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Cumulants in noncommutative probability theory IV. Noncrossing cumulants: De Finetti's theorem and L^p-inequalities ☆

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Abstract

De Finetti's theorem states that any exchangeable sequence of classical random variables is conditionally i.i.d. with respect to some σ-algebra. In this paper we prove a "free" noncommutative analog of this theorem, namely we show that any noncrossing exchangeability system with a faithful state which satisfies a so called weak singleton condition can be embedded into an free product with amalgamation over a certain subalgebra such that the interchangeable algebras remain interchangeable with respect to the operator-valued expectation. Vanishing of crossing cumulants can be verified by checking a certain weak freeness condition and the weak singleton condition is satisfied e.g. when the state is tracial. The proof follows the classical proof of De Finetti's theorem, the main technical tool being a noncommutative L^p-inequality for i.i.d. sums of centered noncommutative random variables in noncrossing exchangeability systems.

Keywords: Cumulants; Partition lattice; Möbius inversion; Free probability; Noncrossing partitions; Noncommutative probability

Contents

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De Finetti's theorem [4,7,12] states that for any exchangeable sequence of random variables there exists a σ-algebra conditional on which the sequence is i.i.d. There are various noncommutative versions of this theorem [1,10,11,16,24], all of which involve tensor product constructions or other commutativity conditions. Indeed there is no hope to obtain a general De Finetti's theorem without imposing additional conditions. In this paper we consider conditions under which a kind of "most noncommutative" version of De Finetti's theorem holds, namely a characterization of exchangeability systems which can be written as an amalgamated free product.

The only prerequisite for this paper is part I of the series [14], where exchangeability systems are introduced and many examples are discussed. In Section I.4.5 of that paper we presented the amalgamated free product as an operator valued exchangeability system with a conditional expectation ψ. Composing this conditional expectation with a state on the amalgamated subalgebra gives rise to a scalar valued exchangeability system.

The question now is, under which conditions can an arbitrary exchangeability system $\mathcal{E}$ be written in this form?

An obvious necessary condition is that crossing cumulants must vanish, because this is the case for the operator-valued amalgamated free cumulants and the $\mathcal{E}$-cumulants are simply the expectations of the latter, see Section I.3.6. Another necessary condition is a certain *weak singleton condition*. The singleton condition introduced in [3] is too strong, because together with the vanishing of crossing cumulants it actually implies freeness. The weak singleton condition to be defined below, however, turns out to be the right one and is automatically satisfied if the state is tracial.

The construction of the conditional expectation essentially follows the classical proof, namely by adjoining the algebra $\mathcal{B}$ of permutation invariant random variables to the initial algebra and extending to it the expectation functional. Moreover, there it is possible to construct a conditional expectation ψ_∞ onto $\mathcal{B}$ as the limit of symmetrizing maps ψ_N. There are certain technical issues regarding the faithfulness of the extension in the non-tracial case. These are solved by a certain Khinchin-type L^p-inequality which is of some independent interest.

Unfortunately we could not find an "application" of the characterization obtained in this paper, except perhaps a new description of freeness with amalgamation (called "weak freeness"), see Section 4.

The paper is organized as follows.

In Section 1 we collect a few definitions and lemmas needed for the statement and the proof of the main result.

In Section 2 we adapt a proof from [1] to the noncrossing situation. It shows that the conditional expectations ψ_N evaluated at words of the form $X_1^{(h(1))}X_2^{(h(2))}\cdots X_n^{(h(n))}$ asymptotically factor according to the connected components of the kernel partition $\ker h$, with an error term of order $1/N$, cf. the analogous commutative result in [7].

In Section 3 we prove a strong law of large numbers in noncrossing exchangeability systems under the assumption of the weak singleton condition.

In Section 4 we discuss a certain weak freeness condition and show that together with the weak singleton condition it implies that crossing cumulants vanish and thus weak freeness is the same as freeness with amalgamation.

1. Preliminaries and statement of main result

In this section we collect the necessary definitions and auxiliary results needed later on. For details we refer to part I [14].

1.1. Exchangeability systems and cumulants

We recall first that a noncommutative probability space is a pair $(\mathcal{A}, \varphi)$ consisting of a complex algebra $\mathcal{A}$ with unit I and a linear functional $\varphi : \mathcal{A} \to \mathbf{C}$ such that $\varphi(I) = 1$. An *exchangeability system* $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, \mathcal{J})$ for the noncommutative probability space $(\mathcal{A}, \varphi)$ consists of another noncommutative probability space $(\mathcal{U}, \tilde{\varphi})$ and an infinite family $\mathcal{J} = (\iota_k)_{k\in\mathbf{N}}$ of state-preserving embeddings $\iota_k : \mathcal{A} \to \mathcal{A}_k \subseteq \mathcal{U}$, which we conveniently denote by $X \mapsto X^{(k)}$, such that the image algebras $\mathcal{A}_j$ are *interchangeable* with respect to $\tilde{\varphi}$: for any family $X_1, X_2, \dots, X_n \in \mathcal{A}$, and for any choice of indices $h(1), \dots, h(n)$ the expectation is invariant under any permutation $\sigma \in \mathfrak{S}_\infty$ in the sense that

$$\tilde{\varphi}\big(X_1^{(h(1))}X_2^{(h(2))}\cdots X_n^{(h(n))}\big) = \tilde{\varphi}\big(X_1^{(\sigma(h(1)))}X_2^{(\sigma(h(2)))}\cdots X_n^{(\sigma(h(n)))}\big). \tag{1.1}$$

Denote by Π_n the lattice of partitions (or equivalence relations) of the set $[n] = \{1, 2, \dots, n\}$. The refinement order $\pi \leqslant \sigma$ means as usual that the partition π is finer than the partition σ. Then permutation invariance means that the value of the expectation (1.1) only depends on the so-called *kernel* of the map h which is the partition $\pi = \ker h \in \Pi_n$ defined by

$$i \sim_\pi j \quad \Leftrightarrow \quad h(i) = h(j)$$

and we denote the common value as

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \tilde{\varphi}\big(X_1^{(\pi(1))}X_2^{(\pi(2))}\cdots X_n^{(\pi(n))}\big). \tag{1.2}$$

Here we consider a partition $\pi \in \Pi_n$ as a function $\pi : [n] \to \mathbf{N}$, mapping each element to the number of the block containing it. This is a canonical example of an index function h with $\ker h = \pi$ and because of condition (1.1) the actual numbering of the blocks does not matter.

Throughout this paper all algebras will be C^*- or pre-C^*-algebras and we will assume that the algebra $\mathcal{U}$ is generated by the algebras $\mathcal{A}_k$ and that the action of $\mathfrak{S}_\infty$ extends to all of $\mathcal{U}$ leaving the state $\tilde{\varphi}$ invariant. For an index set $I \subseteq \mathbf{N}$ we denote by $\mathcal{A}_I$ the algebra generated by $\{\mathcal{A}_i,\ i \in I\}$. While the state φ on $\mathcal{A}$ usually will be assumed to be faithful, this is not always true for the state $\tilde{\varphi}$ on $\mathcal{U}$. Indeed a major part of this paper is dedicated to the proof that a certain GNS-state is at least partially faithful.

The constructions above can be done in the more general situation of an *operator-valued noncommutative probability space*, which is a pair $(\mathcal{A}, \psi)$ consisting of a unital algebra $\mathcal{A}$ and a *conditional expectation* ψ onto some unital subalgebra $\mathcal{B} \subseteq \mathcal{A}$. Here a conditional expectation is a unital positive map $\psi : \mathcal{A} \to \mathcal{B}$, with the property that $\psi(BXB') = B\psi(X)B'$ whenever $B, B' \in \mathcal{B}$ and $X \in \mathcal{A}$. The free amalgamated exchangeability system is an example of this more general concept, see below.

Example 1.1. The most commutative example of an exchangeability system for an arbitrary noncommutative probability space $(\mathcal{A}, \varphi)$ is the infinite tensor product

$$\mathcal{U} = \bigotimes_{i=1}^{\infty} \mathcal{A}_i$$

of infinitely many copies $\mathcal{A}_i$ of $\mathcal{A}$ with the tensor product state $\tilde{\varphi} = \bigotimes_{i=1}^{\infty} \varphi_i$, where $(\mathcal{A}_i, \varphi_i)_{i=1}^{\infty}$ is an infinite family of copies of $(\mathcal{A}, \varphi)$ and the embeddings are

$$\iota_j : X \mapsto X^{(j)} = I \otimes I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots.$$

Then the subalgebras are clearly interchangeable and the partitioned expectation (1.2) evaluates to

$$\varphi_\pi(X_1, X_2, \ldots, X_n) = \prod_{B \in \pi} \varphi\left(\prod_{i \in B} X_i\right)$$

which is familiar from classical probability theory.

Example 1.2. Taking the reduced free product instead of the tensor product leads to the free exchangeability system

$$(\mathcal{U}, \tilde{\varphi}) = \mathop{\bigstar}_{i=1}^{\infty} (\mathcal{A}_i, \varphi_i).$$

More generally, if $(\mathcal{A}, \varphi)$ in addition comes with a conditional expectation ψ onto some subalgebra $\mathcal{B}$ such that $\varphi \circ \psi = \varphi$, then one can construct the amalgamated free exchangeability system

$$(\mathcal{U}, \tilde{\psi}) = \mathop{\bigstar_{\mathcal{B}}}_{i=1}^{\infty} (\mathcal{A}_i, \psi_i).$$

While $(\mathcal{U}, \tilde{\psi}, \mathcal{J})$ is an operator valued exchangeability system for the operator valued noncommutativity space $(\mathcal{A}, \psi)$, it becomes a scalar exchangeability system $(\mathcal{A}, \varphi \circ \psi)$ for any state φ on $\mathcal{B}$ by letting $\tilde{\varphi} = \varphi \circ \tilde{\psi}$.

More examples are listed in [14]. In some sense (made precise in [19]) the reduced free product and the tensor product together with Boolean independence are the only universal exchangeability systems. The emphasis in [19], however, lies on "universality" in the sense that the partitioned moment functionals $\varphi_\pi(X_1, X_2, \dots, X_n)$ depend on the individual distributions of the X_i in a universal way (i.e., as a polynomial formula). This already excludes the free amalgamated exchangeability system constructed above; our approach is less constructive as we assume that an exchangeability system is given a priori and we do not assume universality. The concept of "identical distribution" becomes more involved, as explained below.

Subalgebras $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are called *$\mathcal{E}$-exchangeable* or, more suggestively, *$\mathcal{E}$-independent* if for any choice of random variables $X_1, X_2, \dots, X_n \in \mathcal{B} \cup \mathcal{C}$ and subsets $I, J \subseteq \{1, \dots, n\}$ such that $I \cap J = \emptyset$, $I \cup J = \{1, \dots, n\}$, $X_i \in \mathcal{B}$ for $i \in I$ and $X_i \in \mathcal{C}$ for $i \in J$, we have the identity

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_{\pi'}(X_1, X_2, \dots, X_n)$$

whenever $\pi, \pi' \in \Pi_n$ are partitions with $\pi|_I = \pi'|_I$ and $\pi|_J = \pi'|_J$. Two families of random variables (X_i) and (Y_j) are called $\mathcal{E}$-exchangeable if the algebras they generate have this property.

We say that two random variables X and $Y \in \mathcal{A}$ *have the same distribution given $\mathcal{E}$*, if for any word $W = W_1 W_2 \cdots W_n$ with $W_i \in \{X^{(1)}\} \cup \bigcup_{i \geqslant 2} \mathcal{A}_i$ the expectation $\tilde{\varphi}(W)$ does not change if we replace each occurrence of $X^{(1)}$ by $Y^{(1)}$. We call X and Y *$\mathcal{E}$-i.i.d.* if in addition they are $\mathcal{E}$-independent. Similarly a sequence $(X_i)_{i \in \mathbf{N}} \subseteq \mathcal{A}$ of $\mathcal{E}$-independent random variables is called *$\mathcal{E}$-i.i.d.* if for any word $W = W_1 W_2 \cdots W_n$ with $W_i \in \{X_i^{(1)}\colon\ i \in \mathbf{N}\} \cup \bigcup_{i \geqslant 2} \mathcal{A}_i$ the expectation $\tilde{\varphi}(W)$ does not change if we apply a permutation $\sigma \in \mathfrak{S}_\infty$ to the indices of X_i, i.e., if we replace each occurrence of X_i by $X_{\sigma(i)}$.

Then it is possible to define cumulant functionals, indexed by set partitions $\pi \in \Pi_n$, via

$$K_\pi^{\mathcal{E}}(X_1, X_2, \dots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \mu(\sigma, \pi),$$

where $\mu(\sigma, \pi)$ is the Möbius function of the lattice of set partitions, cf. part I. The use of the probabilistic terminology "independence" and "cumulants" is justified by the following proposition which establishes the analogy to classical probability.

