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Chessboard squares

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Abstract

In this paper we consider the problem posed by Häggkvist on finding $n \times n$ arrays which are avoidable. An array is said to be *avoidable* if an $n \times n$ latin square on the same symbols can be found which differs from the given array in every cell. We describe a family of arrays, known as chessboard arrays, and classify these arrays as avoidable or non-avoidable.

1. Introduction

In this paper we classify by order all $n \times n$ chessboard squares which are avoidable. A *chessboard square* is an array with cells coloured in the form of a chessboard with at most one entry per black cell and no entries in the white cells. The entries are chosen from the symbols $1, 2, \ldots, n$. An array is said to be *avoidable* if an $n \times n$ latin square, on the same symbols, can be found which differs from the array in every cell.

In 1989 Häggkvist asked the general question: Which $n \times n$ arrays are avoidable? It is fairly easy to find some infinite families which are not avoidable [8]. The only published avoidable family is due to Häggkvist [4] in 1989 where he proved the following result.

Theorem 1.1 (Häggkvist). *Let $n = 2^k$ and let P be a partial $n \times n$ column latin square on $1, 2, \ldots, n$ with empty last column. Then there exists an $n \times n$ latin square L, on the same symbols, which differs from P in every cell.*

In Häggkvist's result the structure of the array P is such that each column contains every symbol at most once. Of course if the underlying square is itself a latin square or completable to a latin square then any permutation of the symbols which changes every symbol will give a latin square which avoids the original square.

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Another way we can think of an $n \times n$ array is as the bipartite graph $K_{n,n}$. The general question then becomes: If every edge of $K_{n,n}$ has been assigned a forbidden colour, from the colours $1, 2, \ldots, n$, then is it possible to find a proper edge colouring of $K_{n,n}$ so that every edge is coloured with a colour different from the colour forbidden on it. This is part of a more general problem. Given any graph G with certain colours forbidden on the edges can a suitable colouring be found with $\chi'(G)$ colours. Work related to this can be found in Chetwynd and Häggkvist [3], Bollobás and Hind [1] and Hind [5].

2. Preliminary results

We begin with some definitions. A *partial $n \times n$ latin square* on $1, 2, \ldots, n$ is an array of n rows and n columns, filled with the symbols $1, 2, \ldots, n$ in such a way that every cell contains at most one symbol and every symbol appears at most once in every row and column. It is a latin square if there are no empty cells. The integer n is called the *order* of the latin square. A latin square is said to be in *standard form* if its first row and column contain the elements in their natural order.

An *$n \times n$ chessboard square* is an $n \times n$ array in which all of the cells are coloured black and white alternately (in the form of a chessboard) and which is filled with the symbols $1, 2, \ldots, n$ in such a way that all non-empty cells are black and all white cells are empty. In this paper we allow at most one entry per cell, work on double entry chessboards can be found in Rhodes [8]. Clearly, if n is odd then an $n \times n$ chessboard square can have two distinct forms depending on whether the corner cells are coloured black or white. Note also that a chessboard square is not necessarily a partial latin square.

The following result is in fact an immediate corollary of a theorem proved by Chang [2].

Theorem 2.1. (Chang). *Let P be a partial $n \times n$ latin square on $1, 2, \ldots, n$ whose only non-empty cells are all the main diagonal cells and in which no symbol occurs exactly $n-1$ times. Then P can be completed to an $n \times n$ latin square on $1, 2, \ldots, n$.*

The following theorem, concerning systems of distinct representatives, is due to Hall [7].

Theorem 2.2. (P. Hall) *Let E be a non-empty finite set and let $S_1 S_2, \ldots, S_m$ be non-empty subsets of E. Then $S_1, S_2, \ldots, S_m$ have a system of distinct representatives if and only if the union of any k of the subsets S_i contains at least k elements, for each k: $1 \leqslant k \leqslant m$.*

The next result shows that latin squares of a certain type exist for all $n \geqslant 4$.

