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On optimal k-fold colorings of webs and antiwebs[☆]

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ABSTRACT

A k-fold x-coloring of a graph is an assignment of (at least) k distinct colors from the set $\{1, 2, \ldots, x\}$ to each vertex such that any two adjacent vertices are assigned disjoint sets of colors. The smallest number x such that G admits a k-fold x-coloring is the k-th chromatic number of G, denoted by $\chi_k(G)$. We determine the exact value of this parameter when G is a web or an antiweb. Our results generalize the known corresponding results for odd cycles and imply necessary and sufficient conditions under which $\chi_k(G)$ attains its lower and upper bounds based on clique and integer and fractional chromatic numbers. Additionally, we extend the concept of χ-critical graphs to χ_k-critical graphs. We identify the webs and antiwebs having this property, for every integer $k \geq 1$.

1. Introduction

For any integers $k \geq 1$ and $x \geq 1$, a *k-fold x-coloring* of a graph is an assignment of (at least) k distinct colors to each vertex from the set $\{1, 2, \ldots, x\}$ such that any two adjacent vertices are assigned disjoint sets of colors [22,25]. Each color used in the coloring defines what is called a *stable set* of the graph, *i.e.* a subset of pairwise nonadjacent vertices. We say that a graph G is *k-fold x-colorable* if G admits a k-fold x-coloring. The smallest number x such that a graph G is k-fold x-colorable is called the *k-th chromatic number of* G and is denoted by $\chi_k(G)$ [25]. Obviously, $\chi_1(G) = \chi(G)$ is the conventional *chromatic number of* G. This variant of the conventional graph coloring was introduced in the context of radio frequency assignment problem [17,23]. Other applications include scheduling problems, bandwidth allocation in radio networks, fleet maintenance and traffic phasing problems [2,12,15,18].

Let n and p be integers such that $p \geq 1$ and $n \geq 2p$. As defined by Trotter, the *web* W_p^n is the graph whose vertices can be labeled as $\{v_0, v_1, \ldots, v_{n-1}\}$ in such a way that its edge set is $\{v_i v_j \mid p \leq |i - j| \leq n - p\}$ [26]. The *antiweb* $\overline{W}_p^n$ is defined as the complement of W_p^n. Examples are depicted in Fig. 1, where the vertices are named according to an appropriate labeling (for the sake of convenience, we often name the vertices in this way in the remainder of the text). We observe that these definitions are interchanged in some references (see [21,27], for instance). Webs and antiwebs form subclasses of circulant graphs that play an important role in the context of stable sets and vertex coloring problems [4,5,8,9,11,19–21,27]. Determining 1-fold coloring of circulant graphs is NP-Hard [7]. Therefore, determining chromatic parameters of subclasses of circulant graphs is worthwhile (see, for instance, [1] and references therein).

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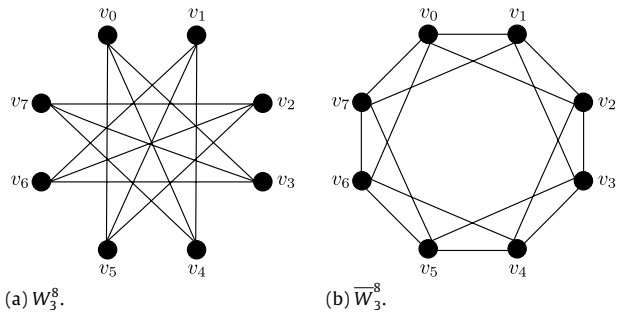


(a) W_3^8. (b) $\overline{W}_3^8$.

Fig. 1. Example of a web and an antiweb.

In this paper, we derive a closed formula for the k-th chromatic number of webs and antiwebs. More specifically, we prove that $\chi_k(W_p^n) = \left\lceil \frac{kn}{p} \right\rceil$ and $\chi_k(\overline{W}_p^n) = \left\lceil \frac{kn}{\left\lfloor \frac{n}{p} \right\rfloor} \right\rceil$, for every $k \in \mathbb{N}$, thus generalizing similar results for odd cycles [25]. The denominator of each of these formulas is the size of the largest stable set in the corresponding graph, *i.e.* the *stability number* of the graph [26]. Indeed, it does not seem surprising that the k-th chromatic number is equal to that value. This claim comes from the idea of, starting with the circular numbering of the vertices, choosing a maximum independent set to describe a color, and shifting the class color appropriately around the circle. However, proving that this process works is not so simple, particularly in the case of antiwebs. Actually, this strategy does not work for general circulant graphs.

Besides its direct relation with the stability number, we also relate the k-th chromatic number of webs and antiwebs with other parameters of the graph, such as the clique, chromatic and fractional chromatic numbers. Particularly, we derive necessary and sufficient conditions under which the classical bounds given by these parameters are tight.

In addition to the value of k-th chromatic number, we also provide optimal k-fold colorings of W_p^n and $\overline{W}_p^n$. Based on the optimal colorings, we analyze when webs and antiwebs are critical with respect to this parameter. A graph G is said to be *χ-critical* if $\chi(G - v) < \chi(G)$, for all $v \in V(G)$. An immediate consequence of this definition is that if v is a vertex of a χ-critical graph G, then there exists an optimal 1-fold coloring of G such that the color of v is not assigned to any other vertex. Not surprisingly, χ-critical subgraphs of G play an important role in several algorithmic approaches to vertex coloring. For instance, they are the core of the reduction procedures of the heuristic of [14] as well as they give facet-inducing inequalities of vertex coloring polytopes explored in cutting-plane methods [3,13,16]. From this algorithmic point of view, odd holes and odd anti-holes are (along with cliques) the most widely used χ-critical subgraphs. It has already been noted that not only odd holes or odd anti-holes, but also χ-critical webs and antiwebs give facet-defining inequalities [3,20].

We extend the concept of χ-critical graphs to χ_k-critical graphs in a straightforward way. Then, we characterize χ_k-critical webs and antiwebs, for any integer $k \geq 1$. The characterization crucially depends on the greatest common divisors between n and p and between n and the stability number (which are equal for webs but may be different for antiwebs). Using Bézout's identity, we show that there exists $k \geq 1$ such that W_p^n is χ_k-critical if, and only if, $\gcd(n, p) = 1$. Moreover, when this condition holds, we determine all values of k for which W_p^n is χ_k-critical. Similar results are derived for $\overline{W}_p^n$, where the condition $\gcd(n, p) = 1$ is replaced by $\gcd(n, p) \neq p$. As a consequence, we obtain that a web or an antiweb is χ-critical if, and only if, the stability number divides $n - 1$. Such a characterization is trivial for webs but it was still not known for antiwebs [20]. More surprisingly, we show that being χ-critical is also a sufficient condition for a web or an antiweb to be χ_k-critical for all $k \geq 1$.

Throughout this paper, we mostly use notation and definitions consistent with what is generally accepted in graph theory. Even though, let us set the grounds for all the notation used from here on. Given a graph G, $V(G)$ and $E(G)$ stand for its set of vertices and edges, respectively. The simplified notation V and E is preferred when the graph G is clear by the context. The complement of G is written as $\overline{G} = (V, \overline{E})$. The edge defined by vertices u and v is denoted by uv.

