JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **184**, 190–199 (1994)

On the Semigroup for Age Dependent Population Dynamics with Spatial Diffusion

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Submitted by E. Stanley Lee

Received December 18, 1992

We continue a population model with age dependence and spatial diffusion in the semigroup framework, in which the assumptions of our previous paper [*Manuscripta Math.* **66** (1990), 161–180] that birth and death rates are independent of spatial variables are removed. The infinitesimal generator is identified and its spectrum studied, and accordingly, by using a positive semigroup theory, we determine its dominant eigenvalue and hence the asymptotic expression is obtained. © 1994 Academic Press, Inc.

1. INTRODUCTION

In our previous paper [1], we established the semigroup framework of age-size structured population dynamics with spatial diffusion in terms of the completeness of the eigenfunctions of the Laplacian operator in the state space. However, its success is based on the crucial assumptions that both birth and death functions are independent of the spatial variable, which is not an exact description in practice. The objective of this paper is to develop the corresponding semigroup framework in a manner different from [1]; in particular, to develop the positive semigroup theory.

Again we consider the deterministic formulation of a McKendrick-type age-dependent population moving in a limited smooth domain Ω in $\mathbb{R}^n$. Let $p(r, t, x)$ be the age density distribution of the population at age $r \geqslant 0$ and time $t \geqslant 0$ and spatial location x in Ω. We still assume in this paper that

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$0 \leqslant r \leqslant r_m$, where r_m is the maximum life expectancy of the species. The evolution of p is governed by the differential equation

$$\begin{cases} \dfrac{\partial p(r,t,x)}{\partial t} + \dfrac{\partial p(r,t,x)}{\partial r} = -\mu(r,x)\, p(r,t,x) + K\, \Delta p(r,t,x), \\ p(r,0,x) = p_0(r,x), \\ p(0,t,x) = \displaystyle\int_0^{r_m} \beta(r,x)\, p(r,t,x)\, dr, \quad p\,|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $\mu(r, x)$ is the death rate function, positive and bounded measurable in any district $[0, r_c] \times \bar{\Omega}$, $r_c < r_m$, and

$$\sup_{r \in [0, r_c]} \int_\Omega \mu^2(r,x)\, dx \text{ continuous with respect to } rt[0, r_m],$$

$$\int_0^r \bar{\mu}(\rho)\, d\rho < \infty, \text{ for } r < r_m \text{ and } \int_0^{r_m} \underline{\mu}(\rho)\, d\rho = \infty, \tag{2}$$

where $\underline{\mu}(r) = \inf_{x \in \Omega} \mu(r,x)$ and $\bar{\mu}(r) = \sup_{x \in \Omega} \mu(r,x)$; $\beta(r,x)$ is the fertility function, bounded nonnegative measurable on $[0, r_m]$, and

$$\operatorname{mes}\{r \mid r \in [0, r_m],\ \beta(r) = \inf_{x \in \Omega} \beta(r,x) > 0\} > 0; \tag{3}$$

$p_0(r, x)$ is an initial distribution, $p_0(r, x) \geqslant 0$; K is a positive constant; and Δ denotes the Laplace operator in $\mathbb{R}^n$.

The paper is organized as follows. In Section 2 the infinitesimal generator $\mathbb{A}$ of the population operator with diffusion is identified. The resolvent operator $R(\lambda, \mathbb{A})$ is constructed and is shown to be compact. Section 3 is dedicated to the existence of a dominant eigenvalue of $\mathbb{A}$ and the compactness of the semigroup associated with $\mathbb{A}$, and finally we give the asymptotic expression of the semigroup.

