Advances in Mathematics **149**, 89–129 (2000)
doi:10.1006/aima.1999.1859, available online at http://www.idealibrary.com on IDEAL®

# Unrectifiable 1-Sets with Moderate Essential Flatness Satisfy Besicovitch's $\frac{1}{2}$-Conjecture

Hany M. Farag

*Mathematics Department, Rice University, Houston, Texas 77005*

Received December 7, 1998; accepted March 4, 1999

In this paper we show that for a wide class of totally unrectifiable 1-sets in the plane (or even a Hilbert space) satisfying a mild measure-theoretic flatness condi-



ture, which states that for all totally unrectifiable 1-sets in the plane (or possibly even in $\mathbf{R}^n$, or a Hilbert space), the lower spherical density is bounded above by $\frac{1}{2}$ at almost every point. 

## 1. INTRODUCTION

Recall that $E$ is called a 1-set if $0 < \mathscr{H}^1(E) < \infty$, where $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure, and that $E$ is said to be totally unrectifiable if $\mathscr{H}^1(E \cap \Gamma) = 0$ for every rectifiable curve $\Gamma$. One of the fundamental characterizations of rectifiability is that of density. For a rectifiable set $E$ in $\mathbf{R}^n$ (say), we have

$$\Theta^1(E, x) \equiv \lim_{r \to 0^+} \left( \frac{\mathscr{H}^1(B(x, r) \cap E)}{2r} \right) = 1, \tag{1.1}$$

for almost every $x \in E$, where $B(x, r)$ is the closed ball with center at $x$, and radius $r$. This is basically due to Besicovitch [Be1], and for subsets of metric spaces it is due to Kirchheim [Ki]. The situation for totally unrectifiable 1-sets on the other hand is drastically different. In fact, the density $\Theta^1(E, x)$ does not exist for a.e. $x \in E$, as first proved in [Be2] and, more recently, for $m$-sets in $\mathbf{R}^n$ in Preiss' remarkable paper [Pr]. Recall that



0001-8708/00 $35.00

$\sigma_1(\mathbf{R}^n)$ is defined to be the smallest number such that if $E$ is a totally unrectifiable 1-set in $\mathbf{R}^n$, then the lower spherical density

$$\Theta^1_*(E, x) \equiv \lim_{r \to 0+} \inf \left( \frac{\mathscr{H}^1(B(x, r) \cap E)}{2r} \right)$$
$$\leqslant \sigma_1(\mathbf{R}^n) \tag{1.2}$$

for a.e. $x \in E$ (see e.g., [Pr], [Ma]). In 1928, Besicovitch [Be1] proved that

$$\sigma_1(\mathbf{R}^2) \leqslant 1 - 10^{-2576}, \tag{1.3}$$

thereby characterizing rectifiability by means of the lower spherical density. In 1938 [Be2], he showed that

$$\sigma_1(\mathbf{R}^2) \leqslant \tfrac{3}{4}. \tag{1.4}$$

He also gave an example showing that

$$\sigma_1(\mathbf{R}^2) \geqslant \tfrac{1}{2}, \tag{1.5}$$

and conjectured that $\sigma_1(\mathbf{R}^2) = \frac{1}{2}$. The upper estimates on $\sigma_1(\mathbf{R}^2)$ were shown to hold for $\sigma_1(\mathbf{R}^n)$ by Moore [Mo], and to metric spaces by Preiss and Tišer [PT]. Thus the conjecture is just as reasonable for $\sigma_1(H)$, where $H$ can even be a Hilbert space (my guess is that it may not hold for general metric spaces).

Only recently has there been any progress on this problem, namely in [PT] where it was shown that, for a metric space,

$$\sigma_1(M) \leqslant \frac{2 + \sqrt{46}}{12} \approx 0.732. \tag{1.6}$$

It is hoped that the present paper will shed some light on the nature of this problem. Treatments of Besicovitch's results can be found in [Fal], and [Far1]. The rest of this paper is organized as follows:

Section 2: we provide the appropriate definitions that we need to state the main results.

Section 3: we give the statement of the main results.

Section 4: we present a technical reduction of the problem using standard techniques in geometric measure theory.

Section 5: this is devoted to the proof of the main results.

Section 6: we remark on some possible extensions of the method, state a corresponding theorem in the setting of a Hilbert space, and comment on a strategy for possible further progress.

## 2. THE NOTION OF ESSENTIAL FLATNESS

Our goal is to show that a certain class of totally unrectifiable 1-sets which look somewhat flat, or thin, at sufficiently small scales, satisfies Besicovitch's $\frac{1}{2}$-conjecture. In this section we make precise the notion of flatness that will be needed to state the main results.

DEFINITION 2.1. If $E \subset \mathbf{R}^2$, $x \in \mathbf{R}^2$, and $r > 0$, we let $\beta_E(x, r)$ be the smallest number such that $E \cap B(x, r)$ is contained in a strip $S$ of width $w$, with $w/r \leqslant \beta_E(x, r)$.

Note that for any set $E$, $\beta_E(x, r) \leqslant 2$, for all $x \in \mathbf{R}^2$, $r > 0$. See [Jo] where the systematic use of such numbers to characterize rectifiability first appeared, and quickly opened numerous applications and developments (see e.g., [DS]).

Our setting will be measure-theoretic in nature, and we will be able to use a less restrictive notion.

DEFINITION 2.2. We let $\beta^1_E(x, r)$ be the smallest number such that there is a strip $S$ of width $w$, so that $w/r \leqslant \beta^1_E(x, r)$, and $\mathscr{H}^1((E \cap B(x, r)) \setminus S) = 0$.

Since the problem is also local in nature, we can generalize even further by defining $\beta^{*1}_E(x)$ via

$$\beta^{*1}_E(x) \equiv \limsup_{r \to 0^+} \beta^1_E(x, r). \tag{2.1}$$

We will also need the following, more general, tools:

DEFINITION 3. For $\varepsilon > 0$, $r > 0$, $x \in \mathbf{R}^2$, let $\gamma^1_E(x, r, \varepsilon)$ be the smallest number such that there is a strip $S$ of width $w$, so that $w/r \leqslant \gamma^1_E(x, r, \varepsilon)$, and $\mathscr{H}^1((E \cap B(x, r)) \setminus S) \leqslant \varepsilon r$.

We now define $\gamma^{*1}_E(x, \varepsilon)$, $\gamma^{*1}_E(x)$ via

$$\gamma^{*1}_E(x, \varepsilon) \equiv \limsup_{r \to 0^+} \gamma^1_E(x, r, \varepsilon), \tag{2.2}$$

$$\gamma^{*1}_E(x) \equiv \sup_{\varepsilon > 0} \gamma^{*1}_E(x, \varepsilon). \tag{2.3}$$

In Section 5 it will be proved that sets with $\gamma_E^{*1}(x)$ somewhat small (in comparison with unity), almost everywhere, satisfy Besicovitch's $\frac{1}{2}$-conjecture. It is such sets that we refer to as *essentially flat* to indicate the measure theoretic sense of that notion. In particular *flat* sets (i.e., those for which $\beta_E(x, r) \ll 1$, for all $x \in E$, $r > 0$) satisfy the conjecture also.

## 3. THE MAIN RESULTS

Theorem 2 below is the main result of this paper, but we first state a simpler version:

THEOREM 1. *Suppose $E \subset \mathbf{R}^2$ is a totally unrectifiable 1-set and that $\beta_E^{*1}(x) \leqslant \frac{1}{4}$ for almost every $x \in E$. Then $\Theta_*^1(E, x) \leqslant \frac{1}{2}$ for almost every $x \in E$.*

A weaker version of this result (when $\beta_E^{*1}(x) \lesssim \frac{1}{10}$ almost everywhere) was obtained in [Far2] using a different method which relied more on the geometry imposed by the essential flatness condition than measure theoretic estimates. Our method below will make use of flatness only at strategic places. There are numerous examples of totally unrectifiable 1-sets satisfying the hypothesis of Theorem 1 (or Theorem 2); say if the set lies on a quasicircle with small constant. Elementary examples of self-similar 1-sets which can be finely tuned for any $\beta_E(x, r)$ desired, may be found in [Far1]. Our main result is slightly stronger and more flexible to apply:

THEOREM 2. *Suppose $E \subset \mathbf{R}^2$ is a totally unrectifiable 1-set, and that $\gamma_E^{*1}(x) \leqslant \frac{1}{4}$ for almost every $x \in E$. Then $\Theta_*^1(E, x) \leqslant \frac{1}{2}$ for almost every $x \in E$.*

*Remark* 3. The bound "$\frac{1}{4}$," on $\gamma_E^{*1}(x)$ or $\beta_E^{*1}(x)$ is not the best possible in our method, but this choice was made as a compromise between strength of the theorem and difficulty of the proof. However, the method seems to break down if one tries to push the bound beyond "$\frac{1}{3}$" for example.

## 4. A TECHNICAL REDUCTION OF THE PROBLEM

For the convenience of the reader we will briefly describe a technical reduction of the problem. Our version is from [PT] which works for general metric spaces. For an elementary version for subsets in the plane, or $\mathbf{R}^n$, see [Far1], or (respectively) [MR], which also appears in an elegant presentation of Besicovitch's results in [Fal]. First we recall a few basic facts about Hausdorff measures.

### 4.1. *Basic Density Properties*

In what follows we assume that $E$ is a 1-set. Let $\bar{D}^1_c(E, x)$ denote the upper convex density of $E$ at the point $x \in \mathbf{R}^n$, i.e.,

$$\bar{D}^1_c(E, x) \equiv \lim_{r \to 0^+} \sup \left( \frac{\mathscr{H}^1(E \cap U)}{diam(U)} \right), \tag{4.1}$$

where the supremum is over all convex sets $U$, with $x \in U$, and $diam(U) \leqslant r$. Since a set and its convex hull have the same diameter, the definition above in not altered if we allow $U$ to be any set. We can now state the following propositions:

PROPOSITION 4. $\bar{D}^1_c(E, x) = 0$ *for a.e.* $x \notin E$.

PROPOSITION 5. $\bar{D}^1_c(E, x) = 1$ *for a.e.* $x \in E$.

For proofs see e.g. [Fal]. Using the regularity of the Hausdorff measure, one can prove the following proposition (see e.g., [Fal] for a proof):

PROPOSITION 4.3. *Suppose* $\Theta^1_*(E, x) > \sigma > 0$, *on a set of positive measure. Then there exist* $\varepsilon, \rho' > 0$, *and a compact* 1-*set* $E' \subset E$, *such that*

$$\mathscr{H}^1(E \cap B(x, r)) \geqslant (\sigma + \varepsilon)\, 2r, \tag{4.2}$$

*for all* $x \in E'$, $0 < r \leqslant \rho'$.

### 4.2. *A Strong Uniformization Principle*

Proposition 6 gave a uniformization of the statement $\Theta^1_*(E, x) > \sigma > 0$ on a set of positive measure and, in principle, this is all that one can usually conclude. If we assume that $E$ is totally unrectifiable however, the statement of Proposition 6 cannot be violated "too often" if we replace $E$, by $E'$ itself. Using the density properties in the previous section, the total unrectifiability of the set $E$, and the compactness properties of Radon measures, one can in fact prove (as in [PT]) the following proposition:

PROPOSITION 7. *Suppose* $E$ *is a totally unrectifiable* 1-*set and* $\Theta^1_*(E, x) > \sigma > 0$ *on a set of positive measure. Then there exist* $\varepsilon > 0$, *a Radon measure* $\mu$ *with support* $F \subset E$, *and disjoint closed sets* $F_1, F_2 \subset F$, *with attained distance, such that*

$$\mu(B(x, r)) \geqslant \left(\sigma + \frac{\varepsilon}{2}\right) 2r, \tag{4.3}$$

*for all* $x \in F$, $r > 0$.

4.3. *The Reduced Form*

By Proposition 7, we see that if $\Theta^1_*(E, x) > \frac{1}{2}$ on a set of positive measure, then we can find $\varepsilon > 0$, a Radon measure $\mu$ with support $F$ contained in $E$, and disjoint closed sets $F_1, F_2 \subset F$, such that

$$\beta(x, r)) \geqslant (1+\varepsilon) r, \tag{4.4}$$

for $x \in F$, $r > 0$. Also, by compactness of measures and the density properties of Section 4.1, we can require that

$$\mu(U) \leqslant diam(U), \tag{4.5}$$

for all $U \subset \mathbf{R}^2$. This is essentially all the information that we have in a general setting. In the next section, we will use (4.4), (4.5), and the flatness hypothesis to get a handle on the problem.

## 5. PROOF OF THE MAIN THEOREM

*Remark* 8. The proof below contains many, delicate (though elementary), nonlinear estimates. This seems to be an unavoidable feature of the problem. We will therefore detail the estimates as necessary, and we will measure angles in degrees to help the reader build an intuition for the geometric picture. Furthermore we will consider all angles to lie between $-180°$, and $180°$, unless otherwise stated (for example geometric angles of the form $\widehat{xyz}$ are understood to be nonnegative).

Assume, to get a contradiction, that $\Theta^1_*(E, x) > \frac{1}{2}$ on a set of positive measure and, consequently, find $\varepsilon > 0$, a measure $\mu$ with support $F$, and compact $F_1, F_2$ with the properties stated in Section 4.3. Recall the setup of Section 4.3. By hypothesis, and compactness of measures again, we can also guarantee that, for every $x \in F$, and $r > 0$,

$$\beta_F(x, r) \leqslant \tfrac{1}{4}. \tag{5.1}$$

The rest of the proof is devoted to obtaining a contradiction using only the three estimates (4.4), (4.5), and (5.1).