Proposition 1.3. [14] *Two subalgebras $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are $\mathcal{E}$-independent if and only if mixed cumulants vanish, that is, whenever $X_i \in \mathcal{B} \cup \mathcal{C}$ are some noncommutative random variables and $\pi \in \Pi_n$ is an arbitrary partition such that there is a block of π which contains indices i and j such that $X_i \in \mathcal{B}$ and $X_j \in \mathcal{C}$, then $K_\pi^{\mathcal{E}}(X_1, X_2, \dots, X_n)$ vanishes.*

With this abstract formalism one can transfer many combinatorial proofs from classical probability to the general situation. One of the most useful results is the product formula of Leonov and Shiryaev.

Proposition 1.4. [14, Proposition 3.3] *Let $(X_{i,j})_{i \in \{1,\dots,m\}, j \in \{1,\dots,n_i\}} \subseteq \mathcal{A}$ be a family of noncommutative random variables containing in total $n = n_1 + n_2 + \cdots + n_m$ variables. Then every partition $\pi \in \Pi_m$ induces a partition $\tilde{\pi}$ on $\{1, \dots, n\} \simeq \{(i, j)\colon\ i \in [m],\ j \in [n_i]\}$ with blocks*

$\tilde{B} = \{(i, j)\colon\ i \in B,\ \ j \in [n_i]\}$, *that is, each block* $B \in \pi$ *is replaced by the union of the intervals* $(\{n_{i-1}+1, n_{i-1}+2, \dots, n_i\})_{i\in B}$. *Then we have*

$$K^{\mathcal{E}}_{\pi}\Bigl(\prod_{j_1} X_{1,j_1}, \prod_{j_2} X_{2,j_2}, \dots, \prod_{j_m} X_{m,j_m}\Bigr) = \sum_{\substack{\sigma\in\Pi_n\\ \sigma\vee\tilde{\hat{0}}_m=\tilde{\pi}}} K^{\mathcal{E}}_{\sigma}(X_{1,1}, X_{1,2}, \dots, X_{m,n_m}).$$

Remark and Definition 1.5. In the sequel we will frequently appeal to the following simple observation in order to reduce the amount of indices, see e.g. Corollary 1.16. We will be dealing with noncommutative polynomials involving variables $X_1^{(h(1))}X_2^{(h(2))}\cdots X_n^{(h(n))}$ for which we want to produce "independent" copies, that is, replacing $X_j^{(h(j))}$ by $X_j^{(h'(j))}$ in such a way that the ranges of the indices h and h' are disjoint. This can be interpreted as follows. Let I be an index set containing all the indices $h(j)$, $j \in \{1, \dots, n\}$. Consider the algebra $\tilde{\mathcal{A}} = \mathcal{A}_I$ generated by $(\mathcal{A}_i)_{i\in I}$ and a sequence $(I_j)_{j\in\mathbf{N}}$ of mutually disjoint index sets $I_j \subseteq \mathbf{N}$ of the same cardinality as I. Then the *extended exchangeability system* $\tilde{\mathcal{E}} = (\mathcal{U}, \tilde{\varphi}, \tilde{\mathcal{J}})$ is an exchangeability system for $\tilde{\mathcal{A}}$, where $\tilde{\iota}_j : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_j = \mathcal{A}_{I_j}$ is the natural permutation isomorphism induced by an arbitrary bijection between I and I_j. The procedure of choosing $X_j^{(h'(j))}$ and $X_j^{(h''(j))}$ in the exchangeability system $\mathcal{E}$ such that the ranges of h' and h'' are disjoint amounts to the same as taking interchangeable copies $\tilde{X}_j^{(1)}$ and $\tilde{X}_j^{(2)}$ of $\tilde{X}_j = X_j^{(h(j))} \in \tilde{A}$ in the exchangeability system $\tilde{\mathcal{E}}$. We will denote these by $X^{(I_j)} = \tilde{X}^{(j)}$. Let us illustrate this idea by a simplified example on the free group $\mathbf{F}_\infty$. The group algebra of $\mathbf{F}_\infty$ is an exchangeability system for the group algebra of $\mathbf{Z}$ and at the same time it is an exchangeability system for the group algebra of $\mathbf{F}_N$ for arbitrary N, because it can be written as

$$\mathbf{F}_\infty = \mathbf{Z} * \mathbf{Z} * \cdots = \mathbf{F}_N * \mathbf{F}_N * \cdots.$$

Thus we will sometimes do proofs for the initial exchangeability system $\mathcal{E}$ and state the results for $\tilde{\mathcal{E}}$ as corollaries. Also cumulants of polynomials $W_j \in \mathcal{A}_I$ are defined in $\tilde{\mathcal{E}}$ as well; the values do not depend on the choice of the index sets I and I_j.

Similarly we will sometimes not distinguish between i.i.d. sequences in $\mathcal{A}$ in the sense defined earlier in this section and sequences of the form $X^{(i)}$.

1.2. Noncrossing partitions and freeness

The lattice of noncrossing partitions, denoted by NC_n, will play a prominent rôle in this paper. We recall that a partition $\pi \in \Pi_n$ is *noncrossing*, if there is no quadruple of indices $i < j < k < l$, such that $i \sim_\pi k$ and $j \sim_\pi l$ and $i \nsim j$. Equivalently, noncrossing partitions can also be characterized recursively by the property that there is always at least one block which is an interval and after removing this block the remaining partition is still noncrossing.

Definition 1.6. We say that *crossing cumulants vanish* in a given exchangeability system $\mathcal{E}$ if for any n and for any choice of random variables $X_1, X_2, \dots, X_n$ we have the identity

$$K^{\mathcal{E}}_{\pi}(X_1, X_2, \dots, X_n) = 0$$

whenever π has a crossing, i.e., $\pi \notin NC_n$. We call such an exchangeability system a *noncrossing exchangeability system.*

A prominent example of a noncrossing exchangeability system is the free exchangeability system, where $(\mathcal{U}, \tilde{\varphi})$ is the reduced free product of an infinite family of copies of a given noncommutative probability space $(\mathcal{A}, \varphi)$, see Section I.4.4. We recall that the free exchangeability system is characterized by the property that

$$\tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = 0$$

whenever $\varphi(X_j) = 0$ and $h(j) \neq h(j+1)$ for every $1 \leqslant j \leqslant n-1$.

Another situation where crossing cumulants vanish is *freeness with amalgamation* [20,25], which is a noncommutative analog of conditional independence. Let $(\mathcal{A}, \psi)$ be a $\mathcal{B}$-valued noncommutative probability space. Then the amalgamated free product $(\mathcal{U}, \tilde{\psi}) = \bigstar_{\mathcal{B}}(\mathcal{A}_i, \psi_i)$ of infinitely many copies of $\mathcal{A}$ with amalgamation over $\mathcal{B}$ is a $\mathcal{B}$-valued noncrossing exchangeability system for $(\mathcal{A}, \psi)$. The amalgamated free exchangeability system is characterized by the property that

$$\tilde{\psi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = 0$$

whenever $\psi(X_j) = 0$ and $h(j) \neq h(j+1)$ for every $1 \leqslant j \leqslant n-1$. As in the case of scalar freeness, the $\mathcal{B}$-valued cumulants $K_\pi^{\mathcal{E},\psi}(X_1, \ldots, X_n)$ vanish for any partition $\pi \notin NC_n$.

Now choose any state φ on $\mathcal{B}$, then

$$\mathcal{E}^\varphi = (\mathcal{U} = \bigstar_{\mathcal{B}} \mathcal{A}_i, \varphi \circ \tilde{\psi}, \mathcal{J}) \tag{1.3}$$

is a scalar-valued exchangeability system for the noncommutative probability space $(\mathcal{A}, \varphi \circ \psi)$ whose cumulants are

$$K_\pi^{\mathcal{E},\varphi\circ\psi}(X_1, \ldots, X_n) = \varphi\big(K_\pi^{\mathcal{E},\psi}(X_1, \ldots, X_n)\big);$$

again cumulants vanish for any partition π with crossings. Other examples of noncrossing exchangeability systems can be constructed by taking conditionally free products [2]. We will see later that these cannot be realized as amalgamated free products, cf. Remark 1.20.

1.3. Multiplicative functions and convolution on the lattice of noncrossing partitions

We refer to [22] or Section I.1.3 for the definition of the incidence algebra of a poset. In the case of noncrossing partitions there is also a reduced incidence algebra of multiplicative functions. Let NCI_n be the set of intervals in NC_n and $NCI = \bigcup_n NCI_n$. Then every interval $[\pi, \sigma]$ has a canonical decomposition

$$[\pi, \sigma] \simeq [\hat{0}_1, \hat{1}_1]^{k_1} \times [\hat{0}_2, \hat{1}_2]^{k_2} \times \cdots \times [\hat{0}_1, \hat{1}_n]^{k_n} \times \cdots, \tag{1.4}$$

where $(k_n)_{n=1}^{\infty}$ is a sequence of integers with finitely many nozero entries and (1.4) is a lattice isomorphism [18]. For example, it is easy to see that for $\pi \in NC_n$ we have

$$[\hat{0}_n, \pi] \simeq \prod_{p=1}^{n} [\hat{0}_p, \hat{1}_p]^{k_p}, \tag{1.5}$$

where k_p is the number of blocks of π of size p.

A function $f : NCI \to \mathbf{C}$ is called *multiplicative* if for any interval $[\pi, \sigma]$ it satisfies

$$f\big([\pi,\sigma]\big) = f\big([\hat{0}_1, \hat{1}_1]\big)^{k_1} f\big([\hat{0}_2, \hat{1}_2]\big)^{k_2} \cdots f\big([\hat{0}_1, \hat{1}_n]\big)^{k_n},$$

where $[\pi, \sigma]$ has the decomposition (1.4). Such a function is determined by its *characteristic sequence* $f_n = f([\hat{0}_n, \hat{1}_n])$ and it is easy to see that the convolution of two multiplicative functions is again multiplicative, i.e., the multiplicative functions constitute an algebra, the so-called *reduced incidence algebra*. For a noncrossing partition π with decomposition as in (1.5) we will denote

$$f_\pi := f\big([\hat{0}_n, \pi]\big) = \prod_{p=1}^{n} f_p^{k_p}.$$

Then the convolution $f \boxstar g$ of two multiplicative functions can be calculated with the aid of the *Kreweras complementation map* [13]. This is a lattice anti-automorphism of NC_n described as follows. Paint n points on a circle and label them clockwise with numbers $1, 2, \ldots, n$. A noncrossing partition $\pi \in NC_n$ can be visualized by drawing inside the circle for each block of π the convex polygon whose vertices are the elements of the block. Now put another n points with labels $\bar{1}, \bar{2}, \ldots, \bar{n}$ on the circle, placing the point with label $\bar{k}$ between the points with label k and $k+1$ and connect the new points with each other by drawing as many lines as possible without intersecting the polygons drawn before. This leads to a noncrossing partition of the set $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ which is called the *Kreweras complement* of π and is denoted $K(\pi)$. It is easy to see that K is an order anti-automorphism, $K(\hat{0}_n) = \hat{1}_n$ and $K(\hat{1}_n) = \hat{0}_n$. It follows that $[\pi, \hat{1}_n] \simeq [\hat{0}_n, K(\pi)]$ and therefore the convolution of multiplicative functions can be written

$$(f \boxstar g)_n = f \boxstar g\big([\hat{0}_n, \hat{1}_n]\big) = \sum_{\pi \in NC_n} f_\pi g_{K(\pi)}.$$

As a consequence the reduced incidence algebra is commutative and there is a "Fourier transform" [15]: Let $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ be the characteristic sequences of two multiplicative functions f and g with $f_1 = g_1 = 1$. The formal power series

$$\varphi_f(z) = \sum_{n=1}^{\infty} f_n z^n$$

is called the *characteristic series* of f. Let $\varphi_f^{\langle -1 \rangle}(z)$ be the compositional inverse of φ_f and

$$\mathcal{F}_f(z) = \frac{1}{z} \varphi_f^{\langle -1 \rangle}(z),$$

then

$$\mathcal{F}_{f \boxstar g}(z) = \mathcal{F}_f(z)\, \mathcal{F}_g(z). \tag{1.6}$$

Two prominent multiplicative functions are the *Zeta function* $\zeta(\pi, \sigma) \equiv 1$ with characteristic series

$$\varphi_\zeta(z) = \frac{z}{1-z}$$

and its inverse, the *Möbius function* μ whose characteristic sequence is given by the signed Catalan numbers $\mu_n = (-1)^{n-1} C_{n-1} = \frac{(-1)^{n-1}}{n}\binom{2n-2}{n-1}$ and the characteristic series is

$$\varphi_\mu(z) = \frac{\sqrt{1+4z}-1}{2}.$$

These functions satisfy $\zeta \boxstar \mu = \delta$, where δ is the unit element of the reduced incidence algebra and has characteristic series $\varphi_\delta(z) = z$. Moreover, functions f and g satisfy $f \boxstar \zeta = g$ if and only if $f = g \boxstar \mu$ and this is the case if and only if

$$\varphi_f\big(z\big(1 + \varphi_g(z)\big)\big) = \varphi_g(z). \tag{1.7}$$

We will encounter applications of these formulae in Section 3.