As already mentioned, a set $S \subseteq V(G)$ is said to be a *stable set* if all vertices in it are pairwise non-adjacent in G, i.e. $uv \notin E\ \forall u, v \in S$. The *stability number* $\alpha(G)$ of G is the size of the largest stable set of G. Conversely, a *clique* of G is a subset $K \subseteq V(G)$ of pairwise adjacent vertices. The *clique number* of G is the size of the largest clique and is denoted by $\omega(G)$. For ease of expression, we frequently refer to the graph itself as being a clique (resp. stable set) if its vertex set is a clique (resp. stable set). The *fractional chromatic number of* G, to be denoted $\bar{\chi}(G)$, is the infimum of $\frac{x}{k}$ among the k-fold x-colorings [24]. It is known that $\omega(G) \leq \bar{\chi}(G) \leq \chi(G)$ and $\frac{n}{\alpha(G)} \leq \bar{\chi}(G)$ [24]. A graph G is *perfect* if $\omega(H) = \chi(H)$, for all induced subgraph H of G.

A *chordless cycle* of length n is a graph G such that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\}$. A *hole* is a chordless cycle of length at least four. An *antihole* is the complement of a hole. Holes and antiholes are odd or even according to the parity of their number of vertices. Odd holes and odd antiholes are minimally imperfect graphs [6]. Observe that the odd holes and odd anti-holes are exactly the webs $W_\ell^{2\ell+1}$ and $W_2^{2\ell+1}$, for some integer $\ell \geq 2$, whereas the cliques are exactly the webs W_1^n.

In the next section, we present general lower and upper bounds for the k-th chromatic number of an arbitrary simple graph. The exact value of this parameter is calculated for webs (Section 3.1) and antiwebs (Section 3.2). Some consequences of this result are presented in the following sections. In Section 4, we relate the k-th chromatic number of webs and antiwebs to their clique, integer and fractional chromatic numbers. In particular, we identify which webs and antiwebs achieve the bounds given in Section 2 and those for which these bounds are strict. The definitions of χ_k-critical and χ_*-critical graphs are introduced in Section 5, as a natural extension of the concept of χ-critical graphs. Then, we identify all webs and antiwebs that have these two properties.

2. Bounds for the k-th chromatic number of a graph

Two simple observations lead to lower and upper bounds for the k-th chromatic number of a graph G. On one hand, every vertex of a clique of G must receive k colors different from any color assigned to the other vertices of the clique. On the other hand, a k-fold coloring can be obtained by just replicating an 1-fold coloring k times. Therefore, we get the following bounds which are tight, for instance, for perfect graphs.

Fact 1. *For every $k \in \mathbb{N}$, $\omega(G) \leq \bar{\chi}(G) \leq \frac{\chi_k(G)}{k} \leq \chi(G)$.*

Another lower bound is related to the stability number, as follows. The lexicographic product of a graph G by a graph H is the graph that we obtain by replacing each vertex of G by a copy of H and adding all edges between two copies of H if and only if the two replaced vertices of G were adjacent. More formally, the *lexicographic product* $G \circ H$ is a graph such that:

1. the vertex set of $G \circ H$ is the Cartesian product $V(G) \times V(H)$; and
2. any two vertices $(u, \hat{u})$ and $(v, \hat{v})$ are adjacent in $G \circ H$ if and only if either u is adjacent to v, or $u = v$ and $\hat{u}$ is adjacent to $\hat{v}$.

As noted by Stahl, another way to interpret the k-th chromatic number of a graph G is in terms of $\chi(G \circ K_k)$, where K_k is a clique with k vertices [25]. It is easy to see that a k-fold x-coloring of G is equivalent to a 1-fold coloring of $G \circ K_k$ with x colors. Therefore, $\chi_k(G) = \chi(G \circ K_k)$. Using this equation we can trivially derive the following lower bound for the k-th chromatic number of any graph.

Fact 2. *For every graph G and every $k \in \mathbb{N}$, $\chi_k(G) \geq \left\lceil \frac{kn}{\alpha(G)} \right\rceil$.*

Proof. If H_1 and H_2 are two graphs, then $\alpha(H_1 \circ H_2) = \alpha(H_1)\alpha(H_2)$ [10]. Therefore, $\alpha(G \circ K_k) = \alpha(G)\alpha(K_k) = \alpha(G)$. We get $\chi_k(G) = \chi(G \circ K_k) \geq \left\lceil \frac{kn}{\alpha(G \circ K_k)} \right\rceil = \left\lceil \frac{kn}{\alpha(G)} \right\rceil$. $\square$

Next we will show that the lower bound given by Fact 2 is tight for two classes of graphs, namely webs and antiwebs. Moreover, some graphs in these classes also achieve the lower and upper bounds stated by Fact 1.

3. The k-th chromatic number of webs and antiwebs

In the remainder, let n and p be integers such that $p \geq 1$ and $n \geq 2p$ and let $\oplus$ stand for addition modulus n, i.e. $i \oplus j = (i + j) \bmod n$ for $i, j \in \mathbb{Z}$. Let $\mathbb{N}$ stand for the set of natural numbers (0 excluded). The following known results will be used later.

Lemma 1 (*Trotter [26]*). *$\alpha(\overline{W}_p^n) = \omega(W_p^n) = \left\lfloor \frac{n}{p} \right\rfloor$ and $\alpha(W_p^n) = \omega(\overline{W}_p^n) = p$.*

Lemma 2 (*Trotter [26]*). *Let n' and p' be integers such that $p' \geq 1$ and $n' \geq 2p'$. The web $W_{p'}^{n'}$ is a subgraph of W_p^n if, and only if, $np' \geq n'p$ and $n(p' - 1) \leq n'(p - 1)$.*

3.1. Web

We start by defining some stable sets of W_p^n. For each integer $i \geq 0$, define the following sequence of integers:

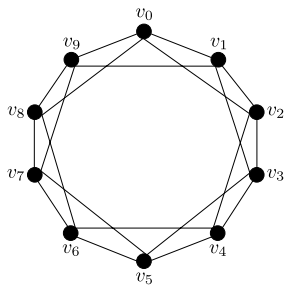
$$S_i = \langle i \oplus 0, i \oplus 1, \ldots, i \oplus (p - 1) \rangle. \tag{1}$$

Lemma 3. *For every integer $i \geq 0$, S_i indexes a maximum stable set of W_p^n.*

Proof. By the symmetry of W_p^n, it suffices to consider the sequence S_0. Let j_1 and j_2 be in S_0. Notice that $|j_1 - j_2| \leq p - 1 < p$. Then, $v_{j_1} v_{j_2} \notin E(W_p^n)$, which proves that S_0 indexes an independent set with cardinality $p = \alpha(W_p^n)$. $\square$

Using the above lemma and the sets S_i, we can now calculate the k-th chromatic number of W_p^n. The main idea is to build a cover of the graph by stable sets in which each vertex of W_p^n is covered at least k times.

Theorem 1. *For every $k \in \mathbb{N}$, $\chi_k(W_p^n) = \left\lceil \frac{kn}{p} \right\rceil = \left\lceil \frac{kn}{\alpha(W_p^n)} \right\rceil$.*



(a) $\overline{W}_3^{10}$.

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
$A(\ell,0)$	0	1	2	3	4	5	6	$\cdots$
$A(\ell,1)$	4	5	6	7	8	9	0	$\cdots$
$A(\ell,2)$	7	8	9	0	1	2	3	$\cdots$

$\ell = 1$

$\ell = 2$

(b) $C(1)$ in blue, $C(2)$ in red.

