Solvability of a Multi-Point Boundary Value Problem at Resonance

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This paper is concerned with the problem of existence of a solution for the multi-point boundary value problem,

\begin{align*}
x''(t) &= f(t, x(t), x'(t)), \quad 0 < t < 1, \quad x(0) = 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i),
\end{align*}

with \( \sum_{i=1}^{k} \xi_i \eta_i = 1 \), in the absence of growth conditions on \( f \). The multi-dimensional version of the problem is also considered.

**Key Words:** Mawhin continuation theorem; resonance; multi-point boundary value problem.

1. INTRODUCTION

Let \( f: [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) be a continuous function, and let \( \xi_i \in (0, \infty), \eta_i \in (0, 1), i = 1, \ldots, k \), with \( 0 < \eta_1 < \cdots < \eta_k < 1 \), be given.

Let us consider the multi-point boundary value problem

\begin{align*}
x''(t) &= f(t, x(t), x'(t)), \quad 0 < t < 1, \\
x(0) &= 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i).
\end{align*}

If \( \sum_{i=1}^{k} \xi_i \eta_i \neq 1 \) then problem (1) is non-resonant, i.e., the linear problem

\begin{align*}
x''(t) &= 0, \\
x(0) &= 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i),
\end{align*}

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has only zero solution, and the differential operator, with boundary conditions taken into account, is invertible. In this case many results have been obtained (cf. [8, 10–12, 20]).

We study the case where \( \sum_{i=1}^k \xi_i \eta_i = 1 \) (the problem is at resonance; linear functions \( x(t) = at \) are solutions of (2)). It is more delicate. Some authors (cf. [3, 4, 6]) have proved the existence of solutions of (1), assuming that nonlinearity \( f \) is sublinear with respect to the second and third variables. We relax this assumption—\( f \) has only square growth with respect to the third variable. In the case where \( k = 1 \) and \( n = 1 \) (three-point boundary value problem, scalar case) the existence of a solution could be obtained from [18], where an upper and lower solution technique has been applied, since our sign condition implies that functions \( x_-(t) = -a_0 t \) and \( x_+(t) = a_0 t \) are respectively lower and upper solutions of (1). It is known [8, 13] that the existence of solutions to the multi-point problem (1) can be studied via the existence of a solution for some three-point boundary value problem, provided that all \( \xi_i \) are either nonnegative or nonpositive. We consider the general case, where \( k, n \) can be arbitrary, positive integers.

A similar kind of equations but with Neumann condition at 0 has been considered in [2, 3, 9, 14]. Moreover, the existence of positive solutions has been established in [1, 19] under some additional assumptions.

2. SCALAR CASE

Let \( Y, X \) denote Banach spaces \( Y = C[0, 1], X = C^1[0, 1] \), with the norms \( \|x\|_\infty = \sup_{t \in [0,1]} |x(t)|, \|x\|_1 = \|x\|_\infty + \|x'\|_\infty \), respectively. We will denote the ball of radius \( a > 0 \) and center \( x \) in a given space by \( B(x, a) \). Define \( Lx = x'' \), where \( \text{dom } L = \{x \in C^2[0, 1]: x(0) = 0, \ x(1) = \sum_{i=1}^k \xi_i x(\eta_i)\} \). Then

\[
\ker L = \{R \ni t \mapsto at \in R : a \in R\},
\]

and \( \text{Im } L = \ker u \) for the functional \( u: Y \to R, \ u(x) = \int_0^1 \tilde{u}(s)x(s)ds \), where \( \tilde{u} \) is a nonnegative function defined by the formula

\[
\tilde{u}(s) = (1 - s) - \sum_{i=1}^k \xi_i (\eta_i - s) \chi_{[0, \eta_i]}(s),
\]

where \( \chi_{[0, \eta_i]} \) stands for the characteristic function of the interval \( [0, \eta_i] \). Moreover, one can notice that for \( s \in (0, 1) \),

\[
\tilde{u}(s) > 0. \quad (3)
\]
If we define $N: X \to Y$ by $N_x(t) = f(t, x(t), x'(t))$, problem (1) can be expressed in the form

$$Lx = N(x).$$

We can decompose the spaces $X, Y$ as topological direct sums: $X = \ker L \oplus \ker Q, Y = \text{im} L \oplus \text{im} Q$, where $Qx(t) = u(x)t$.