Lemma 2.3. *For any $n \geqslant 4$, there exists an $n \times n$ latin square, on $1, 2, \ldots, n$, which has symbol n in each main diagonal cell, and has the entries of the last row in the same order as the last column.*

Proof. By Theorem 2.1, for $n \geqslant 4$ there exists an $(n-1) \times (n-1)$ latin square L whose main diagonal cells contain symbols $1, 2, \ldots, n-1$ in natural order. Let M be the partial $(n-1) \times (n-1)$ latin square obtained by deleting all the main diagonal entries of L and let N be partial $n \times n$ latin square which has M as its upper left $(n-1) \times (n-1)$ subsquare and all remaining cells empty.

Then, by placing symbol n in each main diagonal cell and, for each i: $1 \leqslant i \leqslant n-1$, adding symbol i to cells (i, n) and (n, i) of N we obtain the required $n \times n$ latin square. □

We now give a lemma which shows that an array of a particular form is avoidable.

Lemma 2.4. *Let A be an $n \times n$ array on $1, 2, \ldots, n$. Let A' be an $(n+1) \times (n+1)$ array which has A as its upper left $n \times n$ subsquare and in which, for each i: $1 \leqslant i \leqslant n$, cells $(i, n+1)$ and $(n+1, i)$ both contain the same symbol (if any) as the diagonal cell (i, i) of A. Now, if there exists an $(n+1) \times (n+1)$ latin square L' on $1, 2, \ldots, n+1$ which differs from A' in every cell, which has symbol $n+1$ in each main diagonal cell and in which, for each i: $1 \leqslant i \leqslant n$, cells $(i, n+1)$ and $(n+1, i)$ contain the same symbol, then A is avoidable using symbols $1, 2, \ldots, n$.*

Proof. From L' we can form an $n \times n$ latin square L, on $1, 2, \ldots, n$, which differs from A in every cell, as follows. For each i: $1 \leqslant i \leqslant n$, replace the symbol $n+1$ in cell (i, i) of L' by whichever symbol appears in cell $(i, n+1)$ (and also in cell $(n+1, i)$) of L'.

Then, by deleting the last row and column of L' we obtain an $n \times n$ latin square L, which differs from A in every cell. □

3. Chessboard squares of even order

We being by considering squares of even order. We assume throughout this paper that the upper left corner cell of a chessboard square of even order will be coloured black, thus determining the positions of all other black cells.

Clearly, a 2×2 chessboard square is not always avoidable since the example in Fig. 1 is unavoidable using symbols 1 and 2.

However, this is not the case for larger such squares of even order.

Theorem 3.1. *Let $k \geqslant 2$ and let C be a $2k \times 2k$ chessboard square on symbols $1, 2, \ldots, 2k$ in which any black cell contains at most one symbol. Then C is avoidable.*

1	
	2

Fig. 1.

Proof. We will show that it is possible to construct a $2k \times 2k$ latin square L which differs from C in every cell. Divide C into k^2 2×2 subsquares and label each subsquare with one of the symbols $X_1, X_2, \ldots, X_k$ in the form of a latin square. Let S be the set of symbols $\{1, 2, \ldots, 2k\}$. We show that there exists an ordered partition $S_1, S_2, \ldots, S_k$ of S into k pairs of symbols in such a way that, for all i: $1 \leqslant i \leqslant k$, the pair of symbols S_i is suitable for any 2×2 subsquare of C which is labelled X_i.

Consider a particular 2×2 subsquare X of C. Any non-empty cells in X will appear in the main diagonal. If X has only one entry, say a, then any pair of symbols chosen from S is suitable for X, since any pair not containing symbol a is clearly suitable, whilst any pair which includes a can be used in a 2×2 latin square by putting a in the anti-diagonal cells of X. Similarly, if X contains two entries of symbol a, then any pair of symbols chosen from S is suitable for X. However, if X contains two different entries, say a and b, then the pair $\{a, b\}$ is unsuitable for X since neither a nor b can be put in the main diagonal cells. This means that for each 2×2 subsquare of C there is at most one unsuitable pair of symbols.