Fig. 2. Example of a 2-fold 7-coloring of $\overline{W}_3^{10}$. Recall that $\alpha(\overline{W}_3^{10}) = 3$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. By Fact 2, we only have to show that $\chi_k(W_p^n) \le \left\lceil \frac{kn}{p} \right\rceil$, for an arbitrary $k \in \mathbb{N}$. For this purpose, we show that $\Xi(k) = \langle S_0, S_p, \ldots, S_{(x-1)p} \rangle$ gives a k-fold x-coloring of W_p^n, with $x = \left\lceil \frac{kn}{p} \right\rceil$. We have that

$$\Xi(k) = \Big\langle \underbrace{0 \oplus 0, 0 \oplus 1, \ldots, 0 \oplus p-1}_{S_0}, \underbrace{p \oplus 0, \ldots, p \oplus (p-1)}_{S_p}, \ldots, \underbrace{(x-1)p \oplus 0, \ldots, (x-1)p \oplus (p-1)}_{S_{(x-1)p}} \Big\rangle.$$

Since the first element of $S_{(\ell+1)p}$, $0 \le \ell < x-1$, is the last element of $S_{\ell p}$ plus 1 (modulus n), we have that $\Xi(k)$ is a sequence (modulus n) of integer numbers starting at 0. Also, it has $\left\lceil \frac{kn}{p} \right\rceil p \ge kn$ elements. Therefore, each element between 0 and $n-1$ appears at least k times in $\Xi(k)$. By Lemma 3, this means that $\Xi(k)$ gives a k-fold $\left\lceil \frac{kn}{p} \right\rceil$-coloring of W_p^n, as desired. □

3.2. Antiweb

As before, we proceed by determining stable sets of $\overline{W}_p^n$ that cover each vertex at least k times. Now, we need to be more judicious in the choice of the stable sets of $\overline{W}_p^n$. We start by defining the following sequences (illustrated in Fig. 2):

$$S_0 = \left\langle \left\lceil t \frac{n}{\alpha(\overline{W}_p^n)} \right\rceil : t = 0, 1, \ldots, \alpha(\overline{W}_p^n) - 1 \right\rangle \tag{2}$$

$$\begin{aligned} S_i &= \langle j \oplus 1 : j \in S_{i-1} \rangle, \quad i \in \mathbb{N} \\ &= \langle j \oplus i : j \in S_0 \rangle, \quad i \in \mathbb{N}. \end{aligned}$$

We claim that each S_i indexes a maximum stable set of $\overline{W}_p^n$. This will be shown with the help of the following lemma.

Lemma 4. *If $x, y \in \mathbb{R}$ and $x \ge y$, then $\lfloor x-y \rfloor \le \lceil x \rceil - \lceil y \rceil \le \lceil x-y \rceil$.*

Proof. It is clear that $x - \lceil x \rceil \le 0$ and $\lceil y \rceil - y < 1$. By summing up these inequalities, we get $x - y + \lceil y \rceil - \lceil x \rceil < 1$. Therefore, $\lfloor x-y \rfloor + \lceil y \rceil - \lceil x \rceil \le 0$ and $\lfloor x-y \rfloor \le \lceil x \rceil - \lceil y \rceil$. To get the second inequality, recall that $\lceil x-y \rceil + \lceil y \rceil \ge \lceil x-y+y \rceil = \lceil x \rceil$. □

We now get the counterpart of Lemma 3 for antiwebs.

Lemma 5. *For every integer $i \ge 0$, S_i indexes a maximum stable set of $\overline{W}_p^n$.*

Proof. By the symmetry of an antiweb and the definition of the S_i's, it suffices to show the claimed result for S_0. Let j_1 and j_2 belong to S_0. We have to show that $p \le |j_1 - j_2| \le n-p$. For the upper bound, note that

$$|j_1 - j_2| \le \left\lceil \frac{(\alpha(\overline{W}_p^n) - 1)n}{\alpha(\overline{W}_p^n)} \right\rceil = \left\lceil n - \frac{n}{\alpha(\overline{W}_p^n)} \right\rceil \le n - p,$$

where the last inequality is due to $\alpha(\overline{W}_p^n) = \lfloor n/p \rfloor$. On the other hand,

$$|j_1 - j_2| \ge \min_{\ell \ge 1} \left(\left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{(\ell-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \right).$$

By Lemma 4 and $\alpha(\overline{W}_p^n) = \lfloor n/p \rfloor$, it follows that $|j_1 - j_2| \geq p$. Therefore, S_0 indexes an independent set of cardinality $\alpha(\overline{W}_p^n)$. $\square$

The above lemma is the basis to give the expression of $\chi_k(\overline{W}_p^n)$. We proceed by choosing an appropriate family of S_i's and, then, we show that it covers each vertex at least k times. We first consider the case where $k \leq \alpha(\overline{W}_p^n)$.

Lemma 6. *Let be given positive integers n, p, and $k \leq \alpha(\overline{W}_p^n)$. The index of each vertex of $\overline{W}_p^n$ belongs to at least k of the sequences $S_0, S_1, \ldots, S_{x(k)-1}$, where $x(k) = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$.*

Proof. Let $\ell \in \{1, 2, \ldots, k\}$ and $t \in \{0, 1, \ldots, \alpha(\overline{W}_p^n) - 1\}$. Define $A(\ell, t)$ as the sequence comprising the $(t+1)$-th elements of $S_0, S_1, \ldots, S_{x(\ell)-1}$, that is,

$$A(\ell, t) = \left\langle \left\lceil t \frac{n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus i : \ i = 0, 1, \ldots, \left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil - 1 \right\rangle.$$

Since $\ell \leq \alpha(\overline{W}_p^n)$, $A(\ell, t)$ has $\left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil$ distinct elements. Fig. 2 illustrates these sets for $\overline{W}_3^{10}$.

Let $B(\ell, t)$ be the subsequence of $A(\ell, t)$ formed by its first $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{tn}{\alpha(\overline{W}_p^n)} \right\rceil \leq \left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil$ elements (the inequality comes from Lemma 4). In Fig. 2(b), $B(1, t)$ relates to the numbers in blue whereas $B(2, t)$ comprises the numbers in blue and red. Notice that $B(\ell, t)$ comprises consecutive integers (modulus n), starting at $\left\lceil \frac{tn}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$ and ending at $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1)$. Consequently, $B(\ell, t) \subseteq B(\ell + 1, t)$.

Let $C(1, t) = B(1, t)$ and $C(\ell + 1, t) = B(\ell + 1, t) \setminus B(\ell, t)$, for $\ell < k$. Similarly to $B(\ell, t)$, $C(\ell, t)$ comprises consecutive integers (modulus n), starting at $\left\lceil \frac{(\ell+t-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$ and ending at $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1)$. Observe that the first element of $C(\ell, t+1)$ is the last element of $C(\ell, t)$ plus 1 (modulus n). Then, $C(\ell) = \langle C(\ell, 0), C(\ell, 1), \ldots, C(\ell, \alpha(\overline{W}_p^n) - 1) \rangle$ is a sequence of consecutive integers (modulus n) starting at the first element of $C(\ell, 0)$, that is $\left\lceil \frac{(\ell-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$, and ending at the last element of $C(\ell, \alpha(\overline{W}_p^n) - 1)$, that is

$$\left\lceil \frac{(\alpha(\overline{W}_p^n) + \ell - 1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1) = \left\lceil \frac{(\ell - 1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1).$$

This means that $C(\ell) \equiv \langle 0, 1, \ldots, n-1 \rangle$. Therefore, for each $\ell = 1, 2, \ldots, k$, $C(\ell)$ covers every vertex once. In addition, by definition of $C(\ell, t)$, $\ell > 1$, the sequences S_i's from which the elements of $C(\ell, t)$ are taken are different from those of $C(1, t), C(2, t), \ldots, C(\ell - 1, t)$. Consequently, every vertex is covered k times by $C(1), C(2), \ldots, C(k)$, and so is covered at least k times by $S_0, S_1, \ldots, S_{x(k)-1}$. $\square$

Now we are ready to prove our main result for antiwebs of this section.

