Combinatorial Stokes formulae

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Abstract

For an \( n \)-dimensional pseudomanifold whose vertices get labels from a finite set, there is a “combinatorial Stokes” formula, found by Ky Fan, which links the number of simplices getting \( n \) different labels on the boundary with the number of simplices getting \( n + 1 \) different labels. In 1998, a generalization of this formula was proved by Lee and Shih taking into account the possibility of putting several labels on each vertex. We re-prove and generalize this latter combinatorial Stokes formula in a rather simple and natural way. Furthermore, some applications of the combinatorial Stokes formula of Fan are given; one of them provides a new combinatorial proof of Schrijver’s theorem about Kneser graphs.

1. Introduction

In 1967, Ky Fan proved a combinatorial formula [3] in an attempt to generalize both Sperner’s lemma [10] and Tucker’s lemma [11]. In 1974, Kuhn re-proved Fan’s formula in the two-dimensional case in order to give a constructive proof of the fundamental theorem of algebra [5]. Kuhn called his formula the combinatorial Stokes theorem because it relies the number of fully labeled simplices of the boundary of a triangulation to the number of fully labeled simplices inside the triangulation, reminding us of the classical Stokes formula.

Instead of a unique label, we can put several labels on each vertex of the triangulation. Such a labeling is called a multilabeling, and, to our knowledge, Bapat was the first to have considered these: in 1989, he gave a multilabeling generalization of Sperner’s lemma [1] in order to provide a constructive proof of Gale’s generalization [4] of the KKM lemma.

In 1998, Lee and Shih found the same kind of generalization for Fan’s formula [6].

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2. Tools and notation

2.1. General notation

For a positive integer \( n \), \([n]\) will denote the set \{1, 2, \ldots, n\}. For a finite set \( A \), \( \binom{A}{k} \) is the set of \( k \)-subsets of \( A \) and \( \binom{A}{\leq k} \) the set of subsets of \( A \) whose cardinality is less or equal to \( k \). For a sequence \( a_0, \ldots, a_i, \ldots, a_k \), the sequence \( a_0, \ldots, \hat{a}_j, \ldots, a_k \) is the same sequence with the \( a_j \) missing.

2.2. Chain complexes

A chain complex \( C \) is a (finite or infinite) sequence of free abelian groups \((H_i)_i\) and a sequence of homomorphisms \((\partial_i : H_i \to H_{i-1})_i\) such that \( \partial_{i-1} \circ \partial_i = 0 \) for any \( i \). We write

\[
C : \cdots \xrightarrow{\partial_{i+1}} H_i \xrightarrow{\partial_i} H_{i-1} \xrightarrow{\partial_{i-1}} \cdots
\]

The sequence \((\partial_i)\) is called the boundary operator. We often omit the index, writing \( \partial \) instead of \( \partial_i \).

Given two chain complexes \( C = (H_i, \partial_i)_i \) and \( C' = (H'_i, \partial'_i)_i \), a chain map \( f : C \to C' \) is a sequence \((f_i)_i\) of morphisms \( f_i : H_i \to H'_i \) such that \( \partial'_i f_i = f_i \partial_{i-1} \).

2.3. Simplices and simplicial complexes

We give here a short introduction to the notions of simplices, simplicial complexes and chains for simplicial complexes. For a more complete study of this subject, see the book of Munkres [8].

2.3.1. Simplices and simplicial complexes

An (abstract) simplicial complex is a family \( K \) of subsets of a finite groundset \( X \) with the property that \( \sigma' \subseteq \sigma \in K \) implies \( \sigma' \in K \). We define the dimension of \( K \): \( \text{dim}(K) = \max\{|\sigma| - 1 : \sigma \in K\} \). The sets in \( K \) are called (abstract) simplices and the dimension of a simplex \( \sigma \) is \( \dim(\sigma) = |\sigma| - 1 \). If \( \dim(\sigma) = d \), we say that \( \sigma \) is a \( d \)-simplex.

The element of \( K \) (resp. the subsets of a simplex \( \sigma \)) are called faces. A \( p \)-face of \( K \) is a face of \( K \) of dimension \( p \). The set of \( p \)-faces is denoted by \( K_p \). For a \( p \)-simplex \( \sigma \), the facets are the simplices \( \sigma' \subseteq \sigma \) of dimensions \( p - 1 \). The vertices are the \( 0 \)-faces of \( K \); their set is denoted by \( V(K) \). The edges are the \( 1 \)-faces of \( K \); their set is denoted by \( E(K) \). We have thus \( V(K) = K_0 \) and \( E(K) = K_1 \).

A simplicial complex is said to be pure if all the maximal simplices for inclusion have the same dimension. A \( k \)-regular simplicial complex \( K \) is a pure simplicial complex such that any \((\text{dim}(K) - 1)\)-simplex of \( K \) is contained in exactly \( k \ \text{dim}(K) \)-simplices.

An \( n \)-pseudomanifold \( M \) is an \( n \)-dimensional simplicial complex whose \((n - 1)\)-simplices are each contained in at most two \( n \)-simplices. \( \partial M \), the boundary of the pseudomanifold \( M \), is
those \((n - 1)\)-simplices that are contained in exactly one \(n\)-simplex. A pseudomanifold having no boundary is a 2-regular simplicial complex.