2. Properties of the Infinitesimal Generator

Introducing the state space $X = L^2((0, r_m) \times \Omega)$ with the usual norm and defining the operator $\mathbb{A}: X \to X$ as

$$\begin{cases} \mathbb{A}\phi(r,x) = -\dfrac{\partial \phi(r,x)}{\partial r} - \mu(r,x)\, \phi(r,x) + K\, \Delta\phi(r,x), \qquad \forall \phi \in D(\mathbb{A}), \\ D(\mathbb{A}) = \left\{ \phi(r,x) \mid \phi,\ \mathbb{A}\phi \in X,\ \phi\,|_{\partial\Omega} = 0,\ \phi(0,x) = \displaystyle\int_0^{r_m} \beta(r,x)\, \phi(r,x)\, dr \right\}, \end{cases} \tag{4}$$

we can write Eq. (1) as an evolutionary equation on the state space X:

$$\begin{cases} \dfrac{dp(r,t,x)}{dt} = \mathbb{A}p(r,t,x), \\ p(r,0,x) = p_0(r,x). \end{cases} \tag{5}$$

We need the following

LEMMA 1. *For any* $0 \leqslant s_0 < r_m$, *there exists a unique mild solution* $u(s,x)$, $0 \leqslant \tau \leqslant s \leqslant r_m - s_0$, *to the evolution equation on* X *for any initial function* $\phi(x) \in L^2(\Omega)$

$$\begin{cases} \dfrac{\partial u(s,x)}{\partial s} = [-\mu(s_0+s,x) + \mathbb{B}]\, u(s,x), \\ u(\tau,x) = \phi(x), \end{cases} \tag{6}$$

where the operator $\mathbb{B}_0$ *is considered to be the Laplace operator with Dirichlet boundary condition. Define solution operators of the initial value problem* (6) *by*

$$\mathscr{T}(s_0,\tau,s)\,\phi(x) = u(s,x), \qquad \forall \phi \in L^2(\Omega); \tag{7}$$

then $\mathscr{T}(s_0,\tau,s)$, $0 \leqslant \tau \leqslant s \leqslant r_m - s_0$, *is a family of uniformly linear bounded compact positive operators on* X *and is strongly continuous about* τ, s. *Furthermore,*

$$e^{-\int_\tau^s \bar{\mu}(s_0+\rho)\,d\rho} e^{\mathbb{B}(s-\tau)} \leqslant \mathscr{T}(s_0,\tau,s) \leqslant e^{-\int_\tau^s \underline{\mu}(s_0+\rho)\,d\rho} e^{\mathbb{B}(s-\tau)}, \tag{8}$$

where $e^{\mathbb{B}s}$ *is the positive analytic semigroup generated by the operator* $\mathbb{B}$.

Proof. We first consider (6) in the interval $[0,\bar{s}]$. $\bar{s} < r_m - s_0$ is any given positive number. Define a mapping $\mathscr{F}(s_0)$ from $C([\tau,\bar{s}]; L^2(\Omega))$ into itself by

$$\mathscr{F}(s_0)\,u(s,x) = e^{\mathbb{B}s}\phi(x) - \int_\tau^s e^{\mathbb{B}(s-\sigma)}\mu(s_0+\sigma,x)\,u(\sigma,x)\,d\sigma.$$

Denoting $\|u\|_\infty = \max_{\tau \leqslant s \leqslant \bar{s}} \|u(t,x)\|$ and $M = \sup_{\tau \leqslant s \leqslant \bar{s};\, x \in \Omega} \mu(s_0+s,x)$, it is easy to check that

$$\|\mathscr{F}^n(s_0)u - \mathscr{F}^n(s_0)v\|_\infty \leqslant \frac{M^n(\bar{s}-\tau)^n}{n!}\|u-v\|_\infty,$$

$$\text{for} \quad u, v \in C([\tau,\bar{s}]; L^2(\Omega)), \quad n \geqslant 1.$$

For n large enough, $M^n(t-s)^n/n! < 1$, and by a well known generalization of the Banach contraction principle, $\mathscr{F}(s_0)$ has a unique fixed point u in $C([\tau, \bar{s}]; L^2(\Omega))$ for which

$$u(s, x) = e^{\mathbb{B}s}\phi(x) - \int_\tau^s e^{\mathbb{B}(s-\sigma)}\mu(s_0+\sigma, x)\, u(\sigma, x)\, d\sigma.$$

By assumption (2) on μ and the analytical property of $e^{\mathbb{B}s}$, the right-hand side of the above expression is differentiable. Thus u is differentiable and its derivative satisfies Eq. (6). Equation (8) is the direct consequence of the comparison principle when one notes that

$$\mathscr{T}(s_0, \tau, s) = e^{-\int_\tau^s \mu(s_0+\rho)\, d\rho} e^{\mathbb{B}(s-\tau)} \qquad \text{as} \quad \mu(r, x) = \mu(r).$$

Our claim is concluded by (8) if it is considered that $\mathscr{T}(s_0, \tau, r_m - s_0) = 0$. The positivity and compactness follow from the corresponding properties of $e^{\mathbb{B}s}$.