Theorem 2. *For every $k \in \mathbb{N}$, $\chi_k(\overline{W}_p^n) = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$.*

Proof. By Fact 2, we only need to show the inequality $\chi_k(\overline{W}_p^n) \leq \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$. Let us write $k = \ell\alpha(\overline{W}_p^n) + i$, for integers $\ell \geq 0$ and $0 \leq i < \alpha(\overline{W}_p^n)$. By Lemmas 5 and 6, it is straightforward that the stable sets $S_0, S_1, \ldots, S_{x-1}$, where $x = \left\lceil \frac{in}{\alpha(\overline{W}_p^n)} \right\rceil$, induce an i-fold x-coloring of $\overline{W}_p^n$. The same lemmas also give an $\alpha(\overline{W}_p^n)$-fold n-coloring via sets $S_0, \ldots, S_{n-1}$. One copy of the first coloring together with ℓ copies of the second one yield a k-fold coloring with $\ell n + \left\lceil \frac{in}{\alpha(\overline{W}_p^n)} \right\rceil = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$ colors. $\square$

4. Relation with other parameters

The strict relationship between $\chi_k(G)$ and $\alpha(G)$ established for webs (Theorem 1) and anti-webs (Theorem 2) naturally motivates a similar question with respect to other parameters of G known to be related to the chromatic number. Particularly, we determine in this section when the bounds presented in Fact 1 are tight or strict. In what follows, "a mod b" stands for the remainder of the division of a by b.

Proposition 1. *Let G be W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $\chi_k(G) = k\chi(G)$ if, and only if, $\gcd(n, \alpha(G)) = \alpha(G)$ or $k < \frac{\alpha(G)}{\alpha(G)-r}$, where $r = n \bmod \alpha(G)$.*

Proof. By Theorems 1 and 2, $\chi_k(G) = k\chi(G)$ if, and only if, $\left\lceil \frac{kn}{\alpha(G)} \right\rceil = k \left\lceil \frac{n}{\alpha(G)} \right\rceil$, which is also equivalent to $\left\lceil \frac{kr}{\alpha(G)} \right\rceil = k \left\lceil \frac{r}{\alpha(G)} \right\rceil$. This equality trivially holds if $r = 0$, that is, $\gcd(n, \alpha(G)) = \alpha(G)$. In the complementary case, $\left\lceil \frac{r}{\alpha(G)} \right\rceil = 1$ and, consequently, the equality is equivalent to $\frac{kr}{\alpha(G)} > k - 1$ or still $k < \frac{\alpha(G)}{\alpha(G)-r}$. □

Proposition 2. *Let G be W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $\chi_k(G) = k\omega(G)$ if, and only if, $\gcd(n, p) = p$.*

Proof. Let $s = n \bmod p$. Using Lemma 1, note that $n = \lfloor n/p \rfloor p + s = \omega(G)\alpha(G) + s$. By Theorems 1 and 2, we get

$$\chi_k(G) = \left\lceil \frac{kn}{\alpha(G)} \right\rceil = k\omega(G) + \left\lceil \frac{ks}{\alpha(G)} \right\rceil.$$

The result then follows from the fact that $s = 0$ if, and only if, $\gcd(n, p) = p$. □

As we can infer from Lemma 1, if p divides n, then so does $\alpha(W_p^n)$ and $\alpha(\overline{W}_p^n)$. Under such a condition, which holds for all perfect and some non-perfect webs and antiwebs, the lower and upper bounds given in Fact 1 are equal.

Corollary 1. *Let G be W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $k\omega(G) = \chi_k(G) = k\chi(G)$ if, and only if, $\gcd(n, p) = p$.*

On the other hand, the same bounds are always strict for some webs and antiwebs, including the minimally imperfect graphs.

Corollary 2. *Let G be W_p^n or $\overline{W}_p^n$. If $\gcd(n-1, \alpha(G)) = \alpha(G)$ and $\alpha(G) > 1$, then $\chi_k(G) < k\chi(G)$, for all $k > 1$. Moreover, if $\gcd(n-1, p) = p$ and $p > 1$, then $k\omega(G) < \chi_k(G) < k\chi(G)$, for all $k > 1$.*

Proof. Assume that $\gcd(n-1, \alpha(G)) = \alpha(G)$ and $\alpha(G) \geq 2$. Then, $r := n \bmod \alpha(G) = 1$ and $\frac{\alpha(G)}{\alpha(G)-r} \leq 2$. By Proposition 1, $\chi_k(G) < k\chi(G)$ for all $k > 1$. To show the other inequality, assume that $\gcd(n-1, p) = p$ and $p > 1$. Then, $\gcd(n, p) \neq p$. Moreover, $\alpha(W_p^n) = p > 1$ and $\alpha(\overline{W}_p^n) = \frac{n-1}{p} > 1$ so that $\gcd(n-1, \alpha(G)) = \alpha(G) > 1$. By the first part of this corollary and Proposition 2, the result follows. □

To conclude this section, we relate the fractional chromatic number and the k-th chromatic number. By definition, for any graph G, these parameters are connected as follows:

$$\bar{\chi}(G) = \inf \left\{ \frac{\chi_k(G)}{k} \,\middle|\, k \in \mathbb{N} \right\}.$$

By Theorems 1 and 2, $\frac{\chi_k(G)}{k} \geq \frac{n}{\alpha(G)}$, for every $k \in \mathbb{N}$, and this bound is attained with $k = \alpha(G)$. This leads to

Proposition 3. *If G is W_p^n or $\overline{W}_p^n$, then $\bar{\chi}(G) = \frac{n}{\alpha(G)}$.*

Actually, the above expression holds for a larger class of graphs, namely vertex transitive graphs [24]. The following property readily follows in the case of webs and antiwebs.

Proposition 4. *Let G be W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $\chi_k(G) = k\bar{\chi}(G)$ if, and only if, $\frac{k \gcd(n,\alpha(G))}{\alpha(G)} \in \mathbb{Z}$.*

Proof. Let $\alpha = \alpha(G)$ and $g = \gcd(n, \alpha)$. By Theorems 1 and 2 and Proposition 3, $\chi_k(G) = k\bar{\chi}(G)$ if, and only if, $\frac{kn}{\alpha} \in \mathbb{Z}$. Since n/g and α/g are coprimes, $\frac{kn}{\alpha} = \frac{k(n/g)}{\alpha/g}$ is integer if, and only if, $\frac{k}{\alpha/g} \in \mathbb{Z}$. □

By the above proposition, given any web or antiweb G such that $\alpha(G)$ does not divide n, there are always values of k such that $\chi_k(G) = k\bar{\chi}(G)$ and values of k such that $\chi_k(G) > k\bar{\chi}(G)$.