Let \(\sigma\) be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If \(\dim(\sigma) > 0\), the orderings of the vertices of \(\sigma\) fall into two equivalence classes. Each of these classes is called an orientation of \(\sigma\). An oriented simplex is a simplex \(\sigma\) together with an orientation of \(\sigma\). By \([v_1, \ldots, v_{p+1}]\) we denote the oriented \(p\)-simplex with vertex set \([v_1, v_2, \ldots, v_{p+1}]\) and the equivalence class of the particular ordering \((v_1, \ldots, v_{p+1})\).

By \(-[v_1, \ldots, v_{p+1}]\), we mean the simplex \([v_1, \ldots, v_{p+1}]\) with the opposite orientation.

Hence, we have \([v_1, \ldots, v_{p+1}] = -[v_2, v_1, v_3, v_4, \ldots, v_{p+1}]\) (for \(p \geq 1\)).

The induced orientation of \([v_1, \ldots, v_{p+1}]\) on \([v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}]\) is \((-1)^{i+1}[v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}]\).

An \(n\)-pseudomanifold is said to be oriented if

- each \(n\)-simplex is oriented and
- for any \((n - 1)\)-simplex \(\tau\) contained in two \(n\)-simplices, the induced orientations on \(\tau\) from the two \(n\)-simplices are opposed.

Let \(K\) and \(L\) be two abstract simplicial complexes. A simplicial map of \(K\) into \(L\) is a mapping \(f : V(K) \to V(L)\) such that \(f(\sigma) \in L\) whenever \(\sigma \in K\).

Two abstract simplicial complexes \(K\) and \(L\) are said to be isomorphic if there is a simplicial map between them which is bijective and whose inverse is also simplicial. In this case, we write \(K \cong L\); the two simplicial complexes differ only in the names of the vertices.

### 2.3.2. Chains for simplicial complexes

Let \(G\) be an abelian group and let \(K\) be an abstract simplicial complex. The chain complex \(C(K)\) is

\[
\cdots \to C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \to \cdots,
\]

where \(C_p(K)\) is the free abelian group of all formal linear combinations of \(p\)-faces of \(K\) with coefficients in \(G\). Any element \(c\) of \(C_k(K)\) is called a \(k\)-chain.

We define the boundary operator \(\partial_K\) for a simplicial complex \(K\) as follows: if \(\sigma\) is a \(p\)-simplex \([v_1, \ldots, v_{p+1}]\), \(p \geq 1\), \(\partial_K \sigma := \sum_{i=1}^{p+1} (-1)^{i+1}[v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}]\). The relation \(\partial_K \partial_K = 0\) is true for simplices since \([\ldots, \hat{v}_i, \ldots, v_j, \ldots]\) arises twice with opposite signs. Thus \(C(K)\) is indeed a chain complex.

If \(f\) is a simplicial map of \(K\) to \(L\), we define a collection \(f_\#\) of homomorphisms \(f_\#_p : C_p(K) \to C_p(L)\) by defining it on simplices as follows: for \(\sigma\) a \(p\)-simplex, we have

\[
f_\#(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = p \\ 0 & \text{otherwise}. \end{cases}
\]

\(f_\#\) is then a chain map (it can be easily checked, again, first for simplices).

### 2.3.3. Labelings and Sperner’s lemma

A labeling of a simplicial complex \(K\) is a map from the vertex set \(V(K)\) into a set called the label set. Such a map can always be seen as a simplicial map (at least, the set of all subsets of the label set is a simplicial complex).

A Sperner labeling of a triangulation \(T\) of an \(n\)-dimensional geometric simplex \(\sigma^n\) is a labeling with \(n + 1\) labels such that (i) each vertex of \(\sigma^n\) is labeled with a distinct label, (ii) a vertex \(v\) of
Fig. 1. The two-dimensional Sperner lemma.

T can only get one of the labels of the vertices of its supporting face (that is, the minimal face of \( \sigma^n \) containing \( v \)).

The first (historically speaking) combinatorial Stokes formula is then the Sperner lemma \([10]\), illustrated in Fig. 1:

**Theorem 1 (Sperner’s Lemma).** Let \( T \) be a triangulation of \( \sigma^n \), an \( n \)-dimensional geometric simplex, and suppose that \( T \) gets a Sperner labeling. Then there is at least one simplex of \( T \) whose vertices are labeled with all labels.

This is a direct consequence of many theorems of this paper.

3. Fan’s formulae. Corollaries and applications

Let \( \lambda : V(M) \rightarrow \{-1, +1, -2, +2, \ldots , -m, +m\} \) a labeling of the vertices of an \( n \)-pseudomanifold \( M \).

**Definitions.** If \( M \) is oriented: For \( n + 1 \) distinct integers \( j_1, \ldots , j_{n+1} \) arranged in this order, denote by \( \alpha_+(j_1, j_2, \ldots , j_{n+1}) \) (resp. \( \alpha_-(j_1, j_2, \ldots , j_{n+1}) \) ) the number of those \( \sigma \) for which

- \( \sigma \) is an \( n \)-simplex of \( M \),
- \( \sigma = +[v_1, v_2, \ldots , v_{n+1}] \) (resp. \( -[v_1, v_2, \ldots , v_{n+1}] \)) and \( \lambda(v_i) = j_i \) for \( i = 1, \ldots , n + 1 \).

For every \((n - 1)\)-simplex of \( \partial M \), we consider the induced orientation from the unique \( n \)-simplex of \( M \) containing it.