We now show that the required ordered partition of S exists. For each i: $1 \leqslant i \leqslant k$ there are k 2×2 subsquares of C labelled X_i. Let B_i be the set of all pairs of symbols which are unsuitable for any of the 2×2 subsquares of C labelled X_i. Then $|B_i| \leqslant k$. The number of ordered partitions of $2k$ symbols into k pairs is $(2k)!/(2!)^k$. Of these we must exclude all those partitions in which $S_i \subseteq B_i$, for any i: $1 \leqslant i \leqslant k$. The number of ordered partitions which have a particular pair of symbols in a particular position is $[2(k-1)]!/(2!)^{k-1}$. We must, therefore, exclude $[2(k-1)]!k^2/(2!)^{k-1}$ partitions, since there are k^2 2×2 subsquares of C. Thus, if P is the number of suitable partitions of S, we have

$$P \geqslant \frac{(2k)!}{(2!)^k} - \frac{[2(k-1)]!k^2}{(2!)^{k-1}}$$

$$= \frac{[2(k-1)]!}{(2!)^{k-1}} \left\{ \frac{2k(2k-1)}{2!} - k^2 \right\}.$$

Now $P \geqslant 1$ if $k(2k-1) > k^2$, i.e. for all $k \geqslant 2$. So there exists the required ordered partition of S. Thus, for each i: $1 \leqslant i \leqslant k$, given any 2×2 subsquare X (of C) which is labelled X_i, we can find a 2×2 latin square (on the pair of symbols S_i from our chosen partition) which differs from X in every cell. By repeating this process for every 2×2 subsquare of C, we find k^2 2×2 latin squares that together make up a $2k \times 2k$ latin square L, which differs from C in every cell. □

1		1
	3	
2		2

1		1
	1	
1		1

Fig. 2.

	1	
2		2
	1	

Fig. 3.

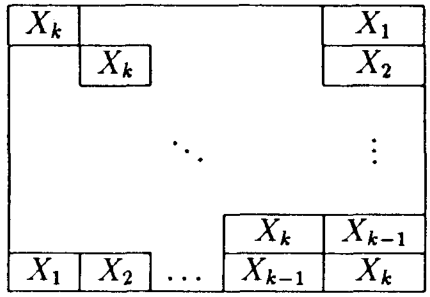
4. Chessboard squares of odd order

We now consider chessboard squares of odd order. As mentioned above, if n is odd then an $n \times n$ chessboard square will have one of two distinct forms depending on whether the corner cells are coloured black or white. Clearly, if the corner cells are coloured black then the total number of black cells in the square is greater than if the corner cells were coloured white.

We shall first consider chessboard squares of odd order in which all corner cells are black. The two 3×3 squares of Fig. 2 are both unavoidable (using symbols $1, 2, 3$). In fact, the second square is one of a whole family of unavoidable squares of odd order. If we take any odd value of n and consider the $n \times n$ chessboard square C (with black corner cells) in which each black cell contains the same symbol, say 1, then C is unavoidable using symbols $1, 2, \ldots, n$. It is interesting to note why this is so. Suppose we rearrange the rows and columns of C to form a new square C' in which all the entries (of symbol 1) appear in the upper left $((n+1)/2) \times ((n+1)/2)$ and lower right $((n-1)/2) \times ((n-1)/2)$ blocks. Then any $n \times n$ latin square L which differed from C' in every cell would clearly have to contain n entries of symbol 1 but the only cells of C' which do not contain symbol 1 are those which lie within the upper right $((n+1)/2) \times ((n-1)/2)$ and lower left $((n-1)/2) \times ((n+1)/2)$ blocks, amongst which we can find only $n-1$ independent cells, Therefore, C' (and thus C) is unavoidable.

We next consider $n \times n$ chessboard squares (where n is odd) in which all corner cells are white. If $n = 1$ a square of this type is clearly avoidable; If $n = 3$ Fig. 3 shows an unavoidable 3×3 square of this type.

The following theorem deals with the odd case where $n \geqslant 7$, leaving just the 5×5 case which we deal with separately.



C'

Fig. 4.