5. χ_k-critical web and antiwebs

We define a χ_k-*critical* graph as a graph G such that $\chi_k(G - v) < \chi_k(G)$, for all $v \in V(G)$. If this relation holds for every $k \in \mathbb{N}$, then G is said to be χ_*-*critical*. Now we investigate these properties for webs and antiwebs. The analysis is trivial in the case where $p = 1$ because W_1^n is a clique. For the case where $p > 1$, the following property will be useful.

Lemma 7. *If G is W_p^n or $\overline{W}_p^n$ and $p > 1$, then $\alpha(G - v) = \alpha(G)$ and $\omega(G - v) = \omega(G)$, for all $v \in V(G)$.*

Proof. Let $v \in V(G)$. Since $p > 1$, v is adjacent to some vertex u. Lemmas 3 and 5 imply that there is a maximum stable set of G containing u. It follows that $\alpha(G - v) = \alpha(G)$. Then, the other equality is a consequence of $\alpha(G) = \omega(\overline{G})$. □

Additionally, the greatest common divisor between n and $\alpha(G)$ plays an important role in our analysis. For arbitrary nonzero integers a and b, Bézout's identity guarantees that the equation $ax + by = \gcd(a, b)$ has an infinite number of integer solutions (x, y). As there always exist solutions with positive x, we can define

$$t(a, b) = \min\left\{t \in \mathbb{N} : \frac{at - \gcd(a, b)}{b} \in \mathbb{Z}\right\}.$$

For our purposes, it is sufficient to consider a and b as positive integers.

Lemma 8. *Let $a, b \in \mathbb{N}$. If $\gcd(a, b) = b$, then $t(a, b) = 1$. Otherwise, $0 < t(a, b) < \frac{b}{\gcd(a,b)}$.*

Proof. If $\gcd(a, b) = b$, then we clearly have $t(a, b) = 1$. Now, assume that $\gcd(a, b) \neq b$. Define the coprime integers $a' = a/\gcd(a, b)$ and $b' = b/\gcd(a, b) > 1$. We have that $t(a, b) = t(a', b')$ because $\gcd(a', b') = 1$ and $\frac{at-\gcd(a,b)}{b} = \frac{a't-1}{b'}$, for all $t \in \mathbb{N}$. By Bézout's identity, there are integers $x > 0$ and y such that $a'x + b'y = 1$. Take $t = x \bmod b'$, that is, $t = x - \left\lfloor \frac{x}{b'} \right\rfloor b'$. Therefore, $0 \leq t < b'$ and $\frac{ta'-1}{b'} = \left(-y - \left\lfloor \frac{x}{b'} \right\rfloor a'\right) \in \mathbb{Z}$. Actually, $t > 0$ since $b' > 1$. These properties of t imply that $0 < t(a, b) = t(a', b') \leq t < b'$. □

5.1. Web

In this subsection, Theorem 1 is used to determine the k-chromatic number of the graph obtained by removing a vertex from W_p^n. For ease of notation, along this subsection let $t^\star = t(n, p) = t(n, \alpha(W_p^n))$.

Lemma 9. *For every $k \in \mathbb{N}$ and every vertex $v \in V(W_p^n)$,*

$$\chi_k(W_p^n - v) = \begin{cases} \left\lceil \dfrac{kn}{p} \right\rceil, & \text{if } \gcd(n, p) \neq 1, \\ \left\lceil \dfrac{kn - \left\lfloor \frac{k}{t^\star} \right\rfloor}{p} \right\rceil, & \text{if } \gcd(n, p) = 1. \end{cases}$$

Proof. Let $q = \gcd(n, p)$. First, suppose that $q > 1$. Using Lemma 2, we conclude that $W_{p/q}^{n/q}$ is a proper subgraph of W_p^n. Then, by the symmetry of W_p^n, it is a subgraph of $W_p^n - v$. By Theorem 1, we have that

$$\chi_k(W_p^n - v) \geq \left\lceil \frac{\frac{n}{q}k}{\frac{p}{q}} \right\rceil = \left\lceil \frac{nk}{p} \right\rceil.$$

The converse inequality follows as a consequence of $\chi_k(W_p^n - v) \leq \chi_k(W_p^n)$.

Now, assume that $q = 1$.

Claim 1. $\chi_k(W_p^n - v) \leq \left\lceil \frac{nk - \left\lfloor \frac{k}{t^\star} \right\rfloor}{p} \right\rceil$.

Proof. By the symmetry of W_p^n, we only need to prove the statement for $v = v_{n-1}$. Since $q = 1$, p divides $nt^\star - 1$. Let us use (1) to define $\Xi = \langle S_0, S_p, \ldots, S_{\left(\frac{nt^\star-1}{p}-1\right)p} \rangle$, which is a sequence (modulus n) of integer numbers starting at 0 and ending at $n - 2$. Notice that it covers $t^\star$ times each integer from 0 to $n - 2$. Using this sequence $\left\lfloor \frac{k}{t^\star} \right\rfloor$ times, we get a $\left(\left\lfloor \frac{k}{t^\star} \right\rfloor t^\star\right)$-fold coloring of $W_p^n - v$ with $\frac{nt^\star-1}{p} \left\lfloor \frac{k}{t^\star} \right\rfloor$ colors. If $t^\star$ divides k, then we are done. Otherwise, by Theorem 1 and the fact that $W_p^n - v \subseteq W_p^n$, we can have an additional $\left(k - \left\lfloor \frac{k}{t^\star} \right\rfloor t^\star\right)$-fold coloring with at most $\left\lceil \frac{n}{p}\left(k - \left\lfloor \frac{k}{t^\star} \right\rfloor t^\star\right) \right\rceil$ colors. Therefore, we obtain a k-fold coloring with at most $\frac{nt^\star-1}{p} \left\lfloor \frac{k}{t^\star} \right\rfloor + \left\lceil \frac{n}{p}\left(k - \left\lfloor \frac{k}{t^\star} \right\rfloor t^\star\right) \right\rceil = \left\lceil \frac{nk - \left\lfloor \frac{k}{t^\star} \right\rfloor}{p} \right\rceil$ colors. □

Claim 2. $\chi_k(W_p^n - v) \geq \left\lceil \frac{(nt^\star-1)k}{pt^\star} \right\rceil$.

Proof. By Theorem 1, it suffices to show that $W_{t^\star}^{n'}$ is a web included in $W_p^n - v$, where $n' = \frac{nt^\star-1}{p} \in \mathbb{Z}$ because $q = 1$. By Lemma 8, $t^\star < p$ implying that $n' < n$. Therefore, we only need to show that $W_{t^\star}^{n'}$ is a subgraph of W_p^n. First, notice that $n \geq 2p + 1$ and so $n' \geq 2t^\star + \frac{t^\star-1}{p} \geq 2t^\star$. Thus, $W_{t^\star}^{n'}$ is indeed a web. To show that it is a subgraph of W_p^n, we apply Lemma 2. On one hand, $nt^\star \geq nt^\star - 1 = n'p$. On the other hand, $n(t^\star - 1) \leq n'(p - 1)$ if, and only if, $n' \leq n - 1$. Therefore, the two conditions of Lemma 2 hold. □

By Claims 1 and 2, we get

$$\left\lceil \frac{nk - \lfloor \frac{k}{t^\star} \rfloor}{p} \right\rceil \geq \chi(W_p^n - v) \geq \left\lceil \frac{nk - \frac{k}{t^\star}}{p} \right\rceil.$$