For \( n \) distinct integers \( j_1, \ldots , j_n \) arranged in this order, denote by \( \beta_+(j_1, j_2, \ldots , j_n) \) (resp. \( \beta_-(j_1, j_2, \ldots , j_n) \) ) the number of those \( \tau \) such that

- \( \tau \) is an \((n - 1)\)-simplex of \( \partial M \),
- \( \tau = +[v_1, v_2, \ldots , v_n] \) (resp. \( -[v_1, v_2, \ldots , v_n] \)) and \( \lambda(v_i) = j_i \) for \( i = 1, \ldots , n \).

Finally, we define \( \alpha := \alpha_+ - \alpha_- \) and \( \beta := \beta_+ - \beta_- \).

If \( M \) is not oriented: For \( n + 1 \) distinct integers \( j_1, \ldots , j_{n+1} \), we denote by \( \alpha(j_1, j_2, \ldots , j_{n+1}) \) the number modulo 2 of those \( \sigma \) such that

- \( \sigma \) is an \( n \)-simplex of \( M \),
- \( \sigma = \{v_1, v_2, \ldots , v_{n+1}\} \) and \( \lambda(v_i) = j_i \) for \( i = 1, \ldots , n + 1 \).

For \( n \) distinct integers \( j_1, \ldots , j_n \), we denote by \( \beta(j_1, j_2, \ldots , j_n) \) the number modulo 2 of those \( \tau \) such that

- \( \tau \) is an \((n - 1)\)-simplex of \( \partial M \),
We write the proof for the oriented case. The other case is proved similarly.

There is only one 2-simplex of type “α” with labels: −1, 3, −4; and there is only one 1-simplex of type “β” on the boundary with labels 2, −4, in this order.

\[ \tau = \{v_1, v_2, \ldots, v_n\} \text{ and } \lambda(v_i) = j_i \text{ for } i = 1, \ldots, n. \]

**Theorem 2 (Fan’s Formula [3]).** Let \( m \) and \( n \) be two positive integers and let \( M \) be an \( n \)-pseudomanifold. Let \( \lambda : V(M) \to \{-1, +1, -2, +2, \ldots, -m, +m\} \) be a labeling of the vertices of \( M \) such that no edge is labeled by \(-i, +i\) for some \( i \) (there is no antipodal edge). Then if \( M \) is oriented we have

\[
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m} \alpha(-j_1, +j_2, -j_3, +j_4, \ldots, (-1)^{n+1}j_{n+1}) \\
+ (-1)^n \alpha(+j_1, -j_2, +j_3, -j_4, \ldots, (-1)^n j_{n+1}) \\
= \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).
\]

If \( M \) is not oriented, the formula holds modulo 2.

This formula is illustrated in Fig. 2.

**Proof.** We write the proof for the oriented case. The other case is proved similarly.

\( \lambda \) is a simplicial map going from \( M \) into the \((m - 1)\)-dimensional simplicial complex \( C \), whose faces are the subsets of \{-1, +1, -2, +2, \ldots, -m, +m\} containing no pair \{-i, +i\} for some \( i \in [m] \) (\( C \) is the boundary of the crosspolytope).

Consider the following chain complex, which we denote by \( C(\mathbb{Z}^2) \):

\[ \cdots \to \mathbb{Z}^2 \xrightarrow{\delta_i} \mathbb{Z}^2 \xrightarrow{\delta_{i-1}} \mathbb{Z}^2 \to \cdots, \]

where \( \delta_i(x, y) = ((-1)^i x + y, x + (-1)^i y) \) (we have, indeed, \( \delta_{i-1} \circ \delta_i = 0 \)).

We define the collection of homomorphisms \( \mu : C(C) \to C(\mathbb{Z}^2) \) as follows:

\[
\mu_p(\sigma) = \begin{cases} 
(1, 0) & \text{if } \sigma = [-j_1, +j_2, -j_3, +j_4, \ldots, (-1)^{p+1}j_{p+1}] \\
& \text{with } 1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq m \\
(0, 1) & \text{if } \sigma = [+j_1, -j_2, +j_3, -j_4, \ldots, (-1)^p j_{p+1}] \\
& \text{with } 1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq m \\
0 & \text{otherwise.}
\end{cases}
\]

We check now that \( \mu \) is a chain map, that is: \( \delta_p \mu_p = \mu_{p-1} \partial C \), for all \( p \geq 1 \). By linearity, there are only three cases to check:
1. If $\sigma$ is of the form $[-j_1, +j_2, \ldots, (-1)^{p+1}j_{p+1}]$ with $1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq m$:
   - $\delta_p \mu_p(\sigma) = \delta_p(1, 0) = ((-1)^p, 1)$ and
   - $\mu_{p-1}(\partial \sigma) = \mu_{p-1}\left(\sum_i (-1)^i[-j_1, +j_2, \ldots, (-1)^{i-1}j_i, \ldots, (-1)^{p+1}j_{p+1}]\right)$
     
     \begin{align*}
     &\quad = \mu_{p-1}(\{+j_2, -j_3, \ldots, (-1)^{p+1}j_{p+1}\}) \\
     &\quad + \mu_{p-1}((-1)^p[-j_1, +j_2, \ldots, (-1)^pj_p]) \\
     &\quad = (0, 1) + ((-1)^p, 0) = ((-1)^p, 1).
     \end{align*}