THEOREM 1. *The operator* $\mathbb{A}$ *defined by* (4) *is the infinitesimal generator of a* C_0*-semigroup* $\mathbb{T}(t)$ *on the state space* X. *Thus there exists a unique mild solution to Eq.* (1) *such that*

$$p(r, t, x) = \mathbb{T}(t)\, p_0(r, x) \in C([0, \infty); X), \qquad \textit{if} \quad p_0(r, x) \in X,$$

and the classical solution

$$p(r, t, x) = \mathbb{T}(t)\, p_0(r, x) \in C^1([0, \infty); X), \qquad \textit{if} \quad p_0(r, x) \in D(\mathbb{A}).$$

Proof. The second part is the direct consequence of the first [2]. For the first, note that

$$\begin{aligned}
\langle \mathbb{A}\phi(r, x), \phi(r, x)\rangle &\leqslant -\int_0^{r_m}\int_\Omega \frac{\partial\phi(r, x)}{\partial r}\phi(r, x)\, dr\, dx \\
&\leqslant \frac{1}{2}\int_\Omega \phi^2(0, x)\, dx = \frac{1}{2}\int_\Omega \left[\int_0^{r_m} \beta(r, x)\,\phi(r, x)\, dr\right]^2 dx \\
&\leqslant \frac{1}{2}\int_\Omega \left[\int_0^{r_m} \beta^2(r, x)\, dr \int_0^{r_m} \phi^2(r, x)\, dr\right] dx \\
&\leqslant \frac{1}{2}\sup_{x\in\Omega}\int_0^{r_m} \beta^2(r, x)\, dr \langle \phi(r, x), \phi(r, x)\rangle,
\end{aligned}$$

i.e., $\mathbb{A}$ is an m-dissipative operator. The conclusion will be established when it is verified that $\lambda \in \rho(\mathbb{A})$ for all sufficiently large $\lambda > 0$. In fact, if the claim holds, $\mathbb{A}$ is a closed operator, and then together with the m-dissiptiveness of $\mathbb{A}$ we know that, for all sufficiently large λ, $\mathbb{A}-\lambda$ is dissipative and $R(I-(\mathbb{A}-\lambda))$, the range of the operator $\mathbb{A}-\lambda$, equals the whole space X; therefor, from Theorem 4.6 in [2, p. 16], we know that $D(\mathbb{A}-\lambda)$ is dense in X and so is $D(\mathbb{A})$, since X is a Hilbert space. Thus $\mathbb{A}$ generates a C_0-semigroup. To this end, we solve the resolvent equation

$$(\lambda-\mathbb{A})\phi=\psi, \qquad \forall\psi\in X;$$

i.e.,

$$\begin{cases} \dfrac{\partial\phi(r,x)}{\partial r}=-(\lambda+\mu(r,x))\,\phi(r,x)+K\,\Delta\phi(r,x)+\psi(r,x), \\ \phi(0,x)=\displaystyle\int_0^{r_m}\beta(r,x)\,\phi(r,x)\,dr, \quad \phi(r,x)|_{\partial\Omega}=0. \end{cases} \tag{9}$$

Letting $\mathscr{T}(0,\tau,s)=\mathscr{T}(\tau,s)$, we have

$$\phi(r,x)=e^{-\lambda r}\mathscr{T}(0,r)\,\phi(0,x)+\int_0^r e^{-\lambda(r-\sigma)}\mathscr{T}(\sigma,r)\,\psi(\sigma,x)\,d\sigma,$$

and accordingly

$$\phi(0,x)-\int_0^{r_m}\beta(r,x)\,e^{-\lambda r}\mathscr{T}(0,r)\,\phi(0,x)\,dr$$
$$=\int_0^{r_m}\beta(r,x)\int_0^r e^{-\lambda(r-\sigma)}\mathscr{T}(\sigma,r)\,\psi(\sigma,x)\,d\sigma.$$

