b and c are arbitrary constants.

The associated Lie vector fields are

$$V_1 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v} - 3w\frac{\partial}{\partial w}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = \frac{\partial}{\partial t}.$$

The non-zero commutation relation between the vector fields are

$$[V_1, V_2] = -V_2, \qquad [V_1, V_2] = -3V_3.$$

Solving the characteristic equation associated with the infinitesimal symmetries (1.4) one obtains

$$\begin{aligned}
z &= \frac{ax+b}{(3at+c)^{\frac{1}{3}}},\\
w_1 &= (3at+c)^{\frac{2}{3}} u,\\
w_2 &= (3at+c)^{\frac{4}{3}} v,\\
w_3 &= (3at+c) w.
\end{aligned} \tag{1.5}$$

We wish to note that while deriving the above similarity reductions under similarity transformation (1.5), one can reduce the system of PDEs (1.1) to the system of ODEs of the form:

$$\begin{aligned}
&2w_1 + zw_1' + w_2' = 0,\\
&-4w_2 - zw_2' + 2a^2 w_2''' + 6w_1 w_2' + 6w_2 w_1' + 12w_3 w_3' = 0,\\
&3w_3 + zw_3' + a^2 w_3''' - 3w_1 w_3' = 0
\end{aligned} \tag{1.6}$$

where prime denotes differentiation with respect to z.

2. Painleve (P) analysis

In order to verify whether the reduced system of ODEs (1.6) is integrable or not we apply Ablowitz–Ramani–Segur (ARS) algorithm [6–8] to the system of ODEs (1.6). Since the independent variable appeared explicitly in the above system of ODEs (1.6), first let us rewrite Eq. (1.6) of the form:

$$\begin{aligned}
&2w_1 + \tau w_1' + z_0 w_1' + w_2' = 0,\\
&-4w_2 - \tau w_2' - z_0 w_2' + 2a^2 w_2''' + 6w_1 w_2' + 6w_2 w_1' + 12w_3 w_3' = 0,\\
&3w_3 + \tau w_3' + z_0 w_3' + a^2 w_3''' - 3w_1 w_3' = 0
\end{aligned} \tag{2.1}$$

where $\tau = z - z_0$ and z_0 is a movable singular point.

Now let us represent the solution to the system of ODEs (2.1) locally as a Laurent series and let the leading order be of the form

$$w_1 = a_0 \tau^{\alpha}, \qquad w_2 = b_0 \tau^{\beta}, \qquad w_3 = c_0 \tau^{\gamma} \tag{2.2}$$

where a_0, b_0 and c_0 are arbitrary constants and α, β and γ are integers to be determined.

Substituting (2.2) into the system of ODEs (2.1) and balancing the dominant terms, we obtain, $\alpha = \beta, \beta - 3 = \beta + \alpha - 1$ and $\beta - 3 = 2\gamma - 1$. Then $\alpha = \beta = \gamma = -2$. Now let us consider the Laurent expansion of the form

$$w_1 = a_0 \tau^{-2} + \beta_1 \tau^{r-2}, \qquad w_2 = b_0 \tau^{-2} + \beta_2 \tau^{r-2}, \qquad w_3 = c_0 \tau^{-2} + \beta_3 \tau^{r-2}. \tag{2.3}$$

Substituting the system of equations (2.3) into the system of ODEs (2.1) and balancing the most singular terms again, we obtain $r = -1, 2, 4, 6$. Let us assume the Laurent series of the form

$$w_1 = \sum_{j=0}^{6} a_j \tau^{j-2}, \qquad w_2 = \sum_{j=0}^{6} b_j \tau^{j-2}, \qquad w_3 = \sum_{j=0}^{6} c_j \tau^{j-2}. \tag{2.4}$$

Substituting the system of equations (2.4) into the system of ODEs (2.1) and equating various powers of τ^n and solving the resultant equations, then we obtain

$$\Big\{\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = \beta_0 = \beta_1 = \beta_5 = \beta_6 = \gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0,$$
$$\alpha_2 = \frac{1}{3}z_0, \alpha_3 = \frac{1}{3}, \beta_2 = -\frac{1}{2}z_0^2, \beta_3 = -z_0, \beta_4 = -\frac{1}{2}\Big\}.$$

The Laurent series solution (2.4) is meromorphic ; consequently the similarity reduced system of ODEs (2.1) also possesses the Painleve property. Even though the system of ODEs (1.6) is integrable in general it is very difficult to integrate it explicitly and obtain a general solution. However, one can obtain a number of physically interesting solutions by considering certain special choices of the infinitesimal symmetries (1.2) which we present, few of them, in the following sections.