Since the problem is resonant ($L$ is not invertible), we shall use

**MAWHIN CONTINUATION THEOREM** [17]. Let $\Omega$ be a bounded open subset of $X$. If the equations

$$Lx = \lambda N(x), \quad \lambda \in (0, 1],$$

have no solution on the boundary $\partial \Omega$ of $\Omega$ and the Brouwer degree

$$\deg(QN \mid \ker L, \Omega \cap \ker L, 0)$$

is defined and does not vanish, then Eq. (4) has a solution in $\Omega$.

To apply the Mawhin continuation theorem we have to show that the topological degree of the operator $QN \mid \ker L$ on the set $B(0, a) \cap \ker L$ for a large enough is not zero and that the solutions of the problem

$$x''(t) = \lambda f(t, x(t), x'(t)), \quad 0 < t < 1$$
$$x(0) = 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i)$$

are a priori bounded in the $\| \cdot \|$ norm independently of $\lambda \in (0, 1]$.

**THEOREM 1.** Suppose that there exist such continuous functions $c, d: \mathbb{R} \to \mathbb{R}$ that

$$|f(t, x, y)| \leq c(x) + d(x)y^2$$

for any $t \in [0, 1], x, y \in \mathbb{R}$, and there exists a positive number $a_0$ such that

$$af(t, at, \text{sgn}(a)y) \geq 0$$

for any $t \in [0, 1], |a| \geq y \geq a_0$. Then the multi-point boundary value problem (1) has at least one solution in $C^2[0, 1]$

**Proof.** Suppose first that the inequality (7) is strong.

First we will prove that solutions of the problems (5) are a priori bounded; more precisely, $|x(t)| \leq a_0$ for $t \in [0, 1]$. Let us suppose that, on the contrary, there exists a solution $x$ of (5) for some $\lambda > 0$ such that $x(t) > a_0$ in a point $t \in (0, 1)$. Consider the function $z(t) = x(t) -$
max\{x(1), a_0\}t, 0 \leq t \leq 1. Then either \( z(t) \leq 0 \) for all \( t \in (0, 1) \), or there exists such \( t_0 \in (0, 1) \) that \( z(t_0) > 0 \).

In the first case we have

\[
x(1) > a_0 \quad \text{and} \quad x(t) \leq x(1)t, \quad t \in (0, 1).
\]

Therefore \( x(\eta_i) \leq x(1)\eta_i \) for all \( i \), so the boundary condition \( x(1) = \sum_{i=1}^k \xi_i x(\eta_i) \) can be satisfied only if \( x(\eta_i) = x(1)\eta_i \) for all \( i \). But then by (8), \( x'(\eta_i) = x'(1) > a_0 \), \( x'(\eta_i) \leq 0 \), which contradicts the condition (7) (with strong inequality).

In the second case the function \( z \) must vanish at the ends of some interval \([0, t_1]\) (if \( x(1) \geq a_0 \) then \( t_1 = 1 \), otherwise \( t_1 \in (0, 1) \)). Without loss of generality we can assume that \( z \) takes the global maximum at \( t_0 \). One has \( z(t_0) > 0 \), \( z'(t_0) = 0 \), and \( z''(t_0) \leq 0 \). Hence \( x'(t_0) = \max\{x(1), a_0\} =: a_i \geq a_0 \), \( x(t_0)/t_0 \geq a_1 \), and

\[
0 \geq z''(t_0) = x''(t_0) = \lambda f(t_0, x(t_0), a_1),
\]

which contradicts (7) with the strong inequality (with \( a = x(t_0)/t_0 \) and \( y = a_1 \)).

Thus we have proved that solutions of (5) have to be bounded from above by \( a_0 \). Analogously, we get that they are bounded from below by \(-a_0 \). So we have that, for any solution \( x \) of (5), \( \|x\|_\infty \leq a_0 \) holds. As we deal with the space \( C^1([0, 1]) \) we need a similar estimate for \( x' \). Denote \( c = \sup_{|x| \leq a_0} d(x) \) and \( d = \sup_{|x| \leq a_0} d(x) \). For solutions of (5) we have \( |x'(t)| \leq c + d|x'(t)|^2 \). Let us divide an interval \([0, 1]\) on subintervals where \( x' \) is of a constant sign.