2. If $\sigma$ is of the form $[+j_1, -j_2, \ldots, (-1)^p j_{p+1}]$ with $1 \leq j_1 < j_2 < \cdots < j_{p+1} \leq m$:
   - $\delta_p \mu_p(\sigma) = \delta_p(0, 1) = (1, (-1)^p)$ and
   - $\mu_{p-1}(\partial \sigma) = \mu_{p-1}\left(\sum_i (-1)^i[+j_1, -j_2, \ldots, (-1)^{i-1}j_i, \ldots, (-1)^p j_{p+1}]\right)$
     
     \begin{align*}
     &\quad = \mu_{p-1}(\{-j_2, +j_3, \ldots, (-1)^{p+1}j_{p+1}\}) \\
     &\quad + \mu_{p-1}((-1)^p[+j_1, -j_2, \ldots, (-1)^p j_p]) \\
     &\quad = (1, 0) + (0, (-1)^p) = (1, (-1)^p).
     \end{align*}

3. If $\sigma = [v_1, v_2, \ldots, v_{p+1}]$, with $|v_1| < |v_2| < \cdots < |v_{p+1}|$, and if at least a $k$ is such that $v_k$ and $v_{k+1}$ have the same sign:
   - $\delta_p \mu_p(\sigma) = \delta_p 0 = 0$ and
   - $\mu_{p-1}(\partial \sigma) = \mu_{p-1}\left(\sum_i (-1)^i[v_1, v_2, \ldots, \hat{v}_i, \ldots, v_{p+1}]\right)$
     
     \begin{align*}
     &\quad = \mu_{p-1}((-1)^{k-1}[v_1, \ldots, \hat{v}_k, \ldots, v_{p+1}]) \\
     &\quad + \mu_{p-1}((-1)^k[v_1, \ldots, \hat{v}_{k+1}, \ldots, v_{p+1}]) = 0.
     \end{align*}

Thus $\mu \circ \lambda_{\#}$ is a chain map, and Fan’s formula is a direct consequence of the commutation:

$$\delta_n(\mu_n \circ \lambda_{\#}) = (\mu_{n-1} \circ \lambda_{\#n-1}) \partial_M$$

applied on the formal sum of all oriented $n$-simplices of $M$. ■

We see now some applications of this formula. The first one is Tucker’s lemma [11], which is the combinatorial counterpart of the Borsuk–Ulam theorem. The second one is a purely combinatorial proof of Schrijver’s theorem [9] concerning the chromatic number of stable Kneser graphs, shorter than Ziegler’s proof of [12]. The third application is Fan’s generalization of Sperner’s lemma. Note that the first and the third applications are well known (see Fan’s papers [2,3]).

**Corollary 1** (Tucker’s Lemma). Let $n$ be a positive integer and $T$ be an antipodal triangulation of the $n$-sphere $S^n$ (that is: if $\sigma \in T$, then $-\sigma \in T$, where $-\sigma$ is the symmetric of $\sigma$ with respect the center of $S^n$). Moreover, we ask that $T$ refines the natural triangulation of $S^n$ induced by the coordinate hyperplanes.

Suppose that $V(T)$ is labeled by labels in $\{-1, +1, -2, +2, \ldots, -n, +n\}$, such that for $v \in V(T)$ $v$ and $-v$ are labeled by opposite numbers (such a labeling is called an antipodal labeling). Then there exists an edge in $T$ whose vertices are labeled by opposite numbers.
Theorem 2

The inequality

We prove in fact the following proposition:

The boundary of \( S \) be the positive hemisphere of signed subsets of the set of all integers, \( \mathbb{Z} \), \( n \) is a positive integer, in such a way that there is no edge whose vertices are labeled by opposite numbers.

Then there is an odd number of \( n \)-simplices labeled by \( n + 1 \) integers \( j_1, j_2, \ldots, j_{n+1} \), with \( 1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m \).

In particular, this implies that \( m > n \), and Tucker’s lemma follows.

The proof works by induction. For \( n = 0 \), the proof is trivial.

For \( n > 0 \), the proof is a direct consequence of Fan’s formula in the oriented case: let \( S^n_+ \) be the positive hemisphere of \( S^n \): \( x \in S^n_+ \Rightarrow x_{n+1} \geq 0 \). \( T \) induces a triangulation \( T_+ \) on \( S^n_+ \).

The boundary of \( T_+ \) is a triangulation of an \( (n-1) \)-sphere \( S^{n-1} \) which satisfies the conditions of the proposition above. Thus induction applies and applying Theorem 2, we know that

\[
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1}j_{n+1})
\]

for \( T_+ \). By antipodality, this number is precisely the number of \( n \)-simplices labeled by \( n + 1 \) integers \( +j_1, -j_2, \ldots, (-1)^{n}j_{n+1} \), with \( 1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m \) in \( T \). \( \blacksquare \)

Second application: a short combinatorial proof of Schrijver’s theorem. We recall the context.

Let \( \mathcal{H} \) be a hypergraph. The associated Kneser graph \( KG(\mathcal{H}) \) is the graph whose vertices are the hyperedges of \( \mathcal{H} \), and in which two vertices are connected by an edge if and only if the two corresponding hyperedges are disjoint.