1.4. The weak singleton condition

One more ingredient is needed for the formulation of the main result. We have already seen that the vanishing of crossing cumulants is a necessary condition for an exchangeability system to come from an amalgamated free product. This condition, however, is not sufficient as will be shown below, namely a so called *weak singleton condition* is also necessary.

Definition 1.7. [3]

(a) An exchangeability system $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, \mathcal{J})$ satisfies the *singleton condition* if

$$\tilde{\varphi}\big(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}\big) = 0$$

whenever one of the indices i_j appears only once and the corresponding random variable X_j satisfies $\varphi(X_j) = 0$.

(b) An exchangeability system $\mathcal{E}$ satisfies the *weak singleton condition* (WSC) if

$$\tilde{\varphi}\big(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}\big) = 0$$

whenever one of the indices i_j appears only once and the corresponding random variable X_j satisfies $\tilde{\varphi}(X_j^{(1)*} X_j^{(2)}) = 0$.

(c) An exchangeability system $\mathcal{E}$ satisfies the *extended weak singleton condition* ($\widetilde{\mathrm{WSC}}$) if the extended exchangeability system $\tilde{\mathcal{E}}$ of Definition 1.5 satisfies (WSC), i.e., for any finite index set $I \subseteq \mathbf{N}$ and any n-tuple of polynomials $W_1, W_2, \dots, W_n \in \mathcal{A}_I$ we have

$$\tilde{\varphi}\big(W_1^{(I_{i_1})} W_2^{(I_{i_2})} \cdots W_n^{(I_{i_n})}\big) = 0$$

whenever one of the index sets I_{i_j} appears only once and the corresponding polynomial W_j satisfies $\tilde{\varphi}(W_j^{(I_1)*} W_j^{(I_2)}) = 0$.

Remark 1.8. The weak singleton condition is indeed weaker than the singleton condition, because it follows from Corollary 1.15 that the condition $\tilde{\varphi}(X^{(1)*}X^{(2)}) = 0$ implies $\varphi(X) = 0$.

The extended WSC is introduced for technical reasons and needed only for Corollary 3.2. We were not able to prove or disprove that it follows from (WSC) in general, however, it is automatically implied by (WSC) if the initial algebra contains "enough" independent random variables, i.e., if for any n-tuple $(X_1, \dots, X_n)$ of elements of $\mathcal{A}$ there exist arbitrary many i.i.d. copies inside $\mathcal{A}$. This is the case in all examples known to the author.

The next proposition shows that a weak singleton condition holds also for cumulants, if it holds for moments.

Proposition 1.9. *Let $\mathcal{E}$ be an exchangeability system in which the weak singleton condition holds. Then $K_\pi^{\mathcal{E}}(X_1, X_2, \dots, X_n) = 0$ whenever the partition π contains a singleton $\{j\}$ such that $\tilde{\varphi}(X_j^{(1)*} X_j^{(2)}) = 0$.*

Proof. Indeed,

$$K_\pi^{\mathcal{E}}(X_1, X_2, \dots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \mu(\sigma, \pi)$$

and all terms vanish, because each $\sigma \leqslant \pi$ contains the singleton $\{j\}$. □

The starting point of this paper is the following observation and its corollary.

Lemma 1.10. *Let $\pi \in \Pi_n$ be an alternating partition, i.e., a partition in which neighbouring elements are in different blocks. Then any noncrossing partition $\sigma \leqslant \pi$ contains at least one singleton.*

Proof. Any noncrossing partition σ contains at least one interval block and the condition $\sigma \leqslant \pi$ implies that this interval block has length 1, i.e., it is a singleton. □

Corollary 1.11. *A noncrossing exchangeability system which satisfies the singleton condition is given by a reduced free product.*

Proof. Let $X_1, \dots, X_n \in \mathcal{A}$ with $\varphi(X_j) = 0$ and let $h(1), h(2), \dots, h(n)$ be indices such that $h(j) \neq h(j+1)$. We have to show that $\tilde{\varphi}(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}) = 0$. The singleton condition implies that $\varphi_\pi(X_1, X_2, \ldots, X_n) = 0$ whenever π contains a singleton and consequently for any such π the corresponding cumulant $K_\pi^{\mathcal{E}}(X_1, X_2, \ldots, X_n)$ vanishes. Now

$$\tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = \sum_{\pi \leqslant \ker h} K_\pi^{\mathcal{E}}(X_1, X_2, \ldots, X_n)$$

and by assumption the sum extends over noncrossing partitions only. By Lemma 1.10 any such partition contains a singleton and the corresponding cumulant vanishes because of Proposition 1.9. □

1.5. Conditional expectations

The proof of 1.11 stays essentially the same if the singleton condition is replaced by $(\widetilde{\mathrm{WSC}})$, resulting in an amalgamated free product. The main technical problem is the construction of the conditional expectation ψ onto a certain algebra $\mathcal{B}$ and to prove faithfulness of an extension of $\tilde{\varphi}$ on $\mathcal{B}$ in order to apply Lemma 1.10. The construction of ψ is the same as in the commutative case, namely as the limit of symmetrizing maps.

Definition 1.12. Let $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, J)$ be an exchangeability system for some noncommutative probability space $(\mathcal{A}, \varphi)$. We define the conditional expectations ψ_N, $N \in \mathbf{N}$, by

$$\psi_N(X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \sigma(X);$$

for polynomials, i.e., elements of the form $X = X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}$ this is

$$\psi_N\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} X_1^{(\sigma(h(1)))} X_2^{(\sigma(h(2)))} \cdots X_n^{(\sigma(h(n)))}.$$

Note that if h is fixed and N large enough, this only depends on $\ker h$.

We collect a few elementary properties of ψ_N. Proofs are easy and can be found in [1].

Proposition 1.13.

(1) $\tilde{\varphi} \circ \psi_N = \tilde{\varphi}$.
(2) $\psi_N(X) = X$ *if and only if* $\sigma(X) = X$ $\forall \sigma \in \mathfrak{S}_N$.
(3) $\psi_N \circ \psi_M = \psi_N$ *for* $M \leqslant N$.
(4) $\psi_N \circ \iota_k = \frac{1}{N} \sum_{j=1}^N \iota_j$ *if* $k \leqslant N$.

In contrast to finite exchangeability systems, there is a nonnegative bilinear form available in the infinite case.

Proposition 1.14. *The sesquilinear form*

$$\langle X, Y \rangle = \tilde{\varphi}\big(Y^{(1)*} X^{(2)}\big)$$

is nonnegative on $\mathcal{A}$.

Proof. Consider for fixed $N \in \mathbf{N}$ the nonnegative expectation

$$\tilde{\varphi}\big(\psi_N\big(X^{(1)}\big)^*\psi_N\big(X^{(1)}\big)\big) = \left(\frac{1}{N!}\right)^2 \sum_{\sigma,\sigma'} \tilde{\varphi}\big(X^{(\sigma(1))*}X^{(\sigma'(1))}\big) = \frac{1}{N^2}\sum_{i,j=1}^{N} \tilde{\varphi}\big(X^{(i)*}X^{(j)}\big)$$
$$= \frac{N(N-1)}{N^2}\tilde{\varphi}\big(X^{(1)*}X^{(2)}\big) + \frac{1}{N}\tilde{\varphi}\big(X^{(1)*}X^{(1)}\big).$$

Now letting $N \to \infty$ yields the claim. □

Positivity implies the Cauchy–Schwarz inequality.

Corollary 1.15. *For any $X, Y \in \mathcal{A}$*

$$\big|\tilde{\varphi}\big(Y^{(1)*}X^{(2)}\big)\big| \leqslant \tilde{\varphi}\big(X^{(1)*}X^{(2)}\big)^{1/2}\tilde{\varphi}\big(Y^{(1)*}Y^{(2)}\big)^{1/2}.$$

Similarly one can prove a multivariable Cauchy–Schwarz inequality, cf. Remark 1.5.

Corollary 1.16. *Let $\{g(1), g(2), \dots, g(m)\}$ and $\{h(1), h(2), \dots, h(n)\}$ be disjoint sets of indices and $P = P(X_1^{(g(1))}, X_2^{(g(2))}, \dots, X_m^{(g(m))})$ and $Q = P(Y_1^{(h(1))}, Y_2^{(h(2))}, \dots, Y_m^{(h(n))})$ noncommutative polynomials in X_i and Y_i, then*

$$\big|\tilde{\varphi}\big(Q^*P\big)\big|^2 \leqslant \tilde{\varphi}\big(P^*P'\big)\tilde{\varphi}\big(Q^*Q'\big),$$

where $P' = P(X_1^{(g'(1))}, X_2^{(g'(2))}, \dots, X_m^{(g'(m))})$ and $Q' = P(Y_1^{(h'(1))}, Y_2^{(h'(2))}, \dots, Y_m^{(h'(n))})$ such that h' (respectively g') is an index function whose range is disjoint from the range of h (respectively g).

After these preparations we can consider two situations in which the weak singleton condition holds.

Proposition 1.17. *Each of the following two conditions implies* (WSC) (*and* ($\widetilde{\text{WSC}}$)).

(a) *The state is faithful and the exchangeability system comes from an amalgamated free product as described in* (1.3).
(b) *The state is tracial.*

Proof. (a) Let $\tilde{\psi}$ be the conditional expectation with respect to which the algebras $\mathcal{A}_i$ are free. Then by Proposition 2.2 we have

$$0 = \tilde{\varphi}\big(X^{(1)*}X^{(2)}\big) = \tilde{\varphi}\big(\tilde{\psi}\big(X^{(1)*}X^{(2)}\big)\big) = \tilde{\varphi}\big(\tilde{\psi}\big(X^{(1)}\big)^*\tilde{\psi}\big(X^{(1)}\big)\big)$$

and by faithfulness this implies that $\tilde{\psi}(X^{(1)}) = 0$. Now if X appears as a singleton in some word, then the expectation of the word vanishes. Indeed, if $X_j = X$ and the index $h(j)$ appears only once in the range of the index function h, then we may condition on $\mathcal{A}_{h(j)}$ (see Proposition 2.1) and obtain

$$\tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = \tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots \tilde{\psi}\big(X_j^{(h(j))}\big) \cdots X_n^{(h(n))}\big) = 0.$$

(b) Let $X_1, X_2, \ldots, X_n \in \mathcal{A}$ and $h : [n] \to \mathbf{N}$ be an index function such that $h(j)$ is a singleton and assume that $\tilde{\varphi}(X_j^{(1)*} X_j^{(2)}) = 0$. We have to show that $\tilde{\varphi}(X_1^{(h(1))} \cdots X_n^{(h(n))})$ vanishes. By traciality we may assume without loss of generality that $j = n$. Then we may apply the Cauchy–Schwarz inequality of Corollary 1.16 and with any index function h' whose range is disjoint from that of h we obtain

$$\begin{aligned}\big|\tilde{\varphi}\big(X_1^{(h(1))} \cdots X_n^{(h(n))}\big)\big|^2 &\leqslant \tilde{\varphi}\big(X_1^{(h(1))} \cdots X_{n-1}^{(h(n-1))} X_{n-1}^{(h'(n-1))*} \cdots X_1^{(h'(1))*}\big) \tilde{\varphi}\big(X_n^{(h'(n))*} X_n^{(h(n))}\big) \\ &= 0. \qquad \square\end{aligned}$$

1.6. Statement of main result

The first part of Proposition 1.17 shows that the weak singleton condition is a necessary condition for an exchangeability system to come from an amalgamated free product. We can now state the main theorem of this paper.

Theorem 1.18. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, \mathcal{J})$ a noncrossing exchangeability system for $(\mathcal{A}, \varphi)$ with faithful state $\tilde{\varphi}$ which satisfies $(\widetilde{\mathrm{WSC}})$. Then $\mathcal{E}$ can be embedded into a $\mathcal{B}$-valued exchangeability system $\tilde{\mathcal{E}} = (\tilde{U}, \mathcal{J}, \psi)$ such that the interchangeable algebras $\mathcal{A}_i$ are free with amalgamation over $\mathcal{B}$ and interchangeable with respect to ψ.*

Remark 1.19. While it is true that any exchangeability system can be embedded into an amalgamated free product (the trivial one, where $\mathcal{B}$ coincides with the full algebra), it is not always true that this can be done in such a way that the $\mathcal{A}_i$ are still interchangeable. Therefore the preceding theorem is nontrivial. This is like in the commutative case, where an arbitrary exchangeable sequence of random variables is trivially conditionally independent with respect to the full σ-algebra, but they are certainly not conditional i.i.d., unless they are identical.

