Preservation Properties by Bernstein-Type Operators. A Probabilistic Approach

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Abstract—In this paper, we are concerned with preservation properties of first- and second-order by Bernstein-type operators which preserve monotone functions. We obtain characterizations of the preservation of nondecreasing right-continuous functions, first- and second-order modulus of smoothness, Lipschitz classes of first- and second-order, uniform and absolute continuity, and convexity. These kinds of problems lead us to consider the notions of dual and derived operators. We give a simple unified approach based on stochastic orders and probabilistic coupling techniques, in the sense that we represent the operators under consideration in terms of stochastic processes. The preceding results are illustrated by considering well-known Bernstein-type operators, such as generalized Bernstein-Kantorovich, generalized Szász-Kantorovich, Gamma, and Beta operators. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let T be either the set of positive integers or the set of positive real numbers and let I be a subinterval of the real line. We shall be concerned with families of Bernstein-type operators, that is, families \( L := \{L_t, t \in T\} \) of positive linear operators of the form

\[
L_tf(x) := \int_I f(y)\mu_{t,x}(dy), \quad (t, x) \in T \times I,
\]

where \( (\mu_{t,x}, (t, x) \in T \times I) \) is a family of probability measures concentrated on I such that, for each \( x \in I, \mu_{t,x} \) converges weakly to the Dirac measure \( \delta_x \), as \( t \to \infty \), and \( f \) is any real measurable function defined on I for which the right-hand side in (1) makes sense. Classical examples of Bernstein-type operators can be found in the books by Ditzian and Totik [1] and by Altomare and Campiti [2] (see also [3] and Section 5).

One of the main topics dealt with in the literature about Bernstein-type operators is concerned with rates of convergence of \( L_t \) to \( f \), as \( t \to \infty \). Usually, such rates of convergence have been given in terms of the first modulus of continuity. In this sense, we mention the classical papers of

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Popoviciou [4] and Voronovskaja [5], about the Bernstein operator. Baskakov [6], Mählbach [7], among other authors, have obtained rates of convergence for other families of operators. More recently, a study of the order of convergence for Bernstein-type operators in terms of weighted-second modulus of continuity has been considered by Totik [8], Ditzian and Ivanov [9], etc.

When it comes to approximating a function \( f \) by \( L_t f \), another interesting problem is to determine which properties of \( f \) are retained by the approximants \( L_t f \). A classical result in this direction can be found in [4], where we can observe that the Bernstein polynomials transform continuous and convex functions into continuous and convex functions. Later papers [3,10,11] show that many common families of Bernstein-type operators preserve different properties, such as monotonicity, convexity, \( \phi \)-variation, etc.

In this paper, we consider Bernstein-type operators which preserve monotonicity. In this setting, attention is focused on first- and second-order preservation properties such as right-continuity, absolute and uniform continuity, convexity, and the global smoothness measured by the usual first and second modulus of continuity. One of the main goals is to go deeper into the behaviour of these kinds of operators, by means of the above-mentioned preservation properties. We shall show that the properties of certain sets of functions are translated, in a natural way, into a positive linear operator setting. For instance, an operator \( L \) which preserves monotonicity produces at most a countable set of discontinuities, or an operator \( L \) which preserves monotonicity and convexity has a derived operator preserving monotonicity.

On the other hand, another aim is to give a simple unified approach based on ideas taken from the theory of stochastic orders (see the monographs by Stoyan [21] or Shaked and Shanthikumar [22]) as well as on probabilistic coupling techniques which date back to Lindvall's paper [15] concerning the classical Bernstein polynomials. Such a probabilistic approach could possibly be useful in dealing with preservation properties of arbitrary order. In this sense, observe that formula (1) can be rewritten as

\[
L_t f(x) = E f(Z_t(x)), \quad (t, x) \in T \times I,
\]

where \( E \) denotes mathematical expectation and \( Z := (Z_t(x), (t, x) \in T \times I) \) is a family of \( I \)-valued random variables, not necessarily defined on the same probability space, such that each \( Z_t(x) \) has probability distribution \( \mu_{t,x} \). However, the assumption that, for fixed \( t \in T \), \( L_t f \) retains certain smoothness properties of \( f \) within a certain set of functions, seems to be equivalent to the fact that we can find a family \( Z^*_t := (Z^*_t(x), x \in I) \) of random variables defined on the same probability space having nice dependence properties (that is, a stochastic process) and such that each \( Z^*_t(x) \) has the same probability distribution as \( Z_t(x) \), thus defining the same operator \( L_t \). This coupling construction allows us to translate the stochastic properties of \( Z^*_t \) into preservation properties of \( L_t \).

Since a stochastic process is not determined by its marginal distributions, such a coupling
construction is not unique. Actually, when dealing with particular families of Bernstein-type operators, an important ingredient in the probabilistic approach is to find a suitable coupling representation for the operators under consideration.

The contents of this paper are organized as follows. In Section 2, we introduce the necessary notations and collect some auxiliary results. In Section 3, we characterize the preservation of nondecreasing right-continuous (and left-continuous) functions by an operator $L$ in terms of analogous path properties of the underlying stochastic process $Z$. This kind of problem leads us to introduce the notion of duality (see Theorem 1). Roughly speaking, if an operator $L$ preserves the set of nondecreasing right-continuous functions satisfying certain boundary conditions, then $L$ has a (left) dual operator $\tilde{L}$ preserving the set of nondecreasing left-continuous functions satisfying analogous boundary conditions. The operator $\tilde{L}$ is determined by $L$ and vice versa. When applied to a family $L$ of Bernstein-type operators, this result provides a way to construct a new (dual) family $\tilde{L}$ of Bernstein-type operators. Indeed, some operators usually considered in the literature are related by duality. For instance, the duals of the Bernstein and the Szåsz operators are, respectively, the Beta and the Gamma operators (see Section 5). We also characterize the preservation of the first modulus of continuity, uniform continuity and Lipschitz classes of first order by an operator $L$ in terms of the expectation function of the process $Z$ involved in its representation (Theorem 2).