If \( x'(t) > 0 \) for \( t \in (t_1, t_2) \) and \( x'(t_1) = 0 \), one can integrate the inequality

\[
\frac{x''(t)x'(t)}{c + dx'(t)^2} = \frac{\lambda f(t, x(t), x'(t))x'(t)}{c + dx'(t)^2} \leq x'(t)
\]

and obtain

\[
\int_0^{x'(t)} \frac{u}{c + du^2} du = \int_{t_1}^{t_2} \frac{x''(s)x'(s)}{c + dx'(s)^2} ds \leq x(t) - x(t_1) \leq 2a_0.
\]

The integration leads to an upper bound on \( x'(t) \). Analogously, we deal with the case \( x'(t) > 0 \) on \((t_1, t_2)\) and \( x(t_2) = 0 \), using the inequality

\[
\frac{x''(t)x'(t)}{-c - dx'(t)^2} \leq x'(t).
\]
If \( x'(t) > 0 \) for any \( t \in [0, 1] \), then, by the Mean Value Theorem, there exists \( t_0 \) such that \( x'(t_0) = x(1) - x(0) \leq a_0 \). For \( t > t_0 \), we apply the calculations as in the first case, for \( t < t_0 \)—as in the second one.

The calculations for intervals where \( x' < 0 \) are very similar. Therefore, we obtain an a priori bound for the norm of all solutions to (5),

\[
\|x\|_1 < M
\]

(\( M \) is a certain constant).

Now we will show that a topological degree of the operator \( QN \mid \ker L \) on the set \( B(0, M) \cap \ker L \) is not zero. But \( \deg(QN \mid \ker L, B(0, a) \cap \ker L, 0) = \deg(r, (-a, a), 0) \), where \( r: \mathbb{R} \to \mathbb{R} \) is defined by

\[
 r(a) = \int_0^1 \hat{u}(s)f(s, as, a)ds.
\]

The Brouwer degree \( \deg(r, (-a, a), 0) \) is non-zero if \( r(-a)r(a) < 0 \). But from the assumption (7) and from (3) we get that \( r(a) \) is positive and \( r(-a) \) is negative.

Therefore, due to the Mawhin continuation theorem, we get a solution to (1).

Now, we pass to the general case of (7). Consider a problem

\[
\begin{align*}
x^{\mu}(t) &= f(t, x(t), x'(t)) + \frac{1}{n}x, \\
x(0) &= 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i)
\end{align*}
\]

(9)

for any \( n \in \mathbb{N} \). Since the function on the left-hand side of Eq. (9) satisfies the strong inequality (7) and the remaining assumptions of the theorem for any \( n \in \mathbb{N} \), we have a sequence of solutions \((x_n, a)\) to (9). From the proof of the first case, one gets that functions \( x_n \) and \( x_n' \) are equibounded, and hence, by the differential equation (9), \( x_n' \) are equibounded as well. Thus the family \( x_n, n \in \mathbb{N} \), is relatively compact in \( C^1[0, 1] \), and the limit of a convergent subsequence is a solution of the boundary value problem (1).

The assumption (6) is obviously weaker than this from [3, 4, 6] since we allow quadratic growth of \( f \) w.r.t the derivative and no growth condition is imposed with respect to the function. Also the assumption (7) is weaker if compared with the same, though it cannot be inverted as in [1].

In the subsequent examples the assumptions of our theorem are satisfied, but not those from [3, 4, 6].

**Example 1.** Consider the boundary value problem of the form (1) with

\[
f(t, x, y) = g(t, x)h(y),
\]
where \( g: [0, 1] \times R \rightarrow R \) is a continuous function such that
\[
xg(t, x) \geq 0, \quad t \in [0, 1], \quad |x| \geq a_0 t,
\]
for some positive constant \( a_0 \), and \( h: R \rightarrow R \) is a continuous nonnegative function \( h(y) = O(y^2) \) at \( \pm \infty \). Then all assumptions of Theorem 1 are satisfied, and the problem has a solution.

**Example 2.** Let
\[
f(t, x, y) = f_1(t, x, y) + h(y),
\]
where \( f_1 \) is continuous and bounded and \( h(y) = O(y^2) \) has the property
\[
\lim_{y \to \pm \infty} h(y) = \pm \infty.
\]
Then (1) has a solution due to Theorem 1. Taking, for instance, \( h(y) = y|y| \) (in both examples), we have the nonlinearity with superlinear growth, which cannot satisfy conditions from [3, 4, 6].

### 3. Multi-Dimensional Case

Now we shall consider the multi-dimensional case that is the boundary value problem of the same form as in (1), but when \( x: [0, 1] \rightarrow R^n \) and \( f: [0, 1] \times R^{2n} \rightarrow R^n \). The resonance is not simple—\( \dim \ker L = n > 1 \)—which causes some new difficulties. We get an existence result similar to that for the one-dimensional case but with slightly modified assumptions compared with those from Theorem 1.