Remark 1.20. Other examples where crossing cumulants vanish are Boolean independence [21] and more generally conditional free independence [2]. In these examples, however, Theorem 1.18 does not apply because either the state is not faithful or the weak singleton condition fails. Let $(\mathcal{U}, \tilde{\varphi}, \psi) = \bigstar(\mathcal{A}_i, \varphi_i, \psi_i)$ be the conditionally free exchangeability system for $(\mathcal{A}, \varphi)$, cf. [2] or Section I.4.7. For our purposes it is sufficient to know the defining property

$$\tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = \varphi(X_1)\varphi(X_2)\cdots\varphi(X_n)$$

whenever $\psi(X_j) = 0$ and $h(j) \neq h(j+1)$ for every $1 \leqslant j \leqslant n-1$, and the modified pyramidal law

$$\tilde{\varphi}\big(X_1^{(1)} Y^{(2)} X_2^{(1)}\big) = \varphi(X_1)\varphi(Y)\varphi(X_2) + \psi(Y)\big(\varphi(X_1 X_2) - \varphi(X_1)\varphi(X_2)\big),$$

cf. [2] or Lemma I.4.14. Let us investigate the weak singleton condition for some element $X \in \mathcal{A}$. Denoting $\mathring{X} = X - \psi(X)$ we compute

$$\begin{aligned}
&\tilde{\varphi}\big(X^{(1)*}X^{(2)}\big) \\
&\quad = \tilde{\varphi}\big(\big(\mathring{X}^{(1)*} + \psi\big(X^*\big)\big)\big(\mathring{X}^{(2)} + \psi(X)\big)\big) \\
&\quad = \tilde{\varphi}\big(\mathring{X}^{(1)*}\mathring{X}^{(2)}\big) + \big(\varphi\big(X^*\big) - \psi\big(X^*\big)\big)\psi(X) + \psi\big(X^*\big)\big(\varphi(X) - \psi(X)\big) + \big|\psi(X)\big|^2 \\
&\quad = \big|\varphi(X)\big|^2.
\end{aligned}$$

Now let $X \in \mathcal{A}$ be any element with $\varphi(X) = 0$ but $\psi(X) \neq 0$ (this is possible unless $\varphi = \psi$; in the latter case we have just usual freeness) and find elements Y and Z such that $\varphi(YZ) \neq \varphi(Y)\varphi(Z)$. Then

$$\tilde{\varphi}\big(Y^{(1)}X^{(2)}Z^{(1)}\big) = \varphi(Y)\varphi(X)\varphi(Z) + \psi(X)\big(\varphi(YZ) - \varphi(Y)\varphi(Z)\big)$$

does not vanish as it should if the weak singleton condition were true. It follows from Proposition 1.17 that the conditional free product cannot be embedded into a free amalgamated exchangeability system with a faithful state $\tilde{\varphi}$.

In the remaining sections we will construct a conditional expectation ψ on $\mathcal{U}$ and show that the algebras $\mathcal{A}_i$ are free with respect to this ψ. The latter is constructed by a law of large numbers, namely as limit of the symmetrizing maps ψ_N of Definition 1.12. This is motivated by the following heuristics. Assume that $\mathcal{A}_i$ are free with respect to some conditional expectation ψ, then it is known that for any $X \in \mathcal{A}$ with $\psi(X) = 0$ the norm

$$\left\| \sum_{i=1}^{N} X^{(i)} \right\| \leqslant 2\sqrt{N}\|X\| \tag{1.8}$$

and therefore

$$\frac{1}{N}\sum X^{(i)} = \psi(X) + \frac{1}{N}\sum\big(X^{(i)} - \psi(X)\big)$$

converges to $\psi(X)$ in norm as N tends to infinity. We will prove an inequality similar to (1.8) in Section 3 by combinatorial methods, i.e., without assuming freeness and using only (WSC) and the fact that crossing cumulants vanish.

2. A De Finetti lemma

In this section we prove an asymptotic factorization property of the conditional expectations of Definition 1.12. First we need to review noncrossing partitioned conditional expectations [20].

Definition 2.1. [20] Let ψ be a conditional expectation. For a noncrossing partition $\pi \in NC_n$ let $b = \{k, k+1, \dots, l\}$ be an interval block and define recursively

$$\psi[\pi](X_1, X_2, \dots, X_n) = \psi[\pi \setminus b]\big(X_1, X_2, \dots, X_{k-1}, \psi(X_k X_{k+1} \cdots X_l)X_{l+1}, \dots, X_n\big).$$

These partitioned expectations appear in the calculation of Speicher's amalgamated free cumulants.

Proposition 2.2. [20] *Let $\mathcal{E} = (\bigstar_{\mathcal{B}}\mathcal{A}_i, \tilde{\psi}, \mathcal{J})$ be the amalgamated free exchangeability system for a $\mathcal{B}$-valued noncommutative probability space $(\mathcal{A}, \psi)$. Then for any noncrossing partition ρ and any finite sequence $X_1, X_2, \ldots, X_n \in \mathcal{A}$ the partitioned expectations* (1.2) *coincide with Speicher's partitioned expectations in Definition* 2.1:

$$\psi_\rho(X_1, X_2, \ldots, X_n) = \psi[\rho](X_1, X_2, \ldots, X_n).$$

More generally, for an arbitrary index function h and any noncrossing partition ρ such that $\rho \geqslant \ker h$ we have

$$\tilde{\psi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = \tilde{\psi}[\rho]\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big).$$

We will show that the conditional expectations of Definition 1.12 asymptotically have the same property. For the proof of this fact the following elementary estimate is needed in two places.

Lemma 2.3. *Let j, p, N be positive integers with $j \leqslant N$ and $p \leqslant N$, then*

$$1 - \left(1 - \frac{j}{N}\right)\left(1 - \frac{j}{N-1}\right) \cdots \left(1 - \frac{j}{N-p+1}\right) \leqslant \frac{pj}{N-p+1}. \tag{2.1}$$

Proof. Denote E_p the left-hand side of (2.1). Clearly the sequence E_p satisfies $0 \leqslant E_p \leqslant 1$, is nondecreasing and therefore $E_p \geqslant E_1 = j/N$. Moreover, it satisfies the recursion

$$E_{p+1} = 1 - \left(1 - \frac{j}{N}\right)\left(1 - \frac{j}{N-1}\right) \cdots \left(1 - \frac{j}{N-p}\right) = E_p + \frac{j}{N-p}(1 - E_p).$$

We proceed by induction to show that $C_p = \frac{pj}{N-p+1}$ is an upper bound. Suppose that for E_p we the estimate $E_p \leqslant C_p$ holds. Then

$$\begin{aligned} E_{p+1} &\leqslant C_p + \frac{j}{N-p}\left(1 - \frac{j}{N}\right) = \frac{pj}{N-p+1} + \frac{j}{N-p}\left(1 - \frac{j}{N}\right) \\ &\leqslant \frac{pj}{N-p} + \frac{j}{N-p} \leqslant \frac{(p+1)j}{N-p}. \qquad \square \end{aligned}$$

The proof of the following inequality has been adapted to noncrossing partitions from [1, Lemma 2.6]; the estimate goes back to and is a noncommutative analog of the main result in [7].

Lemma 2.4. *Let $\pi \in \Pi_n$ and $\rho \in NC_n$ such that $\rho \geqslant \pi$ containing $p = |\pi|$ and $r = |\rho|$ blocks, respectively. Then for $N \geqslant p$*

$$\begin{aligned} &\big\| \psi_N\big(X_1^{(\pi(1))} X_2^{(\pi(2))} \cdots X_n^{(\pi(n))}\big) - \psi_N[\rho]\big(X_1^{(\pi(1))}, X_2^{(\pi(2))}, \ldots, X_n^{(\pi(n))}\big) \big\| \\ &\quad \leqslant \frac{(2r-1)p^2}{N-p+1} \prod \|X_i\|. \end{aligned}$$

Proof. Let $\rho = \rho_1 < \rho_2 < \cdots < \rho_r = \hat{1}_n$ be a maximal chain in $[\rho, \hat{1}_n] \cap NC_n$ with the property that the blocks b_j of ρ can be labeled in such a way that

$$\rho_k = \{b_1 \cup b_2 \cup \cdots \cup b_k, b_{k+1}, \dots, b_r\}.$$

Such a chain can be constructed by ordering the chains with respect to their minimal elements and then successively merging the leftmost two blocks. Correspondingly we label the blocks a_j of π in such a way that $a_1, \dots, a_{j_1} \subseteq b_1$, $a_{j_1+1}, \dots, a_{j_2} \subseteq b_2$, etc., $a_{j_{r-1}+1}, \dots, a_{j_r} \subseteq b_r$. Denote $p_k = j_k - j_{k-1}$ the number of blocks of π which are contained in the kth block b_k of ρ. Let $\tilde{\rho} = \rho/\pi$, i.e., the partition of the block set of π induced by ρ: For $i, j \in \{1, \dots, p\}$ we set $i \sim_{\tilde{\rho}} j$ if a_i and a_j are contained in the same block of ρ. We have to compare the first term

$$\begin{aligned}\psi_N\big(X_1^{(\pi(1))} X_2^{(\pi(2))} \cdots X_n^{(\pi(n))}\big) &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} X_1^{(\sigma(\pi(1)))} X_2^{(\sigma(\pi(2)))} \cdots X_n^{(\sigma(\pi(n)))} \\ &= \frac{(N-p)!}{N!} \sum_{\substack{h:[p]\to[N] \\ \ker h = \hat{0}_p}} X_1^{(h(\pi(1)))} X_2^{(h(\pi(2)))} \cdots X_n^{(h(\pi(n)))} \qquad (2.2)\end{aligned}$$

with the second one

$$\begin{aligned}&\psi_N[\rho]\big(X_1^{(\pi(1))} X_2^{(\pi(2))} \cdots X_n^{(\pi(n))}\big) \\ &\quad = \left(\frac{1}{N!}\right)^r \sum_{\sigma_1, \dots, \sigma_r \in \mathfrak{S}_N} X_1^{(\sigma_{\rho(1)}(\pi(1)))} X_2^{(\sigma_{\rho(2)}(\pi(2)))} \cdots X_n^{(\sigma_{\rho(n)}(\pi(n)))} \\ &\quad = \left(\prod_{k=1}^r \frac{(N-p_k)!}{N!}\right) \sum_{\substack{h:[p]\to[N] \\ \ker h \wedge \tilde{\rho} = \hat{0}_p}} X_1^{(h(\pi(1)))} X_2^{(h(\pi(2)))} \cdots X_n^{(h(\pi(n)))}. \qquad (2.3)\end{aligned}$$

As in the proof of [1, Lemma 2.6] we now split (2.3) as

$$\begin{aligned}&\psi_N[\rho]\big(X_1^{(\pi(1))} X_2^{(\pi(2))} \cdots X_n^{(\pi(n))}\big) \\ &\quad = \left(\prod_{k=1}^r \frac{(N-p_k)!}{N!}\right) \left(\sum_{k=1}^{r-1} \sum_{\substack{h:[p]\to[N] \\ \ker h \wedge \tilde{\rho}_k = \hat{0}_p \\ \ker h \wedge \tilde{\rho}_{k+1} > \hat{0}_p}} X_1^{(h(\pi(1)))} X_2^{(h(\pi(2)))} \cdots X_n^{(h(\pi(n)))}\right. \\ &\qquad \left. + \sum_{\substack{h:[p]\to[N] \\ \ker h = \hat{0}_p}} X_1^{(h(\pi(1)))} X_2^{(h(\pi(2)))} \cdots X_n^{(h(\pi(n)))}\right).\end{aligned}$$

Up to a multiplicative constant, the last term is the same as (2.2), and we will show that the constants are asymptotically the same; but first we will bound the remaining $r-1$ terms of the sum. The conditions $\ker h \wedge \tilde{\rho}_k = \hat{0}_p$ and $\ker h \wedge \tilde{\rho}_{k+1} > \hat{0}_p$ mean that $h|_{\{1,\dots,j_k\}}$ is injective, but

$h|_{\{1,\dots,j_{k+1}\}}$ is not, i.e., at least one of the indices $h(j_k+1), h(j_k+2), \dots, h(j_{k+1})$ is contained in $\{h(1),\dots,h(j_k)\}$; remember that $h|_{\{j_k+1,\dots,j_{k+1}\}}$ *is* injective. Thus

$$\sum_{\substack{h:[p]\to[N]\\ \ker h\wedge\tilde{\rho}_k=\hat{0}_p\\ \ker h\wedge\tilde{\rho}_{k+1}>\hat{0}_p}} = \sum_{h(1),\dots,h(k)\text{ distinct}} \quad \sum_{\substack{h(j_k+1),\dots,h(j_{k+1})\text{ distinct}\\ \{h(1),\dots,h(k)\}\cap\{h(j_k+1),\dots,h(j_{k+1})\}\neq\emptyset}} \quad \sum_{h(j_{k+2}),\dots,h(j_r)}.$$