In Section 4, we consider the closely related problems of preservation of absolute continuity, convex functions with a continuous derivative, second-order modulus of continuity, and Lipschitz classes of second order. Under mild assumptions, we hold that for any absolutely continuous function $\Phi$ with derivative $\Phi'$ satisfying a certain integrability condition or being nonnegative, $L\Phi$ is absolutely continuous with derivative $(L\Phi)' = D\Phi'$ (with respect to the Lebesgue-Stieltjes measure $m$ determined by the expectation function of $Z$), where $D$ is an operator of the form (3) below. The operator $D$, determined $m$-a.e., is called the derived operator of $L$. Most of this result is contained in [23, Theorem 1]. This gives a characterization of the preservation of absolute continuity in terms of $m$. It also allows us to characterize the preservation of the remaining mentioned properties in terms of the second-order moment of the process involved, for operators preserving convexity. Moreover, we give preservation inequalities of type (2) for the second modulus obtaining absolute constants for a wide class of Bernstein-type operators (Remark 5). Finally, the last section is devoted to illustrating the preceding results by considering a few examples of well-known families of Bernstein-type operators. Using a suitable coupling representation for these operators, we compute their corresponding dual and derived operators.

## 2. THE SETTING

Let $I$ be a closed interval of the real line with left and right endpoints $a$ and $b$, respectively. We shall consider positive linear operators $L$ of the form

$$Lf(x) := \int_I f(y)\mu_x(dy) = Ef(Z(x)), \quad x \in I,$$

(3)

where $Z := (Z(x), x \in I)$ is a family of $I$-valued random variables, such that each $Z(x)$ has probability distribution $\mu_x$ and $f : I \to \mathbb{R}$ is a measurable function such that

$$L|f|(x) < \infty, \quad x \in I.$$ (4)

In the common examples, $Z$ is integrable, that is, $E|Z(x)| < \infty, x \in I$. A family $Z^* := (Z^*(x), x \in I)$ of $I$-valued random variables is said to be a version of $Z$ if both families have the same marginals, i.e., if $Z^*(x) = (L)Z(x), x \in I$, where $(L)$ stands for equality in distribution. Observe that two versions define the same operator $L$. If $S$ is a set of real functions defined on $I$, we say that $L$ preserves $S$, denoted by $L(S) \subseteq S$, if $Lf \in S$, for any $f \in S$ satisfying the integrability condition (4).
From now on, unless otherwise specified, we shall assume that every real function \( f \) is defined on \( I \) and every random variable \( X \) is \( I \)-valued. Denote, respectively, by \( \mathcal{T} \), \( \mathcal{R} \), and \( \mathcal{L} \) the sets of nondecreasing, nondecreasing right-continuous, and nondecreasing left-continuous functions. For any \( f \in \mathcal{T} \), \( f(x+) \) stands for the right limit of \( f \) at \( x \), with the convention that \( f(b+) = f(b) \), if \( b \) is real. The function \( f(x-) \), \( x \in I \), is defined in a similar way.

In order to define generalized inverses, the following sets of functions are useful. Denote by \( \mathcal{T}_r \) the set of functions \( f : I \to I \), \( f \in \mathcal{T}_r \), such that
\[
 f(b) = b, \text{ if } b \text{ is real},
\]
\[
 f(b-) = b, \text{ if } b = c^-, \text{ and}
\]
\[
 f(a+) = a, \text{ if } a = -c^-. \]

As can be seen in [24] and the references therein, there are no standard definitions of generalized inverses. The preceding definitions are justified by the following result, whose proof is straightforward (although tedious).

**Theorem 1.** The following properties hold.

(a) If \( f \in \mathcal{T}_r \), then \( f(x) \geq y \) if and only if \( x \geq \bar{f}(y) \), \( x, y \in I \). Moreover, \( \bar{f} \in \mathcal{L}_r \) and \( \bar{f} = f \).

(b) If \( f \in \mathcal{L}_r \), then \( f(x) \leq y \) if and only if \( x \leq \bar{f}(y) \), \( x, y \in I \). Moreover, \( \bar{f} \in \mathcal{R}_r \) and \( \bar{f} = f \).

(c) If \( f \in \mathcal{R}_r \) and \( f(a) = a \), whenever \( a \) is real, then
\[
 (\bar{f})(y+) = \bar{f}(y^-), \quad y \in I \setminus \{b\}. \]

Finally, we recall that a random variable \( X \) is said to be smaller than \( Y \) in the usual stochastic order, denoted by \( X \leq_{st} Y \), if \( P(X > x) \leq P(Y > x) \), \( x \in \mathbb{R} \). A family \( Z = (Z(x), x \in I) \) of random variables is said to be stochastically ordered if \( Z(x) \leq_{st} Z(y) \), whenever \( x \leq y \), \( x, y \in I \). We shall need the following well-known result (cf. [25-27] for its extension to a more general setting).

**Lemma 2.** Let \( L \) be an operator of the form (3) represented by a family \( Z \). The following assertions are equivalent.

(a) \( L(I) \subseteq I \).

(b) \( Z \) is stochastically ordered.

(c) There exists a version of \( Z \) with paths in \( I \).

3. FIRST-ORDER PRESERVATION PROPERTIES

In this section, we shall be concerned with operators \( L \) preserving monotonicity, that is, operators \( L \) such that \( L(I) \subseteq I \). For any \( f \in I \), denote by \( D^+(f) \) the set of its right-discontinuity points. We shall first show that an operator \( L \) preserving monotonicity produces at most a countable set of right-discontinuity points.

**Proposition 1.** Let \( L \) be an operator of the form (3) represented by a family \( Z \). Assume that \( L(I) \subseteq I \). Then, there exists a countable set \( N \subseteq I \) such that \( D^+(Lf) \subseteq N \), for any \( f \in \mathcal{R} \) satisfying (4). Furthermore, \( D^+(Lf) = N \) for any strictly increasing function \( f \in \mathcal{R} \) satisfying (4).

**Proof.** By Lemma 2, it can be assumed that \( Z \) has paths in \( I \). Let \( f \in \mathcal{R} \) satisfying (4). Since the process \( Z_+ := (Z(x+), x \in I) \) has paths in \( \mathcal{R} \) and \( Z(x+) \geq Z(x), x \in I \), we have by monotone convergence,
\[
 \lim_{y \uparrow x} Ef(Z(y)) = Ef(Z(x+)) \geq Ef(Z(x)), \quad x \in I. \]
In particular, choosing \( f \) as the indicator function \( 1_{[q, \infty) \cap I}, q \in \mathbb{Q} \) (the set of rational numbers), we obtain

\[
\lim_{y \downarrow x} P(Z(y) \geq q) = P(Z(x) \geq q) \geq P(Z(x) \geq q), \quad x \in I, \quad q \in \mathbb{Q}.
\]  

Define the sets

\[
N := \{x \in I : Z(x) \neq \mathcal{L}(Z(x))\} \quad \text{and} \quad N_q := D^+ (L1_{[q, \infty) \cap I}), \quad q \in \mathbb{Q}.
\]

From (5), it is clear that \( D^+(L^f) \subseteq N \). This, together with (6), implies that \( N = \bigcup_{q \in \mathbb{Q}} N_q \). Since \( L \) preserves monotonicity, \( N_q \) is countable, and therefore, \( N \) is also countable. The last assertion in Proposition 1 is an immediate consequence of (5). This completes the proof of Proposition 1.