5.1. *The Idea behind the Proof*

In this section we give the main geometric idea behind the proof by showing how one can obtain the $\frac{1}{2}$ from a very crude picture assuming that, for all $x \in F$, $r > 0$, $\beta_F(x, r) \leqslant \beta$ for some (very small) $\beta > 0$. The actual proof in Section 5.2 is then devoted to handling the various obstructions that arise when we try to achieve an optimal result using this idea.

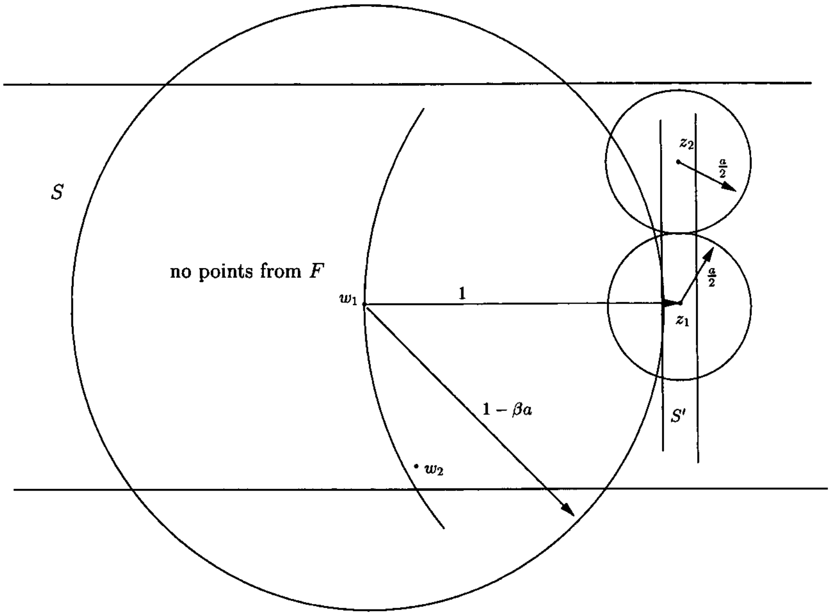


FIGURE 1

Recall the setup of Section 4.3. We have two disjoint pieces $F_1, F_2 \subset F$. Let $w \in F_1$, $w' \in F_2$, be such that $s \equiv dist(F_1, F_2) = |w - w'|$. We first observe that there exists a strip $S$ of width $\beta s$ such that $F_1 \cap B(w, s) \subset S$. All of our arguments will take place in $S$. See Fig. 1. Now consider the ball $B(w, s/2)$. By (4.4), (4.5), there exists a pair of points $w_1, z_1 \in F_1 \cap B(w, s/2)$, with (by choice of label) $w_1$ closer to $w$, and such that

$$|w_1 - z_1| \equiv diam\left(F_1 \cap B\left(w, \frac{s}{2}\right)\right)$$
$$\geqslant (1+\varepsilon)\frac{s}{2}.$$

Let us now rescale and set $|w_1 - z_1| \equiv 1$. We now consider the ball $B(w_1, 1)$. By the same argument, we must have $w_2, z_2 \in F_1 \cap B(w_1, 1)$, labeled as before, such that

$$|w_1 - z_1| \equiv diam(F_1 \cap B(w_1, 1))$$
$$\geqslant (1+\varepsilon).$$

Note that we also have

$$\beta_F(w_1, 1) \leqslant \beta.$$

Set $a=|z_1-z_2|$. As we will see in Section 5.2, the Pythagorean Theorem implies that

$$|z_2-w_2| \lesssim 1+\beta a.$$

For the sake of our crude argument, let us ignore the curvature of the circle (which is reasonable when $\beta \ll 1$). Let us also assume that the line segments $\overline{w_1 z_1}$ is horizontal and $\overline{z_1 z_2}$ is vertical. We also assume that $B(z_1, a)\cap F_1$ is contained in a vertical strip $S'$ of width $\beta a$. Consider now the three balls $B(w_1, 1-\beta a)$, $B(z_1, a/2)$, $B(z_2, a/2)$. While the last two have disjoint interior, the first one may overlap them *except* when intersected with $F_1$. By (4.4), if $U=B(w_1, 1-\beta a)\cup B(z_1, a/2)\cup B(z_2, a/2)$, we then have

$$\mu(U)>1-\beta a+a, \tag{5.2}$$

whereas by definition of $z_2, w_2$, the estimate on their distance, and the triangle inequality, we get

$$diam(U)\leqslant 1+\beta a+\frac{a}{2}, \tag{5.3}$$

which contradicts (4.5), if $\beta \lesssim \frac{1}{4}$.

We must warn the reader that the bound on $\beta$ just obtained is rather deceptive. In fact, if we believe the above simplistic picture, we can get a much better bound for $\beta$ (close to $\frac{1}{2}$), but the reason that we have a bound like $\frac{1}{4}$ in our main result really comes from the details of the proof in Section 5.2. A more realistic picture would have to allow for all possible positions and orientations of the strip $S'$, the curvature of the circle, and we cannot always use $w_1$ as the center for the big ball we took in the construction of $U$. We will have to allow $w_2$ to also be the center in some cases. This however introduces new points from $F_1$ into the picture, and we must chase a sequence of quadruplets like $z_1, z_2, w_1, w_2$, to get a contradiction like the one illustrated above.

## 5.2. *The Body of the Proof*

We now give the details of the proof. The next lemma is a statement of the Pythagorean Theorem in circles which will play an essential role in our analysis.

LEMMA 5.2. *Suppose* $w_1, z_1 \in \mathbf{R}^2$, $|w_1-z_1|=1$. *Suppose also that* $w_2, z_2 \in B(w_1, 1)\cap B(z_1, 1)$, *let* $f_1=1-|w_1-z_2|$, $f_2=1-|w_2-z_1|$, $\theta=\widehat{z_1 w_1 z_2}$. *Define* $\alpha$, *such that* $|\alpha|=\widehat{w_2 z_1 w_1}$, *and* $\alpha$ *is positive whenever* $w_2, z_2$ *lie on*

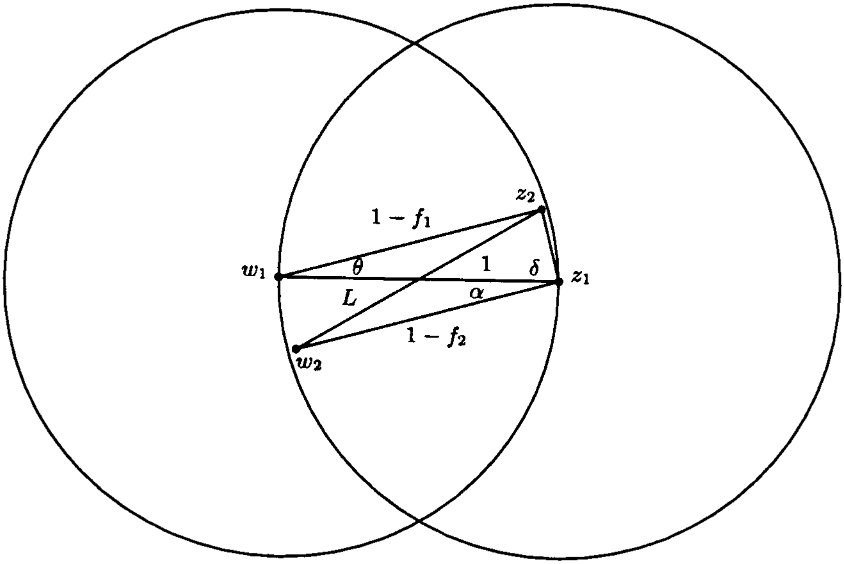


FIGURE 2

*opposite sides of the line segment $\overline{w_1 z_1}$, and negative if they lie on the same side. Let $L = |w_2 - z_2|$, then*

$$\begin{aligned} L^2 = 1 &+ 2\sin(\theta)\sin(\alpha) + 8\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right) + f_1^2 + f_2^2 \\ &- 2f_1\left(1 + \sin(\theta)\sin(\alpha) - 2\sin^2\left(\frac{\alpha}{2}\right) + 4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)\right) \\ &- 2f_2\left(1 + \sin(\theta)\sin(\alpha) - 2\sin^2\left(\frac{\theta}{2}\right) + 4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)\right) \\ &+ 2f_1 f_2\left(1 + \sin(\theta)\sin(\alpha) - 2\sin^2\left(\frac{\alpha}{2}\right) - 2\sin^2\left(\frac{\theta}{2}\right)\right. \\ &\left. + 4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)\right) \end{aligned} \tag{5.4}$$

*Proof.* See Fig. 2. Let $a = |z_1 - z_2|$, $\delta = \widehat{w_1 z_1 z_2}$. By the Pythagorean Theorem,

$$L^2 = ((1 - f_2)\cos(\alpha) - a\cos(\delta))^2 + ((1 - f_2)\sin(\alpha) + a\sin(\delta))^2. \tag{5.5}$$

By projecting on, and orthogonal to, the line segment $\overline{w_1 z_1}$, we find that

$$(1-f_1)\cos(\theta)=1-a\cos(\delta) \tag{5.6}$$

and

$$(1-f_1)\sin(\theta)=a\sin(\delta). \tag{5.7}$$

Upon substituting and simplifying some trigonometric identities we can cas-t (5.5) in the advertized form. ∎

The geometry of Lemma 9 will repeat throughout the rest of the proof and we will *repeatedly* use the notation of this lemma without further warning whenever such a geometry is present. The labeling of the points will always be made clear so that no confusion should arise. The next lemma introduces the role that (5.1) will play in our method.

LEMMA 10. *Let* $x, y, z \in \mathbf{R}^2 \cap S_\beta$, *where* $S_\beta$ *is a strip of width* $\beta\,|x-y|$. *Suppose* $|y-z| \geqslant |x-y| \geqslant |x-z| > 0$. *If* $\theta = \widehat{xyz}$, *then* $\sin(\theta) \leqslant \beta$.

*Proof.* See Fig. 3. We observe (by an elementary argument) that the best possible strip (i.e., one of minimal width) must have one of the sides of the triangle $xyz$ on one of its boundary components and the opposite vertex on the other component. Comparing the widths of these three strips

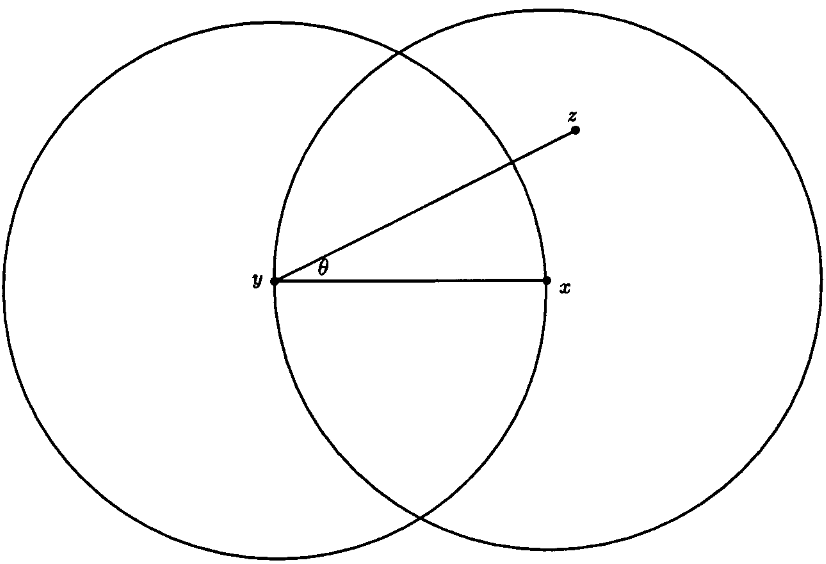


FIGURE 3

while keeping in mind that $|z-y| \geqslant |x-y| \geqslant |x-z|$, we find that the best strip is the one with $\overline{yz}$ on one of its boundary components and $x$ on the other component. Thus, we conclude that $|x-y| \sin(\theta) \leqslant \beta |x-y|$. ∎

Now we consider the ball $B(w, s/2)$. By (4.4) and (4.5), there must exist $w_1, z_1 \in B(w, s/2) \cap F$, such that

$$|w_1 - z_1| = diam\left(B\left(w, \frac{s}{2}\right) \cap F\right) \geqslant (1+\varepsilon)\frac{s}{2}. \tag{5.8}$$

We observe here that $|z_1 - w_1|$ satisfies the following estimate

$$|z_1 - w_1| \leqslant 1.2\left(\frac{s}{2}\right), \tag{5.9}$$

which is a consequence of the fact that, by the law of cosines,

$$\frac{|z_1 - w_1|^2}{s^2}$$

is (crudely) dominated by $1 + (\frac{3}{2})^2 - 3\cos(A+B)$, where $A$ and $B$ are acute angles such that $\sin(A) + \sin(B) \leqslant \frac{1}{4}$. We leave the simple details to the reader. Next, consider the ball $B(w_1, |w_1 - z_1|)$. By (4.4) and (4.5), there must exist $w_2, z_2 \in B(w_1, |w_1 - z_1|) \cap F$, so that

$$|w_2 - z_2| \geqslant diam(B(w_1, |w_1 - z_1|) \cap F) \geqslant (1+\varepsilon)\,|w_1 - z_1|. \tag{5.10}$$

We also observe that $z_2, w_2 \in B(w_1, |w_1 - z_1|) \cap B(z_1, |w_1 - z_1|)$. It will now be convenient to rescale and set $|w_1 - z_1| \equiv 1$, and we will use the notation of Lemma 9.