To conclude the proof, we show that equality holds everywhere above. Let us write $k = \lfloor \frac{k}{t^\star} \rfloor t^\star + r$, where $0 \leq r < t^\star$. By the definition of $t^\star$, we have that $\frac{nt^\star - 1}{p} \in \mathbb{Z}$ but $\frac{nr-1}{p} \notin \mathbb{Z}$. It follows that

$$\begin{aligned}\left\lceil \frac{nk - \frac{k}{t^\star}}{p} \right\rceil &\geq \left\lceil \frac{nk - \lfloor \frac{k}{t^\star} \rfloor - 1}{p} \right\rceil = \frac{nt^\star - 1}{p} \left\lfloor \frac{k}{t^\star} \right\rfloor + \left\lceil \frac{nr - 1}{p} \right\rceil \\ &= \left\lceil \frac{nt^\star - 1}{p} \left\lfloor \frac{k}{t^\star} \right\rfloor + \frac{nr}{p} \right\rceil = \left\lceil \frac{nk - \lfloor \frac{k}{t^\star} \rfloor}{p} \right\rceil. \quad \square\end{aligned}$$

Remark 1. The proof of Lemma 9 provides the alternative equality $\chi_k(W_p^n - v) = \left\lceil \frac{kn - \frac{k}{t^\star}}{p} \right\rceil$ when $\gcd(n, p) = 1$.

Removing a vertex from a graph may decrease its k-th chromatic number by a value varying from 0 to k. For webs, the expressions of $\chi_k(W_p^n)$ and $\chi_k(W_p^n - v)$ given above together with Lemma 4 bound this decrease as follows.

Corollary 3. *Let $k \in \mathbb{N}$ and $v \in V(W_p^n)$. If $\gcd(n, p) \neq 1$, then $\chi_k(W_p^n) = \chi_k(W_p^n - v)$. Otherwise, $\left\lfloor \frac{k}{pt^\star} \right\rfloor \leq \chi_k(W_p^n) - \chi_k(W_p^n - v) \leq \left\lceil \frac{k}{pt^\star} \right\rceil$.*

Remark 2. An important feature of a χ-critical graph G is that, for every vertex $v \in V(G)$, there is always an optimal coloring where v does not share its color with the other vertices. Such a property makes it easier to show that inequalities based on χ-critical graphs are facet-defining for 1-fold coloring polytopes [3,16,20]. For $k \geq 2$, Corollary 3 establishes that cliques are the unique webs for which there exists an optimal k-fold coloring where a vertex does not share any of its k colors with the other vertices. Indeed, for $p \geq 2$ and $k \geq 2$, the upper bound given in Corollary 3 leads to $\chi_k(W_p^n) - \chi_k(W_p^n - v) \leq \left\lceil \frac{k}{2} \right\rceil < k$.

Next, we identify the values of n, p, and k for which the lower bound given in Corollary 3 is nonzero. In other words, we characterize the χ_k-critical webs, for every $k \in \mathbb{N}$.

Theorem 3. *Let $k \in \mathbb{N}$. If $\gcd(n, p) \neq 1$, then W_p^n is not χ_k-critical. Otherwise, the following assertions are equivalent:*

(i) *W_p^n is χ_k-critical;*
(ii) *$k \geq pt^\star$ or $0 < \frac{nk}{p} - \left\lfloor \frac{nk}{p} \right\rfloor \leq \frac{k}{pt^\star}$;*
(iii) *$k \geq pt^\star$ or $k = at^\star + bp$ for some integers $a \geq 1$ and $b \geq 0$.*

Proof. The first part is an immediate consequence of Corollary 3. For the second part, assume that $\gcd(n, p) = 1$, which means that $\frac{nt^\star - 1}{p} \in \mathbb{Z}$. Let $r = kn \bmod p$, i.e. $\frac{r}{p} = \frac{kn}{p} - \left\lfloor \frac{kn}{p} \right\rfloor$. So, assertion (ii) can be rewritten as

$$k \geq pt^\star \quad \text{or} \quad k \geq rt^\star \quad \text{with } r > 0. \tag{3}$$

On the other hand, by Theorem 1 and Remark 1, it follows that

$$\chi_k(W_p^n) = \left\lfloor \frac{kn}{p} \right\rfloor + \left\lceil \frac{r}{p} \right\rceil \quad \text{and} \quad \chi_k(W_p^n - v) = \left\lfloor \frac{kn}{p} \right\rfloor + \left\lceil \frac{r - \frac{k}{t^\star}}{p} \right\rceil.$$

Therefore, W_p^n is χ_k-critical if, and only if, $\left\lceil \frac{r}{p} \right\rceil > \left\lceil \frac{r - \frac{k}{t^\star}}{p} \right\rceil$. If $r = 0$, this means that $\left\lceil \frac{-\frac{k}{t^\star}}{p} \right\rceil \leq -1$ or, equivalently, $k \geq pt^\star$. If $r \geq 1$, then the condition is equivalent to $\left\lceil \frac{r - \frac{k}{t^\star}}{p} \right\rceil \leq 0$ or still $k \geq rt^\star$. As $r < p$, we can conclude that W_p^n is χ_k-critical if, and only if, condition (3) holds.

To show that (3) implies assertion (iii), it suffices to show that $k \geq rt^\star$ and $r > 0$ imply that there exist integers $a \geq 1$ and $b \geq 0$ such that $k = at^\star + bp$. Indeed, notice that $\frac{kn - r}{p} \in \mathbb{Z}$. Then, $\frac{knt^\star - rt^\star}{p} = \frac{(nt^\star - 1)k + (k - rt^\star)}{p} \in \mathbb{Z}$. We can deduce that $\frac{k - rt^\star}{p} \in \mathbb{Z}$ or, equivalently, $k = rt^\star + bp$ for some $b \in \mathbb{Z}$. Since $k \geq rt^\star$ and $r \geq 1$, the desired result follows.

Conversely, let us assume that $k = at^\star + bp$ for some integers $a \geq 1$ and $b \geq 0$. If $a \geq p$, then we trivially get condition (3). So, assume that $a < p$. We claim that $r = a$. Indeed,

$$r = (nat^\star) \bmod p = nat^\star - \left\lfloor \frac{(nt^\star - 1)a + a}{p} \right\rfloor p = a - \left\lfloor \frac{a}{p} \right\rfloor p = a.$$

Since $a \geq 1$ and $b \geq 0$, we have that $k \geq rt^\star$ and $r > 0$. □

As an immediate consequence of Theorem 3(iii), we have the characterization of χ_*-critical webs.

Theorem 4. *The following assertions are equivalent:*

(i) W_p^n *is* χ_**-critical;*
(ii) W_p^n *is* χ*-critical;*
(iii) $\alpha(W_p^n)$ *divides* $n - 1$.