**Remark 1.** We have shown that the processes \( Z_+ \) and \( Z \) differ in a countable set. The same property is satisfied by \( Z_- := (Z(x-), x \in I) \) and \( Z \). Therefore, analogous statements to those given in Proposition 1 hold true with respect to the set of left-discontinuity points of \( L^f \), for any \( f \in \mathcal{L} \). Also, the set of discontinuity points of \( L^f \), for any bounded continuous function \( f \), is contained within a countable set.

Proceeding as in the proof of Proposition 1, we obtain the following.

**Corollary 1.** Let \( L \) be an operator of the form (3) represented by a family \( Z \). Then we have the following.

(a) \( L(S) \subseteq S \) if and only if there exists a version of \( Z \) with paths in \( S \), where \( S = \mathbb{R} \) or \( \mathcal{L} \).

(b) If \( I = [a, b] \subseteq \mathbb{R} \), then \( L(S) \subseteq S \) if and only if there exists a version of \( Z \) with paths in \( S \), where \( S = \mathcal{R} \) or \( \mathcal{L} \).

Assume that \( I = [a, b] \subseteq \mathbb{R} \). If \( L \) preserves \( \mathcal{R} \), we define the left-dual operator of \( L \) by

\[
\tilde{L}f(y) := Ef (\tilde{Z}(y)), \quad y \in I,
\]

where \( \tilde{Z} := (\tilde{Z}(y), y \in I) \) is the left-continuous inverse process of \( Z \), as defined in Section 2. Similarly, if \( L \) preserves \( \check{Z} \), we define the right-dual operator of \( L \) by

\[
\check{L}f(y) := Ef (\check{Z}(y)), \quad y \in I,
\]

where \( \check{Z} := (\check{Z}(y), y \in I) \) is the right-continuous inverse process of \( Z \). Observe that, by Lemma 1(a),

\[
P (\tilde{Z}(y) \leq x) = P(Z(x) \geq y), \quad x, y \in I,
\]

and therefore, the definition of \( \check{L} \) does not depend upon the version of the process \( Z \) used to represent the operator \( L \). The same is true with regard to \( \check{L} \).

We enunciate the following.

**Theorem 1. Duality.** Assume that \( I = [a, b] \subseteq \mathbb{R} \). Then we have the following.

1. Let \( L \) be an operator of the form (3) represented by a family \( Z \). We have the following.

(a) If \( L \) preserves \( \mathcal{R} \), then

\[
\tilde{L}1_{(-\infty, x) \cap I}(y) = L1_{[y, \infty) \cap I}(x), \quad x, y \in I.
\]

Moreover, \( \check{L} \) preserves \( \mathcal{E} \) and \( \check{L} = L \).

(b) If \( L \) preserves \( \mathcal{L} \), then

\[
\check{L}1_{[x, \infty) \cap I}(y) = L1_{(-\infty, y) \cap I}(x), \quad x, y \in I.
\]

Moreover, \( \check{L} \) preserves \( \mathcal{R} \) and \( \check{L} = L \).
(c) If \( L \) preserves \( \mathcal{R} \) and \( \bar{L} \), there exists a countable set \( N \subseteq I \) such that, for any bounded measurable function \( f \), we have
\[
\bar{L}f(y) = \bar{L}f(y), \quad y \in I \setminus N.
\]

(II) Let \( L := (L_t, t \in T) \) be a family of Bernstein-type operators of the form (1). If, for any \( t \in T \), \( L_t \) preserves \( \mathcal{R} \), (respectively, \( \mathcal{E} \)), then the left-dual family \( \bar{L} := (\bar{L}_t, t \in T) \) (respectively, the right-dual family \( \bar{L} := (\bar{L}_t, t \in T) \)) is a family of Bernstein-type operators.

**Proof of I.**

(a) The first assertion immediately follows from (7) and (9). By Lemma 1(a), the process \( \hat{Z} \) has paths in \( \mathcal{L} \), and therefore, \( \bar{L} \) preserves \( \mathcal{L} \), as it follows from Corollary 1(b). Again by Lemma 1(a), \( \hat{Z}(x) = Z(x) \), \( x \in I \), which shows that \( \bar{L} = L \).

(b) Statement (b) is proved in a similar way.

(c) By assumption and Corollary 1(b), \( L \) is representable by a process \( Z \) with paths in \( \mathcal{R} \) such that \( Z(a) = a \), whenever \( a \) is real. Since \( L \) preserves \( \mathcal{L} \), \( L \) is also representable by the process \( \hat{Z} \), and therefore, \( Z(x) = (\mathcal{L})\hat{Z}(x-) \), \( x \in I \). Hence, we have from (7), (8), and Lemma 1(c),
\[
\bar{L}f(y) = Ef(\hat{Z}(y)), \quad y \in I,
\]
and
\[
\bar{L}f(y) = Ef(\hat{Z}(y-)) = Ef((\hat{Z})(y+)), \quad y \in I \setminus \{b\}.
\]

Recalling the proof of Proposition 1, the set
\[
N = \{y \in I : (\hat{Z})(y+) \neq (\mathcal{L})\hat{Z}(y)\}
\]
is countable. This shows (c).

**Proof of II.** For each \( t \in T \), let \( Z_t := (Z_t(x), x \in I) \) and \( \hat{Z}_t := (\hat{Z}_t(y), y \in I) \) be the stochastic processes representing the operators \( L_t \) and \( \bar{L}_t \), respectively. From the equality
\[
P(Z_t(x) \geq y) = P(\hat{Z}_t(y) \leq x), \quad x, y \in I,
\]
it is readily seen that \( Z_t(x) \) converges weakly to \( x \), as \( t \to \infty \), for any \( x \in I \), if and only if \( \hat{Z}_t(y) \) converges weakly to \( y \), as \( t \to \infty \), for any \( y \in I \). A similar statement holds true with regard to the process \( \hat{Z}_t := (\hat{Z}_t(y), y \in I) \) representing the operator \( \bar{L}_t \). The proof of Theorem 1 is complete.