There are $N(N-1)\cdots(N-j_k+1)$ different choices for $h(1), h(2), \dots, h(j_k)$,

$$N(N-1)\cdots(N-p_{k+1}+1)-(N-j_k)(N-j_k-1)\cdots(N-j_k-p_{k+1}+1)$$

choices for $h(j_k+1), h(2), \dots, h(j_{k+1})$, and

$$\prod_{s=k+2}^{r}\frac{N!}{(N-p_s)!}$$

possibilities to choose the remaining indices $h(j_{k+1}+1), \dots, h(p)$. The kth term can therefore be estimated by

$$\left(\prod_{s=1}^{k+1}\frac{(N-p_s)!}{N!}\right)\frac{N!}{(N-j_k)!}\left(\frac{N!}{(N-p_{k+1})!}-\frac{(N-j_k)!}{(N-j_k-p_{k+1})!}\right)\prod_{j=1}^{n}\|X_j\|.$$

By Lemma 2.3

$$\begin{aligned}
&\frac{(N-p_{k+1})!}{N!}\left(\frac{N!}{(N-p_{k+1})!}-\frac{(N-j_k)!}{(N-j_k-p_{k+1})!}\right)\\
&\quad=1-\frac{(N-j_k)(N-j_k-1)\cdots(N-j_k-p_{k+1}+1)}{N(N-1)\cdots(N-p_{k+1}+1)}\\
&\quad=1-\left(1-\frac{N-j_k}{N}\frac{N-j_k-1}{N-1}\cdots\frac{N-j_k-p_{k+1}+1}{N-p_{k+1}+1}\right)\\
&\quad=1-\left(1-\frac{j_k}{N}\right)\left(1-\frac{j_k}{N-1}\right)\cdots\left(1-\frac{j_k}{N-p_{k+1}+1}\right)\\
&\quad\leqslant\frac{p_{k+1}j_k}{N-p_{k+1}+1}
\end{aligned}$$

and therefore the kth term is smaller than

$$\begin{aligned}
&\frac{N(N-1)\cdots(N-j_k+1)}{N(N-1)\cdots(N-p_1+1)\cdots N(N-1)\cdots(N-p_k+1)}\frac{p_{k+1}j_k}{N-p_{k+1}+1}\prod\|X_i\|\\
&\quad\leqslant\frac{p_{k+1}j_k}{N-p_{k+1}+1}\prod\|X_i\|
\end{aligned}$$

and

$$\sum_{k=1}^{r-1} \frac{p_{k+1} j_k}{N - p_{k+1} + 1} \leqslant (r-1) \frac{\bar{p} p}{N - \bar{p} + 1},$$

where $\bar{p} = \max p_k$. Now we come to the difference between the final term and (2.2).

$$\begin{aligned}
&\left\| \left(\prod_{k=1}^{r} \frac{(N-p_k)!}{N!} - \frac{(N-p)!}{N!} \right) \sum_{\ker h = \hat{0}_p} X_1^{(h(\pi(1)))} X_2^{(h(\pi(2)))} \cdots X_n^{(h(\pi(n)))} \right\| \\
&\quad \leqslant \left(\prod_{k=1}^{r} \frac{(N-p_k)!}{N!} - \frac{(N-p)!}{N!} \right) N(N-1)\cdots(N-p+1) \prod \|X_i\| \\
&\quad = \left(1 - \prod_{k=1}^{r} \frac{(N-p_k)!}{N!} N(N-1)\cdots(N-p+1) \right) \prod \|X_i\| \\
&\quad = \left(1 - \prod_{k=1}^{r} \frac{(N-j_{k-1})(N-j_{k-1}-1)\cdots(N-j_k+1)}{N(N-1)\cdots(N-p_k+1)} \right) \prod \|X_i\| \\
&\quad = \left(1 - \prod_{k=1}^{r} \left(1 - \frac{j_{k-1}}{N}\right)\left(1 - \frac{j_{k-1}}{N-1}\right) \cdots \left(1 - \frac{j_{k-1}}{N-p_k+1}\right) \right) \prod \|X_i\| \\
&\quad \leqslant \left(1 - \prod_{k=1}^{r} \left(1 - \frac{p_k j_{k-1}}{N-p_k+1}\right) \right) \prod \|X_i\| \\
&\quad \leqslant \left(1 - \left(1 - \frac{p\bar{p}}{N-\bar{p}+1}\right)^r \right) \prod \|X_i\| \\
&\quad \leqslant r \frac{p\bar{p}}{N-\bar{p}+1} \prod \|X_i\|
\end{aligned}$$

by Lemma 2.3. □

The conditional expectations ψ_N need not converge but we can construct a limit by extending the algebra with the help of the GNS-construction as in [1]; the price of this is a possible loss of faithfulness, which will be repaired in the next section. Let $\pi : \mathcal{U} \to B(\mathfrak{H})$ be the GNS representation of $\mathcal{U}$ on $\mathfrak{H} = L^2(\mathcal{U}, \tilde{\varphi})$. By assumption it is faithful and cyclic with cyclic vector ξ_0, i.e., $\{\pi(X)\xi_0\colon X \in \mathcal{U}\}$ is a dense subspace of $\mathfrak{H}$ and $\tilde{\varphi}(X) = \langle \pi(X)\xi_0, \xi_0 \rangle$. Since we assumed that $\mathcal{U}$ is generated by $(\mathcal{A}_i)_{i \in I}$, the action of $\mathfrak{S}_\infty$ on $\mathcal{U}$ can be extended to a representation U_σ on $\mathfrak{H}$ which is characterized by

$$U_\sigma \pi(X)\xi_0 = \pi\big(\sigma(X)\big)\xi_0 \quad \forall \sigma \in \mathfrak{S}_\infty,\ \forall X \in \mathcal{U}.$$

Let

$$\mathfrak{H}_\infty = \{\xi \in \mathfrak{H}\colon U_\sigma \xi = \xi\ \forall \sigma \in \mathfrak{S}_\infty\}$$

be the subspace of U-invariant elements and $P_\infty : \mathfrak{H} \to \mathfrak{H}_\infty$ the orthogonal projection, and define

$$\psi_\infty(X) = P_\infty \pi(X) P_\infty.$$

Similarly the projection P_N onto

$$\big[\pi\big(\psi_N(\mathcal{U})\big)\xi_0\big] = \{\xi \in \mathfrak{H}\colon\ U_\sigma \xi = \xi\ \forall \sigma \in \mathfrak{S}_N\}$$

is characterized by the property

$$P_N \pi(X)\xi = \pi\big(\psi_N(X)\big)\xi \quad \forall \sigma \in \mathfrak{S}_N,\ \forall X \in \mathcal{U}.$$

Clearly every $P_N \geqslant P_\infty$ and the sequence P_N is monotonically decreasing to its strong limit P_∞. We continue our work in the extended noncommutative probability space $(\tilde{\mathcal{U}}, \tilde{\tilde{\varphi}})$ generated by $\pi(\mathcal{U})$ and P_∞ and where the state $\tilde{\tilde{\varphi}}(X) = \langle X\xi_0, \xi_0\rangle$ is the GNS-extension of $\tilde{\varphi}$. We may also consider it as an operator-valued noncommutative probability space with the conditional expectation

$$\psi_\infty : \tilde{\mathcal{U}} \to \mathcal{B} = P_\infty \tilde{\mathcal{U}} P_\infty$$

and we have $\tilde{\tilde{\varphi}} = \tilde{\tilde{\varphi}} \circ \psi_\infty$. Moreover,

$$\tilde{\tilde{\varphi}}\big(\psi_\infty(X_1)\psi_\infty(X_2)\cdots\psi_\infty(X_n)\big) = \lim_{N\to\infty} \tilde{\varphi}\big(\psi_N(X_1)\psi_N(X_2)\cdots\psi_N(X_n)\big);$$

in particular, $\tilde{\tilde{\varphi}}$ is a trace if $\tilde{\varphi}$ is a trace. As a corollary to Lemma 2.4 we have the following generalization of [1, Lemma 3.1].

Lemma 2.5. *Let $h : [n] \to \mathbf{N}$ be an index function and let ρ be any noncrossing partition such that $\rho \geqslant \ker h$. Then for any sequence $X_1, X_1, \ldots, X_n \in \mathcal{A}$ we have the factorization*

$$\begin{aligned}\tilde{\varphi}\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) &= \lim_{N\to\infty} \big\langle \psi_N[\rho]\big(X_1^{(h(1))}, X_2^{(h(2))}, \ldots, X_n^{(h(n))}\big)\xi_0, \xi_0\big\rangle \\ &= \big\langle \psi_\infty[\rho]\big(X_1^{(h(1))}, X_2^{(h(2))}, \ldots, X_n^{(h(n))}\big)\xi_0, \xi_0\big\rangle.\end{aligned}$$

First part of the proof of Theorem 1.18. Let $\mathcal{U}_0 \subseteq \tilde{\mathcal{U}}$ be the algebra of polynomials, i.e., the (non-closed) algebra generated by $(\mathcal{A}_i)_{i\in I}$ and $\mathcal{B}_0 = \psi_\infty(\mathcal{U}_0)$ its image under ψ (as a vector space). Let us assume for a moment that $\tilde{\tilde{\varphi}}$ is faithful on $\mathcal{B}_0$ in the sense that for any element $W \in \mathcal{B}_0$ the equation $\tilde{\tilde{\varphi}}(W^*W) = 0$ implies that $W = 0$. We show that the images $\tilde{\mathcal{A}}_i = \pi(\mathcal{A}_i)$ under the GNS representation of $\mathcal{U}$ are free with amalgamation over $\mathcal{B}$. To this end let $X_j \in \mathcal{A}$, $1 \leqslant j \leqslant n$ be an arbitrary finite sequence with $\psi_\infty(X_j) = 0$ and let $h : [n] \to \mathbf{N}$ be an index function with $h(j) \neq h(j+1)$. We have to show that

$$\psi_\infty\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big) = 0.$$

By the assumed faithfulness of $\tilde{\tilde{\varphi}}$ on $\mathcal{B}_0$ it suffices to show that

$$\tilde{\tilde{\varphi}}\big(\psi_\infty\big(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\big)Y\big) = 0 \quad \forall Y \in \mathcal{B}_0$$

and it is enough to consider monomials of the form

$$Y = \psi_\infty\bigl(Y_1^{(g(1))} Y_2^{(g(2))} \cdots Y_m^{(g(m))}\bigr)$$

with $Y_j \in \mathcal{A}$ and g an arbitrary index function. Indeed, any element of $\mathcal{B}_0$ is a sum of products of elements like this, and for products we have for any $X \in \mathcal{B}_0$

$$\begin{aligned}
&\tilde{\tilde{\varphi}}\bigl(X\psi_\infty\bigl(Y_1^{(f(1))} Y_2^{(f(2))} \cdots Y_p^{(f(p))}\bigr)\psi_\infty\bigl(Z_1^{(g(1))} Z_2^{(g(2))} \cdots Z_q^{(g(q))}\bigr)\bigr) \\
&\quad = \tilde{\tilde{\varphi}}\bigl(X\psi_\infty\bigl(Y_1^{(f(1))} Y_2^{(f(2))} \cdots Y_p^{(f(p))} Z_1^{(g'(1))} Z_2^{(g'(2))} \cdots Z_q^{(g'(q))}\bigr)\bigr),
\end{aligned}$$

where g' is an index function with $\ker g' = \ker g$ and whose range is disjoint from the range of f. Thus consider

$$\begin{aligned}
&\tilde{\tilde{\varphi}}\bigl(\psi_\infty\bigl(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\bigr)\psi_\infty\bigl(Y_1^{(g(1))} Y_2^{(g(2))} \cdots Y_m^{(g(m))}\bigr)\bigr) \\
&\quad = \lim_{N\to\infty} \tilde{\varphi}\bigl(\psi_N\bigl(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}\bigr)\psi_N\bigl(Y_1^{(g(1))} Y_2^{(g(2))} \cdots Y_m^{(g(m))}\bigr)\bigr) \\
&\quad = \lim_{N\to\infty} \tilde{\varphi}\bigl(\psi_N\bigl(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))} Y_1^{(g(1))} Y_2^{(g(2))} \cdots Y_m^{(g(m))}\bigr)\bigr),
\end{aligned}$$

where we assume without loss of generality that h and g have disjoint range,

$$\begin{aligned}
&= \tilde{\varphi}\bigl(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))} Y_1^{(g(1))} Y_2^{(g(2))} \cdots Y_m^{(g(m))}\bigr) \\
&= \sum_{\substack{\rho_1 \leqslant \ker h \\ \rho_2 \leqslant \ker g}} K^{\mathcal{E}}_{\rho_1 \cup \rho_2}(X_1, \ldots, X_n, Y_1, \ldots, Y_m);
\end{aligned}$$

by assumption the sum runs over all noncrossing partitions only. By Lemma 1.10 any noncrossing partition $\rho_1 \leqslant \ker h$ contains a singleton, say $\{j\}$. Now

$$\begin{aligned}
0 = \tilde{\tilde{\varphi}}\bigl(\psi_\infty(X_j)^* \psi_\infty(X_j)\bigr) &= \lim_{N\to\infty} \tilde{\varphi}\bigl(\psi_N(X_j)^* \psi_N(X_j)\bigr) \\
&= \lim_{N\to\infty} \tilde{\varphi}\bigl(\psi_N\bigl(X_j^{(1)*} X_j^{(2)}\bigr)\bigr) = \tilde{\varphi}\bigl(X_j^{(1)*} X_j^{(2)}\bigr)
\end{aligned}$$

and we may apply Proposition 1.9 to every term of the sum to see that it vanishes. □

The main problem is now to prove faithfulness of $\tilde{\tilde{\varphi}}$ on $\mathcal{B}_0$. In the tracial case we may dispose of this problem as follows.