Lemma 11. $\max\{\sin(\alpha), \sin(\theta)\} \leqslant \frac{1}{4}$.

*Proof.* This is just an application of Lemma 10 to each of the triplets $\{z_1, w_1, z_2\}$, and $\{w_1, z_1, w_2\}$. ∎

Proposition 12. *Suppose* $Q = \{w_1, w_2, z_1, z_2\} \subset \mathbf{R}^2$, *such that*

1. $|w_1 - z_1| = 1$,
2. $w_2, z_2 \in B(w_1, 1) \cap B(z_1, 1)$,
3. $L \equiv |w_2 - z_2| \geqslant 1$,
4. $\beta_Q(x, r) \leqslant \frac{1}{4}$ *for* $x \in Q$, $r > 0$.

*If $z_2, w_2$, are labeled so that $|z_2 - z_1| \leqslant |z_2 - w_1|$, then, for $i = 1, 2$,*

$$L^2 \leqslant 1 + 2\left((1.017)\sin(\theta)\sin(\alpha) - \frac{f_i}{(1.05)}\right), \tag{5.11}$$

$$\sin(\theta) \leqslant |z_1 - z_2| \leqslant (1.05)\sin(\theta), \tag{5.12}$$

*and*

$$f_i \leqslant (1.07)\sin(\alpha)\sin(\theta). \tag{5.13}$$

*Proof.* First observe that (3) implies that $\alpha$ is positive according to our convention (i.e., $z_2, w_2$, must lie on opposite sides of $\overline{w_1 z_1}$). By (2)–(4), we can assume (without loss of generality that) $z_2 \in B(z_1, \frac{1}{2})$, $w_2 \in B(w_1, \frac{1}{2})$. Thus

$$\max_i \{f_i\} \leqslant \tfrac{1}{2}. \tag{5.14}$$

Using (5.14) and Lemma 11 we find that, for any $0 \leqslant f_1 \leqslant \frac{1}{2}$, the right-hand side of (5.1) is decreasing with respect to $f_2$ (it helps to check the linear terms). By (3), we conclude that

$$\begin{aligned} 0 &\leqslant 2\sin(\theta)\sin(\alpha) + 8\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right) + f_1^2 \\ &\quad - 2f_1\left(1 + \sin(\theta)\sin(\alpha) + 4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right) - 2\sin^2\left(\frac{\alpha}{2}\right)\right). \end{aligned} \tag{5.15}$$

Let

$$b(\theta, \alpha) = \sin(\theta)\sin(\alpha) + 4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\alpha}{2}\right), \tag{5.16}$$

$\theta_0 = \alpha_0 \equiv \arcsin(\frac{1}{4})$, and $b_0 = b(\theta_0, \alpha_0)$. Noting that for any $f_1 \leqslant \frac{1}{2}$, the right-hand side of (5.15) is maximized when $\theta = \alpha = \theta_0$, we can obtain the crude estimate

$$\begin{aligned} f_1 &\leqslant b_0 - \frac{b_0^2}{2} + 2b_0\sin^2\left(\frac{\alpha_0}{2}\right) - 2\sin^4\left(\frac{\alpha_0}{2}\right) \\ &\quad + \frac{1}{8}\left(1 + b_0^2 - 4b_0\sin^2\left(\frac{\alpha_0}{2}\right) - 4\sin^2\left(\frac{\alpha_0}{2}\right) + 4\sin^4\left(\frac{\alpha_0}{2}\right)\right)^{-3/2} \\ &\quad \times \left(b_0^2 - 4b_0\sin^2\left(\frac{\alpha_0}{2}\right) - 4\sin^2\left(\frac{\alpha_0}{2}\right) + 4\sin^4\left(\frac{\alpha_0}{2}\right)\right)^2 \\ &\leqslant \frac{1.017}{16}. \end{aligned} \tag{5.17}$$

Similarly,

$$f_2 \leqslant \frac{1.017}{16}. \tag{5.18}$$

Substituting (5.17), (respectively, (5.18)) back into (5.4), and using Lemma 11, we get that for $i = 1, 2$,

$$L^2 \leqslant 1 + 2 \left\{ \sin(\theta) \sin(\alpha) \left( 1 + 4 \frac{\sin^2\left(\frac{\theta_0}{2}\right) \sin^2\left(\frac{\alpha_0}{2}\right)}{\sin(\theta_0) \sin(\alpha_0)} \right) - \frac{f_i}{(1.05)} \right\}, \tag{5.19}$$

which is (5.11). Using (3) and Lemma 11 we get (5.13). The left-most inequality of (5.12) is obvious. Finally, using the law of cosines, we have (recall $a = |z_1 - z_2|$)

$$\begin{aligned} a^2 &= (1 - f_1)^2 + 1 - 2(1 - f_1) \cos(\theta) \\ &= 4 \sin^2\left(\frac{\theta}{2}\right) (1 - f_1) + f_1^2. \end{aligned} \tag{5.20}$$

Using (5.13), we get

$$\begin{aligned} a^2 &\leqslant 4 \sin^2\left(\frac{\theta}{2}\right) + (1.07)^2 \sin^2(\alpha) \sin^2(\theta) \\ &\leqslant \sin^2(\theta) \left( \frac{4 \sin^2\left(\frac{\theta_0}{2}\right)}{\sin^2(\theta_0)} + \left(\frac{1.07}{4}\right)^2 \right) \\ &\leqslant (1.1) \sin^2(\theta), \end{aligned} \tag{5.21}$$

which implies the right-most inequality of (5.12). This concludes the proof of the lemma. ∎

We now need a further improvement on Lemma 11:

PROPOSITION 13. $\max\{\sin(\alpha), \sin(\theta)\} \leqslant \frac{3}{16}$.

*Proof.* We will only prove that $\sin(\theta) \leqslant \frac{3}{16}$, since the proof for $\sin(\alpha)$ is identical. See Fig. 4. Suppose $a \geqslant \sin(\theta) > \frac{3}{16}$. We choose natural coordinates for the line segment $\overline{z_1 z_2}$. By a translation, and a rotation, we can assume that the center line of the minimal strip $S$ which contains $B(w_1, 1) \cap F$, is horizontal, and that $z$ is at the origin. We let $|\lambda|$ be the acute angle that $\overline{z_1 z_2}$ makes with the vertical, and we define $\lambda$ to be positive when $z_2$ has a negative real part. Let $h_2$ be the distance from $z_2$ to

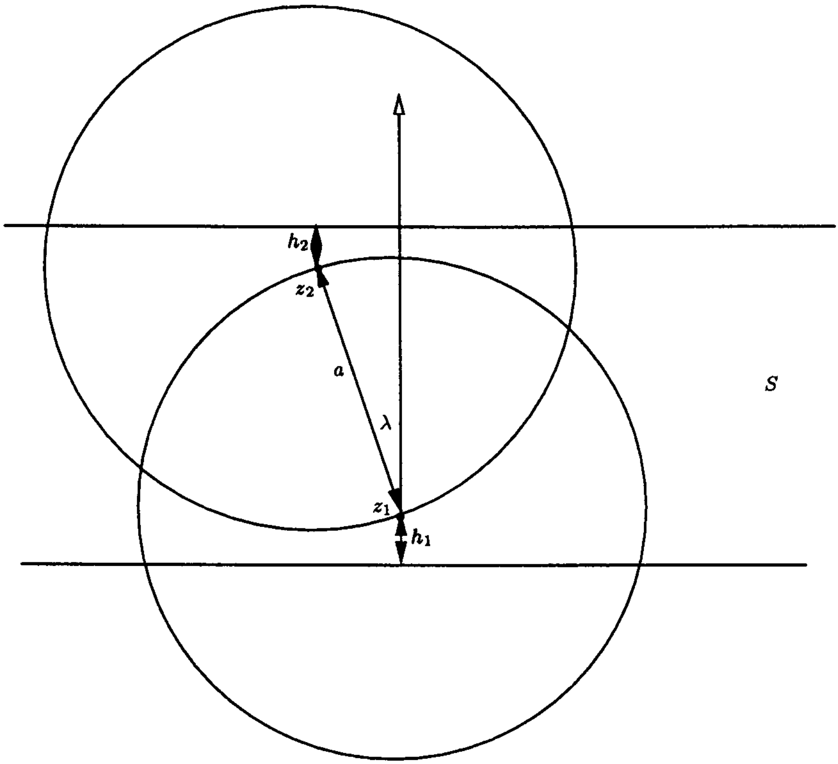


FIGURE 4

the upper boundary of $S$, and $h_1$ the distance from $z_1$ to the lower boundary of $S$. We start by proving:

LEMMA 14. *The following estimates hold*:

$$-9.1^\circ \leqslant \lambda \leqslant 14.9^\circ, \tag{5.22}$$

$$a\cos(\lambda) + h_1 + h_2 \leqslant \tfrac{1}{4}, \tag{5.23}$$

*and*

$$\max\{0, (1-f_2)\sin(\lambda - 10.6^\circ)\} \leqslant h_1. \tag{5.24}$$

*Proof.* We first record some crude estimates. Since $z_2 \in B(z_1, 1) \cap B(w_1, 1)$, then

$$\delta \leqslant 90^\circ - \frac{\theta}{2}. \tag{5.25}$$

By (5.7), (5.12), (5.18), and our hypothesis, we must have

$$\delta \geqslant \arcsin\left(\frac{\left(1-\frac{1.017}{16}\right)}{(1.05)}\right) \geqslant 63.1°. \tag{5.26}$$

Since the maximum acute angle that $\overline{w_1 z_1}$ makes with the horizontal cannot exceed $\arcsin(\frac{1}{4})$, we conclude that

$$\begin{aligned} \lambda &\leqslant 90° - \delta + \arcsin(\tfrac{1}{4}) \\ &\leqslant 41.4°. \end{aligned} \tag{5.27}$$

Using (5.12), (5.25), and Lemma 11, we get

$$\begin{aligned} \lambda &\geqslant 90° - \{90° - \tfrac{1}{2} \arcsin(\tfrac{3}{16}) + \arcsin(\tfrac{1}{4})\} \\ &\geqslant -9.1°, \end{aligned} \tag{5.28}$$

which is the left inequality of (5.22). Projecting on the vertical axis, we have

$$h_1 + a\cos(\lambda) + h_2 \leqslant \tfrac{1}{4}, \tag{5.29}$$

which is (5.23), and

$$\max\{0, (1-f_2)\sin(\alpha+\delta+\lambda-90°)\} \leqslant h_1, \tag{5.30}$$

which is (5.24). Since $L^2 \geqslant 1$, we have (by the law of cosines),

$$a^2 + (1-f_2)^2 - 2a(1-f_2)\cos(\delta+\alpha) \geqslant 1. \tag{5.31}$$

The left-hand side of (5.31) is decreasing with respect to $f_2$, and thus by (5.12), we get

$$(\tfrac{1.05}{4})^2 + 1 - 2 \times \tfrac{3}{16}\cos(\alpha+\delta) \geqslant 1, \tag{5.32}$$

which gives

$$\alpha + \delta \geqslant 79.4°. \tag{5.33}$$

Substituting in (5.30), we obtain (5.24). Thus, if $\lambda \geqslant 10.6°$, we can combine (5.23), (5.24), and our assumption to get

$$h_2 + \tfrac{3}{16}\cos(\lambda) + (1-f_2)\sin(\lambda - 10.6°) \leqslant \tfrac{1}{4}. \tag{5.34}$$

Using the last summand alone, we get that

$$\lambda \leqslant 25°. \tag{5.35}$$

Substituting $\lambda = 14.9°$ in (5.34) produces a violation and, since the left-hand side of that inequality is increasing for $0 \leqslant \lambda \leqslant 25°$, we conclude

$$\lambda \leqslant 14.9°. \tag{5.36}$$

This concludes the proof of the lemma. ▮

Now let $\mathscr{R}$ be the region defined by

$$\mathscr{R} \equiv ((S_1 \cap B(z_1, a)) \cup (S_2 \cap B(z_2, a))) \cap S, \tag{5.37}$$

where, for $i = 1, 2$, $S_i$ is a minimal strip such that $B(z_i, a) \cap F \subset S_i$. We prove the following intuitively obvious lemma.