Theorem 4.1. *Let $k \geqslant 4$ and let C be a $(2k-1) \times (2k-1)$ chessboard square on symbols $1, 2, \ldots, 2k-1$ in which all corner cells are white and each black cell contains at most one symbol. Then C is avoidable.*

Proof. Let C' be the $2k \times 2k$ square which has C as its upper left $(2k-1) \times (2k-1)$ square and has empty last row and column. Divide C' into k^2 2×2 subsquares and label each subsquare with one of the symbols $X_1, X_2, \ldots, X_k$, in the form of a $k \times k$ latin square, in such a way that the k main diagonal subsquares each have symbol X_k and the kth row and column each have the entries $X_1, X_2, \ldots, X_k$ in that order, as shown in Fig. 4. Such a latin square exists for $k \geqslant 4$ by 2.3.

We show that it is possible to find a $2k \times 2k$ latin square square L', on $1, 2, \ldots, 2k$, which differs from C' in every cell. Moreover, we will choose the entries so that L' has symbol $2k$ in each main diagonal cell and so that the entries of the $2k$th row are in exactly the same order as those of the $2k$th column.

The main diagonal cells of each subsquare of C' labelled X_k are empty, so symbol $2k$ is suitable for each main diagonal cell of C'. We must now find a symbol which is suitable for all the anti-diagonal cells of each subsquare of C' labelled X_k. The subsquare labelled X_k in the lower right corner of C' is totally empty since the last row and column of C' and all corner cells of C are empty. The remaining $k-1$ subsquares of C' labelled X_k may have entries in the anti-diagonal cells. Therefore, at most $2k-2$ of the remaining symbols are unsuitable for the anti-diagonal cells of those subsquares of C' labelled X_k, so there remains at least one symbol which is suitable; we may suppose without loss of generality that symbol $2k-1$ is chosen.

Let S be the set of symbols $\{1, 2, \ldots, 2k-2\}$. We show that there exists an ordered partition $S_1, S_2, \ldots, S_{k-1}$ of S into $k-1$ pairs of symbols in such a way that each pair S_i contains symbols which are suitable for any 2×2 subsquare of C' labelled X_i; these pairs of symbols are then used to form the 2×2 latin squares that together will make up L'.

Fig. 5.

For a particular i: $1 \leqslant i \leqslant k-1$, consider the k subsquares of C' labelled X_i. Let X_i^c be the subsquare labelled X_i which intersects the last column of C' and let X_i^r be that which intersects the last row of C'. X_i^c and X_i^r will certainly be assigned the same pair of symbols, but we must also ensure that the pair chosen is suitable to form two identical 2×2 latin squares in the two subsquares; this is to ensure that the entries of the $2k$th row and column of L' will be in the same order.

Subsquares X_i^c and X_i^r of C' each have at most one non-empty cell. Suppose X_i^c has one entry, say a, in its lower left cell and X_i^r has one entry, say b, in its upper right cell, as shown in Fig. 5.

If $a=b$ then any pair of symbols is suitable for subsquares X_i^c and X_i^r of C' since any pair not including symbol a is clearly suitable, whilst any pair which includes symbol a can be used by placing a in the anti-diagonal cells of the subsquares. If, on the other hand, $a \neq b$ the pair $\{a,b\}$ is unsuitable for X_i^c and X_i^r since neither a nor b can be put in the main diagonal cells of these subsquares. Thus there is at most one pair of symbols which is unsuitable for X_i^c and X_i^r together.

Any of the $k-2$ remaining subsquares of C' labelled X_i may have two entries, say c and d, one in each anti-diagonal cell. If $c \neq d$ then the pair $\{c,d\}$ is unsuitable for that particular subsquare of C' but otherwise any pair of symbols is suitable.

Thus, for each i: $1 \leqslant i \leqslant k-1$, there is a total of at most $k-1$ pairs of symbols which are unsuitable for the subsquares of C' labelled X_i. Let B^i be the set of all such unsuitable pairs of symbols. Then $|B_i| \leqslant k-1$.