**Remark 2.** Assume in Corollary 1(b) that \( I = [a, \infty) \). Then if \( Z \) has paths in \( \mathcal{R} \), \( L \) preserves the set \( \mathcal{R} \). However, the converse implication is not true. A simple counter-example is given by the operator \( L \) defined as \( Lf(x) = 1/2(f(x) + f(0)) \), \( x \geq 0 \), represented by a family \( Z \) such that \( P(Z(x) = x) = P(Z(x) = 0) = 1/2, x \geq 0 \). Notwithstanding, if \( L \) is an operator represented by a family \( Z \) with paths in \( \mathcal{R} \), we can define its left-dual operator \( \bar{L} \) as in (7) and Theorem 1 still holds with minor modifications. The preceding applies if \( Z \) has paths in \( \mathcal{L} \) or if \( I = (-\infty, b] \) or \( I = \mathbb{R} \).

Denote by
\[
\omega_1(f; \delta) := \sup\{|f(x) - f(y)| : x, y \in I, |x - y| \leq \delta\}, \quad \delta \geq 0,
\]
the usual first modulus of continuity of \( f \). Let \( \mathcal{F}_1 \) be the set of functions having finite first modulus of continuity and let \( \mathcal{U} \subseteq \mathcal{F}_1 \) be the set of uniformly continuous functions. Also, for any \( \alpha \in (0, 1] \), let \( \text{Lip}_1(\alpha) \) be the Lipschitz class of first order defined as the set of functions \( f \in \mathcal{U} \) such that
\[
\omega_1(f; \delta) \leq A\delta^\alpha, \quad \delta \geq 0, \quad \text{for some} \ A > 0.
\]
If \( Z \) is an integrable family, the preservation of these sets of functions can be characterized in terms of the expectation function \( m(x) := EZ(x), x \in I \), as the following theorem shows.
THEOREM 2. Let \( L \) be an operator of the form (3) represented by an integrable family \( Z \). Assume that \( L(I) \subseteq I \). Then we have the following.

(a) \( L(S) \subseteq S \) if and only if the expectation function \( m \in S \), where \( S = \mathcal{R}, \mathcal{L}, \mathcal{F}_1, \) or \( \mathcal{U} \).

(b) \( L(\text{Lip}_1(\alpha)) \subseteq \text{Lip}_1(\alpha) \), for any \( \alpha \in (0, 1] \) if and only if \( m \in \text{Lip}_1(1) \).

PROOF. The "only if" part being trivial, we shall show the "if" part. The cases \( S = \mathcal{R} \) and \( S = \mathcal{L} \) immediately follow from Proposition 1 and Remark 1. Let \( \delta > 0 \) and let \( x, x + h \in I \) with \( 0 \leq h \leq \delta \). By Lemma 2, it can be assumed that the process \( Z \) has paths in \( I \). Therefore,

\[
|Lf(x + h) - Lf(x)| \leq E|f(Z(x + h)) - f(Z(x))| \leq E\omega_1(f; Z(x + h) - Z(x)).
\]  

(10)

Using (10) and the well-known inequality

\[
\omega_1(f; ah) \leq (1 + a)\omega_1(f; h), \quad a, h \geq 0,
\]

with \( a = \omega_1(m; \delta) \), we obtain

\[
\omega_1(Lf; \delta) \leq 2\omega_1(f; \omega_1(m; \delta)).
\]

The remaining cases follow from this inequality. This completes the proof of Theorem 2.

REMARK 3. Let \( L := (L_t, t \in T) \) be a family of Bernstein-type operators preserving the sets \( \mathcal{F}_1 \) and \( \text{Lip}_1(\alpha) \). The question of finding the best absolute constants \( C \) in inequalities of the form

\[
\omega_1(L_t f; \delta) \leq C \omega_1(f; \delta), \quad f \in \mathcal{F}_1 \text{ or } f \in \text{Lip}_1(\alpha), \quad \delta \geq 0, \quad t \in T,
\]

has been considered in \([13, 28]\) for particular families of operators \( L \).

4. SECOND-ORDER PRESERVATION PROPERTIES

In this section, we consider the preservation of absolutely continuous functions, convex functions with a continuous derivative, functions having finite second modulus of continuity, and Lipschitz classes of second order, by an operator \( L \) represented by an integrable family \( Z \). For any Borel set \( A \subseteq \mathbb{R}^n \), denote by \( \mathcal{B}(A) \) the Borel \( \sigma \)-field in \( A \). If the expectation function \( m \in \mathcal{R} \), we also denote by \( m \) the Lebesgue-Stieltjes measure on \((I, \mathcal{B}(I))\) determined by \( m((x, y]) = m(y) - m(x), x, y \in I, x < y \). On the other hand, let \( M \) be the set of measurable functions, let \( A \) be the set of absolutely continuous functions, and finally, let \( A_+ \subseteq \mathcal{A} \) be the set of absolutely continuous functions \( \Phi \) having a nonnegative derivative \( \Phi' \). The following theorem is concerned with the preservation of the sets \( A_+ \) and \( A \). Most of it is a consequence of a more general result given in \([23]\) (see Theorem 1), and therefore, we omit the proof.

THEOREM 3. Let \( L \) be an operator of the form (3) represented by an integrable family \( Z \). Assume that \( L(\mathcal{R}) \subseteq \mathcal{R} \). Then we have the following.

(a) There exists an operator \( D \) of the form (3) such that

(i) \( D(\mathcal{M}) \subseteq \mathcal{M} \);

(ii) for any \( \Phi \in A_+ \) satisfying (4) or \( \Phi \in A \) verifying the following integrability condition,

\[
\int \left| \Phi'(t) \right| \left( P(Z(y) \geq t) - P(Z(x) \geq t) \right) dt < \infty, \quad x, y \in I, \quad x \leq y,
\]

(11)

we have

\[
L\Phi(y) - L\Phi(x) = \int_{(x, y]} D\Phi'(z) m(dz), \quad x, y \in I, \quad x \leq y.
\]

(12)

(b) If \( D^* \) is an operator of the form (3) satisfying (i) and (ii), then there exists a set \( N \in \mathcal{B}(I) \) with \( m(N) = 0 \) such that, for any integrable \( f \in \mathcal{M} \),

\[
D^* f(z) = D f(z), \quad z \in I \setminus N.
\]
As a consequence, $L(A_+) \subseteq A_+$ if and only if $m \in A_+$. In particular, if $m$ is the Lebesgue measure on $I$, we have for any $\Phi \in A_+$ satisfying (4),

$$(L\Phi)' = D\Phi'.$$

**Remark 4.** With the assumptions of Theorem 3, we have: $L\Phi \in A$ for any $\Phi \in A$ satisfying (11) if and only if $m \in A_+$.