End of the proof of Theorem 1.18 in the tracial case. If $\tilde{\varphi}$ is a trace, so is $\tilde{\tilde{\varphi}}$ and its kernel is a two-sided ideal. Since $\tilde{\varphi}$ is faithful, the intersection of $\pi(\mathcal{U})$ with $\ker\tilde{\tilde{\varphi}}$ is trivial and therefore $\mathcal{U}$ is faithfully embedded into the quotient algebra $\tilde{U}/\ker\tilde{\tilde{\varphi}}$, on which the trace is faithful. Now we can apply the arguments of the proof above with $\tilde{U}$ replaced by the quotient $\tilde{U}/\ker\tilde{\tilde{\varphi}}$. □

In the non-tracial case there is more work to do, namely we will show that the extended state $\tilde{\tilde{\varphi}}$ is indeed faithful on $\mathcal{B}_0$. To this end we need a very strong law of large numbers for noncrossing exchangeability systems, which we prove in the next section.

3. A noncommutative L^p-inequality in the case of noncrossing cumulants

Our aim is to show that $\psi_\infty(X) = 0$ if $\tilde{\varphi}(X^{(1)*}X^{(2)}) = 0$. That is, such random variables satisfy a very strong law of large numbers. We need a combinatorial proof in order to use the combinatorial information about cumulants, that is, we will use the fact that for a faithful state $\tilde{\varphi}$ we have

$$\left\|\sum X_i\right\| = \lim_{p\to\infty} \tilde{\varphi}\Bigl(\Bigl(\Bigl(\sum X_i\Bigr)^*\Bigl(\sum X_i\Bigr)\Bigr)^p\Bigr)^{1/2p}. \tag{3.1}$$

The proof is somewhat in the spirit of [17] where it is shown that the noncommutative L^p-norms of so-called p-orthogonal sums (of which our situation is a special case) can be estimated

$$\left\|\sum X_i\right\|_{L^{2p}(\tau)} \leqslant \frac{3\pi}{2}\, p\, S(X, p),$$

where

$$S(X, p) = \max\Bigl\{\Bigl\|\Bigl(\sum X_i^* X_i\Bigr)^{1/2}\Bigr\|, \Bigl\|\Bigl(\sum X_i X_i^*\Bigr)^{1/2}\Bigr\|\Bigr\}.$$

However, in order to get something useful out of (3.1) we will need constants which stay bounded as p tends to infinity. This is related to the question in [17, Remark 0.3] whether there are uniform constants for free martingale inequalities, owing to the fact that the size of the lattice NC_n of noncrossing partitions is of order 4^n, while the size of the lattice of all partitions Π_n is much bigger. The tracial version in Proposition A.1 gives further evidence for a positive answer to this question. For our purposes, however, we need a variant of the inequality for i.i.d. sequences also in the nontracial case.

Proposition 3.1. *Assume that a noncrossing exchangeability system $\mathcal{E}$ satisfies the weak singleton condition and has a faithful state. Then for any selfajoint random variable X with $\tilde{\varphi}(X^{(1)}X^{(2)}) = 0$ the interchangeable sequence $X^{(i)}$ satisfies the inequality*

$$\left\|\sum_{i=1}^N X^{(i)}\right\| \leqslant \frac{2\sqrt{N}}{1 - 1/\sqrt{N}}\|X\|$$

for every $N \geqslant 2$.

Proof. We give three estimates with increasing difficulty and accuracy. Roughly the idea is as follows. We assume that $X^{(i)}$ are as in the statement of the proposition. By faithfulness of $\tilde{\varphi}$, we can use (3.1) although the "L^p-norm" associated to $\tilde{\varphi}$ is not really a norm. We can expand the latter in terms of cumulants:

$$\begin{aligned}\tilde{\varphi}\Bigl(\Bigl(\sum X^{(i)}\Bigr)^p\Bigr) &= \sum_{\pi\in NC_p} K_\pi\Bigl(\sum X^{(i)}\Bigr) = \sum_{\pi\in NC_p}\sum_{\ker h\geqslant\pi} K_\pi\bigl(X^{(h(1))}, X^{(h(2))}, \ldots, X^{(h(p))}\bigr)\\ &= \sum_\pi N^{|\pi|} K_\pi(X).\end{aligned}$$

First estimate. Because of the weak singleton condition only partitions without singletons contribute. Any such partition has at most $\frac{p}{2}$ blocks and therefore the sum is of order $N^{p/2}$ times the number of noncrossing partitions:

$$\leqslant N^{p/2}\frac{1}{p+1}\binom{2p}{p}\max_{\pi}\bigl|K_\pi(X)\bigr|.$$

Each cumulant K_π, in turn, can be estimated by

$$\begin{aligned}\bigl|K_\pi(X)\bigr| &= \biggl|\sum_{\sigma\leqslant\pi}\varphi_\sigma(X)\mu_{NC}(\sigma,\pi)\biggr| \leqslant \frac{1}{p+1}\binom{2p}{p}\|X\|^p\max_{\sigma,\pi}\bigl|\mu_{NC}(\sigma,\pi)\bigr| \\ &\simeq 16^p\|X\|^p.\end{aligned}$$

Thus by this first rough estimate we obtain the inequality

$$\Bigl|\tilde{\varphi}\Bigl(\Bigl(\sum X^{(i)}\Bigr)^p\Bigr)\Bigr| \leqslant 64^p N^{p/2}\|X\|^p$$

and taking limits

$$\Bigl\|\sum X^{(i)}\Bigr\| \leqslant 64\sqrt{N}\|X\|.$$

Second estimate. With a little effort, we can improve on the constant considerably. First note that we can evaluate $a_\pi = \sum_{\sigma\leqslant\pi}|\mu_{NC}(\sigma,\pi)|$ explicitly. Since μ_{NC} is a multiplicative function, so are $|\mu_{NC}|$ and $a = |\mu_{NC}| \boxtimes \zeta$. By applying the Kreweras complementation map we have

$$a_n = \sum_{\sigma\in NC_n}\bigl|\mu_{NC}(\sigma,\hat{1}_n)\bigr| = \sum_{\sigma\in NC_n}\bigl|\mu_{NC}(\hat{0}_n,\sigma)\bigr|, \tag{3.2}$$

i.e., $a = |\mu_{NC}| \boxtimes \zeta$ and we can use (1.7). The characteristic series of $|\mu_{NC}|$ is

$$\varphi_{|\mu|} = \frac{1}{2}\bigl(1-\sqrt{1-4z}\bigr)$$

and $\varphi_a(z)$ satisfies the equation

$$\frac{1}{2}\Bigl(1-\sqrt{1-4z\bigl(1+\varphi_a(z)\bigr)}\Bigr) = \varphi_a(z).$$

Together with the condition $\varphi_a(0) = 0$ this yields the solution

$$\varphi_a(z) = \frac{1}{2}\bigl(1-z-\sqrt{1-6z+z^2}\bigr) = z+2z^2+6z^3+22z^4+\cdots.$$

This is the generating function of the "large Schröder numbers" [6,23]; they show up in a similar context in [8].

We have to estimate

$$\left|\sum_{\pi\in NC_p^{\geqslant 2}} N^{|\pi|}K_\pi(X)\right| \leqslant \sum_{\pi\in NC_p^{\geqslant 2}} N^{|\pi|}a_\pi \|X\|^p,$$

where $NC_p^{\geqslant 2}$ is the set of noncrossing partitions without singletons. The sequence

$$b_n = \sum_{\pi\in NC_n^{\geqslant 2}} N^{|\pi|}a_\pi$$

is the characteristic sequence of the convolution of the multiplicative function $N\cdot\mathring{a}$ with characteristic sequence $(N\mathring{a}_n)_n$ with the ζ-function, where

$$\mathring{a}_n = \begin{cases} 0 & n=1,\\ a_n & n\geqslant 2,\end{cases}$$

and

$$\varphi_{\mathring{a}}(z) = \frac{1}{2}\bigl(1-3z-\sqrt{1-6z+z^2}\bigr).$$

The characteristic series $\varphi_b(z)$ can be found by yet another appeal to (1.7), namely it satisfies the equation

$$N\varphi_{\mathring{a}}\bigl(z\bigl(1+\varphi_b(z)\bigr)\bigr) = \varphi_b(z)$$

and the relevant solution is

$$\varphi_b(z) = \frac{2(N+1)}{N+2+3Nz+N\sqrt{1-6z+(1-8N)z^2}} - 1.$$

The dominant singularity comes from the radical $1-6z+(1-8N)z^2$. The zeros of the latter are $\frac{1}{8N-1}(\pm\sqrt{2(N+1)}-3)$ and therefore

$$b_n \sim \left(\frac{8N-1}{2\sqrt{2(N+1)}-3}\right)^n;$$

it follows that

$$\left\|\sum X^{(i)}\right\| \leqslant \frac{8N-1}{2\sqrt{2(N+1)}-3}\|X\|.$$

The constant tends to $2\sqrt{2}$ as $N\to\infty$, which is not bad, as the best possible constant is 2.

Third estimate. With even some more effort, one can obtain the optimal constant (at least as $N\to\infty$) as follows. The previous estimate was done using the numbers a_n from (3.2) and we neglected the fact that for the calculation of the cumulants $K_\pi(X)$ partitions with singletons do not contribute. Thus it will be more accurate to work with the numbers

$$\tilde{a}_n = \sum_{\pi \in NC_n^{\geqslant 2}} \left|\mu_{NC}(\pi, \hat{1}_n)\right| \tag{3.3}$$

which constitute the characteristic sequence of the multiplicative function $\mathring{\zeta} \boxtimes |\mu_{NC}|$ where

$$\mathring{\zeta}_n = \begin{cases} 0, & n = 1, \\ 1, & n \geqslant 2, \end{cases}$$

is the Zeta function on the poset of noncrossing partitions without singletons. This convolution can be carried out with the aid of (1.6). The "Fourier transforms" of the functions

$$\varphi_{\mathring{\zeta}}(z) = \sum_{n=2}^{\infty} z^n = \frac{z^2}{1-z} \quad \text{and} \quad \varphi_{|\mu|}(z) = \frac{1}{2}\bigl(1 - \sqrt{1-4z}\bigr)$$

are

$$\mathcal{F}_{\mathring{\zeta}} = \frac{\pm\sqrt{z^2+4z} - z}{2z} \quad \text{and} \quad \mathcal{F}_{|\mu|}(z) = 1 - z,$$

respectively. Therefore

$$\mathcal{F}_{\mathring{\zeta} \boxtimes |\mu|}(z) = \frac{\pm\sqrt{z^2+4z} - z}{2z}(1-z),$$

i.e., $y = y(z) = \varphi_{\mathring{\zeta} \boxtimes |\mu|}(z)$ satisfies the algebraic equation

$$y(1-y)(1-y-z) = z^2.$$

We are interested in the asymptotics of the numbers

$$\tilde{b}_n = \sum_{\pi \in NC_n} N^{|\pi|} \tilde{a}_\pi$$

whose generating function can be determined by (1.7), namely

$$N\varphi_{\tilde{a}}\bigl(z\bigl(1 + \varphi_{\tilde{b}}(z)\bigr)\bigr) = \varphi_{\tilde{b}}(z).$$