We will now show that a suitable ordered partition of S exists. The number of ordered partitions of $2k-2$ symbols into $k-1$ pairs is $(2k-2)!/(2!)^{k-1}$. Of these we must exclude all those partitions in which $S_i \subseteq B_i$, for any i: $1 \leqslant i \leqslant k-1$. The number of ordered partitions which have a particular pair of symbols in a particular position is $[2(k-2)]!/(2!)^{k-2}$. We must, therefore, exclude at most $[2(k-2)]!(k-1)^2/(2!)^{k-2}$ partitions, since $|B_i| \leqslant k-1$ for each i: $1 \leqslant i \leqslant k-1$. Thus, if P is the number of suitable partitions of S, we have

$$P \geqslant \frac{(2k-2)!}{(2!)^{k-1}} - \frac{[2(k-2)]!(k-1)^2}{(2!)^{k-2}}$$

$$= \frac{[2(k-2)]!}{(2!)^{k-2}} \left\{ \frac{(2k-2)(2k-3)}{2!} - (k-1)^2 \right\}.$$

1	2	3		
4	1	5		
			1	*
			*	1
			*	*

Fig. 6.

Now

$$\frac{(2k-2)(2k-3)}{2!}-(k-1)^2=(k-1)(k-2),$$

so we have $P \geqslant 1$ whenever $(k-1)(k-2)>0$, i.e. when $k \geqslant 3$. So we can find the required partition of S. Thus, for any i: $1 \leqslant i \leqslant k-1$, given a 2×2 subsquare X of (C') which is labelled X_i, we can find a 2×2 latin square (on the pair of symbols S_i), which differs from X in every cell. Moreover, we can find a particular 2×2 latin square on S_i which differs from X_i^c and from X_i^r in every cell.

Recall also that the pair $\{2k-1, 2k\}$ is suitable for all the subsquares of C' which are labelled X_k (with symbol $2k$ being suitable for all the main diagonal cells). We can thus find a $2k\times 2k$ latin square L' on symbols $1,2,\ldots,2k$ which differs from C' in every cell and, moreover, has symbol $2k$ in each main diagonal cell and the entries of the $2k$th row and column in the same order. Thus, by Lemma 2.4, there exists a $(2k-1)\times(2k-1)$ latin square L, on $1,2,\ldots,2k-1$, which differs from C in every cell. □

For the sake of completeness, we give a theorem to show that all 5×5 chessboard squares (with white corner cells) are avoidable. We first need the following lemma.

Lemma 4.2. *Let P be a partial 5×5 latin square on $1,2,3,4,5$ in which all non-empty cells appear within the upper left 2×3 rectangle R_1 and lower right 3×2 rectangle R_2 of P. Suppose that R_1 and R_2 each contain two entries of symbol 1 and one entry of each of symbols $2,3,4$ and 5 and that P satisfies the condition that if any pair of symbols $\{x,y\}$: $\{x,y\}\subset\{2,3,4,5\}$ appear together in the same row of R_1 then they do not appear together in any column of R_2. Then P can be completed to a 5×5 latin square on $1,2,3,4,5$.*

Proof. We consider all the possible forms for P. We may assume, without loss of generality, that P has the form of the square of Fig. 6. It then remains to consider the possible entries for the remaining four non-empty cells of P (marked * in the figure). We first consider cells (3, 5) and (4, 4). There are $\binom{4}{2}=6$ possible pairs of symbols which could be placed in these cells; the order in which they appear is immaterial since the two cells can be permuted, leaving the arrangement of the entries of symbol 1

(1)

1	2	3		
4	1	5		
			1	3
			2	1
			4	5

(2)

1	2	3		
4	1	5		
			1	3
			2	1
			5	4

(3)

1	2	3		
4	1	5		
			1	4
			2	1
			5	3

(4)

1	2	3		
4	1	5		
			1	5
			2	1
			4	3

(5)

1	2	3		
4	1	5		
			1	5
			3	1
			4	2

Fig. 7.

unchanged, by permuting rows and permuting columns of P. Also if we allow relabling of the symbols the pair $2,3$ is the same as the pair $4,5$ and the pair $2,5$ is the same as the pair $3,4$. Hence with out loss of generality, we only need to consider the pairs $2,3$; $2,4$; $2,5$ or $3,5$. Once the entries for cells $(3,5)$ and $(4,4)$ have been chosen, there are either one or two different possibilities for the remaining non-empty cells of P, depending on the choices made. Given that symbols 2 and 3 or symbols 4 and 5 cannot appear in the same column of R_2, we conclude that P will be of the form of one of the partial 5×5 latin squares of Fig. 7, each of which can be completed to a 5×5 latin square on $1,2,3,4,5$. □

We are now in a position to prove our next theorem.