Then we obtain the solution of the system of ODEs

$$w_1 = \frac{1}{3}z, \qquad w_2 = -\frac{1}{2}z^2, \qquad w_3 = 0$$

and hence the solution of the system of PDEs (1.1) are

$$u = \frac{(ax+b)}{3(3at+c)}, \qquad v = -\frac{(a^2x^2 + 2abx + b^2)}{2(3at+c)^2}, \qquad w = 0. \tag{2.5}$$

Generic subcases: Let us recall that in the derivation of the general similarity reductions we made an assumption that $a \neq 0$ and $c \neq 0$. Now let us consider another cases and the possible similarity reductions.

2.1. Case 1: $a = c = 0$, b is arbitrary

The similarity variables take the form:

$$z = t,$$
$$w_1 = u,$$
$$w_2 = v,$$
$$w_3 = w. \tag{2.6}$$

The similarity reduced system of ODEs turns out to be

$$w_1' = 0, \qquad w_2' = 0, \qquad w_3' = 0.$$

Thus the solution obtained will be stationary.

2.2. Case 2: $a = b = 0$, c is arbitrary

The similarity variables take the form:

$$z = x,$$
$$w_1 = u,$$
$$w_2 = v,$$
$$w_3 = w. \tag{2.7}$$

The similarity reduced system of ODEs turns out to be

$$w_2' = 0,$$
$$2w_2''' + 6w_1w_2' + 6w_2w_1' + 12w_3w_3' = 0,$$
$$w_3''' - 3w_1w_3' = 0. \tag{2.8}$$

From the first equation of the system of ODEs (2.8) we obtain $w_2 = k_1$ and from the second one we obtain $w_1 = \frac{k_2 - 6w_3^2}{6k_1}$. When we substitute w_1 into the third equation and integrate it twice we obtain

$$w_3' = \sqrt{k_4 + \frac{k_3}{k_1}w_3 + \frac{k_2}{2k_1}w_3^2 - \frac{1}{2k_1}w_3^4}.$$

2.2.1. Subcase 1: $k_2 = k_3 = 0$, k_1 and k_4 are arbitrary

we obtain

$$w = \frac{\sqrt{2}sn\left(\frac{\sqrt{2\sqrt{2k_1k_4}}x}{2\sqrt{k_1}}, I\right)\sqrt{\sqrt{k_1k_4}}}{\sqrt{\sqrt{2}}},$$

and

$$u = -\frac{\sqrt{2}sn\left(\frac{\sqrt{2\sqrt{2k_1k_4}}x}{2\sqrt{k_1}}, I\right)\sqrt{k_4}}{\sqrt{k_1}},$$
$$v = k_{1.}$$

2.2.2. Subcase 2: $k_3 = k_4 = 0$, k_1 and k_2 are arbitrary

we obtain

$$w = \sqrt{\frac{2k_1k_2 - x^2}{2k_1}},$$

and

$$u = \frac{-5k_1k_2 + 3x^2}{6k_1^2},$$
$$v = k_{1.}$$

2.2.3. Subcase 3: $k_3 = 0$, k_1, k_2 and k_4 are arbitrary

we obtain

$$w = \frac{2sn\left(A_1x,\ \frac{1}{2}\sqrt{-\frac{4k_1k_4+k_2^2+k_2A_2}{k_1k_4}}\right)}{\sqrt{-\frac{k_2-A_2}{k_1k_4}}},$$

and

$$u = \frac{k_2}{2k_1} + \frac{12sn\left(A_1x,\ \frac{1}{2}\sqrt{-\frac{4k_1k_4+k_2^2+k_2A_2}{k_1k_4}}\right)^2 k_4}{k_2 - A_2},$$
$$v = k_{1.}$$

where $A_1 = \frac{\sqrt{-k_1(k_2-A_2)}}{2\sqrt{k_1}}$ and $A_2 = \sqrt{8k_1k_4 + k_2^2}$.

If we use the Painleve analysis we obtain

$$u = \frac{2}{x_2}, \qquad v = -\frac{1}{2}c_0^2, \qquad w = \frac{c_0}{x}. \tag{2.9}$$

2.3. Case 3: $b = c = 0$, a is arbitrary

The similarity variables take the form:

$$z = \frac{x}{t^{\frac{1}{3}}},$$
$$w_1 = t^{\frac{2}{3}}u,$$
$$w_2 = t^{\frac{4}{3}}v,$$
$$w_3 = tw. \tag{2.10}$$

The similarity reduced system of ODEs turns out to be

$$\begin{aligned}&2w_1 + zw_1' + 3w_2' = 0,\\ &-4w_2 - zw_2' + 6w_2''' + 18w_1w_2' + 18w_2w_1' + 36w_3w_3' = 0,\\ &3w_3 + zw_3' + 3w_3''' - 9w_1w_3' = 0\end{aligned} \tag{2.11}$$

where prime denotes differentiation with respect to z.