Thus $x = x(z) = \varphi_{\tilde{b}}(z)$ satisfies the equations

$$N\varphi_{\tilde{a}}\bigl(z(1+x)\bigr) = x, \qquad \frac{x}{N}\left(1 - \frac{x}{N}\right)\left(1 - \frac{x}{N} - z(1+x)\right) = z^2(1+x)^2,$$

therefore $x = x(z)$ is the solution of the equation

$$g(x,z) = \frac{x}{N}\left(1 - \frac{x}{N}\right)\left(1 - \frac{x}{N} - z(x+1)\right) - z^2(x+1)^2 = 0. \tag{3.4}$$

If $x(z)$ has a singularity at z, then both $g(x,z)=0$ and $\partial_x g(x,z)=0$ and therefore z is a zero of the resultant

$$\begin{aligned}
&\operatorname{Res}\bigl(g(x,z),\partial_z g(x,z)\bigr)\\
&\quad=\frac{1}{N^9}(Nz+1)\,(N+1)^2\,z^2\\
&\qquad\times\bigl(8z-2Nz+32z^2-4z^3+26Nz^2+16Nz^3-N^2z^2+10N^2z^3-N^2z^4+4N^3z^4-5\bigr),
\end{aligned}$$

cf. [5,9]. By Pringsheim's theorem we know that the dominant singularity is positive and therefore it must be a root of the last factor

$$r(z)=-5+(8-2N)z+\bigl(32+26N-N^2\bigr)z^2+\bigl(-4+16N+10N^2\bigr)z^3+\bigl(-N^2+4N^3\bigr)z^4.$$

We claim that

$$r(z)\neq 0\quad\text{for } 0\leqslant z\leqslant\frac{1}{2\sqrt{N}}\left(1-\frac{1}{\sqrt{N}}\right).$$

Indeed, let

$$z=\frac{\alpha}{2\sqrt{N}}\left(1-\frac{1}{\sqrt{N}}\right)\quad\text{with } 0\leqslant\alpha\leqslant 1,$$

then it is tedious but not difficult to verify that

$$\begin{aligned}
&r\left(\frac{\alpha}{2\sqrt{N}}\left(1-\frac{1}{\sqrt{N}}\right)\right)\\
&\quad=N\left(\frac{\alpha^4}{4}-\frac{\alpha^2}{4}\right)+N^{1/2}\left(-\alpha+\frac{\alpha^2}{2}+\frac{5}{4}\alpha^3-\alpha^4\right)\\
&\qquad-5+\alpha+\frac{25}{4}\alpha^2-\frac{15}{4}\alpha^3+\frac{23}{16},\alpha^4+N^{-1/2}\left(4\alpha-13\alpha^2+\frac{23}{4}\alpha^3-\frac{3}{4}\alpha^4\right)\\
&\qquad+N^{-1}\left(-4\alpha+\frac{29}{2}\alpha^2-\frac{29}{4}\alpha^3-\frac{1}{8}\alpha^4\right)+N^{-3/2}\left(-16\alpha^2+\frac{11}{2}\alpha^3+\frac{1}{8}\alpha^4\right)\\
&\qquad+N^{-2}\left(8\alpha^2-\frac{1}{2}\alpha^3-\frac{1}{16}\alpha^4\right)-N^{-5/2}\frac{3}{2}\alpha^3+N^{-3}\frac{1}{2}\alpha^3\\
&\quad=-\frac{N}{4}\alpha^2\bigl(1-\alpha^2\bigr)-N^{1/2}\left(\frac{1}{4}\left(\frac{1}{4}-\left(\alpha-\frac{1}{2}\right)^2\right)+(1-\alpha)+\alpha\left(\alpha-\frac{1}{2}\right)^2\right)\\
&\qquad-\bigl(1-N^{-1/2}\bigr)\left(\frac{1}{16}+(1-\alpha)\left(\frac{47}{16}+\frac{29}{8}\alpha+2(1-\alpha)^2+\frac{23}{16}\bigl(1-(1-\alpha)^3\bigr)\right)\right)\\
&\qquad-\bigl(N^{-1/2}-N^{-1}\bigr)\left(\frac{877}{256}+\frac{3}{32}(1-\alpha)+\frac{27}{8}\left(\frac{1}{8}-\left(\alpha-\frac{1}{2}\right)^3\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\alpha-\frac{1}{2}\right)^2\left(\frac{685}{256}+\frac{11}{16}\left(\frac{1}{4}-\left(\alpha-\frac{1}{2}\right)^2\right)\right)\bigg)\bigg) \\
&-N^{-1}\left(\frac{15}{16}+3(1-\alpha)\big(1-(1-\alpha)^2\big)+\frac{37}{8}(1-\alpha)^2-\frac{9}{16}(1-\alpha)^4\right) \\
&-\big(N^{-3/2}-N^{-2}\big)\alpha^2\left(16-\frac{11}{2}\alpha-\frac{1}{4}\alpha^2\right)-N^{-2}\alpha^2\left(8-5\alpha-\frac{3}{16}\alpha^2\right) \\
&-N^{-5/2}\big(3-N^{-1/2}\big)\frac{1}{2}\alpha^3
\end{aligned}
$$

which is strictly negative for $0 \leqslant \alpha \leqslant 1$. Therefore asymptotically as n tends to infinity we have

$$
\tilde{b}_n \leqslant \left(\frac{2\sqrt{N}}{1-1/\sqrt{N}}\right)^n. \qquad \square
$$

Using Remark 1.5 we obtain the following corollary.

Corollary 3.2. *Assume that a noncrossing exchangeability system $\mathcal{E}=(\mathcal{U},\tilde{\varphi},\mathcal{J})$ satisfies* $(\widetilde{\mathrm{WSC}})$ *and has a faithful state. Let $I \subseteq \mathbf{N}$ be a finite index set and (I_j) a sequence of disjoint index sets of the same cardinality as I, cf. Remark* 1.5. *Let $X \in \mathcal{A}_I$ be a selfadjoint polynomial with $\tilde{\varphi}(X^{(I_1)}X^{(I_2)})=0$, then the interchangeable sequence $X^{(I_j)}$ satisfies the inequality*

$$
\left\|\sum_{j=1}^{N} X^{(I_j)}\right\| \leqslant \frac{2\sqrt{N}}{1-1/\sqrt{N}}\|X\|
$$

for every $N \geqslant 2$. In particular, $\psi_N(X^{(I)})$ is $\mathcal{O}(1/\sqrt{N})$ and converges to zero as N tends to infinity.

Remark 3.3. This is the only place where $(\widetilde{\mathrm{WSC}})$ is needed rather than (WSC). While $(\widetilde{\mathrm{WSC}})$ holds in all examples known to us, we were not able to decide whether it follows from (WSC).

End of the proof of Theorem 1.18 in the nontracial case. It remains to prove faithfulness of the state $\tilde{\varphi}$ on $\mathcal{B}_0$. Let $X \in \mathcal{B}_0$ such that $\tilde{\varphi}(X^*X)=0$, i.e., $X=\psi_\infty(W)$, where $W \in \mathcal{U}_0$ is some polynomial, say $W \in A_I$ for some finite index set I, and let I_1 and I_2 be disjoint copies of I, cf. Remark 1.5. By assumption

$$
0=\tilde{\varphi}\big(\psi_\infty(W)^*\psi_\infty(W)\big)=\tilde{\varphi}\big(\psi_\infty\big(W^{(I_1)*}W^{(I_2)}\big)\big)=\tilde{\varphi}\big(W^{(I_1)*}W^{(I_2)}\big)
$$

and by Corollary 3.2 this implies that $\psi_\infty(W)=\lim_{N\to\infty}\psi_N(W)=0$. $\square$

4. Weak freeness

In this section we discuss the notion of weak freeness, which together with $(\widetilde{\mathrm{WSC}})$ implies vanishing of crossing cumulants.

Definition 4.1. Let $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, \mathcal{J})$ be an exchangeability system for a noncommutative probability space $(\mathcal{A}, \varphi)$. For an index set $I \subseteq \mathbf{N}$ denote $\mathcal{A}_I$ the subalgebra of $\mathcal{U}$ generated by $(\mathcal{A}_i)_{i\in I}$. We say that $\mathcal{E}$ satisfies *weak freeness* if

$$\tilde{\varphi}(X_1 X_2 \cdots X_n) = 0$$

whenever $\tilde{\varphi}(X_j^{(1)*} X_j^{(2)}) = 0$, I_k are disjoint index sets and $X_j \in \mathcal{A}_{(I_{i_j})}$ with $i_j \neq i_{j+1}$ for every j. Here $X_j^{(1)}$ and $X_j^{(2)}$ refer to copies of X_j in $A_{I'_{i_j}}$ and $A_{I''_{i_j}}$, where I'_{i_j} and I''_{i_j} are disjoint copies of I_{i_j}.

It will be convenient to adapt the exchangeability system as indicated in Remark 1.5. Decompose $\mathbf{N}$ into an infinite union of disjoint copies of itself $\mathbf{N} = \bigcup_{j=0}^{\infty} I_j$. Then relabel the indices and consider the exchangeability system with embeddings $\iota_{ij} : \mathcal{A} \to \mathcal{A}_{i,j} \subseteq \mathcal{U}$, $i, j \in \mathbf{N}$. Thus $\mathcal{U}$ is also an exchangeability system for $\tilde{A} = \bigvee_{j\in\mathbf{N}} \mathcal{A}_{0j}$ and we will work with this interpretation in this section, i.e., our random variables X are elements of $\tilde{\mathcal{A}}$ and $X^{(i)}$ are elements of $\tilde{\mathcal{A}}_i = \bigvee_{j\in\mathbf{N}} \mathcal{A}_{ij}$. Thus if $X = X_1^{(0,j_1)} X_2^{(0,j_2)} \cdots X_n^{(0,j_n)}$, then $X^{(i)} = X_1^{(i,j_1)} X_2^{(i,j_2)} \cdots X_n^{(i,j_n)}$. The weak freeness condition of Definition 4.1 can be rephrased more clearly as follows, namely

$$\tilde{\varphi}\bigl(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}\bigr) = 0$$

whenever $X_j \in \tilde{\mathcal{A}}$ with $\tilde{\varphi}(X_j^{(1)*} X_j^{(2)}) = 0$ and $i_j \neq i_{j+1}$ for all $1 \leqslant j \leqslant n-1$.

We need this regrouping in order to define an asymptotic conditional expectation which is used to transfer proofs from the amalgamated free situation. As in Section 1.5, we define symmetrizing maps

$$\begin{aligned} \psi_N : \tilde{\mathcal{A}} &\to \tilde{\mathcal{A}} \\ X &\mapsto \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \sigma(X) \end{aligned}$$

which will allow us to construct asymptotically ψ-centered random variables.

Lemma 4.2. *Let $X_j \in \tilde{\mathcal{A}}$ be polynomials, that is, linear combinations of elements of the form*

$$Z_1^{(0,j_1)} Z_2^{(0,j_2)} \cdots Z_m^{(0,j_m)}$$

with $Z_j \in \mathcal{A}$. Then

$$K_\pi\bigl(X_1, \ldots, X_{k-1}, \psi_N(X_k), X_{k+1}, \ldots, X_n\bigr) \underset{N\to\infty}{\longrightarrow} 0$$

unless $\{k\}$ is a singleton of π. In other words, $\psi_N(X_k)$ is asymptotically independent from the rest.

Proof. Indeed,

$$\begin{aligned}
&K_\pi\bigl(X_1,\dots,X_{k-1},\psi_N(X_k),X_{k+1},\dots,X_n\bigr)\\
&\quad=\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}K_\pi\bigl(X_1,\dots,X_{k-1},\sigma(X_k),X_{k+1},\dots,X_n\bigr).
\end{aligned}$$

Let s be the maximal superscript appearing in the polynomials X_j. If σ maps each $1\leqslant j\leqslant s$ to some index strictly greater than s, then $\sigma(X_k)$ is independent from the other X_j and by assumption the cumulant vanishes. The number of permutations σ of this type is

$$(N-s)(N-s-1)\cdots(N-2s)\cdot(N-s)!,$$

i.e., almost all permutations, because the ratio of the rest is

$$\frac{N!-(N-s)(N-s-1)\cdots(N-2s)\cdot(N-s)!}{N!}\underset{N\to\infty}{\longrightarrow}0. \qquad \square$$

When calculating cumulants of elements $X_j\in\tilde{\mathcal{A}}$, we can thus replace X_j by $X_j-\psi_N(X_j)$ for each non-singleton index j and then let N tend to infinity. Replacing X_j by $X_j-\psi_N(X_j)$ allows to apply the weak singleton condition, because due to the permutation invariance of the state we have

$$\tilde\varphi\bigl(\sigma(X)^{(1)*}X^{(2)}\bigr)=\tilde\varphi\bigl(X^{(1)*}X^{(2)}\bigr)$$

and therefore

$$\begin{aligned}
&\tilde\varphi\bigl(\bigl(X^{(1)}-\psi_N(X)^{(1)}\bigr)^*\bigl(X^{(2)}-\psi_N(X)^{(2)}\bigr)\bigr)\\
&\quad=\tilde\varphi\bigl(X^{(1)*}(X)^{(2)}\bigr)-\frac{1}{N!}\sum_\sigma\tilde\varphi\bigl(\sigma(X)^{(1)*}X^{(2)}\bigr)\\
&\qquad-\frac{1}{N!}\sum_\sigma\tilde\varphi\bigl(X^{(1)*}\sigma(X)^{(2)}\bigr)+\frac{1}{(N!)^2}\sum_{\sigma,\tau}\tilde\varphi\bigl(\sigma(X)^{(1)*}\tau(X)^{(2)}\bigr)\\
&\quad=0.
\end{aligned}$$

Theorem 4.3. *Let $\mathcal{E}=(\mathcal{U},\tilde\varphi,\mathcal{J})$ be an exchangeability system for a noncommutative probability space $(\mathcal{A},\varphi)$ with faithful state $\tilde\varphi$ such that both weak freeness and the weak singleton condition holds. Then crossing cumulants vanish. In particular, $\mathcal{E}$ can be embedded into an amalgamated free product.*

Proof. The proof consists of three parts by reducing an arbitrary crossing partition to an alternating partition without singletons.