Define the operator $\mathscr{B}_\lambda\colon L^2(\Omega)\to L^2(\Omega)$ by

$$\mathscr{B}_\lambda\phi(x)=\int_0^{r_m}\beta(r,x)\,e^{-\lambda r}\mathscr{T}(0,r)\,\phi(x)\,dr, \qquad \forall\phi\in L^2(\Omega). \tag{10}$$

Then

$$\lambda\in\rho(\mathbb{A}) \qquad \text{iff} \quad 1\in\rho(\mathscr{B}_\lambda), \tag{11}$$

and for $\lambda\in D(\mathbb{A})$,

$$\begin{aligned} R(\lambda,\mathbb{A})\,\psi(r,x)&=e^{-\lambda r}\mathscr{T}(0,r)(I-\mathscr{B}_\lambda)^{-1}\int_0^{r_m}\beta(r,x)\,dr \\ &\quad\times\int_0^r e^{-\lambda(r-\sigma)}\mathscr{T}(\sigma,r)\,\psi(\sigma,x)\,d\sigma \\ &\quad+\int_0^r e^{-\lambda(r-\sigma)}\mathscr{T}(\sigma,r)\,\psi(\sigma,x)\,d\sigma. \end{aligned} \tag{12}$$

From (7) we know that

$$\lim_{\lambda \to \infty} \|\mathscr{B}_\lambda\| = 0;$$

hence for all sufficiently large $\lambda > 0$, $(I - \mathscr{B}_\lambda)^{-1}$ exists and so does the operator $(\lambda - \mathbb{A})^{-1}$. The proof is complete.

From the last statements and expression (12), $R(\lambda, \mathbb{A})$ is compact for all $\lambda \in \rho(\mathbb{A})$. Hence $\sigma(\mathbb{A})$, the spectrum of operator $\mathbb{A}$, consists of only the eigenvalues of $\mathbb{A}$. When $\lambda \in \sigma(\mathbb{A})$, its corresponding eigenfunction $\phi(r, x)$ can be expressed as

$$\phi(r, x) = e^{-\lambda r} \mathscr{T}(0, r)\, \phi_0(x), \tag{13}$$

where $\phi_0(x)$ is the nonzero solution of

$$\phi(x) - \int_0^{r_m} \beta(r, x)\, e^{-\lambda r} \mathscr{T}(0, r)\, \phi(x)\, dr = 0. \tag{14}$$

3. The Properties of the Semigroup

In this section we determine the dominant eigenvalue of the operator $\mathbb{A}$ and therefore find the asymptotic expression of the semigroup $\mathbb{T}(t)$ generated by $\mathbb{A}$.

From (10)

$$\mathscr{B}_\lambda \geqslant \mathscr{C}_\lambda, \tag{15}$$

where $\mathscr{C}_\lambda$ is defined to be

$$\mathscr{C}_\lambda = \int_0^{r_m} \underline{\beta}(r)\, e^{-\lambda r - \int_0^r \bar{\mu}(\rho)\, d\rho} e^{\mathbb{B} r}\, dr.$$

Given any nonnegative functions $\phi(x)$, $\psi(x)$ on $L^2(\Omega)$, both not identical to zero, then [3, 4]

$$\langle e^{\mathbb{B} r} \phi, \psi \rangle > 0, \qquad \text{for all} \quad r > 0.$$

From assumption (3) and the expression of $\mathscr{C}_\lambda$, we know that

$$\langle \mathscr{B}_\lambda \phi, \psi \rangle \geqslant \langle \mathscr{C}_\lambda \phi, \psi \rangle > 0, \qquad \text{for all real } \lambda. \tag{16}$$

For the case of $(\beta(r, x), \mu(r, x))$ instead of $(\underline{\beta}(r), \bar{\mu}(r))$, our results of [1] together with the conclusion in Section 2 imply that there exists a real number $\hat{\lambda}_0$ such that

$$\gamma(\mathscr{B}_{\hat{\lambda}_0}) \geq \gamma(\mathscr{C}_{\hat{\lambda}_0}) = 1.$$