Lemma 15. $\partial\mathscr{R} \subset \partial S_1 \cup \partial S_2 \cup \partial S$

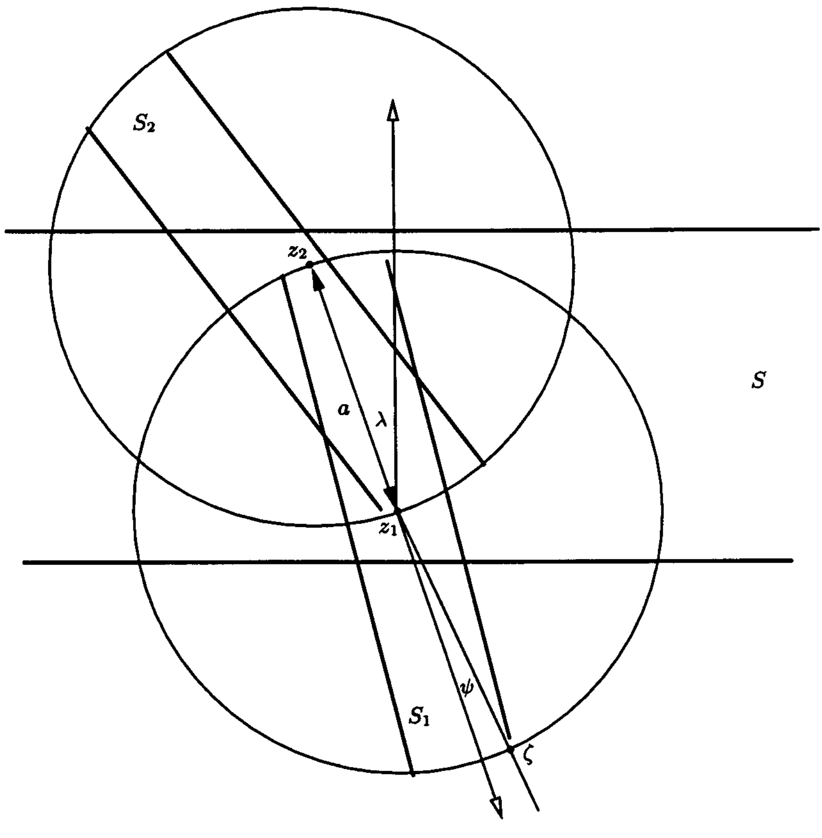


FIGURE 5

*Proof.* See Fig. 5. Suppose $\zeta \in (\partial B(z_1, a) \cap S_1) \backslash B(z_2, a)$. Let $\psi \in (-180°, 180°]$, be the angle between the directed line segment $\overrightarrow{z_1 \zeta}$, and $\overrightarrow{z_2 z_1}$, which we define positive when $\zeta$, $w_2$, are in different components of $\mathbf{R}^2 \backslash \mathscr{L}$, where $\mathscr{L}$ is the line containing $z_1, z_2$. We observe first (with an elementary argument using Lemma 11) that $\psi$ must satisfy

$$|\psi| \geqslant 2\theta_0, \tag{5.38}$$

and thus, if $\zeta$ is to lie in $S$, we must have (by a similar argument as leading to (5.23)),

$$|\zeta - z_1| \leqslant \frac{\frac{1}{4} - \frac{3}{16}\cos(\lambda)}{\cos(|\psi| + \lambda)}. \tag{5.39}$$

By (5.22) and (5.41), we get

$$|\zeta - z_1| \leqslant \tfrac{1.53}{16}, \tag{5.40}$$

which is absurd. Similarly, we cannot have a point $\zeta \in (\partial B(z_2, a) \cap S_2) \backslash B(z_1, a)$, which would also lie in $S$.

We now wish to estimate $diam(\mathscr{R})$ for any choice of $S_1$, $S_2$, and $\lambda$.

LEMMA 16. $diam(\mathscr{R}) \leqslant \frac{1.322}{4}$.

*Proof.* Let $S_0$ be a strip of width $\frac{1}{4}a$, with centerline making acute angle $\theta_0$, with $\overline{z_1 z_2}$, so that the acute (positive) angle it makes with the horizontal is $(90° - \lambda - \theta_0)$. Let $\zeta_1'$, $\zeta_2'$, be the points of intersection of the upper component of $\partial S$, with $\partial S_0$, and $\zeta_1$, $\zeta_2$, the corresponding intersections of $\partial S_0$ with the lower component of $\partial S$. We choose the labeling in such a way that $\mathrm{Re}(\zeta_1) < \mathrm{Re}(\zeta_2)$, and $\mathrm{Re}(\zeta_1') < \mathrm{Re}(\zeta_2')$. It is an elementary exercise (using the law of cosines) to see that $diam(\mathscr{R}) \leqslant |\zeta_1' - \zeta_2|$, for any choice of $0 \leqslant \lambda \leqslant 14.9°$, $S_1$, $S_2$. Now a computation (using the law of cosines) gives

$$\begin{aligned}
(diam(\mathscr{R}))^2 &\leqslant |\zeta_1' - \zeta_2|^2 \\
&\leqslant \left(\frac{1}{4\cos(14.9° + \theta_0)}\right)^2 + \left(\frac{a}{4\cos(14.9° + \theta_0)}\right)^2 \\
&\quad + 2\left(\frac{1}{4\cos(14.9° + \theta_0)}\right)^2 a \sin(14.9° + \theta_0) \\
&= \left(\frac{1}{4\cos(14.9° + \theta_0)}\right)^2 (a^2 + 1 + 2a\sin(14.9° + \theta_0)).
\end{aligned} \tag{5.41}$$

Hence, by (5.12), we get

$$diam(\mathscr{R}) \leqslant \tfrac{1.322}{4}. \quad \blacksquare \tag{5.41}$$

Now let $\tilde{\zeta}_i \in B(z_i, a) \cap F \cap \mathscr{R}$, be such that $diam(\mathscr{R} \cap F) = |\tilde{\zeta}_1 - \tilde{\zeta}_2|$. By Lemma 16

$$|\tilde{\zeta}_1 - \tilde{\zeta}_2| \leqslant \tfrac{1.322}{4}. \tag{5.43}$$

Consider $\mathscr{G} \equiv F \cap (B(\tilde{\zeta}_1, |\tilde{\zeta}_1 - \tilde{\zeta}_2|/2) \cup B(\tilde{\zeta}_2, |\tilde{\zeta}_1 - \tilde{\zeta}_2|/2))$. By (4.4) and (4.5), there must exist $\tilde{\zeta}'_1 \in B(\tilde{\zeta}_1, \ |\tilde{\zeta}_1 - \tilde{\zeta}_2|/2) \cap F$, $\tilde{\zeta}'_2 \in B(\tilde{\zeta}_2, |\tilde{\zeta}_1 - \tilde{\zeta}_2|/2) \cap F$, so that

$$\begin{aligned} |\tilde{\zeta}'_1 - \tilde{\zeta}'_2| &\equiv diam(\mathscr{G}) \\ &\geqslant (1+\varepsilon)\, |\tilde{\zeta}_1 - \tilde{\zeta}_2|. \end{aligned} \tag{5.44}$$

However, by construction, for at least some $i, \tilde{\zeta}'_i$, must be in $(F \cap B(\tilde{\zeta}_i, |\tilde{\zeta}_1 - \tilde{\zeta}_2|/2)) \backslash B(z_i, a)$. Assume this happens for $i = 1$. We now wish to show that this is impossible.

LEMMA 17. $\max\{|\tilde{\zeta}'_1 - z_1|, |\tilde{\zeta}'_2 - z_2|\} \leqslant \tfrac{1.13}{4}$.

*Proof.* It is not difficult (using the law of cosines) to see that

$$\begin{aligned} \max_{\tilde{\zeta}_1} |\tilde{\zeta}_1 - z_1|^2 &\leqslant \left( \frac{\frac{1}{4} - a\cos(\lambda)}{\cos(|\lambda| + \theta_0)} \right)^2 + \left( \frac{a}{4\cos(|\lambda| + \theta_0)} \right)^2 \\ &\quad + \left( \frac{2a}{4\cos^2(|\lambda| + \theta_0)} \right) \left( \frac{1}{4} - a\cos(\lambda) \right) \sin(|\lambda| + \theta_0) \\ &= (4\cos(|\lambda| + \theta_0))^{-2}\, (1 - 8a\cos(\lambda) + 16a^2\cos^2(\lambda) \\ &\quad + a^2 + 2a\sin(|\lambda| + \theta_0) - 8a^2\cos(\lambda)\sin(|\lambda| + \theta_0)). \end{aligned} \tag{5.45}$$

This is decreasing in $a$, and a computation using the estimates on $a$, $\lambda$, yields

$$\max |\tilde{\zeta}_1 - z_1| \leqslant \tfrac{1.9}{16}. \tag{5.46}$$

Similarly,

$$\max |\tilde{\zeta}_2 - z_2| \leqslant \tfrac{1.9}{16}. \tag{5.47}$$

Combining (5.46) and (5.47) with (5.43) we conclude the proof of the lemma.

Now by Lemma 17, (5.3), and a similar argument as in Lemma 10, we conclude that the acute angle between the directed line segments $\overrightarrow{z_2 z_1}$, $\overrightarrow{z_2 \tilde{\zeta}'_1}$, cannot exceed

$$\arcsin(\tfrac{1.2}{4} \times \tfrac{16}{3} \times \tfrac{1}{4}) \leqslant 22.2°.$$

Let $Y$ be the interSection of the half line which starts at $z_2$, and passes through $\tilde{\zeta}'_1$, with the bottom component of $\partial S$. We have

$$|z_2 - Y| \leqslant \frac{1}{4\cos(14.9° + 22.2°)} \leqslant \frac{1.26}{4}. \tag{5.48}$$

It is not difficult to see (using the law of cosines) that

$$|z_1 - \tilde{\zeta}'_1| \leqslant |z_1 - Y|. \tag{5.49}$$

By the law of cosines and (5.47) we get

$$|z_1 - \tilde{\zeta}'_1|^2 \leqslant a^2 + (\tfrac{1.26}{4})^2 - 2a\,(\tfrac{1.26}{4})\cos(22.2°). \tag{5.50}$$

This is decreasing in $a$ for $\frac{3}{16} \leqslant a \leqslant \frac{1.1}{4}$, and hence

$$|z_1 - \tilde{\zeta}'_1| \leqslant \tfrac{2.7}{16} \tag{5.51}$$

which contradicts the assumption that $\tilde{\zeta}'_1 \notin B(z_1, a)$. This concludes the proof of Proposition 13. ▮

Using Proposition 13, we can obtain a sharper bound on $f_1 + f_2$.

PROPOSITION 5.11. $(f_1 + f_2) \leqslant (1.12)\,\frac{3}{16}$.

*Proof.* By (5.11), we see that if $L \geqslant 1$, then

$$\max_i \{f_i\} \leqslant (1.07)\sin(\theta)\sin(\alpha). \tag{5.52}$$

Using (5.15), (5.16), with $b = b(\theta, \alpha)$, we get (recall (5.16))

$$\begin{aligned} 0 \leqslant\; & 2b + (f_1 + f_2)^2 - 2(f_1 + f_2)(1 + b) \\ & + 2f_1 f_2 \left(b - 2\sin^2\left(\frac{\alpha}{2}\right) - 2\sin^2\left(\frac{\theta}{2}\right)\right) \\ & + 4f_1 \sin^2\left(\frac{\alpha}{2}\right) + 4f_2 \sin^2\left(\frac{\theta}{2}\right), \end{aligned} \tag{5.53}$$

which in turn implies

$$0 \leqslant 2b + (f_1 + f_2)^2 - 2(f_1 + f_2)(1 + b)$$
$$+ 2f_1 f_2 \left( b - 4 \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\theta}{2}\right)\right) + 4f_1 \sin^2\left(\frac{\alpha}{2}\right) + 4f_2 \sin^2\left(\frac{\theta}{2}\right). \quad (5.54)$$

Now (5.52), (5.54), give

$$0 \leqslant b \left( 2 + (1.07) \left( 4 \sin^2\left(\frac{\alpha}{2}\right) + 4 \sin^2\left(\frac{\theta}{2}\right)\right)\right)$$
$$+ 8(1.07)^2\, b \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\theta}{2}\right) + (f_1 + f_2)^2 - 2(f_1 + f_2)(1 + b). \quad (5.55)$$

Upon substituting for $\theta$ and $\alpha$ from Proposition 13, we get

$$0 \leqslant 2.16b + (f_1 + f_2)^2 - 2(f_1 + f_2)(1 + b), \quad (5.56)$$

which gives the estimate

$$f_1 + f_2 \leqslant 1 + b - \sqrt{(1 + b)^2 - 2.16b}$$
$$\leqslant (1.06)\, b \leqslant (1.081) \sin(\theta) \sin(\alpha), \quad (5.57)$$

and also

$$(f_1 + f_2) \leqslant 0.21a \leqslant (1.12)(\tfrac{3}{16})\, a, \quad (5.58)$$

where we used $\sin(\theta) \leqslant a$. This concludes the proof of Proposition 18. ∎

The remaining part of the proof is devoted to the construction of a set like $U$ in Section 5.1, which will produce a contradiction between (4.4) and (4.5). The reader may find it helpful to reread Section 5.1 at this point. As we mentioned in Section 5.1, we will have to chase a sequence of quadruplets like $\{z_1, z_2, w_1, w_2\}$, and we will have to show that at some stage we will find the right one! For $i = 1, 2$, we let

$$z_i^{(1)} = z_i, \qquad w_i^{(1)} = w_i,$$
$$a^{(1)} = |z_1^{(1)} - z_2^{(1)}|, \quad (5.59)$$

and

$$G_i^{(1)} = \left( B\left(z_1^{(1)}, \frac{a^{(1)}}{2}\right) \cup B\left(z_2, \frac{a^{(1)}}{2}\right) \cup B\left(w_i^{(1)}, R_i^{(1)}\right)\right) \cap F, \quad (5.60)$$

where $R_i^{(1)}$ is the maximum possible radius that keeps the overlap of the above balls containing, at most, a subset of $F$ having measure zero. See Fig. 6. Also, let

$$d_i^{(1)} = diam(G_i^{(1)}), \tag{5.61}$$

and $X_i^{(1)}$, $Y_i^{(1)} \in F \cap G_i^{(1)}$ defined via

$$|X_i^{(1)} - Y_i^{(1)}| = diam(G_i^{(1)}), \tag{5.62}$$

and labeled such that

$$\max\{|X_i^{(1)} - w_i^{(1)}|, |Y_i^{(1)} - z_i^{(1)}|\} \leqslant \max\{|X_i^{(1)} - z_i^{(1)}|, |Y_i^{(1)} - w_i^{(1)}|\}. \tag{5.63}$$