Theorem 4.3. *Let C be a 5×5 chessboard square on symbols $1,2,3,4,5$ in which all corner cells are white and each black cell contains at most one symbol. Then C is avoidable.*

Proof. We begin by rearranging the rows and columns of C, as shown in Fig. 8, to form a new square C'.

We show that there exists a 5×5 latin square L' on $1,2,3,4,5$ which differs from C' in every cell; we can then find the latin square L which differs from C in every cell by simply rearranging the rows and columns of L'.

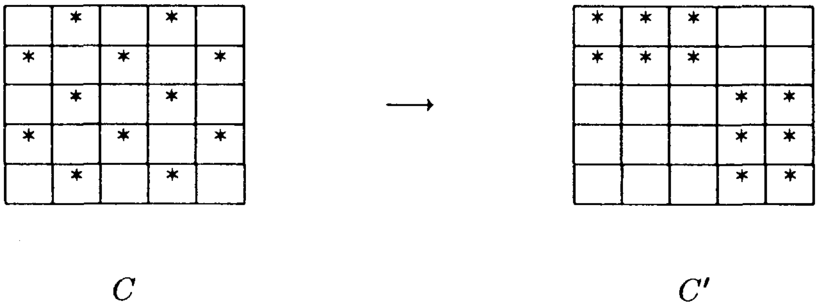


Fig. 8.

1	1	1		
1	1	1		
			1	1
			1	1
			1	1

1	1	1		
1	1	1		
			2	2
			2	2
			2	2

Fig. 9.

In C', let B_1 be the upper left 2×3 block of black cells and let B_2 be the lower right 3×2 block. Let $M(B_i)$ be the maximum number of entries of any one symbol in block B_i of C', for each i. We will assume that blocks B_1 and B_2 each contain a total of six entries (we may fill in any empty black cells if necessary) so that $2\leqslant M(B_i)\leqslant 6$ for each i.

Now we have, without loss of generality, one of the following three cases:

(1) $M(B_1)=M(B_2)=6$,

(2) $M(B_1)=6$ and $M(B_2)<6$,

(3) $M(B_1)<6$ and $M(B_2)<6$.

Case 1: If case 1 holds then C' will have the form of one of the two squares of Fig. 9; both of these are avoidable using symbols 1, 2, 3, 4, 5.

Case 2: If we have case 2 then we may assume that block B_1 has six entries of symbol 1, say. We must therefore place symbol 1 in both cells of one of the diagonals of the upper right 2×2 subsquare of L'.

We must then consider block B_2 of C'. If each of the two columns of this block has three repeats of the same symbol then C' will have the form of one of the squares of Fig. 10, each of which is avoidable using the symbols 1, 2, 3, 4, 5. So we will assume that at most one column of block B_2 has three repeats of any one symbol. Now we will find the symbol (other than symbol 1) which appears most in block B_2 of C' and place this symbol in the remaining diagonal cells of the upper right 2×2 square of L'. Suppose symbol 2 is this symbol; then we wish to fill in blocks B_1 and B_2 of L' with symbols 3,

1	1	1		
1	1	1		
			1	2
			1	2
			1	2

1	1	1		
1	1	1		
			2	3
			2	3
			2	3

Fig. 10.

3	4	5	1	2
5	3	4	2	1
			3	5
			4	3
			5	4

$\longrightarrow$

3	4	5	1	2
5	3	4	2	1
4	1	2	3	5
2	5	1	4	3
1	2	3	5	4

Fig. 11.