Proof. Since any χ_*-critical graph is χ-critical, we only need to prove that (ii) implies (iii), and (iii) implies (i). Moreover, (iii) is equivalently to $t^\star = \gcd(n, p) = 1$. To show the first implication, we apply Theorem 3(iii) with $k = 1$. It follows that $\gcd(n, p) = 1$ and $at^\star \leq 1$ for $a \geq 1$. Therefore, $t^\star = \gcd(n, p) = 1$. For the second part, notice that any $k \in \mathbb{N}$ can be written as $k = at^\star + bp$ for $a = k \geq 1$ and $b = 0$, whenever $t^\star = 1$. The result follows again by Theorem 3(iii). □

Corollary 4. *Cliques, odd holes and odd anti-holes are all* χ_**-critical.*

5.2. Antiwebs

Now, we turn our attention to $\overline{W}_p^n$. Similarly to the previous subsection, Theorem 2 is used to determine the k-chromatic number of the graph obtained by removing a vertex from $\overline{W}_p^n$. In this subsection, let $t^\star = t(n, \alpha(\overline{W}_p^n))$.

Lemma 10. *For every* $k \in \mathbb{N}$ *and every vertex* $v \in V(\overline{W}_p^n)$,

$$\chi_k(\overline{W}_p^n - v) = \begin{cases} \left\lceil \dfrac{kn}{\alpha(\overline{W}_p^n)} \right\rceil & \text{if } \gcd(n, p) = p, \\ \left\lceil \dfrac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil & \text{if } \gcd(n, p) \neq p. \end{cases}$$

Proof. First assume that p divides n. Using Fact 1 and Corollary 1, we get

$$k\omega(\overline{W}_p^n - v) \leq \chi_k(\overline{W}_p^n - v) \leq \chi_k(\overline{W}_p^n) = k\omega(\overline{W}_p^n).$$

By Lemma 7, $\omega(\overline{W}_p^n) = \omega(\overline{W}_p^n - v)$ if $p > 1$. The same equality trivially holds when $p = 1$ since $\overline{W}_1^n$ has no edges. These facts and the above expression show that $\chi_k(\overline{W}_p^n - v) = \chi_k(\overline{W}_p^n) = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$.

Now assume that $\gcd(n, p) \neq p$. Then, $p > 1$ and $n > 2p$. By Fact 2 and Lemma 7, we have that $\chi_k(\overline{W}_p^n - v) \geq \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil$. Now, we claim that $\overline{W}_p^n - v$ is a subgraph of $\overline{W}_p^{n-1}$. First, notice that this antiweb is well-defined because $n - 1 \geq 2p$. Now, let $v_iv_j \in E(\overline{W}_p^n - v) \subset E(\overline{W}_p^n)$. Then $|i - j| < p$ or $|i - j| > n - p > (n - 1) - p$. Therefore, $v_iv_j \in E(\overline{W}_p^{n-1})$. This proves the claim. Then, Theorem 2 implies that $\chi_k(\overline{W}_p^n - v) \leq \chi_k(\overline{W}_p^{n-1}) = \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^{n-1})} \right\rceil$. Moreover, since p does not divide n, it follows that $\alpha(\overline{W}_p^{n-1}) = \left\lfloor \frac{n-1}{p} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = \alpha(\overline{W}_p^n)$. This shows the converse inequality $\chi_k(\overline{W}_p^n - v) \leq \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil$. □

Using again Lemma 4, we can now bound the difference between $\chi_k(\overline{W}_p^n)$ and $\chi_k(\overline{W}_p^n - v)$.

Corollary 5. *Let* $k \in \mathbb{N}$ *and* $v \in V(\overline{W}_p^n)$. *If* p *divides* n, *then* $\chi_k(\overline{W}_p^n - v) = \chi_k(\overline{W}_p^n)$. *Otherwise,* $\left\lfloor \frac{k}{\alpha(\overline{W}_p^n)} \right\rfloor \leq \chi_k(\overline{W}_p^n) - \chi_k(\overline{W}_p^n - v) \leq \left\lceil \frac{k}{\alpha(\overline{W}_p^n)} \right\rceil$.

Remark 3. For $k \geq 2$, no antiweb has an optimal k-fold coloring where a vertex does not share any of its k colors with other vertices. Since $\alpha(\overline{W}_p^n) \geq 2$, Corollary 5 establishes that $\chi_k(\overline{W}_p^n) - \chi_k(\overline{W}_p^n - v) \leq \left\lceil \frac{k}{2} \right\rceil < k$, whenever $k \geq 2$.

The above results also allow us to characterize the χ_k-critical antiwebs, as follows.

Theorem 5. *Let $k \in \mathbb{N}$. If $\gcd(n, p) = p$, then $\overline{W}_p^n$ is not χ_k-critical. Otherwise, the following assertions are equivalent:*

(i) *$\overline{W}_p^n$ is χ_k-critical;*

(ii) *$k \geq \alpha(\overline{W}_p^n)$ or $0 < \frac{nk}{\alpha(\overline{W}_p^n)} - \left\lfloor \frac{nk}{\alpha(\overline{W}_p^n)} \right\rfloor \leq \frac{k}{\alpha(\overline{W}_p^n)}$;*

(iii) *$k \geq \alpha(\overline{W}_p^n)$ or $k = at^\star + bq$ for some integers $a \geq 1$ and $b \geq \frac{a(\gcd(n,\alpha(\overline{W}_p^n))-t^\star)}{q}$, where $q = \alpha(\overline{W}_p^n)/\gcd(n, \alpha(\overline{W}_p^n))$.*

Proof. We use Theorem 2 and Lemma 10 to get the expressions of $\chi_k(\overline{W}_p^n)$ and $\chi_k(\overline{W}_p^n - v)$. Then, the first part of the statement immediately follows. Now assume that $\gcd(n, p) \neq p$. Let $\alpha = \alpha(\overline{W}_p^n)$ and $r = kn \bmod \alpha$ so that $\frac{r}{\alpha} = \frac{kn}{\alpha} - \left\lfloor \frac{kn}{\alpha} \right\rfloor$. It follows that

$$\chi_k(\overline{W}_p^n) = \left\lfloor \frac{kn}{\alpha} \right\rfloor + \left\lceil \frac{r}{\alpha} \right\rceil \quad \text{and} \quad \chi_k(\overline{W}_p^n - v) = \left\lfloor \frac{kn}{\alpha} \right\rfloor + \left\lceil \frac{r-k}{\alpha} \right\rceil.$$

Therefore, $\overline{W}_p^n$ is χ_k-critical if, and only if, $\left\lceil \frac{r}{\alpha} \right\rceil > \left\lceil \frac{r-k}{\alpha} \right\rceil$. If $r = 0$, this means that $\left\lceil -\frac{k}{\alpha} \right\rceil \leq -1$ or, equivalently, $k \geq \alpha$. If $r \geq 1$, then the condition is equivalent to $\left\lceil \frac{r-k}{\alpha} \right\rceil \leq 0$ or still $k \geq r$. As $r < \alpha$, we can conclude that $\overline{W}_p^n$ is χ_k-critical if, and only if,

$$k \geq \alpha, \quad \text{or} \quad k \geq r \text{ and } r > 0. \tag{4}$$

Notice that this is exactly assertion (ii).

To show the remaining equivalence, we use again (4). Let $g = \gcd(n, \alpha)$. By the definitions of r and $t^\star$, we have that $\frac{gk-rt^\star}{\alpha} = \frac{nk-r}{\alpha}t^\star - \frac{nt^\star-g}{\alpha}k \in \mathbb{Z}$. It follows that $k = at^\star + bq$ for some $b \in \mathbb{Z}$ and $a = r/g \in \mathbb{Z}$. Therefore, the second alternative of (4) implies the second alternative of assertion (iii). This leads to one direction of the desired equivalence.