In order to verify whether the reduced system of ODEs (2.9) is integrable or not we apply Ablowitz–Ramani–Segur (ARS) algorithm [6–8] to the system of ODEs (2.9). Since the independent variable appeared explicitly in the above system of ODEs (2.9) first let us rewrite the system of ODEs (2.9) of the form:

$$\begin{aligned}&2w_1 + \tau w_1' + z_0w_1' + 3w_2' = 0,\\ &-4w_2 - \tau w_2' - z_0w_2' + 6w_2''' + 18w_1w_2' + 18w_2w_1' + 36w_3w_3' = 0,\\ &3w_3 + \tau w_3' + z_0w_3' + 3w_3''' - 9w_1w_3' = 0\end{aligned} \tag{2.12}$$

where $\tau = z - z_0$ and z_0 is a movable singular point.

Then we obtain

$$w_1 = \frac{1}{9}z, \qquad w_2 = -\frac{1}{18}z^2, \qquad w_3 = 0$$

and hence the solution of the system of PDEs (1.1) are

$$u = \frac{x}{9t}, \qquad v = -\frac{x^2}{18t^2}, \qquad w = 0. \tag{2.13}$$

2.4. Case 4: $a = 0$, b and c are arbitrary (leads travelling wave solution)

The similarity variables take the form:

$$\begin{aligned}&z = cx - bt,\\ &w_1 = u,\\ &w_2 = v,\\ &w_3 = w.\end{aligned} \tag{2.14}$$

The similarity reduced system of ODEs turns out to be

$$\begin{aligned}&bw_1' + cw_2' = 0,\\ &-bw_2' + 2c^3w_2''' + 6cw_1w_2' + 6cw_2w_1' + 12cw_3w_3' = 0,\\ &bw_3' + c^3w_3''' - 3cw_1w_3' = 0\end{aligned} \tag{2.15}$$

where prime denotes differentiation with respect to z.

Let us assume the Laurent series of the form

$$w_1 = \sum_{j=0}^{6} a_j\tau^{j-2}, \qquad w_2 = \sum_{j=0}^{6} b_j\tau^{j-2}, \qquad w_3 = \sum_{j=0}^{6} c_j\tau^{j-2}. \tag{2.16}$$

Then, we obtain the solution of the system of PDEs (1.1) are

$$u = -\frac{2c^3}{(cx - bt)^2} + a_2, \qquad v = \frac{2bc}{(cx - bt)^2} - \frac{b(-6ca_2 + b)}{6c^2}, \qquad w = c_2 \tag{2.17}$$

or

$$u = \frac{4c^3}{(cx - bt)^2} + \frac{b}{3c}, \qquad v = -\frac{4bc}{(cx - bt)^2} - \frac{b^2 - 6c_2\sqrt{6bcc}}{6c^2}, \qquad w = \frac{2\sqrt{6bcc}}{(cx - bt)^2} + c_2. \tag{2.18}$$

2.5. Case 5: $b = 0$, a and c are arbitrary

The similarity variables and similarity functions take the form

$$\begin{aligned} z &= \frac{ax}{(3at+c)^{\frac{1}{3}}}, \\ w_1 &= (3at+c)^{\frac{2}{3}} u, \\ w_2 &= (3at+c)^{\frac{4}{3}} v, \\ w_3 &= (3at+c) w. \end{aligned} \tag{2.19}$$

We wish to note that while deriving the above similarity reductions under similarity transformation (2.19), one can reduce the system of PDEs (1.1) to the system of ODEs of the form:

$$\begin{aligned} &2w_1 + zw_1' + w_2' = 0, \\ &-4w_2 - zw_2' + 2a^2 w_2''' + 6w_1 w_2' + 6w_2 w_1' + 12 w_3 w_3' = 0, \\ &3w_3 + zw_3' + a^2 w_3''' - 3w_1 w_3' = 0 \end{aligned} \tag{2.20}$$

where prime denotes differentiation with respect to z.