In the setting of Theorem 3, an operator $D$ of the form (3) satisfying (i) and (ii) is called a derived operator of $L$. According to Part (b), it can be said that an operator $L$ essentially determines “its” derived operator. However, the converse implication does not hold. That is, given an operator $D$ of the form (3) satisfying (i) and (ii), then formula (12) does not define a positive linear operator $L$. To see this, we give the following.

**Counter Example.** Let $I = [0, 1]$ and let $D$ be the operator given by

$$Df(z) = Ef(zV), \quad z \in I,$$

where $V$ is a random variable having the uniform distribution on $[0, 1]$. Then, the formula

$$Lf(x) = f(0) + \int_0^x \frac{f(z) - f(0)}{z} \, dz, \quad x \in I,$$

defines a linear operator acting on the set of measurable functions $f$ for which the right-hand side in (13) makes sense. For any $\Phi \in A_+$, we obviously have

$$L\Phi(y) - L\Phi(x) = \int_x^y D\Phi(z) \, dz, \quad x, y \in I, \quad x \leq y.$$

Thus, $D$ is an operator of form (3) satisfying (i) and (ii) in Theorem 3. Notwithstanding, $L$ is not a positive operator. Actually, $L$ cannot be written as

$$Lf(x) = \int_I f(y)\mu_x(dy), \quad x \in I,$$

for a signed measure $\mu_x$. For, if this was the case, we would have for any $x \in I \setminus \{0\}$ and $r \in [0, x]$,

$$\mu_x([0, r]) = 1 + \ln r - \ln x, \quad \mu_x(\{0\}) = -\infty \quad \text{and} \quad \mu_x((0, r]) = +\infty,$$

which is a contradiction.

Denote by $C_x$ the set of convex functions which are continuous at the endpoints of $I$, whenever they are real. The following result, which is contained in [23] (see Theorem 2), deals with the preservation of the set $C_x$ (we include it for the sake of clarity).

**Corollary 2.** Let $L$ be an operator of the form (3) represented by an integrable family $Z$ of nonidentically distributed random variables. Assume that $L(\mathcal{R}) \subseteq \mathcal{R}$. Then, $L(C_x) \subseteq C_x$ if and only if:

(a) $m(x) = cx + d, \; x \in I$, for some constants $c > 0$ and $d$;

(b) $L$ has a derived operator $D$ such that $D(\mathcal{R}) \subseteq \mathcal{R}$.

In such a case, we have for any $\Phi \in C_x$ satisfying (4),

$$L\Phi(y) - L\Phi(x) = c \int_x^y D\Phi'(z) \, dz, \quad x, y \in I, \quad x \leq y.$$  \hfill (14)

Let $C$ be the set of real continuous functions defined on $I$ and let $C^{(1)}$ be the set of real functions with derivative $f' \in C$. Recall that the second modulus of continuity of $f \in C$ is given by

$$\omega_2(f; \delta) := \sup \{ |f(x+h) - f(x)| : x - h, \; x + h \in I, \; 0 \leq h \leq \delta \}, \quad \delta \geq 0,$$
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where

\[ \Delta_h^2 f(x-h) := f(x-h) - 2f(x) + f(x+h). \]

Analogous to the class \( \mathcal{F}_1 \) given in Section 3, we define \( \mathcal{F}_2 \) as the set of continuous functions having finite second modulus of continuity. Also, for any \( \alpha \in (0, 2] \), we denote by \( \text{Lip}_2(\alpha) \) the Lipschitz class of second order given by

\[ \text{Lip}_2(\alpha) := \{ f \in C : \omega_2(f; \delta) \leq A \delta^\alpha, \delta \geq 0, \text{ for some } A > 0 \}. \]

If \( Z \) is a family such that \( E(Z(x))^2 < \infty, x \in I \), the preservation of these sets of functions can be characterized in terms of the second-order moment, as is shown in Theorem 4 below. Previous to this, we need the following result (cf. [29]).

**Lemma 3.** (See [29].) Let \( f \in C \) and \( 0 < \delta \leq (b-a)/2 \). Then, there exists a function \( P_\delta f \) satisfying,

(a) for almost everywhere \( x \in I \),

\[ |(P_\delta f)^\prime(x)| \leq \frac{3}{2} \delta^{-2} \omega_2(f; \delta); \]

(b) for every \( x \in I \),

\[ |P_\delta f(x) - f(x)| \leq \frac{3}{4} \omega_2(f; \delta). \]

**Theorem 4.** Let \( L \) be an operator of the form (3) represented by an integrable square family \( Z \) of nonidentically distributed random variables, with \( m_2(x) := E(Z(x))^2 \), \( x \in I \). Suppose that \( L(\mathcal{R}) \subseteq \mathcal{R} \) and \( L(\mathcal{C}_x) \subseteq \mathcal{C}_x \). Then we have the following.

(a) \( L(\mathcal{S}) \subseteq \mathcal{S} \) if and only if \( m_2 \in \mathcal{S} \), where \( \mathcal{S} = \mathcal{C}_x \cap \mathcal{C}^{(1)} \) or \( \mathcal{F}_2 \).

(b) \( L(\text{Lip}_2(\alpha)) \subseteq \text{Lip}_2(\alpha) \), for any \( \alpha \in (0, 2] \), if and only if \( m_2 \in \text{Lip}_2(2) \).

Moreover, \( m_2 \in \text{Lip}_2(2) \) if and only if

\[ \omega_2(Lf; \delta) \leq C \omega_2(f; \delta), \quad f \in \mathcal{F}_2, \quad \delta > 0, \quad (15) \]

for a constant \( C > 0 \) independent of \( f \) and \( \delta \).