Conversely, let us assume that assertion (iii) holds, that is, there exist integers $a \geq 1$ and b such that $k = at^\star + bq$ and $bq \geq ag - at^\star$. Then, $k \geq ag$. If $ag \geq \alpha$, then we trivially get item (ii). So, assume that $ag < \alpha$. We will show that $r = ag$. Indeed,

$$\begin{aligned} r &= \left(nat^\star + \frac{nb}{g}\alpha\right) \bmod \alpha = (nat^\star) \bmod \alpha \\ &= nat^\star - \left\lfloor \frac{(nt^\star - g)a + ag}{\alpha} \right\rfloor \alpha = ag - \left\lfloor \frac{ag}{\alpha} \right\rfloor \alpha = ag. \end{aligned}$$

Since $a \geq 1$ and $k \geq ag$, we have that $k \geq r$ and $r > 0$, showing the converse implication. □

The counterpart of Theorem 4 for antiwebs can be stated now.

Theorem 6. *The following assertions are equivalent:*

(i) *$\overline{W}_p^n$ is χ_*-critical;*

(ii) *$\overline{W}_p^n$ is χ-critical;*

(iii) *$\alpha(\overline{W}_p^n)$ divides $n - 1$.*

Proof. Let $\alpha = \alpha(\overline{W}_p^n)$, $g = \gcd(n, \alpha)$ and $q = \alpha/g$. It is trivial that (i) implies (ii). Now assume that $\overline{W}_p^n$ is χ-critical. By applying Theorem 5(iii) with $k = 1$, we have that $at^\star + bq = 1$ and $bq \geq ag - at^\star$, for some integers $a \geq 1$ and b. Then, $ag \leq 1$. It follows that $a = g = 1$ and $b = \frac{1-t^\star}{q} \in \mathbb{Z}$. Since $\gcd(n, p) \neq p$, due to Theorem 5, and $1 \leq t^\star < q$, due to Lemma 8, we obtain that $0 \geq b \geq \left\lceil \frac{1-q}{q} \right\rceil = 0$. Therefore, $t^\star = g = 1$ showing that α divides $n - 1$.

Conversely, assume that $\frac{n-1}{\alpha} \in \mathbb{Z}$, i.e. $t^\star = g = 1$. Then, $\alpha \neq \frac{n}{p}$, which implies that $\gcd(n, p) \neq p$. Moreover, any $k \in \mathbb{N}$ can be written as $k = at^\star + bq$ for $a = k \geq 1$ and $b = 0$. Since a and b satisfy the conditions of Theorem 5(iii), $\overline{W}_p^n$ is χ_*-critical. □

Corollary 6. *If p divides $n - 1$ then W_p^n and $\overline{W}_p^n$ are χ_*-critical.*

Proof. By Theorem 4, it remains to show that $\overline{W}_p^n$ is χ_*-critical. Under the hypothesis, we have that $\alpha(\overline{W}_p^n) = \frac{n-1}{p}$. Since $\frac{n-1}{\alpha(\overline{W}_p^n)} = p \in \mathbb{Z}$, the result follows by Theorem 6. □

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References

[1] J. Barajas, O. Serra, On the chromatic number of circulant graphs, Discrete Mathematics 309 (2009) 5687–5696.
[2] V.C. Barbosa, E. Gafni, Concurrency in heavily loaded neighborhood-constrained systems, ACM Transactions on Programming Languages and Systems 11 (1989) 562–584.
[3] M. Campêlo, R. Corrêa, Y. Frota, Cliques, holes and the vertex coloring polytope, Information Processing Letters 89 (4) (2004) 159–164.
[4] E. Cheng, S. de Vries, Antiweb-wheel inequalities and their separation problems over the stable set polytopes, Mathematical Programming 92 (2002) 153–175.
[5] E. Cheng, S. de Vries, On the facet-inducing antiweb-wheel inequalities for stable set polytopes, SIAM Journal on Discrete Mathematics 15 (4) (2002) 470–487.
[6] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Annals of Mathematics 164 (2006) 51–229.
[7] B. Codenotti, I. Gerace, S. Vigna, Hardness results and spectral techniques for combinatorial problems on circulant graphs, Linear Algebra and its Applications 285 (1998) 123–142.
[8] F. Eisenbrand, G. Oriolo, G. Stauffer, P. Ventura, The stable set polytope of quasi-line graphs, Combinatorica 28 (2008) 45–67.
[9] A. Galluccio, A. Sassano, The rank facets of the stable set polytope for claw-free graphs, Journal of Combinatorial Theory, Series B 69 (1) (1997) 1–38.
[10] D. Geller, S. Stahl, The chromatic number and other functions of the lexicographic product, Journal of Combinatorial Theory, Series B 19 (1975) 87–95.
[11] R. Giles, L.E. Trotter, On stable set polyhedra for $K_{1,3}$-free graphs, Journal of Combinatorial Theory, Series B 31 (3) (1981) 313–326.
[12] M. Halldórsson, G. Kortsarz, Tools for multicoloring with applications to planar graphs and partial k-trees, Journal of Algorithms 42 (2002) 334–366.
[13] P. Hansen, M. Labbé, D. Schindl, Set covering and packing formulations of graph coloring: algorithms and first polyhedral results, Discrete Optimization 6 (2) (2009) 135–147.
[14] F. Herrmann, A. Hertz, Finding the chromatic number by means of critical graphs, ACM Journal of Experimental Algorithmics 7 (2002) 1–9.
[15] R. Klasing, N. Morales, S. Pérennes, On the complexity of bandwidth allocation in radio networks, Theoretical Computer Science 406 (3) (2008) 225–239.
[16] I. Méndez-Díaz, P. Zabala, A cutting plane algorithm for graph coloring, Discrete Applied Mathematics 156 (2) (2008) 159–179.
[17] L. Narayanan, Channel Assignment and Graph Multi-Coloring, Wiley, 2002.
[18] R.J. Opsut, F.S. Roberts, On the fleet maintenance, mobile radio frequency, task assignment and traffic phasing problems, in: G. Chartrand, Y. Alavi, D. Goldsmith, L. Lesniak-Foster, D. Lick (Eds.), The Theory and Applications of Graphs, Wiley, 1981, pp. 479–492.
[19] G. Oriolo, G. Stauffer, Clique-circulants and the stable set polytope of fuzzy circular interval graphs, Mathematical Programming 115 (2008) 291–317.
[20] G. Palubeckis, Facet-inducing web and antiweb inequalities for the graph coloring polytope, Discrete Applied Mathematics 158 (2010) 2075–2080.
[21] A. Pêcher, A.K. Wagler, Almost all webs are not rank-perfect, Mathematical Programming 105 (2006) 311–328.
[22] G. Ren, Y. Bu, k-fold coloring of planar graphs, Science China Mathematics 53 (10) (2010) 2791–2800.
[23] F.S. Roberts, On the mobile radio frequency assignment problem and the traffic light phasing problem, Annals of the New York Academy of Sciences 319 (1979) 466–483.
[24] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs, Wiley-Interscience, 1997.
[25] S. Stahl, n-tuple colorings and associated graphs, Journal of Combinatorial Theory, Series B 20 (1976) 185–203.
[26] L.E. Trotter, A class of facet producing graphs for vertex packing polyhedra, Discrete Mathematics 12 (1975) 373–388.
[27] A.K. Wagler, Antiwebs are rank-perfect, 4OR: A Quarterly Journal of Operations Research 2 (2004) 149–152.