For \( n \geq 2k-1 \), we consider the hypergraph \( S \) whose vertex set is \([n]\) and the hyperedges are those \( k \)-subsets \( A \) such that for \( x, y \in A \subseteq [n] \), \( 1 < |x-y| < n-1 \) (putting the elements of \([n]\) on a circle in the natural order, the hyperedges are \( k \)-subsets not having adjacent elements). The Schrijver graph of parameters \( n, k \) is then defined by \( SG(n, k) := KG(S) \).

Denoting by \( \chi(G) \) the chromatic number of a graph \( G \), we have the following theorem, proved by Schrijver [9]:

**Corollary 2 (Schrijver’s Theorem).** \( \chi(SG(n,k)) = n - 2k + 2 \).

We first need some definitions and notations: For a positive integer \( n \), we write \( \{+, -, 0\}^n \) for the set of all signed subsets of \([n]\), that is, the family of all pairs \( (X^+, X^-) \) of disjoint subsets of \([n]\). Indeed, for \( X \in \{+, -, 0\}^n \), we can define \( X^+ := \{i \in [n]: X_i = +\} \) and analogously \( X^- \).

For sign vectors, we use the usual partial order from oriented matroid theory, which is defined componentwise with \( 0 \leq + \), and \( 0 \leq - \). Hence \( X \leq Y \) if and only if \( X^+ \subseteq Y^+ \) and \( X^- \subseteq Y^- \).

By \( \text{alt}(X) \) we denote the length of the longest alternating subsequence of non-zero signs in \( X \). For instance: \( \text{alt}(+0-0+0-) = 4 \), while \( \text{alt}(-+-++0+-) = 5 \).

Finally, for a poset \( P \), \( \Delta(P) \) is the simplicial complex (the order complex) whose vertices are the elements of \( P \), and whose simplices are the chains of \( P \).

The proof is inspired by Matousek’s combinatorial proof of the Lovász–Kneser theorem stating that \( \chi(KG([n]_k)) = n - 2k + 2 \) [7] and one of the main ingredients, namely the poset \( \Sigma_{n,k} \), is taken from Ziegler’s proof [12].

**Proof.** The inequality \( \chi(SG(n,k)) \leq n - 2k + 2 \) is easy to prove (with an explicit coloring) and well known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.
For \( n \geq 2k \), we define
\[
\Sigma_{n,k} := \Delta(\{X \in \{+, -, 0\}^n : \text{alt}(X) \geq 2k\}).
\]

We define two more simplicial complexes, that are subcomplexes of \( \Sigma_{n,k} \):
\[
\Sigma_{n,k}^+ := \Delta(\{X \in V(\Sigma_{n,k}) : X_n \in \{0, +\}\})
\]
and
\[
\Sigma_{n,k}^- := \Delta(\{X \in V(\Sigma_{n,k}) : X_n \in \{0, -\}\}).
\]

We have then the following properties:
1. \( \Sigma_{n,k}^+ \cup \Sigma_{n,k}^- = \Sigma_{n,k} \).
2. \( \dim(\Sigma_{n,k}^+) = \dim(\Sigma_{n,k}^-) = n - 2k \).
3. \( \Sigma_{n,k}^+ \) and \( \Sigma_{n,k}^- \) are pseudomanifolds.
4. \( \partial \Sigma_{n,k}^+ = \partial \Sigma_{n,k}^- = \Delta(\{X \in V(\Sigma_{n,k}) : X_n = 0\}) \cong \Sigma_{n-1,k} \).

All those properties are easy to prove, except maybe the third one. To prove it, we prove that \( \Sigma_{n,k} \) is a pseudomanifold: take a \( (n - 2k - 1) \)-simplex of \( \Sigma_{n,k} \). It has either the form (i) \( X^{2k} \leq X^{2k+1} \leq \cdots \leq X^{i-1} \leq X^i+1 \leq \cdots \leq X^n \), \( 2k + 1 \leq i \leq n - 1 \), or the form (ii) \( X^{2k+1} \leq \cdots \leq X^n \) or the form (iii) \( X^{2k} \leq \cdots \leq X^{n-1} \), where \( X^i \) is an element of \( V(\Sigma_{n,k}) \) having exactly \( i \) nonzero components.

In the case (i), as we add two nonzero components to \( X^{i+1} \) to obtain \( X^{i+1} \), the \( (n - 2k - 1) \)-simplex is exactly in two different \( (n - 2k) \)-simplices of \( \Sigma_{n,k} \).

In the case (ii), either \( \text{alt}(X^{2k+1}) = 2k + 1 \) and there are two \( X < X^{2k+1} \) such that \( \text{alt}(X) \geq 2k \) (delete the first or the last nonzero components of \( X \)), or \( \text{alt}(X^{2k+1}) = 2k \) and there is exactly one \( j \) such that \( X^{2k+1} = X^{2k+1}_j \neq 0 \), implying that there are two \( X < X^{2k+1} \) such that \( \text{alt}(X) \geq 2k \). In both cases, this means that the \( (n - 2k - 1) \)-simplex is exactly in two different \( (n - 2k) \)-simplices of \( \Sigma_{n,k} \).

In the case (iii), let \( j \) be the unique integer such that \( X^{n-1}_j = 0 \); there are exactly two possibilities for \( X^n_j \): \( X^n_j = - \) or \( X^n_j = + \) and the \( (n - 2k - 1) \)-simplex is exactly in two different \( (n - 2k) \)-simplices of \( \Sigma_{n,k} \).