**Proof.** First, we shall show Part (a) with \( \mathcal{S} = \mathcal{C}_x \cap \mathcal{C}^{(1)} \). The “only if” part being obvious, we shall prove the converse implication. Since \( L(\mathcal{R}) \subseteq \mathcal{R} \) and \( L(\mathcal{C}_x) \subseteq \mathcal{C}_x \), we can assert, according to Corollary 2, that \( L \) has a derived operator \( D_1 \) such that \( D_1(\mathcal{R}) \subseteq \mathcal{R} \). The function \( \Phi(x) = x^2 \) belongs to \( \mathcal{C}_x \cap \mathcal{C}^{(1)} \), therefore, we have from (14),

\[ m_2(y) - m_2(x) = c \int_x^y D_1 e_1(z) \, dz, \quad x, y \in I, \quad x \leq y, \]

where \( e_1(x) := x, x \in I \).

By assumption, \( m_2 \in \mathcal{C}_x \cap \mathcal{C}^{(1)} \), so modifying the definition of \( D_1 \) at point \( b \) (if necessary), we can say that \( D_1 e_1(x), x \in I \), is a nondecreasing continuous function. By Theorem 2(a), we have \( D_1(S) \subseteq S \), with \( S = \mathcal{R} \) and \( \mathcal{L} \). Finally, applying formula (14) to any function \( \Phi \in \mathcal{C}_x \cap \mathcal{C}^{(1)} \) satisfying (4), we obtain the claim.

We shall show the preservation of the remaining classes of functions. As in the preceding case, \( L \) has a derived operator \( D_1 \) such that \( D_1(\mathcal{R}) \subseteq \mathcal{R} \). Since \( Z \) is an integrable square family, formula (14) applied to \( \Phi(x) = x^2 \), \( x \in I \), implies that \( D_1 e_1(x) < \infty, x \in I \setminus \{a, b\} \). Consequently, applying Theorem 3 to the interval \( I \setminus \{a, b\} \), \( D_1 \) has a derived operator \( D_2 \).

For a function \( \Phi \) whose second derivative \( \Phi'' \) is bounded almost everywhere, we have, for any \( x - h, x + h \in I, h \geq 0 \),

\[ |\Delta_h^2 L\Phi(x-h)| = c |h| \int_0^1 \, dz \int_{(x-x+h_2,x+h_2)} D_2 \Phi''(u) \, dD_1 e_1(u)|. \quad (16) \]
Indeed, if \( z_0 \) is an interior point of \( I \), an application of the mean valued theorem asserts that
\[
|\Phi'(z)| \leq |\Phi'(z_0)| + M|z - z_0|,
\]
for a constant \( M > 0 \). Since \( Z \) is an integrable square family, we have
\[
\int_I |\Phi'(z)| \left( P(Z(y) \geq z) - P(Z(x) \geq z) \right) \, dz < \infty, \quad x, y \in I, \ x \leq y.
\]

From Theorem 3(a), we can apply formula (12) to the functions \( L\Phi \) and \( D_1 \Phi' \) to conclude that, for any \( x - h, x + h \in I \), with \( h \geq 0 \),
\[
\Delta_h^2 L\Phi(x - h) = c \left( \int_{x-h}^{x+h} D_1 \Phi'(z) \, dz - \int_{x-h}^{x+h} D_1 \Phi'(z) \, dz \right)
\]
\[
= ch \int_0^1 \left( D_1 \Phi'(x + zh) - D_1 \Phi'(x - h + zh) \right) \, dz
\]
\[
= ch \int_0^1 dz \int_{(x-h+zh,x+zh]} D_2 \Phi''(u) \, dD_1e_1(u),
\]
which shows (16).

Now, let \( f \in \mathcal{F}_2 \). Let \( \delta > 0 \) and \( x - h, x + h \in I \), with \( 0 < h \leq \delta \). It is immediate that
\[
|\Delta_h^2 Lf(x - h)| \leq |\Delta_h^2 L(f - P_\delta f)(x - h)| + |\Delta_h^2 LP_\delta f(x - h)|,
\]
where \( P_\delta f \) is the function given in Lemma 3.

Applying (16) to the functions \( \Phi(x) = P_\delta f(x) \), \( \Phi(x) = x^2 \), \( x \in I \), and taking into account Lemma 3(a), we obtain
\[
|\Delta_h^2 P_\delta f(x - h)| \leq ch 3^2 \delta^{-2} \omega_2(f; \delta) \int_0^1 dz \int_{(x-h+zh,x+zh]} dD_1e_1(u)
\]
\[
= \frac{3}{4} \delta^{-2} \omega_2(f; \delta) \Delta_h^2 m_2(x - h)
\]
\[
\leq \frac{3}{4} \delta^{-2} \omega_2(f; \delta) \omega_2(m_2; \delta).
\]
On the other hand, Lemma 3(b) gives us
\[
|\Delta_h^2 L(f - P_\delta f)(x - h)| \leq 4 \frac{3}{4} \omega_2(f; \delta) = 3 \omega_2(f; \delta).
\]
From (17)–(19), we deduce
\[
\omega_2(Lf; \delta) \leq \left( \frac{3}{4} \delta^{-2} \omega_2(m_2; \delta) + 3 \right) \omega_2(f; \delta).
\]
From the last inequality, Part (a) with \( S = \mathcal{F}_2 \), Part (b) and formula (15) are proved. The proof of Theorem 4 is complete.

Remark 5. In the setting of Theorem 4, we have: if \( f \in \mathcal{F}_2 \) and \( \omega_2(m_2; \delta) \leq A\delta^2 \), for a constant \( A > 0 \), then
\[
\omega_2(Lf; \delta) \leq 3 \left( 1 + \frac{A}{4} \right) \omega_2(f; \delta), \quad \delta \geq 0.
\]
For the Bernstein operator, it is easy to see that \( \omega_2(m_2; \delta) \leq 2\delta^2 \), obtaining the constant \( C \) in (15) equals to 4.5. This constant has been given in [19].
5. EXAMPLES

In this section, we consider some well-known families of Bernstein-type operators taken from the literature on approximation theory. We first give a suitable probabilistic representation for the operators under consideration. Some of their preservation and approximation properties can be found in [11,30-32] and the references therein.