In order to verify whether the reduced system of ODEs (2.20) is integrable or not we apply Ablowitz–Ramani–Segur (ARS) algorithm [6–8] to the system of ODEs (2.20). Since the independent variable appeared explicitly in the above system of ODEs (2.20) first let us rewrite the system of ODEs (2.20) of the form:

$$\begin{aligned} &2w_1 + \tau w_1' + z_0 w_1' + w_2' = 0, \\ &-4w_2 - \tau w_2' - z_0 w_2' + 2a^2 w_2''' + 6w_1 w_2' + 6w_2 w_1' + 12 w_3 w_3' = 0, \\ &3w_3 + \tau w_3' + z_0 w_3' + a^2 w_3''' - 3w_1 w_3' = 0 \end{aligned} \tag{2.21}$$

where $\tau = z - z_0$ and z_0 is a movable singular point.

Let us assume the Laurent series of the form

$$w_1 = \sum_{j=0}^{6} a_j \tau^{j-2}, \qquad w_2 = \sum_{j=0}^{6} b_j \tau^{j-2}, \qquad w_3 = \sum_{j=0}^{6} c_j \tau^{j-2}. \tag{2.22}$$

Then we obtain

$$w_1 = \frac{1}{3} z, \qquad w_2 = -\frac{1}{2} z^2, \qquad w_3 = 0$$

and hence the solution of the system of PDEs (1.1) are

$$u = \frac{ax}{3(3at+c)}, \qquad v = -\frac{a^2 x^2}{2(3at+c)^2}, \qquad w = 0. \tag{2.23}$$

2.6. Case 6: $c = 0$, a and b are arbitrary

$$\begin{aligned} z &= \frac{ax+b}{t^{\frac{1}{3}}}, \\ w_1 &= t^{\frac{2}{3}} u, \\ w_2 &= t^{\frac{4}{3}} v, \\ w_3 &= t w. \end{aligned} \tag{2.24}$$

We wish to note that while deriving the above similarity reductions under similarity transformation (2.24) one can reduce the system of PDEs (1.1) to the system of ODEs of the form:

$$\begin{aligned} &2w_1 + zw_1' + 3w_2' = 0, \\ &-4w_2 - zw_2' + 6a^2 w_2''' + 18 w_1 w_2' + 18 w_2 w_1' + 36 w_3 w_3' = 0, \\ &3w_3 + zw_3' + a^2 w_3''' - 3w_1 w_3' = 0 \end{aligned} \tag{2.25}$$

where prime denotes differentiation with respect to z.

In order to verify whether the reduced system of ODEs (2.25) is integrable or not we apply Ablowitz–Ramani–Segur (ARS) algorithm [6–8] to the system of ODEs (2.25). Since the independent variable appeared explicitly in the above system of ODEs (2.25) then we can rewrite the system of ODEs (2.25) in the form:

$$\begin{aligned}&2w_1+\tau w_1'+z_0w_1'+3w_2'=0,\\&-4w_2-\tau w_2'-z_0w_2'+6a^2w_2'''+18w_1w_2'+18w_2w_1'+36w_3w_3'=0,\\&3w_3+\tau w_3'+z_0w_3'+a^2w_3'''-3w_1w_3'=0\end{aligned}\tag{2.26}$$

where $\tau=z-z_0$ and z_0 is a movable singular point.

Let us assume the Laurent series of the form

$$w_1=\sum_{j=0}^{6}a_j\tau^{j-2},\qquad w_2=\sum_{j=0}^{6}b_j\tau^{j-2},\qquad w_3=\sum_{j=0}^{6}c_j\tau^{j-2}.\tag{2.27}$$

Then we obtain

$$w_1=\frac{1}{9}z,\qquad w_2=-\frac{1}{18}z^2,\qquad w_3=0$$

and hence the solution of the system of PDEs (1.1) are

$$u=\frac{ax+b}{9t},\qquad v=-\frac{(ax+b)^2}{18t^2},\qquad w=0.\tag{2.28}$$

3. Painleve (P) analysis for the original PDE:

Methodology: Consider a system of M polynomial differential equations

$$F_i(u(z),u'(z),u''(z),\ldots,u^{(m_i)}(z))=0,\quad i=1,2,\ldots,M.\tag{3.1}$$

Step 1 (Determine the dominant behavior). It is sufficient to substitute

$$u_i(z)=x_ig^{\alpha_i}(z),\quad i=1,2,\ldots,M,$$

where x_i is a constant, into (3.1) to determine the leading exponents α_i. In the resulting polynomial system, equating every two or more possible lowest exponents of $g(z)$ in each equation gives a linear system for α_i. The linear system is then solved for α_i, and each solution branch is investigated. The traditional Painleve test requires that all the α_i's are integers and that at least one is negative. An alternative approach is to use the "weak" Painleve test, which allows certain rational α_i's and resonances; see [10–12] for more information on the weak Painleve test.

If one or more exponents α_i remain undetermined, we assign integer values to the free α_i so that every equation in (3.1) has at least two different terms with equal lowest exponents.

For each solution α_i we substitute

$$u_i(z)=u_{i,0}(z)g^{\alpha_i}(z),\quad i=1,2,\ldots,M,$$

into (3.1). We then solve the (typically) nonlinear equation for $u_{i,0}(z)$, which is found by balancing the leading terms. By leading terms, we mean those terms with the lowest exponent of $g(z)$.

Step 2 (Determine the resonances). For each α_i and $u_{i,0}(z)$, we calculate the $r_1\le r_2\le r_3\le\cdots\le r_m$ for which $u_{i,0}(z)$ is an arbitrary function in

$$u_i(z)=g^{\alpha_i}(z)\sum_{k=0}^{\infty}u_{i,k}(z)g^k(z),\quad i=1,2,\ldots,M.$$

To do this, we substitute $u_i(z)=u_{i,0}(z)g^{\alpha_i}(z)+u_{i,r}(z)g^{\alpha_i+r}(z)$ into (3.1), and keep only the lowest order terms in $g(z)$ that are linear in $u_{i,r}$. This is carried out by computing the solutions for r of $\det(Q_r)=0$, where the $M\times M$ matrix Q_r satisfies $Q_ru_r=0$, $u_r=(u_{1,r}\ u_{2,r}\ u_{3,r}\cdots u_{\mathrm{M},r})^T$.

If any of the resonances are non-integer, then the Laurent series solutions of (3.1) have a movable algebraic branch point and the algorithm terminates. If r_m is not a positive integer, then the algorithm terminates.

Step 3 (Find the constants of integration and check compatibility conditions). For the system to possess the Painleve property, the arbitrariness of $u_{i,r}(z)$ must be verified up to the highest resonance level. This is carried out by substituting

$$u_i(z)=g^{\alpha_i}(z)\sum_{k=0}^{r_m}u_{i,k}(z)g^k(z)$$

into (3.1), where r_m is the largest positive integer resonance. To simplify step 3, we can use Weiss–Kruskal simplification.

The manifold defined by $g(z) = 0$ is noncharacteristic, that means $g_{zl}(z) \neq 0$ for some l on the manifold $g(z) = 0$. By the implicit function theorem, we can then locally solve $g(z) = 0$ for z_l, so that

$$g(z) = z_l - h(z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_N)$$

for some arbitrary function h.

Now we apply the method into our problem (the Ito coupled system).

Let

$$u = u_0(\varphi(x,t))^{\alpha_1}, \qquad v = v_0(\varphi(x,t))^{\alpha_2}, \qquad w = w_0(\varphi(x,t))^{\alpha_3} \tag{3.2}$$

where u_0, v_0 and w_0 are functions in x, t then system (1.1) becomes

$$\frac{\partial u_0}{\partial t}(\varphi(x,t))^{\alpha_1} + \alpha_1 u_0(\varphi(x,t))^{\alpha_1-1}\frac{\partial \varphi(x,t)}{\partial t} = \frac{\partial v_0}{\partial x}(\varphi(x,t))^{\alpha_2} + \alpha_2 v_0(\varphi(x,t))^{\alpha_2-1}\frac{\partial \varphi(x,t)}{\partial x}, \tag{3.3}$$

$$\begin{aligned}
&\frac{\partial v_0}{\partial t}(\varphi(x,t))^{\alpha_2} + \alpha_2 v_0(\varphi(x,t))^{\alpha_2-1}\frac{\partial \varphi(x,t)}{\partial t}\\
&= -2\frac{\partial^3 v_0}{\partial x^3}(\varphi(x,t))^{\alpha_2} - 6\alpha_2\frac{\partial^2 v_0}{\partial x^2}(\varphi(x,t))^{\alpha_2-1}\frac{\partial \varphi(x,t)}{\partial x} - 6\alpha_2^2\frac{\partial v_0}{\partial x}(\varphi(x,t))^{\alpha_2-2}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^2\\
&\quad - 6\alpha_2\frac{\partial v_0}{\partial x}(\varphi(x,t))^{\alpha_2-1}\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right) + 6\alpha_2\frac{\partial v_0}{\partial x}(\varphi(x,t))^{\alpha_2-2}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^2\\
&\quad - 2v_0(\varphi(x,t))^{\alpha_2-3}\alpha_2^3\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^3 - 6v_0(\varphi(x,t))^{\alpha_2-2}\alpha_2^2\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right)\\
&\quad + 6v_0(\varphi(x,t))^{\alpha_2-3}\alpha_2^2\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^3 - 2v_0(\varphi(x,t))^{\alpha_2-1}\alpha_2\left(\frac{\partial^3 \varphi(x,t)}{\partial x^3}\right)\\
&\quad + 6v_0(\varphi(x,t))^{\alpha_2-2}\alpha_2\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right) - 4v_0(\varphi(x,t))^{\alpha_2-3}\alpha_2\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^3\\
&\quad - 6\frac{\partial u_0}{\partial x}(\varphi(x,t))^{\alpha_1+\alpha_2}v_0 - 6u_0(\varphi(x,t))^{\alpha_1+\alpha_2-1}\alpha_1\left(\frac{\partial \varphi(x,t)}{\partial x}\right)v_0\\
&\quad - 6u_0(\varphi(x,t))^{\alpha_1+\alpha_2}\frac{\partial v_0}{\partial x} - 6u_0(\varphi(x,t))^{\alpha_1+\alpha_2-1}v_0\alpha_2\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\\
&\quad - 12w_0(\varphi(x,t))^{2\alpha_3}\frac{\partial w_0}{\partial x} - 12w_0^2(\varphi(x,t))^{2\alpha_3-1}\alpha_3\left(\frac{\partial \varphi(x,t)}{\partial x}\right)
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\left(\frac{\partial w_0(x,t)}{\partial t}\right)(\varphi(x,t))^{\alpha_3} + \alpha_3 w_0(x,t)(\varphi(x,t))^{\alpha_3-1}\left(\frac{\partial \varphi(x,t)}{\partial t}\right)\\
&= \left(\frac{\partial^3 w_0(x,t)}{\partial x^3}\right)(\varphi(x,t))^{\alpha_3} + 3\alpha_3\left(\frac{\partial^2 w_0(x,t)}{\partial x^2}\right)(\varphi(x,t))^{\alpha_3-1}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\\
&\quad + 3\alpha_3^2\left(\frac{\partial w_0(x,t)}{\partial x}\right)(\varphi(x,t))^{\alpha_3-2}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^2 + 3\alpha_3\left(\frac{\partial w_0(x,t)}{\partial x}\right)(\varphi(x,t))^{\alpha_3-1}\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right)\\
&\quad - 3\alpha_3\left(\frac{\partial w_0(x,t)}{\partial x}\right)(\varphi(x,t))^{\alpha_3-2}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^2 + \alpha_3^3 w_0(x,t)(\varphi(x,t))^{\alpha_3-3}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^3\\
&\quad + 3\alpha_3^2 w_0(x,t)(\varphi(x,t))^{\alpha_3-2}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right) - 3\alpha_3^2 w_0(x,t)(\varphi(x,t))^{\alpha_3-3}\left(\frac{\partial \varphi(x,t)}{\partial x}\right)^3\\
&\quad + \alpha_3 w_0(x,t)(\varphi(x,t))^{\alpha_3-1}\left(\frac{\partial^3 \varphi(x,t)}{\partial x^3}\right) - 3\alpha_3 w_0(x,t)(\varphi(x,t))^{\alpha_3-2}\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right)\left(\frac{\partial \varphi(x,t)}{\partial x}\right)\\
&\quad + 2\alpha_3 w_0(x,t)(\varphi(x,t))^{\alpha_3-3}\left(\frac{\partial \varphi(x,t)}{\partial}\right)^3 - 3u_0(x,t)(\varphi(x,t))^{\alpha_1+\alpha_3}\left(\frac{\partial w_0(x,t)}{\partial x}\right)\\
&\quad - 3\alpha_3 u_0(x,t)w_0(x,t)(\varphi(x,t))^{\alpha_1+\alpha_3-1}\left(\frac{\partial \varphi(x,t)}{\partial x}\right).
\end{aligned} \tag{3.5}$$

From Eq. (3.3) we obtain $\alpha_1 = \alpha_2$ and from Eq. (3.4) we obtain $\alpha_2 - 3 = \alpha_2 + \alpha_1 - 1$. Hence $\alpha_1 = \alpha_2 = -2$.

From Eq. (3.4) we obtain $\alpha_2 - 3 = 2\alpha_3 - 1$. Hence $\alpha_3 = -2$.