We prove now the following proposition:

Suppose that the vertices of \( \Sigma_{n,k} \) are labeled with integers in the set \( \{-1, +1, -2, +2, \ldots, -d, +d\} \) such that no edge is labeled \( -i, +i \) for any \( 1 \leq i \leq d \), and such that the label of vertex \( X \) is the opposite of the label of \( -X (\lambda(-X) = -\lambda(X)) \) for any vertex \( X \) of \( \Sigma_{n,k} \). Then there is an odd number of \( (n - 2k) \)-simplices labeled by \( n - 2k + 1 \) integers \(+j_1, -j_2, \ldots, (-1)^{n-2k}j_{n-2k+1}\), with \( j_1 < j_2 < \cdots < j_{n-2k+1} \).

The proof works by induction on \( n \), for fixed \( k \).

If \( n = 2k \): \( \Sigma_{2k,k} = \{(+, -, +, -, \ldots,+, -, -), (-, +, -, +, \ldots, -, +, -)\} \), and the proof is straightforward.

If \( n > 2k \): \( \Sigma_{n,k}^+ \) is a pseudomanifold (property 3). Its boundary is \( \Sigma_{n-1,k} \) (property 4). By induction and by Theorem 2, we have, for \( \Sigma_{n,k}^+ \):
\[
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n-2k+1} \leq d} \alpha(-j_1, +j_2, \ldots, (-1)^{n-2k+1}j_{n-2k+1})
\]
\[
+ \alpha(+j_1, -j_2, \ldots, (-1)^{n-2k}j_{n-2k+1}) \equiv 1 \mod 2.
\]
The conclusion of the proposition follows from the fact that the number of \((n - 2k)\)-simplices of \(\Sigma_{n,k}^-\) whose labeling is of the form \(+j_1, -j_2, \ldots, (-1)^{n-2k} j_{n-2k+1}\) with \(j_1 < j_2 < \cdots < j_{n-2k+1}\) is the number of \((n - 2k)\)-simplices of \(\Sigma_{n,k}^+\) whose labeling is of the form \(-j_1, +j_2, \ldots, (-1)^{n-2k+1} j_{n-2k+1}\) with \(j_1 < j_2 < \cdots < j_{n-2k+1}\) (by antipodal).

The proposition is proved. We apply it to achieve the proof of Schrijver’s theorem:

Let \(c\) be a coloring of \(SG(n, k)\) with \(t\) colors: \(c : V(SG(n, k)) \to [t]\). Our aim is to prove that \(t \geq n - 2k + 2\).

Let \(\lambda : \{+, -, 0\}^n \to \{-1, 0, 0, \ldots, -(t - 1), +(t - 1)\}\) be a labeling of the vertices of \(\Sigma_{n,k}^+\) defined as follows: \(\lambda(X^+, X^-) = \pm c(S)\), where \(S\) is the stable \(k\)-subset of \([n]\), contained in \(X^+\) or in \(X^-\), having the smallest color. The sign indicates which of \(X^+\) or \(X^-\) we took \(S\) from. As we require that \(\text{alt}(X) \geq 2k\), both \(X^+\) and \(X^-\) contain a stable \(k\)-subset and thus \(|\lambda(X^+, X^-)| \leq t - 1\). By definition of a coloring, there is no antipodal edge.

We can thus apply the proposition just proved, for \(d := t - 1\): there is at least one \((n - 2k)\)-simplex of \(\Sigma_{n,k}\) labeled with \(1 \leq j_1 < j_2 < \cdots < j_{n-2k+1} \leq t - 1\). Hence, \(n - 2k + 1 \leq t - 1\), or equivalently \(n - 2k + 2 \leq t\).

**Corollary 3** (Fan’s Generalization of Sperner’s Lemma). Let \(n\) be a positive integer and let \(M\) be an \(n\)-pseudomanifold. Let \(\lambda : V(M) \to [n + 1]\) be a labeling of the vertices of \(M\).

We have the following relations:

\[
\alpha(1, 2, \ldots, n + 1) = (-1)^n \beta(1, 2, \ldots, n)
\]

if \(M\) is oriented, and

\[
\alpha(1, 2, \ldots, n + 1) \equiv \beta(1, 2, \ldots, n) \mod 2
\]

if not.

**Proof.** Apply Fan’s formula for the set of labels \(\{+1, -2, +3, \ldots, (-1)^n (n + 1)\}\).

The classical Sperner lemma is a direct consequence of these relations (actually, we get much more, namely that the classical Sperner lemma holds with \(\alpha = 1\)).

4. Multilabelings

We consider now generalizations of the following kind: instead of one label for each vertex of a pseudomanifold, we suppose that there are many labels on each vertex.

4.1. Bapat’s theorem, Lee and Shih’s theorem

**Definitions.** Let \(n\) and \(q\) be two positive integers. Let \(M\) be an \(n\)-pseudomanifold, and let \(\lambda : v \in V(M) \mapsto (\lambda_1(v), \ldots, \lambda_q(v))\) be such that each \(\lambda_i, i = 1, \ldots, q\), is a labeling of \(M\). Such a \(\lambda\) is called a \(q\)-multiple labeling.