On the other hand, $\lim_{\lambda \to \infty} \gamma(\mathscr{B}_\lambda) = 0$, and hence by continuity there exists a real λ_0 such that $\gamma(\mathscr{B}_{\lambda_0}) = 1$. Since $\mathscr{B}_{\lambda_0}$ is a compact positive operator, by the well known Krein and Rutman Theorem there exists a nonnegative $\phi_0(x) \in L_2(\Omega)$ such that

$$\mathscr{B}_{\lambda_0} \phi_0 = \lambda_0 \phi_0, \tag{17}$$

i.e., $\sigma(\mathscr{B}_{\lambda_0}) \neq \varnothing$. By a result of Marek [5], $\gamma(\mathscr{B}_\lambda)$ is strictly monotone decreasing with respect to real λ. This is equivalent to the uniqueness of the real eigenvalue of operator $\mathbb{A}$. Fairly, $\sigma(\mathbb{A}) \neq \varnothing$.

When $\lambda > \lambda_0$ and $\gamma(\mathscr{B}_\lambda) < \gamma(\mathscr{B}_{\lambda_0}) = 1$, $(I - \mathscr{B}_\lambda)^{-1}$ exists and is positive, and hence $R(\lambda, \mathbb{A})$ is positive from (12). Therefore, $\mathbb{T}(t)$, the semigroup generated by $\mathbb{A}$, is a positive semigroup and so by [6],

$$\lambda_0 = s(\mathbb{A}) = \omega_0(\mathbb{A}), \tag{18}$$

where $s(\mathbb{A})$, $\omega_0(\mathbb{A})$ denote the spectral bound of $\mathbb{A}$ and the growth bound of the semigroup $\mathbf{T}(t)$, respectively (cf. [6] for the definition).

Integrating along the characteristic, we obtain

$$p(r, t, x) = \begin{cases} \mathscr{T}(r-t, 0, t)\, p_0(r-t, x), & r \geq t, \\ \mathscr{T}(t-r, 0, r) \int_0^{r_m} \beta(s, x)\, p(s, t-r, x)\, ds, & r < t. \end{cases} \tag{19}$$

When $t \geq r_m$,

$$\mathbb{T}(t)\, \phi(r, x) = \mathscr{T}(t-r, 0, r) \int_0^{r_m} \beta(s, x)[\mathbb{T}(t-r)\phi](s, x)\, ds.$$

Let $\phi_n(s, x)$ weakly converge to $\phi(s, x)$ in X; then for any fixed r,

$$\int_0^{r_m} \beta(s, x)[\mathbb{T}(t-r)\phi_n](s, x)\, ds \xrightarrow{w} \int_0^{r_m} \beta(s, x)[\mathbb{T}(t)\phi](s, x)\, ds \quad \text{in } L^2(\Omega).$$

By the compactness of $\mathscr{T}(t-r, 0, r)$, one has

$$g_n(r) = \left\| \mathscr{T}(t-r, 0, r) \int_0^{r_m} \beta(s, x)[\mathbb{T}(t-r)(\phi_n - \phi)](s, x)\, ds \right\|_{L^2(\Omega)} \to 0.$$

On the other hand,

$$\left\| \mathscr{T}(t-r, 0, r) \int_0^{r_m} \beta(s, x)[\mathbb{T}(t-r)(\phi_n - \phi)](s, x)\, ds \right\|_{L^2(\Omega)}$$
$$\leqslant \left[\int_\Omega \int_0^{r_m} \beta^2(s, x)\, ds\, dx \int_0^{r_m} [\mathbb{T}(t-r)(\phi_n - \phi)]^2 (s, x)\, ds\, dx \right]^{1/2}$$
$$\leqslant M \|\phi_n - \phi\|$$

is bounded. Using the dominant control convergence theorem, we have that

$$\lim_{n \to \infty} \|\mathbb{T}(t)(\phi_n - \phi)\| = 0;$$

i.e., $\mathbb{T}(t)\phi_n$ strongly converges to $\mathbb{T}(t)\phi$, so $\mathbb{T}(t)$ is compact.