We now prove some estimates on $R_i^{(1)}$:

LEMMA 19. *For $i = 1, 2$,*

$$|w_i^{(1)} - z_i^{(1)}| - \frac{a^{(1)}}{2} \geqslant R_i^{(1)} \geqslant 1 - \frac{a^{(1)}}{2} - (1.07)\left(\frac{3}{16}\right) a^{(1)}. \tag{5.64}$$

*Proof.* This is just a consequence of (5.13), Proposition 13, and the definition of $R_i^{(1)}$. ∎

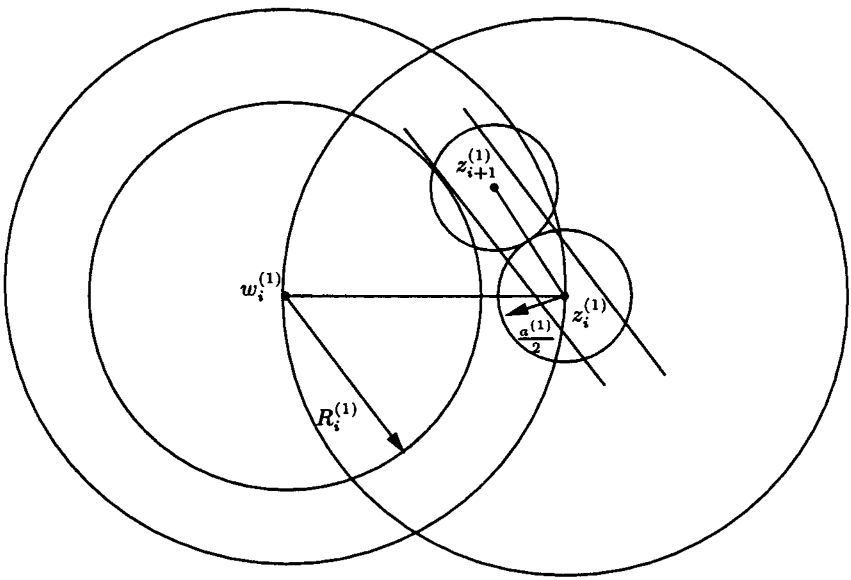


FIGURE 6

We also need some estimates on $diam(G_i^{(1)})$.

LEMMA 20. *For* $i=1, 2$,

$$diam(G_i^{(1)}) \geqslant 1 + \frac{a^{(1)}}{2} - (1.07)\left(\frac{3}{16}\right) a^{(1)}. \tag{5.65}$$

*Proof.* This follows from Lemma 19, (4.4), and (4.5). ∎

The next two lemmas give useful information if a certain quadruplet does not do the job.

LEMMA 5.14. *If* $Y_1^{(1)} \in B(w_1^{(1)}, R_1^{(1)})$, *then*

$$|Y_1^{(1)} - z_1^{(1)}| \geqslant \left(\frac{\frac{1}{2} - (1.07)\left(\frac{3}{16}\right)}{(1.017)\left(\frac{3}{16}\right)}\right) a^{(1)} + \left(\frac{1}{(1.017)(1.05)\left(\frac{3}{16}\right)}\right) \frac{a^{(1)}}{2}$$
$$> 3.6a^{(1)}. \tag{5.66}$$

*Proof.* This is just a consequence of (5.11) and Lemma 20, when we observe that in our case $f_1 \equiv 1 - |Y_1^{(1)} - w_1^{(1)}| \geqslant a^{(1)}/2$. ∎

Similarly, we can prove

LEMMA 22. *If* $Y_2^{(1)} \in B(w_2^{(1)}, R_2^{(1)})$, *then*

$$|Y_2^{(1)} - z_2^{(1)}| \geqslant \left(1 + (1.017)\left(\frac{3}{16}\right)^2\right)^{-1}$$
$$\times \left(\frac{\frac{a^{(1)}}{2} - (1.07)\left(\frac{3}{16}\right) a^{(1)}}{(1.017)} + \frac{a^{(1)}}{2(1.05)(1.017)\left(\frac{3}{16}\right)}\right)$$
$$> 3.5a^{(1)}. \tag{5.67}$$

*Proof.* Here we observe that $f_2 \equiv 1 - |w_2^{(1)} - Y_2^{(1)}| \geqslant a^{(1)}/2$. The only difference from Lemma 21 is that we had to rescale by dividing by the crude bound on $|w_2^{(1)} - z_2^{(1)}|$ in order for (5.11) to apply. ∎

The next proposition will guarantee that our chase will end at some point in the construction

PROPOSITION 23. *There exists $k_0 \geqslant 1$, such that, for $i=1, 2$,*

$$Y_i^{(k_0)} \notin B(w_i^{(k_0)}, R_i^{(k_0)}). \tag{5.68}$$

*Proof.* Suppose that for some $i_0 \in \{1, 2\}$, we have that

$$Y_{i_0}^{(1)} \in B(w_{i_0}^{(1)}, R_{i_0}^{(1)}). \tag{*}$$

We let

$$w_1^{(2)} = w_{i_0}^{(1)}, \qquad w_2^{(2)} = X_{i_0}^{(1)}, \qquad z_1^{(2)} = z_{i_0}^1, \qquad z_2^{(2)} = Y_{i_0}^{(1)}. \tag{5.69}$$

We then apply all the above arguments except that now

$$a^{(2)} = |z_1^{(2)} - z_2^{(2)}| \geqslant 3.5\, |z_1^{(1)} - z_2^{(1)}| = 3.5a^{(1)}. \tag{5.70}$$

We can repeat this process for as long as $(*)$ holds. If $(*)$ always holds, we can repeat and get a sequence $\{|z_1^{(k)} - z_2^{(k)}|\}_k$. Since we started with $|z_1^{(1)} - w_1^{(1)}| < 0.6s$ (see (5.9)), and by (5.11),

$$|z_1^{(k+1)} - w_1^{(k+1)}| \leqslant |z_1^{(k)} - w_1^{(k)}| + (1.017)(\tfrac{3}{16})\, a^{(k)}. \tag{5.71}$$

Thus for some $k'$, we would have

$$|z_1^{(k')} - w_1^{(k')}| < s, \tag{5.72}$$

whereas

$$|z_1^{(k')} - w_1^{(k')}| + (1.017)(\tfrac{3}{16})\, a^{(k')} \geqslant |z_1^{(k'+1)} - w_1^{(k'+1)}| \geqslant s. \tag{5.73}$$

Hence, by (5.70),

$$a^{(k')} \geqslant \frac{2.5}{(1.017)\left(\frac{3}{16}\right)} \left(0.4s - (1.017)\left(\frac{3}{16}\right) a^{(k')}\right). \tag{5.74}$$

Since by Proposition 12, $a^{(k')} \leqslant (1.05)(\frac{3}{16})s$, this is impossible. Thus, for some $k_0$, $(*)$ does not hold for any $i$. ∎

We now investigate the situation imposed on us by Proposition 23. For convenience we will drop the superscript $k_0$. So, we let

$$\begin{aligned} &z_i = z_i^{(k_0)}, \qquad a = |z_1^{(k_0)} - z_2^{(k_0)}|, \qquad X_i = X_i^{(k_0)},\ Y_i^{(k_0)}, \\ &w_i = w_i^{(k_0)}, \qquad R_i = R_i^{(k_0)}, \end{aligned} \tag{5.75}$$

and we keep the convention $|z_1 - w_1| \equiv 1$. Proposition 23 and (5.3) imply

LEMMA 5.24. *$diam(G_1) = diam(G_2) \equiv d$, and we can take $X_1 = X_2 \equiv X$, $Y_1 = Y_2 \equiv Y$.*

In view of (4.4) and (4.5), we can finish the proof of the theorem by proving the following proposition which guarantees that the contradiction we described in Section 5.1, with the set $U$ can now be obtained using, at least, one of the sets $G_1$ and $G_2$.

PROPOSITION 25. $\min_i\{d - R_i\} \leqslant a$.

*Proof.* See Fig. 7. In order to treat $R_1$, $R_2$ on a similar footing, we let, for $i = 1, 2$, $\alpha_i = \widehat{w_{i+1} z_i w_i}$, $\delta_i = \widehat{w_i z_i z_{i+1}}$, $\theta_i = z_i w_i z_{i+1}$ (notice that $\alpha_1$, $\delta_1$, and $\theta_1$, were previously referred to as $\alpha, \delta, \theta$). We also identify the index $i$, with $i+2$, so that $w_3 = w_1$ etc. Let $S_i$ be the strip of width $a/4$, such that $F \cap B(z_i, a) \subset S_i$. Let $z_i' \in B(z_i, a/2)$, be the point on the boundary of $S_i$, that is closest to $w_{i+1}$. Let $|\lambda_i|$ be the acute angle between the centerline of $S_i$, and $\overline{z_1 z_2}$. We choose for $\lambda_i$ a sign in such a way that, when it is positive, then $\delta_i - \lambda_i$ is the acute angle that the center line of $S_i$ makes with $\overline{w_i z_i}$. Finally, let $\overline{\theta}_i = \overline{\alpha}_i = \arcsin(\frac{3}{16})$. We now have

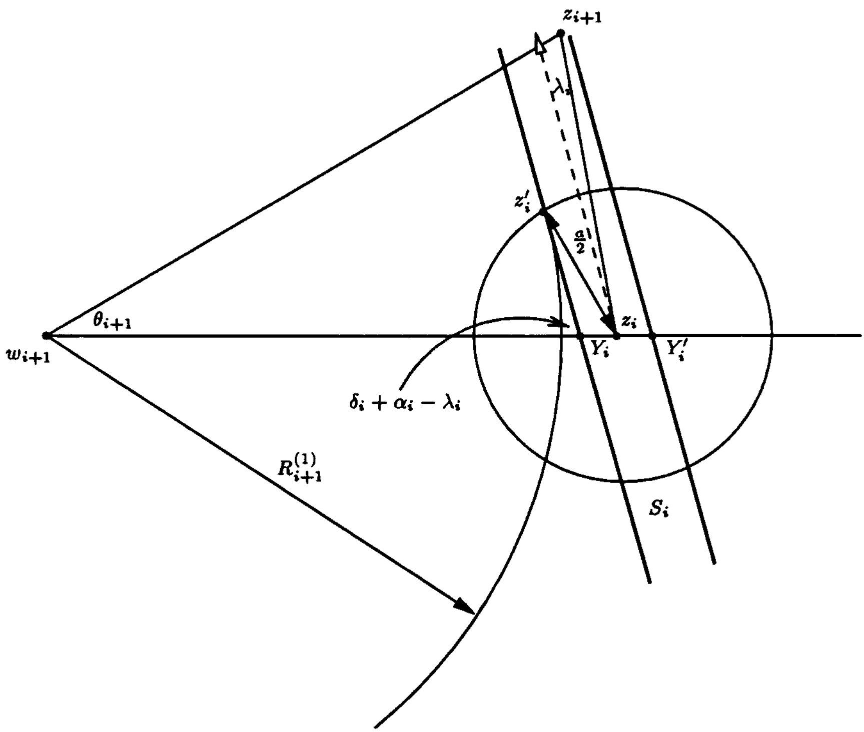


FIGURE 7

LEMMA 26. *Either* $|z_i' - z_i| = a/2$, *or*

$$|\cos(\delta_i + \alpha_i - \lambda_i)| \leqslant \frac{3a}{4}\left(\frac{1}{|w_{i+1} - z_i|}\right). \tag{5.76}$$

*Proof.* Let $Y_i$ the point of intersection of $\partial S_i$ with $\overline{w_{i+1} z_i}$ closest to $w_{i+1}$, and $r_i = |z_i' - z_i|$. Then the law of cosines gives

$$|z_i - w_{i+1}|^2 = r_i^2 + |w_{i+1} - Y_i|^2 - 2r_i\,|w_{i+1} - Y_i|\,|\cos(\delta_i + \alpha_i - \lambda_i)|, \tag{5.77}$$

which is decreasing in $r_i$, for $0 \leqslant r_i \leqslant a/2$, unless

$$|\cos(\delta_i + \alpha_i - \lambda_i)| \leqslant \frac{a}{2\,|w_{i+1} - Y_i|}. \tag{5.78}$$

This implies that, either $|z_i' - z_i| = a/2$, or (5.78) holds. The latter (with an elementary argument) implies

$$|\cos(\delta_i + \alpha_i - \lambda_i)| \leqslant \frac{a}{2\left(|w_{i+1} - z_i| - \dfrac{a}{4\,|\cos(\delta_i + \alpha_i - \lambda_i)|}\right)}, \tag{5.79}$$

which in turn yields the conclusion of the lemma. ∎

We now choose orthogonal coordinates $\xi_i, \eta_i$, such that, in these coordinates, $z_i = (0, 0)$, the $\xi_i$-axis contains the segment $\overline{z_i w_{i+1}}$, and that the $\xi_i$ coordinate of $w_i$ is positive. We define the positive direction of $\eta_i$ in such a way that $z_{i+1}$ has negative $\eta_i$ coordinate. See Fig. 8.

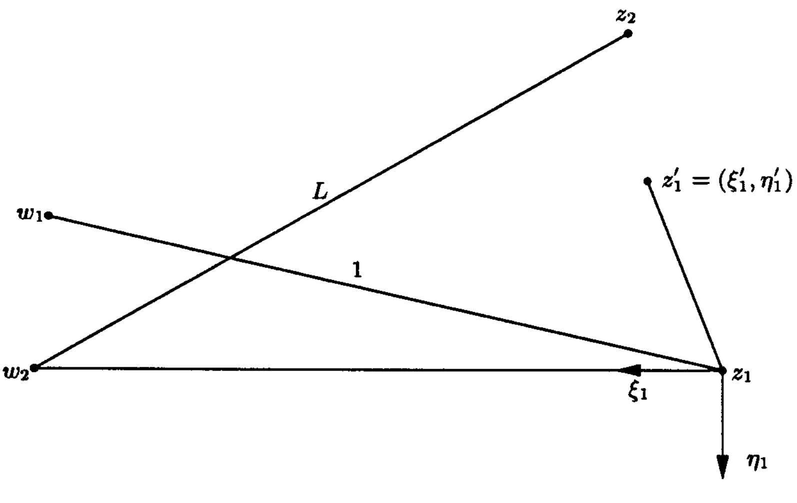


FIGURE 8

LEMMA 27. *For* $i = 1, 2$,

$$\eta'_i \cos(\delta_i + \alpha_i - \lambda_i) \leqslant 0. \tag{5.80}$$

*Proof.* Our convention on the sign of $\eta'_i$, the definition of $z'_i$, and the law of cosines, imply that $\eta'_i$, is negative if and only if $(\delta_i + \alpha_i - \lambda_i) > 90°$, which implies the lemma. ▮

Now let

$$\varepsilon_i = \begin{cases} 1, & \lambda_i \geqslant 0 \\ 0, & \lambda_i < 0, \end{cases} \tag{5.81}$$

and let $z'_i = (\xi'_i, \eta'_i)$, in the $\xi_i, \eta_i$ coordinates. Then, we have

LEMMA 28. *For* $i = 1, 2$,

$$\xi'_i \sin(\delta_i + \alpha_i - \lambda_i) + \eta'_i \cos(\delta_i + \alpha_i - \lambda_i) + \varepsilon_i a \sin(\lambda_i) \leqslant \frac{a}{4}. \tag{5.82}$$

*Proof.* See Fig. 9. The lemma just follows by projecting $\overline{z_i z_{i+1}}$ orthogonal to the center line of the strip $S_i$. ▮

LEMMA 29. *Either* $(\xi'_i)^2 + (\eta'_i)^2 \equiv |z'_i - z_i|^2 = (a/2)^2$, *or*

$$\xi'_i \leqslant \left(\frac{3}{8}\right) \frac{a^2}{\left(1 - (1.07)\left(\frac{3}{16}\right)\right)}. \tag{5.83}$$

*Proof.* As in the proof of Lemma 26, we find from (5.77) that either $|z'_i - z_i| = a/2$, or

$$|z'_i - z_i| = |w_{i+1} - Y_i|\ |\cos(\delta_i + \alpha_i - \lambda_i)|, \tag{5.84}$$

and then

$$\xi'_i = |w_{i+1} - Y_i|\ |\cos(\delta_i + \alpha_i - \lambda_i)|^2. \tag{5.85}$$

Using (5.76), (5.78), and (5.52) we get (5.83). ▮

The following lemma establishes an upper bound on $\xi'_i$, whenever the first alternative of Lemma 29 holds.

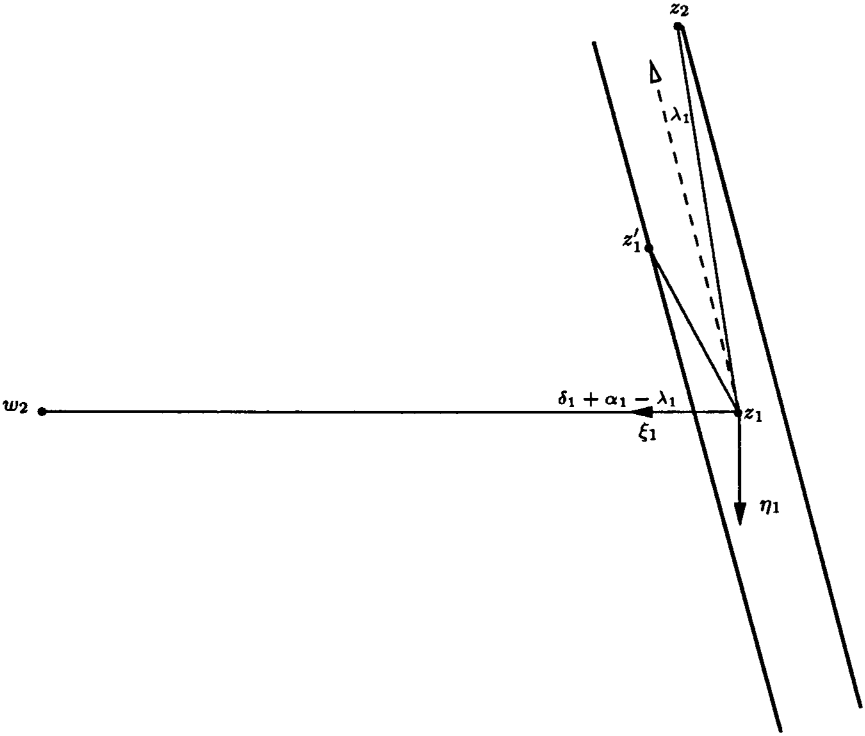


**FIGURE 9**

LEMMA 30. *If* $|z_i' - z_i| = a/2$, *then*

$$\xi_i' \leqslant \frac{a}{2} \sin(\delta_i + \alpha_i - \lambda_i + \varepsilon_{\delta_i + \alpha_i - \lambda_i} m_i), \tag{5.86}$$

*where*

$$m_i = \arc\tan\left(\frac{\left(1 - \left(\frac{1}{2} - 2\varepsilon_i \sin(\lambda_i)\right)^2\right)^{1/2}}{\left(\frac{1}{2} - 2\varepsilon_i \sin(\lambda_i)\right)}\right)$$

$$\geqslant 60^\circ, \tag{5.87}$$

*and*

$$\varepsilon_{\delta_i + \alpha_i - \lambda_i} = \begin{cases} 1, & \delta_i + \alpha_i - \lambda_i \leqslant 90^\circ \\ -1, & \delta_i + \alpha_i - \lambda_i \leqslant 90^\circ \end{cases} \tag{5.88}$$

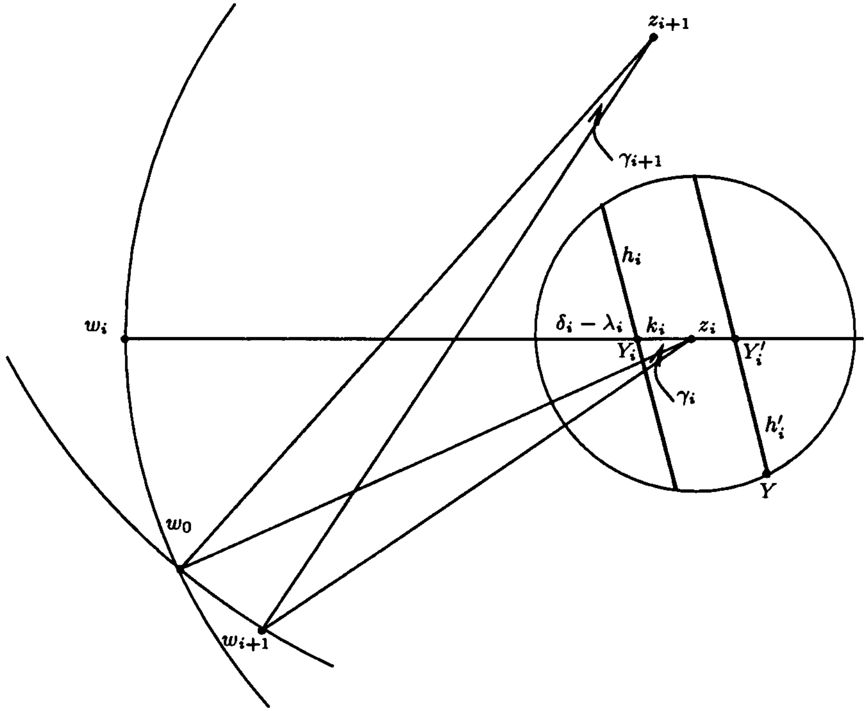


**FIGURE 10**

*Proof.* See Fig. 10. Let $\tau_i \in [0°, 180°]$, be the angle such that

$$\xi_i' = \frac{a}{2} \sin(\tau_i),$$
$$\eta_i' = \frac{a}{2} \cos(\tau_i) \tag{5.89}$$

By Lemma 28, we get

$$\cos(\tau_i - (\delta_i + \alpha_i - \lambda_i)) \leqslant \tfrac{1}{2} - 2\varepsilon_i \sin(\lambda_i). \tag{5.90}$$

The lemma is then concluded when we observe, by (5.7), (5.12), (5.13), that

$$66^0 - \theta_0 \leqslant \text{arc sin}\left(\frac{1-(1.07)\left(\frac{3}{16}\right)^2}{(1.05)}\right) - \theta_0 \leqslant \delta_i + \alpha_i - \lambda_i \leqslant 90° + \theta_0 + \bar{\theta}, \tag{5.91}$$

and that, by Lemma 28, $\tau_i \geqslant 90°$, if and only if, $\delta_i + \alpha_i - \lambda_i \leqslant 90°$. ∎

The next lemma establishes a lower bound on $\xi'_1+\xi'_2$.

LEMMA 31. *At least one of the following statements holds*:

$$\max_i \{|w_{i+1}-z'_i|\} \geqslant L-\frac{a}{2}, \tag{5.92}$$

$$\xi'_1+\xi'_2 \geqslant 0.79a, \tag{5.93}$$

*or*

$$\min_i \{|w_{i+1}-z'_i|\} \geqslant 1-\frac{a}{2}. \tag{5.94}$$

*Proof.* Suppose that the first and last statements do not hold. By definition we have

$$|w_{i+1}-z'_i| = ((1-f_i-\xi'_i)^2+(\eta'_2)^2)^{1/2}. \tag{5.95}$$

Hence,

$$\xi'_1+\xi'_2 \geqslant a-(L-1)-(f_1+f_2). \tag{5.96}$$

Noting that the left-hand side of (5.56) represents $L^2-1$, we get

$$\xi'_1+\xi'_2 \geqslant a-(\tfrac{1}{2})(2.16b+(f_1+f_2)^2-2(f_1+f_2)(1+b))-(f_1+f_2). \tag{5.97}$$

On the other hand, we also have

$$b \leqslant (1.02)\sin(\theta)\sin(\alpha). \tag{5.98}$$

Combining, we get

$$\begin{aligned}\xi'_1+\xi'_2 &\geqslant a-(1.065)(1.02)\left(\frac{3}{16}\right)a+\left((f_1+f_2)\,b-\frac{(f_1+f_2)^2}{2}\right)\\ &\geqslant 0.79a,\end{aligned} \tag{5.99}$$

which is (5.93). ∎

We will now dispose of the three cases in Lemma 31.

*Case* 1. For some $i_0$, $|w_{i+1}-z'_i| \geqslant L-a/2$:

This implies

$$R_{i_0} \geqslant |w_{i_0}-z_{i_0}|-\frac{a}{2}. \tag{5.100}$$

If $i_0 = 2$, then $R_2 \geqslant L - a/2$, whereas by the triangle inequality,

$$d \leqslant L + \frac{a}{2}. \tag{5.101}$$

Therefore,

$$d - R_2 \leqslant a, \tag{5.102}$$

and we are done. If, on the other hand, $i_0 = 1$, then $R_1 \geqslant 1 - a/2$, and this will be covered by Case 3 below.

*Case* 2. $\xi'_1 + \xi'_2 \geqslant 0.79a$:

Using Lemmas 29 and 30, we conclude that for $i = 1, 2$,

$$|z_i - z'_i| = \frac{a}{2}, \tag{5.103}$$

and thus, by Lemma 28 and our assumption, we must have

$$\sum_{i=1}^{2} \sin(\delta_i + \alpha_i - \lambda_i + \varepsilon_{\delta_i + \alpha_i - \lambda_i} m_i) \geqslant 1.58. \tag{5.104}$$

Since

$$\max_i(\sin(\lambda_i)) \leqslant \tfrac{1}{4}, \tag{5.105}$$

$$\max_i(\sin(\theta_i)) \leqslant \tfrac{3}{16}, \tag{5.106}$$

and

$$\max_i(\sin(\alpha_i)) \leqslant \tfrac{3}{16}, \tag{5.107}$$

then a computation shows that this can only hold if, for $i = 1, 2$,

$$\max_i(\sin(\lambda_i)) < 0, \tag{5.108}$$

and

$$\min_i(\delta_i - \lambda_i) > 90^\circ. \tag{5.109}$$

We now use this information to estimate $d$ and $R_i$. We divide this case further into two subcases

1. For $i = 1, 2$, $R_i \geqslant |z'_i - w_{i+1}|$.

In this case we have

$$R_i \geqslant 1 - f_i - \zeta'_i. \tag{5.110}$$

Hence by Lemma 28, we have

$$R_i \geqslant 1 - f_i - \frac{a}{2}\sin(\delta_i + \alpha_i - \lambda_i - 60°). \qquad (5.111)$$

Therefore it suffices to show

$$d^2 - R_{i+1}^2 \leqslant a\{2 - 2f_{i+1} + a(1 - \sin(\delta_i + \alpha_i - \lambda_i - 60°))\}, \qquad (5.112)$$

whenever $Y \in B(z_i, a/2)$.

Let $w_0$ be the point in $\partial B(z_1, 1) \cap \partial B(z_2, L)$, that is closest to $w_1$. Suppose first that $Y \in B(z_1, a/2)$. Let the component of the boundary of $S_1$, closest to $w_2$, intersect the line segment $\overline{w_2 z_1}$, at $Y_1$, a distance $k_1$ from $z_1$, and let its intersection with the circle $\partial B(z_1, a/2)$, closest to $w_2$, be a distance $h_1$, from $Y_1$. See Fig. 10. Let the intersection of the same component with $\overline{w_0 z_1}$ be at a distance $k_1'$ from $z_1$, and the intersection of the other component, with the half line that starts at $w_0$, and passes through $z_1$, be $Y_1'$, and its point of intersection with $\partial B(z_1, a/2)$, furthest from $w_0$, be $\widetilde{Y}_1$, at distance $h_1'$ from $Y_1'$. Let $\gamma_1$ be the acute angle $\widehat{w_0 z_1 w_1}$, and $\gamma_2$ the acute angle $\widehat{w_0 z_2 w_2}$.

We observe by projecting orthogonally to the center line of $S_1$ that

$$k_1' \sin(\delta_1 + \gamma_1 - \lambda_1) = k_1 \sin(\delta_1 + \alpha_1 - \lambda_1). \qquad (5.113)$$

We now observe

LEMMA 32. $d \leqslant |w_0 - \widetilde{Y}_1|$.

*Proof.* This follows from the law of cosines when we maximize $|X - Y|$, noting that $X - Y = X - z_1 + z_1 - Y = X - z_2 + z_2 - Y$, and that $X \in B(z_1, 1) \cap B(z_2, L)$, whereas $Y$ is assumed in $B(z_1, a/2)$. ∎

Using the law of cosines, we get

$$\begin{aligned} d^2 - R_2^2 \leqslant{}& \left(1 + \frac{a}{4\sin(\delta_1 + \gamma_1 - \lambda_1)} - k_1'\right)^2 + (h_1')^2 \\ & - 2h_1' \left(1 + \frac{a}{4\sin(\delta_1 + \gamma_1 - \lambda_1)} - k_1'\right)\cos(\delta_1 + \gamma_1 - \lambda_1) \\ & - (1 - k_1 - f_1)^2 - (h_1)^2 - 2h_1(1 - k_1 - f_1)\cos(\delta_1 + \alpha_1 - \lambda_1). \end{aligned} \qquad (5.114)$$

Since $\widetilde{Y}_1, z'_1 \in \partial B(z_1, a/2)$, we have

$$(h_1)^2 = \left(\frac{a}{2}\right)^2 - (k_1)^2 + 2h_1 k_1 \cos(\delta_1 + \alpha_1 - \lambda_1), \tag{5.115}$$

and

$$(h'_1)^2 = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{4\sin(\delta_1+\gamma_1-\lambda_1)} - k'_1\right)^2 + 2\left(\frac{a}{4\sin(\delta_1+\gamma_1-\lambda_1)} - k'_1\right) h'_1 \cos(\delta_1+\gamma_1-\lambda_1). \tag{5.116}$$

Hence

$$d^2 - R_2^2 \leqslant \left(1 + \frac{a}{4\sin(\delta_1+\gamma_1-\lambda_1)} - k'_1\right)^2 - (1 - k_1 - f_2)^2 - \left(\frac{a}{4\sin(\delta_1+\gamma_1-\lambda_1)} - k'_1\right)^2 + k_1^2 - 2(h'_1\cos(\delta_1+\gamma_1-\lambda_1) + h_1\cos(\delta_1+\alpha_1-\lambda_1) - h_1 f_2 \cos(\delta_1+\gamma_1-\lambda_1)), \tag{5.117}$$

so that

$$d^2 - R_2^2 \leqslant \frac{a}{2\sin(\delta_1+\gamma_1-\lambda_1)} + 2(k_1 - k'_1) + f_2(2 - f_2 - 2h_1\,|\cos(\delta_1+\gamma_1-\lambda_1)|) + 2(h'_1 + h_1)\,|\cos(\delta_1+\alpha_1-\lambda_1)|. \tag{5.118}$$

Using (5.113), (5.13), and maximizing the right-hand side, we get

$$d^2 - R_2^2 \leqslant \frac{a}{2\sin(\delta_1+\gamma_1-\lambda_1)} + \frac{2k_1\,|\cos(\delta_1+\alpha_1-\lambda_1)|\,|\alpha_1-\gamma_1|}{\sin(\delta_1+\alpha_1-\lambda_1)} - 2k_1 f_2 + f_2(2 - f_2 - 2h_1\,|\cos(\delta_1+\gamma_1-\lambda_1)|) + 2(h'_1+h_1)\cos(\delta_1+\alpha_1-\lambda_1). \tag{5.119}$$

Now, maximizing with respect to $f_2$ using (5.13), we get

$$d^2 - R_2^2 \leqslant \frac{a}{2\sin(\delta_1+\gamma_1-\lambda_1)} + \frac{2k_1\,|\cos(\delta_1+\gamma_1-\lambda_1)|}{\sin(\delta_1+\gamma_1-\lambda_1)}\,|\alpha_1-\gamma_1| - 2(0.21)\,ak_1 + 0.21a(2 - 0.21a - 2h_1\,|\cos(\delta_1+\gamma_1-\lambda_1)|) + 2(h'_1+h_1)\,|\cos(\delta_1+\gamma_1-\lambda_1)|. \tag{5.120}$$

Maximizing the right-hand side with respect to $h_1$, $k_1$, $|\cos(\delta_1+\gamma_1-\lambda_1)|$, $|\cos(\delta_1+\alpha_1-\lambda_1)|$, we get

$$d^2-R_2^2 \leqslant \frac{a}{2\sin(25.3°)} + \frac{a}{2\sin(25.3°)}\left(\frac{3}{16}\tan(25.3°)-0.21\left(\frac{3}{16}\right)\right)$$
$$+0.21a(2-0.21a-2h_1|\cos(\delta_1+\gamma_1-\lambda_1)|)$$
$$+2a|\cos(\delta_1+\gamma_1-\lambda_1)| \leqslant 1.85a. \tag{5.121}$$

Hence, by a computation using (5.91) and (5.15), we get

$$d^2-R_2^2+2af_2-a^2(1-\sin(\delta_1+\gamma_1-\lambda_1-60°)) \leqslant 1.9a. \tag{5.122}$$

Suppose now that $Y \in B(z_2, a/2)$. Let the component of the boundary of $S_2$ closest to $w_2$, intersect the line segment $\overline{w_1 z_2}$ at $Y_2$, a distance $k_2$ from $z_2$, and let its intersection with the circle $\partial B(z_2, a/2)$, closest to $w_1$, be at distance $h_2$, from $Y_2$ (see Fig. 10). Let the intersection of the same component with $\overline{w_0 z_1}$ be a distance $k_2'$ from $z_2$, and the intersection of the other component with the half line starting at $w_0$, and passing through $z_2$, be $Y_2'$, and its point of intersection with $\partial B(z_2, a/2)$, furthest from $w_0$, be at distance $h_2'$ from $Y_2'$.

Similar to (5.113), we here have

$$k_2' \sin(\delta_2+\gamma_2-\lambda_2) = k_2 \sin(\delta_2+\alpha_2-\lambda_2), \tag{5.123}$$

and as in Lemma 32, we have

LEMMA 33. $d \leqslant |w_0 - \widetilde{Y}_2|$.

Now similar to (5.114), we have (recall $L \equiv |w_2 - z_2|$)

$$d^2-R_1^2 \leqslant \left(L+\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)}-k_2'\right)^2+(h_2')^2$$
$$-2h_2'\left(L+\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)}-k_2'\right)\cos(\delta_2+\gamma_2-\lambda_2)$$
$$-(1-k_2-f_1)^2-(h_2)^2-2h_2(1-k_2-f_1)\cos(\delta_2+\alpha_2-\lambda_2). \tag{5.124}$$

Also,

$$(h_2)^2=\left(\frac{a}{2}\right)^2-(k_2)^2+2h_2k_2\cos(\delta_2+\alpha_2-\lambda_2), \tag{5.125}$$

and

$$(h_2')^2 = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)} - k_2'\right)^2 + 2\left(\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)} - k_2'\right) h_2' \cos(\delta_2+\gamma_2-\lambda_2). \quad (5.126)$$

Hence

$$d^2 - R_1^2 \leqslant L^2 - 1 + 2L\left(\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)} - k_2'\right) + 2f_1 + 2k_2 - f_1^2 - 2k_2 f_1 - 2(Lh_2'\cos(\delta_2+\gamma_2-\lambda_2) + (1-f_1)\, h_2 \cos(\delta_2+\alpha_2-\lambda_2)). \quad (5.127)$$

Using (5.11) and (5.123), we get

$$d^2 - R_1^2 \leqslant 2(1.017)\left(\frac{3}{16}\right) a + f_1\left(2 - \frac{2}{(1.05)} - 2k_2 - f_1 - h_2\,|\cos(\delta_2+\alpha_2-\lambda_2)|\right) + 2k_2 \frac{|\cos(\delta_2+\alpha_2-\lambda_2)|}{\sin(\delta_2+\alpha_2-\lambda_2)}\,|\alpha_2-\gamma_2| + \frac{a}{2\sin(\delta_2+\alpha_2-\lambda_2)} + 2(h_2\,|\cos(\delta_2+\alpha_2-\lambda_2)| + h_2'\,|\cos(\delta_2+\gamma_2-\lambda_2)|) + 2(L-1)\left(\frac{a}{4\sin(\delta_2+\gamma_2-\lambda_2)} - k_2' + h_2'\,|\cos(\delta_2+\gamma_2-\lambda_2)|\right). \quad (5.128)$$

Let $\Lambda = d^2 - R_1^2 + 2af_1 - a^2(1-\sin(\delta_2+\alpha_2-\lambda_2))$, then

$$\Lambda \leqslant 2(1.017)\left(\frac{3}{16}\right) a + a\left(\frac{(1.07)\,3}{16}\right) \times \left(0.1 + 2a - \frac{2a}{4\cos(25.3°)} - \frac{a}{2}\,|\sin(25.3°)|\right) + a\,\frac{\sin(25.3°)}{2\cos^2(25.3°)} \times \arcsin\left(\frac{3}{16}\right) + \frac{a}{2\cos(25.3°)} + 2a\sin(25.3°) - a^2(1-\sin(55.3°)) \leqslant 1.91a. \quad (5.129)$$

2. $|z_1' - w_2| > R_2$.

This case is trivial since it implies $R_2 \geqslant L - a/2$, which can be eliminated as in Case 1.

3. $|z_2' - w_1| > R_1$.

This implies, in particular, that

$$R_1 \geqslant 1 - \frac{a}{2}, \tag{5.130}$$

and will be covered in Case 3.

*Case* 3. $R_1 \geqslant 1 - a/2$:

Note that (5.94) implies this case, as explained in Case 1. Suppose first that $Y \in B(z_1, a/2)$. Then, by the triangle inequality, we have

$$d \leqslant 1 + \frac{a}{2}, \tag{5.131}$$

and we are done. Suppose however that $Y \in B(z_2, a/2)$, and observe that now it suffices to show that

$$d^2 - R_1^2 \leqslant 2a. \tag{5.132}$$

See Fig. 11. We divide this case into three subcases:

1. $\cos(\delta_2 - \lambda_2 + \alpha_2) > \frac{3}{16}$:

In particular this implies that $\delta_2 - \lambda_2 + \alpha_2 < 90° - \bar{\theta}$. In this case we observe, by the law of cosines, that we must have $w_1$, and $X$, lie on opposite sides of the line segment $\overline{w_2 z_2}$. Let $\rho = \widehat{X z_2 w_2}$. Then it is not difficult to see (using the law of cosines), that the worst-case estimate on $d$ is when

$$\lambda_2 = \theta_0, \tag{5.133}$$

$$\rho + \alpha_2 = \theta_0, \tag{5.134}$$

$$|X - z_2| = L, \tag{5.135}$$

and $Y$ lies on the intersection of $\partial B(z_2, a/2)$ and the component of $\partial S_2$ that is furthest from $w_2$. For the worst-case estimate on $R_1$, we have the same condition on $\rho, \alpha_2, X$, as above, but that $z_1'$ lies on the component of $\partial S_1$ that is closest to $w_1$, and

$$\lambda_1 = -\theta_0. \tag{5.136}$$

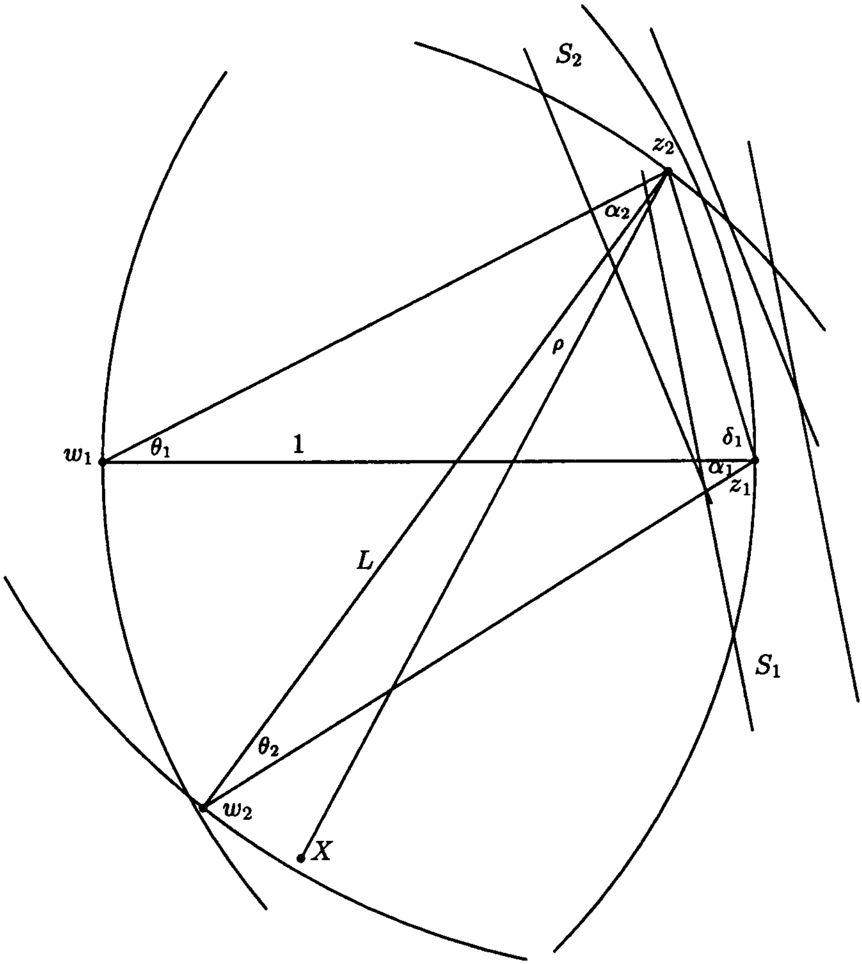


FIGURE 11

Applying (5.3), to $B(z_1, a/2)$, and $B(z_2, a/2)$, we get

$$d^2 - R_1^2 \leqslant L^2 + \left(\frac{a}{2}\right)^2 - aL\cos(180^\circ - (\delta_2 - \lambda_2) + 30^\circ + \rho)$$
$$-1 - \left(\frac{a}{2}\right)^2 + a\cos(180^\circ - (\delta_1 - \lambda_1) - 30^\circ). \quad (5.137)$$

From the geometry, we have, for $i = 1, 2$,

$$\delta_{i+1} + \alpha_{i+1} + \theta_i + \delta_i = 180^\circ, \quad (5.138)$$

and

$$\delta_{i+1}+\alpha_{i+1}\geqslant 90^\circ-\frac{\theta_i}{2}. \tag{5.139}$$

Hence, using (5.25), and maximizing with respect to $\theta_1$, we get

$$\begin{aligned} d^2-R_1^2 &\leqslant a2(1.017)(\tfrac{3}{16})^2+a(1+(1.017)(\tfrac{3}{16})^2) \\ &\quad \times\sin(2\theta_0+30^\circ)+a\sin(\theta_0+30^\circ)\leqslant 1.98a. \end{aligned} \tag{5.140}$$

2. $\cos(\delta_2-\lambda_2+\alpha_2)<-\frac{3}{16}$:

This implies in particular that $\delta_2-\lambda_2>90^\circ$. In this case the worst estimate on $d^2-R_1^2$ is when $X$ coincides with $w_0$, as in Case 2, and the estimate is given by

$$\begin{aligned} d^2-R_1^2 &\leqslant L^2+\left(\frac{a}{2}\right)^2-aL\cos(\delta_2-\lambda_2+\rho+30^\circ) \\ &\quad -1-\left(\frac{a}{2}\right)^2+a\cos(180^\circ-(\delta_1-\lambda_1)-30^\circ), \end{aligned} \tag{5.141}$$

with $\delta_2$, $\rho$, $\lambda_2$, satisfying

$$\begin{aligned} &\rho\leqslant\alpha_2, \\ &\delta_2\leqslant 90^\circ, \\ &\lambda_2\leqslant\theta_0. \end{aligned} \tag{5.142}$$

Hence the estimate on $d^2-R_1^2$ is less than that in (5.140).

3. $|\cos(\delta_2-\lambda_2+\alpha_2)|\leqslant\frac{3}{16}$:

This implies that $|\delta_2-\lambda_2+\alpha_2-90^\circ|\leqslant\bar{\theta}$. If $R_1\geqslant|w_1-z_1'|$, then, by the law of cosines, (5.3), (5.107), and noting that $\widehat{Xz_2w_2}\leqslant\theta_0$, we get

$$\begin{aligned} d^2-R_1^2 &\leqslant L^2+\left(\frac{a}{2}\right)^2-aL\cos(90^\circ+\bar{\theta}+\theta_0+30^\circ) \\ &\quad -(1-f_1)^2-\left(\frac{a}{2}\right)^2+a(1-f_1)\cos(90^\circ-30^\circ-\bar{\theta}) \\ &\leqslant 2(1.017)\left(\frac{3}{16}\right)a-\frac{2f_1}{(1.05)}+2f_1-f_1^2 \\ &\quad +a\left\{\left(1+(1.017)\left(\frac{3}{16}\right)^2\right)\sin(\theta_0+\bar{\theta}+30^\circ)+(1-f_1)\sin(30^\circ+\bar{\theta})\right\} \\ &\leqslant 1.5a. \end{aligned} \tag{5.143}$$

If $R_1 < |w_1 - z_2'|$, then $d$ is the same as in (5.141), whereas $R_1$ has two possibilities. Let $z'$ be the point in $S_1 \cap B(z_1, a/2)$, closest to $w_1$. We have, either

(a) $z' \in S_2 \cap B(z_1, a/2)$:

then, by the law of cosines, we have

$$R_1^2 \geqslant 1 + \left(\frac{a}{2}\right)^2 - a\cos(90° - \bar{\theta} - 30°). \tag{5.144}$$

Or,

(b) $z' \in S_1 \cap (B(z_1, a/2) \backslash B(z_2, a/2))$:

then

$$R_1^2 \geqslant 1 + \left(\frac{a}{2}\right)^2 - a\cos(90° - 30° - \theta_0). \tag{5.145}$$

Thus, the estimate on $d^2 - R_1^2$ is in fact smaller than that in (5.140), and we are done.

We have thus covered all the possibilities for the positions and orientations of the strips $S_1, S_2$. This concludes the proof of the proposition. ∎

The proof of Theorem 2 is now complete.

## 6. REMARKS AND EXTENSIONS

### 6.1. *Sharpness of the Method*

As mentioned in Remark 3, Theorem 2 can be slightly sharpened. In particular, Proposition 13 can be improved upon, and that would allow further improvement. It is interesting to note however that, although (5.1) facilitated our method, it introduced unavoidable difficulties in combining the geometry of circles and strips. On the other hand, the results are rather reassuring that the conjecture should be true, and they shed some light on the underlying issues. Furthermore, as we will explain in Section 6.4, modifications of this method may open the way to further progress.

### 6.2. *Higher Dimensional Settings*

Although we chose to work exclusively in the plane, due to the involved nature of the geometry, the method is in fact dimension independent since it is built upon the Pythagorean Theorem (or the law of cosines). Furthermore, the only tools we really needed were the estimates on the diameters of sets like $G_i$ and radii such as $R_i$ (see (5.60)). These estimates are worst when the 4 points $z_1, z_2, w_1, w_2$ (see Lemma 9) lie in a 2-plane. However, the details of the geometry become more cumbersome when we must study balls and tubes together in higher dimensions. Although we claim that the results are dimension free, we will state a weaker theorem in the case of a Hilbert space which can be verified using similar, but less delicate, estimates, as in the proof above, while we hope to introduce more natural methods based on [Far1] in future papers.

Suppose $H$ is a Hilbert space, and $E \subset H$. We start with

DEFINITION 4. For $\varepsilon > 0$, $r > 0$, and $x \in H$, let $\gamma_E^1(x, r, \varepsilon)$ be the smallest number such that there is a tube $T$ of diameter $D$, so that $D/r \leqslant \gamma_E^1(x, r, \varepsilon)$, and $\mathscr{H}^1((E \cap B(x, r)) \backslash T) \leqslant \varepsilon r$.

We also define $\gamma_E^{*\,1}(x, \varepsilon)$, $\gamma_E^{*\,1}(x)$ according to (2.2) and (2.3). We now have

THEOREM 34. *Suppose* $E \subset H$ *is a totally unrectifiable* 1-*set, and that* $\gamma_E^{*\,1}(x) \leqslant \frac{1}{8}$, *for almost every* $x \in E$. *Then* $\Theta_*^1(E, x) \leqslant \frac{1}{2}$, *for almost every* $x \in E$.

It was pointed out to the author by David Preiss that even in some Banach spaces, our results still hold; namely they hold for those with a sufficiently smooth norm we can run a very similar argument.

### 6.3. *An Opposite Extreme Case*

We now would like to briefly comment on the naturality of the condition of our theorems. It is interesting to observe that if we can find a single point $x \in \mathbf{R}^2$, such that for some $r > 0$, we have

$$\beta_F(x, r) \geqslant 2 - \delta, \tag{6.1}$$

where $0 \leqslant \delta \leqslant \frac{1}{8}$ (say), then we can find finite sequences of points $\{z_i\}_i$, and radii$\{r_i\}_i$, satisfying the same conditions as $x$, $r$, and so that

$$B(z_i, r_i) \cap B(z_j, r_j) = \phi, \tag{6.2}$$

whenever $i \neq j$, and,

$$\sum_i 2r_i \geqslant 2\, diam\left(\left(\bigcup_i B(z_i, r_i)\right) \cap F\right), \tag{6.3}$$

which, when combined with (4.4), contradicts (4.5). We leave the details as an exercise to the reader (the points $z_i$ lie close to the boundary of $B(x, r)$ and form a "loop"). It is interesting to note that the argument is even easier the higher the dimension. Thus the condition imposed in Theorem 1 (or 2) is in fact a natural opposite of this theme and it seems that understanding what happens as we relax the condition of the theorem (i.e., push the bound on $\gamma_E^{*1}(x)$ allowed by our theorem), gives nontrivial insight into the problem.

### 6.4. *The Motivation behind the Method*

As mentioned in Section 6.1, our hypothesis, which led to (5.1), allowed us to get some positive results but introduced some complications. Furthermore, the method resists significant extensions (e.g. pushing the bound on $\gamma_E^{*1}(x)$ beyond $\frac{1}{2}$ for instance). In particular, we used (5.1) at many levels. First, we used it to obtain the big strip $S$ which contains $B(w, s) \cap F$, we then used it at many other points, like $w_1, w_2, z_1, z_2$, and at many scales, which range between $\varepsilon s$, and $s$. It would be much better to obtain a method which only uses the strip $S$, for instance, and no other conditions, other than the general setup of the problem as in Section 4. Although this would not be a complete solution of the problem, it would however allow us to remove the flatness requirement on the rest of the set, since, outside of $B(w, s)$, the set is allowed to be anything. Better yet, we would only be imposing a condition on that scale only (for a given pair $w_1, w_2$). Furthermore, this would start to impose strong geometric requirements on the set $F$, and we can eliminate certain geometric configurations. A method with this approach was obtained in [Far1] but not fully utilized. In a forthcoming paper we hope to develop such a method.

## ACKNOWLEDGMENT

I thank Peter Jones for his challenge, continuous encouragement, and very helpful advice. It was a question of his that initiated the investigations in [Far2] and consequently this work. I am also grateful to Guy David's very helpful remarks on [Far1] which, in turn, helped improve the present work. Special thanks also to the referee for very helpful comments and suggestions. Finally, I am indebted to David Preiss for very enjoyable and stimulating conversations regarding this problem.

## REFERENCES

[Be1] A. S. Besicovitch, On the fundamental geometrical properties of linearly meamsurable plane sets of points, *Math. Ann.* **98** (1928), 422–464.

[Be2] A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points II, *Math. Ann.* **115** (1938), 296–329.

[DS] G. David and S. Semmes, "Analysis of, and on, Uniformly Rectifiable Sets," Surveys and Monographs 38, Amer. Math. Soc., 1993.

[Fal] K. J. Falconer, "Geometry of Fractal Sets," Cambridge University Press, Cambridge, UK, 1985.

[Far1] H. M. Farag, "Some Affirmative Results towards the Besicovitch $\frac{1}{2}$-Conjecture," Thesis, Yale, 1997.

[Far2] H. M. Farag, On the $\frac{1}{2}$-problem of Besicovitch, preprint.

[Fe] H. Federer, "Geometric Measure Theory," Springer, Berlin, 1969.

[Jo] P. W. Jones, Rectifiable sets and the traveling salesman problem, *Invent. Math.* **102** (1990), 1–15.

[Ki] B. Kirchheim, thesis, Prague, 1988.

[Ma] P. Mattila, "Geometry of Sets and Measures in Euclidean Spaces," Cambridge University Press, Cambridge, UK, 1995.

[Mo] E. F. Moore, Density ratios and $(\phi, 1)$ rectifiability in $n$-space, *Trans. Amer. Math. Soc.* **69** (1950), 324–334.

[MR] A. P. Morse and J. F. Randolph, The $\phi$ rectifiable subsets of the plane, *Trans. Amer. Math. Soc.* **55** (1944), 236–305.

[Pr] D. Preiss, Geometry of measures in $R^n$: distribution, rectifiability, and densities, *Ann. of Math.* **125** (1987), 537–643.

[PT] D. Preiss and J. Tišer, On Besicovitch's $\frac{1}{2}$-problem, *J. London Math. Soc. (2)* **45** (1992), 279–287.