If \(M\) is oriented: For \(n + 1\) distinct integers \(j_1, \ldots, j_{n+1}\) arranged in this order, denote by \(\alpha_+(j_1, j_2, \ldots, j_{n+1})\) (resp. \(\alpha_-(j_1, j_2, \ldots, j_{n+1})\)) the number of those pairs \((\sigma, f)\) such that

- \(\sigma\) is an \(n\)-simplex of \(M\) and \(f\) is an injection from \(V(\sigma)\) into \([q]\),
- \(\sigma = [+v_1, v_2, \ldots, v_{n+1}]\) (resp. \([-v_1, v_2, \ldots, v_{n+1}]\)) and \(\lambda_f(v_i) = j_i\) for \(i = 1, \ldots, n + 1\).


For every \((n - 1)\)-simplex of \(\partial M\), we consider the induced orientation from the unique \(n\)-simplex of \(M\) containing it.

For \(n\) distinct integers \(j_1, \ldots, j_n\) arranged in this order, denote by \(\beta_+(j_1, j_2, \ldots, j_n)\) (resp. \(\beta_-(j_1, j_2, \ldots, j_n)\)) the number of those pairs \((\tau, g)\) such that

- \(\tau\) is an \((n - 1)\)-simplex of \(\partial M\) and \(g\) is an injection from \(V(\tau)\) into \([q]\),
- \(\tau = +[v_1, v_2, \ldots, v_n]\) (resp. \(-[v_1, v_2, \ldots, v_n]\)) and \(\lambda_g(v_i) = j_i\) for \(i = 1, \ldots, n\).

Finally, we define \(\alpha := \alpha_+ - \alpha_-\) and \(\beta := \beta_+ - \beta_-\). If \(M\) is not oriented, we define \(\alpha\) and \(\beta\) as in the previous section. Now, we are in position to give Bapat’s theorem:

**Theorem 3 (Bapat’s Theorem).** Let \(T\) be a triangulation of the \(n\)-simplex \(\sigma^n\) and let \(\lambda = (\lambda_i)_{i=1,\ldots,n+1}\) be an \((n + 1)\)-multiple labeling, where the \(\lambda_i, i = 1, 2, \ldots, n + 1\) are \(n + 1\) Sperner labelings of the vertices of \(T\). Then

\[
\alpha(1, 2, \ldots, n + 1) = (n + 1)!
\]

When all \(\lambda_i\) are equal, Sperner’s lemma appears as a special case.

Bapat’s theorem is a direct implication of the following multilabeled version of Theorem 2: as we said in the introduction, such a theorem was established by Lee and Shih in 1998 [6]:

**Theorem 4 (Lee and Shih’s Theorem).** Let \(m, n\) and \(q\) be three positive integers and let \(M\) be an \(n\)-pseudomanifold. Let \(\lambda = (\lambda_i)_{i=1,\ldots,q}\) be a \(q\)-multiple labeling, where for each \(i\lambda_i : V(M) \to \{-1, +1, -2, +2, \ldots, -m, +m\}\) is a labeling of the vertices of \(M\) such that there are no \(j_1 \neq j_2\) and \((v_1, v_2) \in E(M)\) such that \(\lambda_{j_1}(v_1) = -\lambda_{j_2}(v_2)\).

Then we have if \(M\) is oriented:

\[
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1} j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, \ldots, (-1)^n j_{n+1})
\]

\[
= (q - n) \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).
\]

If \(M\) is not oriented, the formula holds modulo 2.

This theorem is a corollary of a more general theorem which will be stated and proved in the next section.

We obtain the formula of Theorem 2 when all \(\lambda_i\) are equal.

**4.2. \(K\)-labelings**

Now, we give a more general version of the definition of multilabelings, in order to obtain a generalization (and a simple proof) of Lee and Shih’s theorem.

**Definitions.** Let \(n\) and \(k\) be two positive integers. Let \(M\) be an \(n\)-pseudomanifold, \(K\) be a \(k\)-regular simplicial complex and let \(\lambda : v \in V(M) \mapsto (\lambda_w(v))_{w \in V(K)}\) be such that each \(\lambda_w, w \in V(K),\) is a labeling of \(M\). Such a \(\lambda\) is called a \(K\)-labeling.

If \(M\) is oriented: For \(n + 1\) distinct integers \(j_1, \ldots, j_{n+1}\) arranged in this order, denote by \(\alpha_+(j_1, j_2, \ldots, j_{n+1})\) (resp. \(\alpha_-(j_1, j_2, \ldots, j_{n+1})\)) the number of those pairs \((\sigma, f)\) such that

- \(\sigma\) is an \(n\)-simplex of \(M\) and \(f(\sigma)\) is an \(n\)-simplex of \(K\),
- \(\sigma = +[v_1, v_2, \ldots, v_{n+1}]\) (resp. \(-[v_1, v_2, \ldots, v_{n+1}]\)) and \(\lambda_{f(v_i)}(v_i) = j_i\) for \(i = 1, \ldots, n+1\).
For every \((n-1)\)-simplex of \(\partial M\), we consider the induced orientation from the unique \(n\)-simplex of \(M\) containing it.

For \(n\) distinct integers \(j_1, \ldots, j_n\) arranged in this order, denote by \(\beta_+(j_1, j_2, \ldots, j_n)\) (resp. \(\beta_-(j_1, j_2, \ldots, j_n)\)) the number of those pairs \((\tau, g)\) such that
- \(\tau\) is an \((n-1)\)-simplex of \(\partial M\) and \(g(\sigma)\) is an \((n-1)\)-simplex of \(K\),
- \(\tau = +[v_1, v_2, \ldots, v_n]\) (resp. \(-[v_1, v_2, \ldots, v_n]\)) and \(\lambda_g(\sigma_i) = j_i\) for \(i = 1, \ldots, n\).