Substituting $\alpha_1 = \alpha_2 = \alpha_3 = -2$ into Eqs. (3.3)–(3.5) and then requiring the leading terms (of $(\varphi(x,t))^{-5}$ in Eq. (3.5)) balance, give $u_0 = -4\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2$, from leading terms (of $(\varphi(x,t))^{-3}$ in Eq. (3.3)) balance and substituting about u_0, gives $v_0 = -4\left(\frac{\partial\varphi(x,t)}{\partial x}\right)\left(\frac{\partial\varphi(x,t)}{\partial t}\right)$ and from leading terms (of $(\varphi(x,t))^{-5}$ in Eq. (3.3)) balance and substituting about u_0 and v_0, gives $w_0 = \pm 2\sqrt{2}I\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^{\frac{3}{2}}\sqrt{\frac{\partial\varphi(x,t)}{\partial t}}$.

We have two cases first when $w_0 = 2\sqrt{2}I\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^{\frac{3}{2}}\sqrt{\frac{\partial\varphi(x,t)}{\partial t}}$ and the second one when $w_0 = -2\sqrt{2}I\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^{\frac{3}{2}}\sqrt{\frac{\partial\varphi(x,t)}{\partial t}}$ we will study these two cases.

Substituting

$$u(x,t) = -4\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2(\varphi(x,t))^{-2} + u_r(x,t)(\varphi(x,t))^{r-2},$$
$$v(x,t) = -4\left(\frac{\partial\varphi(x,t)}{\partial x}\right)\left(\frac{\partial\varphi(x,t)}{\partial t}\right)(\varphi(x,t))^{-2} + v_r(x,t)(\varphi(x,t))^{r-2}$$

and

$$w(x,t) = \pm 2\sqrt{2}I\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^{\frac{3}{2}}\sqrt{\frac{\partial\varphi(x,t)}{\partial t}}(\varphi(x,t))^{-2} + w_r(x,t)(\varphi(x,t))^{r-2}$$

into (1.1) and equating the coefficient of $(\varphi(x,t))^{r-5}$ in Eq. (3.4) we obtain the following characteristic equation for the resonances

$$-8(-8+r)(-6+r)(-4+r)(-3+r)(-2+r)(1+r)(2+r)\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^{\frac{15}{2}}\left(\frac{\partial\varphi(x,t)}{\partial t}\right)^4 = 0.$$

Assuming $\left(\frac{\partial\varphi(x,t)}{\partial x}\right) \neq 0$ and $\left(\frac{\partial\varphi(x,t)}{\partial t}\right) \neq 0$, then

$$r=-2, \quad r=-1, \quad r=2, \quad r=3, \quad r=4, \quad r=6, \quad r=8.$$

We now substitute

$$u = (\varphi(x,t))^{-2}\sum_0^8 u_k(x,t)(\varphi(x,t))^k,$$
$$v = (\varphi(x,t))^{-2}\sum_0^8 v_k(x,t)(\varphi(x,t))^k,$$
$$w = (\varphi(x,t))^{-2}\sum_0^8 w_k(x,t)(\varphi(x,t))^k \tag{3.6}$$

into (1.1) and use the Weiss–Kruskal simplification [13,14] (i.e. $\varphi(x,t) = x - h(t)$) we obtain

$$u_0 = -4, \quad v_0 = 4h_t, \quad w_0 = 2\sqrt{2}I\sqrt{-h'(t)}, \quad u_1 = 0, \quad v_1 = 0, \quad w_1 = 0,$$
$$u_2 = -\frac{1}{3}h'(t), \quad v_2 = c_1(x,t), \quad w_2 = \frac{I\sqrt{-h'(t)}((h'(t))^2 + 2v_2)}{2\sqrt{2}h'(t)},$$
$$u_3 = c_2(x,t), \quad v_3 = \frac{1}{3}(-h''(t) - 3h'(t)u_3), \quad w_3 = -\frac{I\sqrt{-h'(t)}(h''(t) + 12h'(t)u_3)}{6\sqrt{2}h'(t)}$$
$$u_4 = c_3(x,t), \quad v_4 = \frac{1}{2}(u_{3t} - 2h'(t)u_4), \quad w_4 = -\frac{I\sqrt{-h'(t)}u_4}{\sqrt{2}},$$
$$u_5 = \frac{1}{168(h'(t))^2}((h'(t))^2h''(t) + 40h'(t)u_{4t} - 2h'(t)v_{2t} + 30(h'(t))^3u_3 + 6h''(t)v_2 + 60h'(t)u_3v_2),$$
$$v_5 = -\frac{1}{168(h'(t))}((h'(t))^2h''(t) - 16h'(t)u_{4t} - 2h'(t)v_{2t} + 30(h'(t))^3u_3 + 6h''(t)v_2 + 60h'(t)u_3v_2),$$
$$w_5 = -\frac{1}{168\sqrt{2}(h'(t))^2}\Big(I\sqrt{-h'(t)}(5(h'(t))^2h''(t) + 32h'(t)u_{4t} + 4h'(t)v_{2t} + 24(h'(t))^3u_3 + 2h''(t)v_2$$
$$+ 48h'(t)u_3v_2)\Big),$$