(A) GENERALIZED SZÁSZ-KANTOROVICH OPERATOR. This operator has the form

\[ S_{t,m}f(x) := \sum_{k=0}^{\infty} \frac{e^{-tx} (tx)^k}{k!} \int_0^1 \ldots \int_0^1 f \left( \frac{k + u_1 + \ldots + u_m}{t} \right) du_1 \ldots du_m \]

\[ = Ef \left( \frac{N(tx) + U_1 + \ldots + U_m}{t} \right), \quad x \geq 0, \quad t > 0, \quad m = 0,1, \ldots. \]

where \( (N(x), x \geq 0) \) is a standard Poisson process and \( (U_k, k = 1, \ldots, m) \) are \( m \) independent and on the interval \([0,1]\) uniformly distributed random variables, which are also independent of \((N(x), x \geq 0)\). Observe that \( S_t := S_{t,0} \) is the classical Szász operator and \( S_t := S_{t,1} \) is the Szász-Kantorovich operator introduced by Butzer [30].

(B) GAMMA OPERATOR. This operator is defined by

\[ G_t f(x) := \frac{1}{\Gamma(tx)} \int_0^\infty f \left( \frac{\theta}{t} \right) e^{-\theta} d\theta = Ef \left( \frac{G(tx)}{t} \right), \quad x \geq 0, \quad t > 0, \]

where \((G(x), x \geq 0)\) is a gamma process, that is, a process starting at the origin, having independent stationary increments and such that, for each \( x > 0 \), \( G(x) \) has the gamma density.

\[ d_x(\theta) := \frac{\theta^{x-1} e^{-\theta}}{\Gamma(x)}, \quad \theta > 0. \]

(C) GENERALIZED BERNSTEIN-KANTOROVICH OPERATOR. The definition is the following:

\[ B_{n,m} f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 \ldots \int_0^1 f \left( \frac{k + u_1 + \ldots + u_m}{n + m} \right) du_1 \ldots du_m \]

\[ = Ef \left( \frac{S_n(x) + U_1 + \ldots + U_m}{n + m} \right), \quad x \in [0,1], \quad n = 1,2, \ldots, \quad m = 0,1, \ldots, \]

where

\[ S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(X_k), \quad x \in [0,1], \quad n = 1,2, \ldots, \]

and \((U_k, k = 1, \ldots, m)\) and \((X_k, k = 1, 2, \ldots)\) are two independent sequences of independent and on the interval \([0,1]\) uniformly distributed random variables. Observe that \( B_n := B_{n,0} \) is the classical Bernstein polynomial of degree \( n \), whereas \( B_{n,1} \) is the Bernstein-Kantorovich operator.

(D) BETA OPERATOR. This operator is given by

\[ \beta_t f(x) := \int_0^1 f(\theta) \frac{\theta x^{1-x} (1-x)^{1-\theta}}{\beta(tx,t(1-x))} d\theta = Ef \left( \frac{G(tx)}{G(t)} \right), \quad x \in [0,1], \quad t > 0, \]

where \( \beta(\cdot,\cdot) \) is the beta function and \((G(x), x \geq 0)\) is the gamma process.

All the preceding operators preserve monotonicity, since all the processes appearing in their probabilistic representation have nondecreasing paths. As we can find versions of the processes involved having right-continuous or left-continuous paths, all these operators preserve the sets \( \mathcal{R} \) and \( \mathcal{L} \). For \( m \geq 1 \), the generalized Bernstein-Kantorovich operator \( B_{n,m} \) does not preserve the set \( \mathcal{R} \) nor the set \( \mathcal{L} \). For \( m \geq 1 \), the generalized Szász-Kantorovich operator \( S_{t,m} \) does not
preserve $\mathcal{L}$. All the remaining operators preserve both $\mathcal{R}$ and $\mathcal{L}$, as they can be represented by processes with paths in $\mathcal{R}$ and $\mathcal{L}$. However, for the sake of brevity, we shall consider these operators as preserving $\mathcal{R}$, and therefore, evaluate their left-duals preserving $\mathcal{L}$.

In each case, the process $Z$ is integrable. In fact, it can be seen by calculus that $EZ(x) = cx + d$, with $c = 1$ and $d = 0$ in Cases (B) and (D), $c = 1$ and $d = m/2t$ in Case (A) and $c = n/(n + m)$ and $d = m/2(n + m)$ in Case (C). Hence, by Theorem 2, all the operators above preserve the sets $\mathcal{F}_1$, $\mathcal{U}$ and Lip$_1(\alpha)$, $\alpha \in (0, 1]$. Finally, we shall compute their derived operators, determined up to an exceptional set of zero Lebesgue measure. It will be clear that, in each case, the derived operator preserves the set $\mathcal{R}$. Therefore, by Corollary 2, all the operators considered preserve convexity. It is easy to see that the second-order moment of all the processes involved are polynomials of second degree, therefore, as a consequence of Theorem 4, all the operators preserve the sets $C_2 \cap C^0$, $\mathcal{F}_2$ and Lip$_2(\alpha)$ for any $\alpha \in (0, 2]$. In fact, $\omega_2(m; \delta) \leq 2\delta^2$ in all cases considered, obtaining a constant $C = 4, 5$ in inequalities of type (15).

Together with the preceding notations, we shall use the following: for any real $x$, $[x]$ denotes the integral part of $x$ and $[x]$ stands for the ceiling of $x$, i.e., the smallest integer not less than $x$, with the convention that $[x] = 0$, $x \leq 0$. Finally, recall that (cf. [33]) for any $m = 1, 2, \ldots$, the random variable $U_1 + \cdots + U_m$ has density

$$g_m(z) := \frac{1}{(m-1)!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (z-k)^{m-1}, \quad z \in [0, m], \quad (g_0(z) = 1_{\{0\}}(z)).$$

(A) **GENERALIZED SZÁSZ-KANTOROVICH OPERATOR.** Let $x, y \geq 0$, $t > 0$, and $m = 0, 1, 2, \ldots$. Using the well-known formula (cf. [34])

$$P(N(x) \geq y) = P(G([y]) \leq x),$$

we see that

$$P \left( \frac{N(tx) + U_1 + \cdots + U_m}{t} \geq y \right) = P \left( G\left(\left\lfloor \frac{ty - U_1 - \cdots - U_m}{t} \right\rfloor \right) \leq x \right),$$

where $(U_k, k = 1, \ldots, m)$ are independent of the gamma process $(G(x), x > 0)$. Therefore, it follows by calculus that the left-dual operator of $S_{t,m}$ is given by

$$\tilde{S}_{t,m}f(y) = Ef \left( G\left(\left\lfloor \frac{ty - U_1 - \cdots - U_m}{t} \right\rfloor \right) \right) = \int_0^m G_t \left( \left\lfloor \frac{ty - z}{t} \right\rfloor \right) g_m(z) \, dz$$

$$= - \sum_{r=0}^{m} g_{m+1}(ty - \left\lfloor ty \right\rfloor + r + 1)G_t \left( \left\lfloor \frac{ty - r}{t} \right\rfloor \right).$$