4 and 5. This is clearly possible in block B_1, so it remains to consider B_2. Since we have effectively removed symbols 1 and 2 from this block (2 being the most common symbol here), there are at most four cells whose entries we must consider. Suppose symbol 3 appears the most amongst these four cells. If these are three entries of symbol 3 they do not all appear in the same column and, furthermore, they are the only entries in B_2, which is then clearly avoidable using symbols 3, 4, 5. If, however, block B_2 contains two or less entries of symbol 3 then there is certainly a pair of independent cells in the block, in which symbol 3 can be placed. If we ensure that at least one entry of symbol 3 is placed in a cell containing symbol 4 or 5 (if one exists) we then have only one remaining entry to consider and the block can certainly be completed with symbols 4 and 5. The partial latin square we obtain will be of the form of the first square of Fig. 11 and is thus completable to a 5×5 latin square, as shown in the second square.

Case 3: Lastly, if case (3) holds we will show that blocks B_1 and B_2 of L' can be filled with symbols $1, 2, 3, 4, 5$ in such a way that the square satisfies the conditions of Lemma 4.2 and is thus completable to a 5×5 latin square.

We have assumed that C' has exactly twelve non-empty cells. Therefore, there exists a symbol, say symbol 1, which appears no more than twice in C'. We may suppose that block B_1 contains at least as many entries of symbol 1 as block B_2. We show that it is possible to put two entries of symbol 1 in each of blocks B_1 and B_2 of L', along with one entry each of symbols 2, 3, 4, and 5 in such a way that the square satisfies the said conditions. First consider block B_1. Symbol 1 appears at most twice in this block. We find the symbol which occurs most in the block, suppose it is symbol 2, and, if possible, place symbol 1 in two cells of L' which contained symbol 2. Otherwise we

1	2	3
4	1	5

Fig. 12.

ensure that at least one entry of symbol 1 occupies a cell which contained symbol 2. Now we must place the symbols 2, 3, 4 and 5 in the remaining four cells of B_1.

No symbol is repeated more than three times in these four cells and for each cell we have a choice of three possible symbols which are suitable; thus, by Hall's Theorem (2.2) we can find a way to place symbols 2, 3, 4 and 5 in the four cells. We may suppose without loss of generality, that block B_1 of L' has the form shown in Fig. 12.

We will now consider block B_2. This block has at most one entry of symbol 1. We can therefore find a pair of independent cells in block B_2 of L' where symbol 1 can be placed. Again we must make certain that at least one (and if possible two) of these entries are put in cells containing the most common entry in this block. We must now fill the remaining four cells with the symbols 2, 3, 4 and 5 in such a way that neither symbols 2 and 3 nor symbols 4 and 5 appear together in the same column.

Consider the entries appearing in the remaining four cells of block B_2 of C'. Again, no one symbol appears in all four cells. Thus we can find two cells, one in each column of block B_2, containing different entries, say a and b; call these cells c_1 and c_2 in some order. We may suppose that the number of entries of symbol a in the four cells is greater than or equal to the number of entries of symbol b. Now the two remaining cells, c_3 and c_4, one in each column of block B_2 will contain, in some order, one of the following pairs of symbols:

$$\{a,a\},\ \{a,b\},\ \{c,c\},\ \{a,c\} \text{ or } \{c,d\}, \text{ where } a,b,c,d \text{ are all distinct.}$$

Any pair of symbols can be put on top of $\{a,b\}$, $\{a,c\}$ or $\{c,d\}$ by choosing the order correctly and if the case arises where cells c_3 and c_4 contain $\{a,a\}$ (or $\{c,c\}$) we will simply choose to put the pair containing a (or c, resp.) in cells c_1 and c_2. We can therefore put one of the pairs $\{2,3\}$ or $\{4,5\}$ in cells c_1 and c_2 and the other pair in cell c_3 and c_4 in some order.

The resulting partial latin square, with blocks B_1 and B_2 completely non-empty, satisfies the conditions of Lemma 4.2 and can thus be completed to the required 5×5 latin square L', which differs from C' in every cell. $\square$

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