**Theorem 2.** Let us assume that \( f \) is continuous and the conditions
\[
|f_i(t, x, y)| \leq b_i(x) + c_i(x)|y_i|^2 + d_i(x, y_1, \ldots, y_{i-1})
\]
are satisfied for \( i = 1, \ldots, n, \ t \in [0, 1], \ x \in R^n, \ \text{and} \ y \in R^n, \) where \( b_i, c_i, \) and \( d_i \) are continuous nonnegative functions; there exists such a number \( a_0 \) that, for any \( i \in \{1, \ldots, n\}, \)
\[
a_i f_i(t, ta, y) \geq 0,
\]
where \( t \in [0, 1], \ a = (a_1, \ldots, a_n) \) is such that \( |a_i| = \max |a_j| \) and \( y = (y_1, \ldots, y_n) \) has the \( i \)-th coordinate \( y_i \) with the property \( |a_i| \geq \text{sgn}(a_i) y_i \geq a_0 \). Then the boundary value problem (1) has at least one solution.

**Proof.** Without loss of generality we shall assume that inequalities (11) are strong (compare the proof of Theorem 1). It is easily seen that now
\[
\ker L = \{ t \mapsto ta : a \in R^n \}, \quad \im L = \bigcap_{i=1}^n \ker u_i
\]
where
\[ u_i(z) = \int_0^1 \tilde{u}(s)z_i(s)ds. \]

The Mawhin continuation theorem can be applied again.

We will first prove that the solutions of (5) are a priori bounded. Let \( z(t) = x(t) - t(\max(x_i(1), a_0)) \), where \( x \) is a solution of (5). If one of its coordinates is not less than \( a_0 \) at a point, then the corresponding function \( z_i \) takes the global maximum greater than 0,
\[ z_i(t) = \sup_t z_i(t) > 0; \]
choose this coordinate for which \( x_i(t) \) is the greatest number. Then applying the arguments from the proof of Theorem 1, we get a contradiction. Similarly, \( x_i(t) \geq -a_0 \), Thus \( \|x\| \leq na_0 \).

Denote \( b = \max_i \sup_{|x| \leq na_0} b_i(x), c = \max_i \sup_{|x| \leq na_0} c_i(x), d = \max_i \sup_{|x| \leq na_0} d_i(x) \). To estimate the derivative \( x' \) of a solution (5), we use the induction w.r.t. \( i \). For \( i = 1 \), one has condition (11) identical to (7), and the proof of Theorem 1 leads to an estimate. If we find
\[ |x'_i(t)| \leq M_j \]
for \( j < i \), then condition (11) gives
\[ \frac{x_i(t)x'_i(t)}{b + cx'_i(t)} \leq x'_i(t) + \frac{d \sum_j |x'_j(t)|}{b + cx'_i(t)^2} \leq x'_i(t) + \frac{d \sum_j M_j}{b} \]
and
\[ \frac{x_i(t)x''(t)}{-b - cx_i(t)} \leq x'_i(t) - \frac{d \sum_j |x'_j(t)|}{-b - cx'_i(t)^2} \leq x'_i(t) + \frac{d \sum_j M_j}{b} \]
on intervals where \( x'_i > 0 \). By integration similar to that in the one-dimensional case we get boundedness of \( x'_i \) on intervals where \( x'_i \) is positive, whence we obtain the boundedness of \( \|x\| \).

We need to calculate the degree of the mapping \( QN | \ker L \) on a sufficiently large neighborhood of 0. To this end, observe that the same is true of the Browder degree \( \deg(r, (-a_0, a_0)^n, 0) \) where \( r(a) = \int_0^1 \tilde{u}(s)f(s, sa, a)ds \). By condition (11) with strong inequalities, the mapping \( r \) has Miranda’s property on the boundary of the cube,
\[ r_i | H^+_i > 0, \quad r_i | H^-_i < 0, \]
where $H_+^i$ (resp. $H_-^i$) is the top (resp. bottom) $i$th face of the cube. It is obvious that the linear homotopy with the identity mapping has no zero on the boundary; hence the calculated degree is 1.

**Remark.** We expect that (10) can be weakened to the more convenient form,

$$\|f(t, x, y)\| \leq b(x) + c(x)\|y\|^2.$$ 

**REFERENCES**