Finally, we define \(\alpha := \alpha_+ - \alpha_-\) and \(\beta := \beta_+ - \beta_-\). If \(M\) is not oriented, we define \(\alpha\) and \(\beta\) as in the previous section.

**Theorem 5.** Let \(m, n\) and \(k\) be three positive integers, let \(M\) be an \(n\)-pseudomanifold and let \(K\) be a \(k\)-regular simplicial complex. Let \(\lambda = (\lambda_w)_{w \in V(K)}\) be a \(K\)-labeling, where, for each \(w \in K\), \(\lambda_w : V(M) \to \{-1, +1, -2, +2, \ldots, -m, +m\}\) is a labeling of the vertices of \(M\) such that there are no \((w_1, w_2) \in E(K)\) and \((v_1, v_2) \in E(M)\) such that \(\lambda_{w_1}(v_1) = -\lambda_{w_2}(v_2)\).

Then we have, if \(M\) is oriented,
\[
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1}j_{n+1}) + (-1)^n \alpha_+(+j_1, -j_2, \ldots, (-1)^n j_{n+1}) = k \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1}j_n).
\]

If \(M\) is not oriented, the formula holds modulo 2.

**Proof.** We write the proof for the oriented case. The other case is proved similarly.

Let \(L\) be the simplicial complex whose vertex set is defined by \(V(L) := V(M) \times V(K)\) and whose simplices are those subsets \(\sigma = \{(v_1, w_1), \ldots, (v_{p+1}, w_{p+1})\}\) of \(V(L)\) such that \(\{v_1, \ldots, v_{p+1}\}\) is a \(p\)-simplex of \(M\) and \(\{w_1, \ldots, w_{p+1}\}\) is a \(p\)-simplex of \(K\). We give \(\sigma\) the same orientation as \([v_1, \ldots, v_{p+1}]\).

Let \(\lambda\) induce a labeling \(\phi\) of \(L\): \(\phi((v, w)) = \lambda_w(v)\). \(\phi\) is a simplicial map going from \(L\) into the \((m-1)\)-dimensional simplicial complex \(C\), whose faces are the subsets of \([-1, +1, -2, +2, \ldots, -m, +m]\) containing no pair \(-i, +i\) for some \(i \in [m]\) (the boundary of the crosspolytope).

Using the same notation as in the proof of Theorem 2, we have the following equality:
\[
d \mu \cdot \phi_{|\partial L} = \mu \cdot \phi_{|\partial L}. \partial_{\lambda}.\]
Applying this formula to the formal sum of all oriented \(n\)-simplices of \(L\), \(L := \sum_{\sigma \in \sigma_n} \sigma\), and noticing that \(\partial_{\lambda}L = k \sum_{[v_1, \ldots, v_n] \in \partial M, [w_1, \ldots, w_n] \in K_{n-1}} [(v_1, w_1), \ldots, (v_n, w_n)]\), the formula follows.

Lee and Shih’s theorem is the special case when \(K = \binom{[q]}{\leq n+1}\).

4.3. Another formula

Suppose now that we wonder about the number \(\alpha_+(j_1, j_2, \ldots, j_{n+1})\) (resp. \(\alpha_-(j_1, j_2, \ldots, j_{n+1})\)) of those pairs \((\sigma, f)\) such that
- \(\sigma\) is an \(n\)-simplex of \(M\) and \(f\) is any map from \(V(\sigma)\) in \([q]\),
- \(\sigma = +[v_1, v_2, \ldots, v_{n+1}]\) (resp. \(-[v_1, v_2, \ldots, v_{n+1}]\)) and \(\lambda_f(v_i) = j_i\) for \(i = 1, \ldots, n+1\),
using the number $\beta_{+}(j_1, j_2, \ldots, j_n)$ (resp. $\beta_{-}(j_1, j_2, \ldots, j_n)$) of pairs $(\tau, g)$ such that

• $\tau$ is an $(n-1)$-simplex of $M$ and $g$ is any map from $V(\tau)$ in $[q]$, $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.

• $\tau = +[v_1, v_2, \ldots, v_n]$ (resp. $-[v_1, v_2, \ldots, v_n]$) and $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.

We have then, with $\alpha := \alpha_{+} - \alpha_{-}$ and $\beta := \beta_{+} - \beta_{-}$, and with the same kind of proof as before (omitted here):

**Theorem 6.** Let $m, n$ and $q$ be three positive integers and let $M$ be an $n$-pseudomanifold. Let $\lambda = (\lambda_i)_{i=1,\ldots,q}$ be a $q$-multiple labeling, where for each $i$ $\lambda_i : V(M) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ is a labeling of the vertices of $M$ such that there are no $j_1, j_2$ and $(v_1, v_2) \in E(M)$ such that $\lambda_j(v_1) = -\lambda_{j_2}(v_2)$.

Then we have, if $M$ is oriented,

$$
\sum_{1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1}j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, \ldots, (-1)^nj_n)
$$

$$
= q \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^nj_n).
$$

If $M$ is not oriented, the formula holds modulo 2.

**References**