$$u_6 = c_4(x,t), \qquad v_6 = -\frac{1}{672(h'(t))^3}(-h'(t))^3h'''(t) - 30(h'(t))^4u_{3t}$$
$$+40h'(t)h''(t)u_{4t} - 40(h'(t))^2u_{4tt} - 8h'(t)h''(t)v_{2t} + 2(h'(t))^2v_{2tt}$$
$$-30(h'(t))^3h''(t)u_3 - 60(h'(t))^2v_{2t}u_3 + 672(h'(t))^4u_6 + 12(h''(t))^2v_2$$
$$-6h'(t)h'''(t)v_2 - 60(h'(t))^2u_{3t}v_2 + 60h'(t)h''(t)u_3v_2,$$

$$w_6 = -\frac{1}{288\sqrt{2}(h'(t))^2}\Big(I\sqrt{-h'(t)}(h''(t))^2 - 2h'(t)h'''(t) - 24(h'(t))^2u_{3t} - 6h'(t)h''(t)u_3$$
$$+72(h'(t))^2u_3^2 + 288(h'(t))^2u_6\Big),$$

$$u_7 = \frac{1}{24}(u_{4t} - 12u_3u_4), \qquad v_7 = \frac{1}{120}(-5h'(t)u_{4t} + 24u_{6t} + 60h'(t)u_3u_4),$$

$$w_7 = -\frac{1}{10080\sqrt{2}(h'(t))^3}\Big(I\sqrt{-h'(t)}(-30(h'(t))^4h''(t) + 70(h'(t))^2u_{3tt} - 360(h'(t))^3u_{4t} + 1344(h'(t))^2u_{6t}$$
$$-45(h'(t))^3v_{2t} + 45(h'(t))^5u_3 + 1260(h'(t))^2u_{3t}u_3 - 560(h'(t))^2h''(t)u_4 - 1680(h'(t))^3u_3u_4$$
$$-30(h'(t))^2h''(t)v_2 + 120h'(t)u_{4t}v_2 - 90h'(t)v_{2t}v_2 + 180(h'(t))^3u_3v_2 + 60h''(t)v_2^2 + 180h'(t)u_3v_2^2)\Big),$$

$$u_8 = c_5(x,t), \qquad v_8 = \frac{1}{144}(-144u_8h'(t) - 12u_4u_{3t} - 12u_3u_{4t} + u_{4tt}),$$

$$w_8 = -\frac{1}{8064\sqrt{2}(h'(t))^3}\Big(I\sqrt{-h'(t)}792u_3^2v_2(h'(t))^2 + 1008u_4^2(h'(t))^3 + 4032u_8(h'(t))^3$$
$$+396u_3^2(h'(t))^4 + 108u_3v_2h'(t)h''(t) + 30u_3(h'(t))^3h''(t) + 6v_2(h''(t))^2 - 2(h'(t))^2(h''(t))^2$$
$$-2v_2h'(t)h'''(t) - 5(h'(t))^3h'''(t) - 48v_2(h'(t))^2u_{3t} - 24(h'(t))^4u_{3t}$$
$$+528u_3(h'(t))^2u_{4t} + 36h'(t)h''(t)u_{4t} - 24u_3(h'(t))^2v_{2t} - h'(t)h''(t)v_{2t} - 32(h'(t))^2u_{4tt} - 4(h'(t))^2v_{2tt}\Big),$$

where $c_1(x,t)$, $c_2(x,t)$, $c_3(x,t)$, $c_4(x,t)$ and $c_5(x,t)$ are arbitrary functions, then we establish u, v, w by substituting into (3.6) from the two cases.

4. Conclusion

In this paper we have explored only Lie point symmetries. However, in recent years several works have been devoted to study the integrability aspects of coupled systems (1.1) through higher order symmetries [15]. Presently, we are also investigating the existence of higher order symmetries for the nonlinear Ito coupled system.

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