This means that $\tilde{S}_{t,m}$ produces polynomial splines of order $m + 1$ with knots $k/t, k = 0, 1, \ldots$ of multiplicity 1 (see [35]). Similarly, it can be seen that the left- and right-duals of the Szász operator $S_t$ are given, respectively, by

$$\tilde{S}_tf(y) = G_t \left( \left\lfloor \frac{ty}{t} \right\rfloor \right) \quad \text{and} \quad \tilde{S}_tf(y) = G_t \left( \left\lfloor \frac{ty}{t} \right\rfloor + \frac{1}{t} \right).$$

It is therefore clear that $\tilde{S}_t$ and $\tilde{S}_t$ differ in a countable set.

Finally, it can be proven that the derived operator of $S_{t,m}$ is given by

$$D_{t,m} = S_{t,m+1}.$$

(B) **GAMMA OPERATOR.** Let $x, y \geq 0$, and $t > 0$. Let $(\tilde{G}(y), y \geq 0)$ be the left-continuous inverse process of the gamma process $(G(x), x \geq 0)$, which satisfies

$$P \left( \tilde{G}(y) \leq x \right) = P(G(x) \geq y) = \int_y^\infty \frac{g_{x-1} e^{-\theta}}{\Gamma(x)} \, d\theta.$$
Then, the left-dual operator of $G_t$ is given by

$$G_t f(y) = Ef \left( \frac{\tilde{G}(ty)}{t} \right).$$

It is worth noting that $P(G(y) \leq k + 1) = P(N(y) \leq k)$, $k = 0, 1, \ldots$, as it follows from (20). Thus, for each $y > 0$, the random variable $\tilde{G}(y) - 1$ has a continuous distribution function which coincides with the Poisson distribution with mean $y$ on the set of nonnegative integers. Moreover, $\tilde{G}(y) - 1 \leq_{st} N(y) \leq_{st} \tilde{G}(y)$.

As for the derived operator $D_t$ of $G_t$, it is not hard to see that

$$D_t f(x) = Ef \left( \frac{UG(tx + 1) + (1 - U)G(tx)}{t} \right),$$

where $U$ is a random variable independent of $(G(x), x \geq 0)$ having the uniform distribution on $[0, 1]$.

(C) **GENERALIZED BERNSTEIN-KANTOROVICH OPERATOR.** Let $x, y \in [0, 1]$, $n = 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$. The Bernstein operator $B_n$ preserves the set $\mathcal{K}$. Nevertheless, for $m \geq 1$, $B_{n,m}$ does not. Using the equality (cf. [34])

$$(21)$$

we obtain

$$P \left( \frac{S_n(x)}{n} \geq y \right) = P \left( \frac{G([ny] + U_1 + \cdots + U_m)}{G(n + 1) \leq x} \right),$$

where $a \wedge b = \min(a, b)$ and $(U_k, k = 1, \ldots, m)$ are independent of $(G(x), x \geq 0)$.

Consider the operator $B^*_{n,m}$ defined by

$$B^*_{n,m} f(y) = Ef \left( \frac{G([ny + my - U_1 - \cdots - U_m] \wedge n)}{G(n + 1)} \right)$$

$$= \int_0^m \beta_{n+1} f \left( \frac{[ny + my - z] \wedge n}{n + 1} \right) g_m(z) dz$$

$$= \sum_{r=0}^m g_{m+1}(ny + my - [ny + my] + r + 1) \beta_{n+1} f \left( \frac{[ny + my - r] \wedge n}{n + 1} \right),$$

where the last equality follows by calculus. Observe that $B^*_{n,m}$ produces polynomial splines of order $m + 1$ with knots $k/(n + m)$, $k = 0, \ldots, n + m$ of multiplicity 1. In particular, we have for the Bernstein operator,

$$B^* n, f(y) := B^* n, 0 f(y) = \beta_{n+1} f \left( \frac{[ny]}{n + 1} \right).$$

We note the following: $B^*_{n,m}$ preserves the set $\mathcal{L}$. $B^*_{n,m}$ is the left-dual operator of the Bernstein polynomials. For $m \geq 1$, $B^*_{n,m}$ is not the left-dual operator of $B_{n,m}$, as it can be proved that

$$B^*_{n,m} f(x) = B_{n,m} f(x), \quad x \in [0, 1] \quad \text{and} \quad B^*_{n,m} f(1) = f(1) \neq B_{n,m} f(1).$$

Finally, the derived operator of $B_{n,m}$ is given by

$$D_{n,m} = B_{n-1,m+1}, \quad (B_{0,m} \equiv 0).$$
(D) Beta Operator. Let $x, y \in [0,1]$, and $t > 0$. Denote by $(\tau_t(y), 0 \leq y \leq 1)$ the left-continuous inverse process of $(G(tx)/G(t), 0 \leq x \leq 1)$. We have

$$P(\tau_t(y) \leq x) = P\left(\frac{G(tx)}{G(t)} \geq y\right) = \int_0^1 \frac{\theta^{x-1}(1-\theta)^{t(1-x)-1}}{\beta(tx, t(1-x))} d\theta. \quad (22)$$

As in the preceding examples, the left-dual operator of $\tau_t$ is given by

$$\beta_t f(y) = Ef(\tau_t(y)).$$

It follows from (21) and (22) that

$$P\left(\tau_{n+1}(y) \leq \frac{k+1}{n+1}\right) = P(S_n(y) \leq k), \quad k = 0, 1, \ldots, n.$$ 

Therefore, for any $n = 1, 2, \ldots$ and $y \in (0, 1)$, the random variable $(n + 1)\tau_{n+1}(y) - 1$ has a continuous distribution function which coincides with the binomial distribution with parameters $n$ and $y$ on the set $\{0, 1, \ldots, n\}$.

Finally, some tedious computations show that the derived operator $D_t$ of $\beta_t$ is given by

$$D_t f(x) = Ef\left(U \frac{G(tx + 1)}{G(t + 1)} + (1 - U) \frac{G(tx)}{G(t + 1)}\right),$$

where $U$ is a random variable independent of $(G(x), x \geq 0)$ having the uniform distribution on $[0,1]$.

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