We will use the following terminology. A partition $\pi\in\Pi_n$ is called *alternating* if $i\not\sim_\pi i+1$ for all $1\leqslant i\leqslant n-1$, that is, adjacent elements are in different blocks of π. For non-alternating partitions we denote by

$$\operatorname{cn}(\pi)=\#\bigl\{(k,k+1)\colon\ k\sim_\pi k+1,\ 1\leqslant k\leqslant n-1\bigr\}$$

the *number of connected neighbours* of π. Clearly π is alternating if and only if $\operatorname{cn}(\pi) = 0$.

Step 1. Alternating partitions without singletons. First assume that π is an alternating partition without singletons. Let $\varepsilon > 0$. Then by Lemma 4.2 we may find $N > 0$ such that

$$K_\pi(X_1, \ldots, X_n) = K_\pi\big(X_1 - \psi_N(X_1), \ldots, X_n - \psi_N(X_n)\big) + R_n$$

with error term $|R_n| < \varepsilon$. Now a look at the moment-cumulant formula

$$K_\pi\big(X_1 - \psi_N(X_1), \ldots, X_n - \psi_N(X_n)\big) = \sum_{\sigma \leqslant \pi} \varphi_\sigma\big(X_1 - \psi_N(X_1), \ldots, X_n - \psi_N(X_n)\big)\mu(\sigma, \pi)$$

shows that the sum runs over alternating partitions and by weak freeness every term vanishes.

Step 2. Reducing everything to alternating partitions. The aim is now to express an arbitrary cumulant in terms of alternating ones. We will use the product formula of Leonov and Shiryaev from Proposition 1.4 to reduce the number of connected neighbours. Consider a partition $\pi \in \Pi_n$ with crossings and $\operatorname{cn}(\pi) > 0$. Pick an arbitrary element k with $k \sim_\pi k+1$. We will express K_π as a sum of cumulants K_ρ such that $k \nsim_\rho k+1$ and $\operatorname{cn}(\rho) < \operatorname{cn}(\pi)$. Let $\hat{\pi} = \pi/[k = k+1] \in \Pi_{n-1}$ be the partition obtained from π by identifying k and $k+1$. Then by Proposition 1.4 we have with $\nu = \{\{1\}, \{2\}, \ldots, \{k, k+1\}, \ldots, \{n\}\} =$  the decomposition

$$\begin{aligned} K_{\hat{\pi}}(X_1, \ldots, X_k X_{k+1}, \ldots, X_n) &= \sum_{\rho \vee \nu = \pi} K_\rho(X_1, \ldots, X_k, X_{k+1}, \ldots, X_n) \\ &= K_\pi(X_1, \ldots, X_n) + \sum_{\substack{\rho \vee \nu = \pi \\ \rho < \pi}} K_\rho(X_1, \ldots, X_n). \end{aligned}$$

Each contributing partition ρ in the second sum is obtained from π by splitting the block containing k and $k+1$ into two in such a way that k and $k+1$ are separated. In particular, $\operatorname{cn}(\rho) \leqslant \operatorname{cn}(\pi) - 1$ and $\operatorname{cn}(\hat{\pi}) = \operatorname{cn}(\pi) - 1 < \operatorname{cn}(\pi)$ as well. To conclude, we have

$$K_\pi(X_1, \ldots, X_n) = K_{\hat{\pi}}(X_1, \ldots, X_k X_{k+1}, \ldots, X_n) - \sum_{\substack{\rho \vee \nu = \pi \\ \rho < \pi}} K_\rho(X_1, \ldots, X_n)$$

and all partitions appearing on the right-hand side have less connected neighbours than π. Repeating this operation finitely many times we end up with a linear combination of alternating partitions (possibly involving singletons).

Step 3. Getting rid of singletons. Assume that after step 2 we have arrived at an alternating partition π. Then for N large enough, we have by Lemma 4.2 again

$$K_\pi(X_1, \ldots, X_n) \approx K_\pi(\tilde{X}_1, \ldots, \tilde{X}_n),$$

where

$$\tilde{X}_j = \begin{cases} X_j & \text{if } \{j\} \text{ is a singleton of } \pi, \\ X_j - \psi_N(X_j) & \text{if } \{j\} \text{ is not a singleton of } \pi. \end{cases}$$

We may therefore assume without loss of generality that $\psi(X_j^{(1)*}X_j^{(2)}) = 0$ for the non-singleton indices j. If the singleton entries k satisfy $\psi(X_k^{(1)*}X_k^{(2)}) = 0$ as well, then we may proceed as in step one. If, however, there are singletons for which this is not the case, we may eliminate them as follows. Let $\{k\}$ be the first of these critical singletons, then we may write the cumulant as

$$K_\pi(X_1,\dots,X_n) = K_\pi\bigl(X_1,\dots,X_k - \psi_N(X_k),\dots,X_n\bigr) + K_\pi\bigl(X_1,\dots,\psi_N(X_k),\dots,X_n\bigr).$$

The first term has one critical singleton less than the left-hand side and the second term can be treated with the product formula as follows. Let again $\hat\pi = \pi/[k = k+1] \in \Pi_{n-1}$ be the partition obtained from π by identifying k with $k+1$. Moreover, put $\nu = \{\{1\},\{2\},\dots,\{k,k+1\},\dots,\{n\}\} = \mid\ \mid\ \dots\ \mid\ \sqcap\ \mid\ \dots\ \mid$ and let $\tilde\pi = \pi \vee \nu$ be the partition obtained from π by adjoining the singleton $\{k\}$ to the block containing $k+1$. Then we have by Proposition 1.4 again

$$\begin{aligned}
&K_{\hat\pi}\bigl(X_1,X_2,\dots,\psi_N(X_k)X_{k+1},\dots,X_n\bigr)\\
&\quad = K_{\tilde\pi}\bigl(X_1,X_2,\dots,\psi_N(X_k),X_{k+1},\dots,X_n\bigr)\\
&\qquad + \sum_{\substack{\rho\vee\nu=\tilde\pi\\ \rho<\tilde\pi}} K_\rho\bigl(X_1,X_2,\dots,\psi_N(X_k),X_{k+1},\dots,X_n\bigr).
\end{aligned}$$

$K_{\tilde\pi}$ vanishes asymptotically by Lemma 4.2 and so do all K_ρ in which $k \sim_\pi k+1$, and hence the only nontrivial term on the right-hand side is the cumulant indexed by $\rho = \pi$, because this is the only one in which k is a singleton. Thus

$$K_\pi\bigl(X_1,X_2,\dots,\psi_N(X_k),X_{k+1},\dots,X_n\bigr) \approx K_{\hat\pi}\bigl(X_1,X_2,\dots,\psi_N(X_k)X_{k+1},\dots,X_n\bigr)$$

and $\hat\pi$ has one singleton less than π. Repeating this procedure we end up with a linear combination of alternating partitions without singletons and step one of the proof applies. □

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Appendix A. Free L^p inequalities

We consider now the tracial version of Proposition 3.1. In this case Hölder's inequality is available and we can get estimates in terms of the L^p-norms.

Proposition A.1. *Assume that a noncrossing exchangeability system $\mathcal{E} = (\mathcal{U},\tau,\mathcal{J})$ is tracial and faithful. Then for any sequence of $\mathcal{E}$-independent random variables X_i with $\tau(X_i^{(1)*}X_i^{(2)}) = 0$ the inequality*

$$\Bigl\|\sum_{i=1}^N X_i\Bigr\|_{L^{2p}(\tau)} \leqslant C_{2p} S(X,2p)$$

holds with $C_{2p} \leqslant \frac{3\pi}{4z_0} \simeq 9.85859$ as $p \to \infty$ where z_0 is computed in (A.1).

Proof. We expand the L^p-norm in terms of cumulants:

$$\tau\Bigl(\Bigl(\sum X_i^* X_j\Bigr)^p\Bigr) = \sum_{\pi \in NC_{2p}} K_\pi\Bigl(\sum X_i^*, \sum X_i, \dots, \sum X_i^*, \sum X_i\Bigr)$$

and for each π, we estimate the cumulant K_π. Because of the weak singleton condition, only partitions without singletons are involved in the sums.

$$\begin{aligned}
&K_\pi\Bigl(\sum X_i^*, \sum X_i, \dots, \sum X_i^*, \sum X_i\Bigr) \\
&\quad = \sum_{\ker h \geqslant \pi} K_\pi\bigl(X_{h(1)}^*, X_{h(2)}, \dots, X_{h(2p-1)}^*, X_{h(2p)}\bigr) \\
&\quad = \sum_{\ker h \geqslant \pi} \sum_{\sigma \leqslant \pi} \tau_\sigma\bigl(X_{h(1)}^*, X_{h(2)}, \dots, X_{h(2p-1)}^*, X_{h(2p)}\bigr)\mu_{NC}(\sigma, \pi).
\end{aligned}$$

Now for fixed σ we have [17, Lemmas 3.3 and 3.4]:

$$\Biggl|\sum_{\ker h \geqslant \pi} \tau_\sigma(X_{h(1)} X_{h(2)} \cdots X_{h(2p)})\Biggr| \leqslant \Bigl\|\sum \lambda(g_i) \otimes X_i\Bigr\|_{2p}^{2p} \leqslant \left(\frac{3\pi}{4}\right)^{2p} S(X, 2p),$$

where $\lambda(g_i)$ is the left regular representation of the generators of the free group. Indeed, we can find a suitable discrete group G and elements $F_1, \dots, F_{2p}$ in $L^p(\tau_G \otimes \tau)$ such that

$$\|F_k\|_{2p} = \Bigl\|\sum \lambda(g_i) \otimes X_i\Bigr\|_{2p}$$

and

$$\sum_{\ker h \geqslant \pi} \tau_\sigma(X_{h(1)} X_{h(2)} \cdots X_{h(2p)}) = \tau_G \otimes \tau(F_1 F_2 \cdots F_{2p}).$$

The construction is done as in the proof of [17, Lemma 3.3], with the slight modification that X_i is replaced by $X_i^{(\sigma(k))}$ when constructing F_k. With these preparations we continue similarly as before:

$$\tau\Bigl(\Bigl(\sum X_i^* X_j\Bigr)^p\Bigr) \leqslant \sum_{\pi \in NC^{\geqslant 2}} \sum_{\substack{\sigma \in NC_{2p}^{\geqslant 2} \\ \sigma \leqslant \pi}} \bigl|\mu_{NC}(\sigma, \pi)\bigr| \left(\frac{3\pi}{4}\right)^{2p} S(X, 2p).$$

We know the asymptotics of

$$\tilde{b}_n = \sum_{\pi \in NC^{\geqslant 2}} \sum_{\substack{\sigma \in NC_{2p}^{\geqslant 2} \\ \sigma \leqslant \pi}} \bigl|\mu_{NC}(\sigma, \pi)\bigr|$$

from its generating function, which satisfies Eq. (3.4) with $N = 1$. Its dominant singularity is a zero of the resultant

$$r(z) = 3z^4 + 22z^3 + 57z^2 + 6z - 5$$

and the singularity in question is

$$z_0 = -\frac{11}{6} + \frac{1}{6}\sqrt{7+\gamma} + \sqrt{14 - \gamma + \frac{992}{\sqrt{7+\gamma}}} \simeq 0.238999, \tag{A.1}$$

where

$$\gamma = 9\big(207 - 48\sqrt{3}\big)^{1/3} + 9\big(207 + 48\sqrt{3}\big)^{1/3};$$

consequently $\tilde{b}_n \simeq z_0^{-n}$ as $n \to \infty$. $\square$

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