THEOREM 2. *The semigroup $\mathbb{T}(t)$ is compact for $t \geqslant r_m$ but not for $t < r_m$, and hence there is no analytical extension.*

From Theorem 2, $\omega_{\text{ess}}(\mathbb{A})$, the essential spectral radius of operator $\mathbb{A}$ is $-\infty$ (see [6]). We have immediately from Theorem 9.10 in [6, p. 223] that

$$\lambda_0 = \{\lambda \mid \operatorname{Re} \lambda = s(\mathbb{A})\}.$$

The conclusions above show that λ_0 is a pole of the resolvent of $R(\lambda, \mathbb{A})$, which is equivalent by (12) to $\gamma(\mathscr{B}_{\lambda_0}) = 1$ being a pole of the operator $R(\lambda, \mathscr{B}_{\lambda_0})$. On the other hand, the relation claimed in (16) shows that $\mathscr{B}_{\lambda_0}$ is a semisupporting operator introduced by Sawashima [7]. Together with the last statements, it is known from [7] that $\gamma(\mathscr{B}_{\lambda_0}) = 1$ is an algebraically simple eigenvalue of $\mathscr{B}_{\lambda_0}$, and its eigenfunction ϕ_0 defined by (17) is a quasi-interior point of $L^2(\Omega)$; i.e.,

$$\langle \phi_0, \psi \rangle > 0, \qquad \text{for all} \quad \psi(x) \in L^2(\Omega), \quad \psi(x) \geqslant 0, \quad \psi \neq 0.$$

Hence

$$\phi_0(r, x) = e^{-\lambda r} \mathscr{T}(0, r)\, \phi_0(x) \tag{20}$$

is a quasi-interior point of X.

Summarizing, we have [2, 8]

THEOREM 3. (i) *The spectrum $\sigma(\mathbb{A})$ consists of all eigenvalues of operator $\mathbb{A}$, and the intersection of any finite strip paralleling the y axis with $\sigma(\mathbb{A})$ contains at most a finite number of eigenvalues of $\mathbb{A}$.*

(ii) *$\mathbb{A}$ has only one real eigenvalue λ_0 which is algebraically simple, and it is greater than any real part of the other eigenvalues of $\mathbb{A}$.*

(iii) *The asymptotic expression*

$$\mathbb{T}(t)\,\phi(r,x)=e^{\lambda_0 t}\mathscr{T}(0,r)\,\mathbb{C}_{\lambda_0}\int_0^{r_m}\beta(r,x)\,dr\int_0^r e^{-\lambda_0(r-\sigma)}\mathscr{T}(\sigma,r)\,\phi(\sigma,r)\,d\sigma$$
$$+\,o(e^{(\lambda_0-\varepsilon)t}),$$

where

$$\mathbb{C}_{\lambda_0}=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)(I-\mathscr{B}_\lambda)^{-1} \tag{21}$$

and ε *is any positive number such that* $\sigma(\mathbb{A})\cap\{\lambda\mid\lambda_0-\varepsilon\leqslant\operatorname{Re}\lambda\leqslant\lambda_0\}=\lambda_0$, *holds.*

COROLLARY. *For any initial* $p_0(r,x)\in D(\mathbb{A})$, *the semigroup solution of Eq.* (1) *has the following asymptotic expression*:

$$p(r,t,x)=e^{\lambda_0 t}\mathscr{T}(0,r)\,\mathbb{C}_{\lambda_0}\int_0^{r_m}\beta(r,x)\,dr\int_0^r e^{-\lambda_0(r-\sigma)}\mathscr{T}(\sigma,r)\,p_0(\sigma,r)\,d\sigma$$
$$+\,o(e^{(\lambda_0-\varepsilon)t}).$$

We generalize all the results for population operators involving no spatial diffusion to the general case of a population system with spatial diffusion; the former case was investigated fully in [9].

ACKNOWLEDGMENTS

The support of earmarked research grants from RGC of Hong Kong and the Postdoctoral Science Foundation of China are gratefully acknowledged.

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Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis