Journal of Computational and Applied Mathematics 229 (2009) 302–312

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Global attractivity and permanence of a SVEIR epidemic model with pulse vaccination and time delay

Huiming Wei [a], Yu Jiang [b], Xinyu Song [b], G.H. Su [a,*], S.Z. Qiu [a]

[a] State Key Laboratory of Multiphase Flow in Power Engineering, Xi'an Jiaotong University, Xi'an 710049, China

[b] Department of Mathematics, XinYang Normal University, Xinyang 464000, Henan, China

ARTICLE INFO

Article history:
Received 18 June 2008
Received in revised form 19 October 2008

MSC:
34D23
92D30

Keywords:
Epidemic model
Time delay
Pulse vaccination
Permanence
Globally attractive

ABSTRACT

In this study, we propose a new SVEIR epidemic disease model with time delay, and analyze the dynamic behavior of the model under pulse vaccination. Pulse vaccination is an effective strategy for the elimination of infectious disease. Using the discrete dynamical system determined by the stroboscopic map, we obtain an 'infection-free' periodic solution. We also show that the 'infection-free' periodic solution is globally attractive when some parameters of the model under appropriate conditions. The permanence of the model is investigated analytically. Our results indicate that a large vaccination rate or a short pulse of vaccination or a long latent period is a sufficient condition for the extinction of the disease.

1. Introduction

Infectious diseases have tremendous influence on human life. Controlling infectious disease is a very important issue. The epidemic models based on those assumptions in [1,10,12] are customarily called SIR (susceptible, infectious, recovered) and SVIR (susceptible, vaccinees, infectious, recovered) models. Li, Smith and Wang [15] assume that a susceptible individual first goes through a latent period after infection before becoming infectious.

In recent years, pulse vaccination, the repeated application of vaccine over a defined age range is gaining prominence as a strategy for the elimination of childhood viral infectious such as measles, hepatitis, parotitis, small pox and phthisis. To finish a vaccination process, usually there are different schedules for different diseases and vaccines. But for each schedule, some doses should be taken by vaccinees several times and there must be some fixed time interval between two doses. For example, in their experiment, Gabbuti et al. in [9] used three doses (20 μg/dose) of recombinant hepatitis B Vaccine (Engwerix B, Smith Klime Beecharm Biological, Rixensart, Belgium) in [2,3,6,7,13,24] given at 0, 1 and 6 months for vaccination against hepatitis B. One month after the third dose of vaccine, they found that 99.8% of vaccines gained anti-HBS antibody. Eleven years after vaccination, 91.2% of vaccines examined still had a protective level of anti-HBS. Generally, the consideration of the latent period and the immune period gives rise to models with the incorporation of delays. Then death during a latent period and temporary immunity period should be considered, which is called the phenomena of 'time delay', so time delay has important biologic meaning in epidemic models. Therefore, in the present paper, we propose a new delay SVEIR epidemic model with horizontal transmission, and study their dynamic behavior under pulse vaccination.

The organization of this paper is as follows: In the next section, SVEIR epidemic model and preliminary theory are introduced. The existence and global behavior of an 'infection-free' periodic solution are analyzed in Section 3. The

* Corresponding author.
E-mail addresses: ghsu@mail.xitu.edu.cn (G.H. Su), yujiang686@163.com (Y. Jiang).

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.10.046

permanence of the disease is discussed in Section 4. We show the effect of pulse vaccination rate, period of pulsing, latent period of the disease on the dynamical behavior of the model by numerical analysis in Section 5. Finally, we will give the main conclusions in Section 6.

2. SVEIR epidemic model and preliminary information

To eliminate childhood viral infections such as measles and polio, pulse vaccination, the repeated application of vaccine is usually done for children over a defined age range. Let $T > 0$ be the time between two consecutive pulse vaccinations and $0 \le \theta \le 1$ be the fraction of susceptible subjects to whom the vaccine is inoculated. We can write the following system:

$$\begin{cases} \left.\begin{aligned} &\dot{S}(t) = \mu - \mu S(t) - \beta S(t) I(t), \\ &\dot{V}(t) = -\beta_1 V(t) I(t) - \gamma_1 V(t) - \mu V(t), \\ &\dot{E}(t) = \beta S(t) I(t) + \beta_1 V(t) I(t) - \beta e^{-\mu\tau} S(t-\tau) I(t-\tau) - \beta_1 e^{-\mu\tau} V(t-\tau) I(t-\tau) - \mu E(t), \\ &\dot{I}(t) = \beta e^{-\mu\tau} S(t-\tau) I(t-\tau) + \beta_1 e^{-\mu\tau} V(t-\tau) I(t-\tau) - \gamma I(t) - \mu I(t) - \alpha I(t), \\ &\dot{R}(t) = \gamma_1 V(t) + \gamma I(t) - \mu R(t) \end{aligned}\right\}, \quad t \ne nT. \\ \left.\begin{aligned} &S(t^+) = (1-\theta) S(t), \\ &V(t^+) = V(t) + \theta S(t), \\ &E(t^+) = E(t), \\ &I(t^+) = I(t), \\ &R(t^+) = R(t) \end{aligned}\right\}, \quad t = nT, n \in N, \end{cases} \tag{2.1}$$

where S, E, I and R denote the susceptible, exposed, infectious and recovered individuals, respectively. A new group V is divided from S and denotes the density of vaccines who have begin the vaccination process.

Let $\mathcal{C} = C([-\tau, 0]; \mathbb{R}_+)$, the space of continuous functions (or integrable functions) from the interval $[-\tau, 0]$ to the non-negative reals. Then the initial condition for system (2.1) is $S(0), V(0), I(0) \in \mathcal{C}$ and $E(0), R(0) \in \mathbb{R}_+$, where $S(0) = S(\theta), V(0) = V(\theta), I(0) = I(\theta), -\tau \le \theta \le 0$.

Motivated by [4,16,20,23,25] and based on (2.1), we assume that:

(i) Death rate for disease, natural death rate and born rate are α, μ and b, respectively, where $b = \mu$.

(ii) The total population size varies, and suppose that the influx or recruitment of the susceptible and the exposed is constant μ (including newborns and immigration of susceptibles).

(iii) Let β be the transmission rate of disease when susceptible individuals contact with infected individuals.

(iv) The recovery rate of infected individual is γ. The recovered individual are assumed to have immunity (so called natural immunity) against the disease.

(v) Let γ_1 be the average rate (and hence $\frac{1}{\gamma_1}$ is the average time) for vaccines to obtain immunity and move into recovered population.

(vi) The vaccines contact with infected individuals before obtaining immunity has the possibility of infection with a disease transmission rate $\beta_1 (\beta_1 < \beta)$.

(vii) The time delay is introduced in the system describing the dynamics of the disease. At time t only susceptible individuals and the vaccines that have contacted with infected individuals τ time units ago, that is at time $t - \tau$, become infectious, provided that they have survived the incubation period of τ units, given that they were alive at time $t - \tau$ when they contact with infected individuals. Thus the incidence of newly infected individuals is given by the mass action term $\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)$ and $\beta_1 e^{-\mu\tau} V(t-\tau) I(t-\tau)$.

Since the equations for E and R are independent of other equations. The dynamics of (2.1) are determined by the following system:

$$\begin{cases} \left.\begin{aligned} &\dot{S}(t) = \mu - \mu S(t) - \beta S(t) I(t), \\ &\dot{V}(t) = -\beta_1 V(t) I(t) - \gamma_1 V(t) - \mu V(t), \\ &\dot{I}(t) = \beta e^{-\mu\tau} S(t-\tau) I(t-\tau) + \beta_1 e^{-\mu\tau} V(t-\tau) I(t-\tau) - (\gamma + \mu + \alpha) I(t) \end{aligned}\right\}, \quad t \ne nT,\ n \in N, \\ \left.\begin{aligned} &S(t^+) = (1-\theta) S(t), \\ &V(t^+) = V(t) + \theta S(t), \\ &I(t^+) = I(t) \end{aligned}\right\}, \quad t = nT,\ n \in N. \end{cases} \tag{2.2}$$

The initial condition of (2.2) is given as

$$S(\theta) = \phi_1(\theta), I(\theta) = \phi_2(\theta), V(\theta) = \phi_3(\theta), \quad -\tau \le \theta \le 0, \tag{2.3}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in \mathcal{PC}$ and $\mathcal{PC}$ is the space of all piecewise functions $\phi : [-\tau, 0] \to \mathbb{R}^3_+$ with points of discontinuity at $-nT (n \in N)$ of the first kind and which are continuous from the left, i.e., $\phi(-nT^-) = \phi(-nT)$, where

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_i \ge 0, i = 1, 2, 3\}.$$

We designate the norm of an element ϕ in $\mathcal{PC}$ by

$$\|\phi\| = \sup_{-\tau \le \theta \le 0} \{|\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|\}.$$

By biological meaning, we further assume that $\phi_i(0) > 0$ for $i = 1, 2, 3$. The meaningful domain of system (2.2) is

$$\Omega = \left\{(S, V, I) \in \mathbb{R}^3_+ : S + V + I \leq 1\right\}$$

and it is easy to prove that Ω is a positive invariant set.

The solution of system (2.2) is a piecewise continuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}^3_+$, $\Phi(t)$ is continuous on $(nT, (n+1)T]$, $n \in N$, and $\Phi(nT^+) = \lim_{t \to nT^+} \Phi(t)$ exists. In fact, the right hand side of system (2.2) can ensure the existence and uniqueness of solution of system (2.2).

Lemma 2.1. *Suppose $X(t) = (S(t), V(t), E(t), I(t), R(t))$ is any solution of (2.1) with initial conditions (2.3), then $S(t) \leq 1, V(t) \leq 1, E(t) \leq 1, I(t) \leq 1, R(t) \leq 1$ for all t large enough.*

Proof. Let $N(t) = S(t) + V(t) + E(t) + I(t) + R(t)$ be the total size of the population of system (2.1). Since $N(t^+) = N(t^-) = N(t)$ for all $t \geq 0$, then $N(t)$ is continuous on $t \in [0, +\infty)$. Calculating the derivative of $N(t)$ follows from the solution for system (2.1), we have that

$$\dot{N(t)} = \mu - \mu N(t) - \alpha I(t) \leq \mu - \mu N(t)$$

which implies that $\lim_{t \to \infty} \sup N(t) \leq 1$. So $N(t)$ is uniformly ultimately bounded. Hence, by the definition of $N(t)$, we get that there exists positive integer n_1 such that $S(t) \leq 1, V(t) \leq 1, E(t) \leq 1, I(t) \leq 1, R(t) \leq 1$ for all $t \geq n_1 T$. The proof is completed. □

Lemma 2.2. *Let us consider the following impulsive differential equations*

$$\begin{cases} \dot{u}(t) = a - bu(t), & t \neq nT \\ u(t^+) = (1 - \theta)u(t), & t = nT, \end{cases} \tag{2.4}$$

where $a > 0$, $b > 0$, $0 < \theta < 1$. Then exists a unique positive periodic solution of system (2.4) $\tilde{u}_e(t) = \frac{a}{b} + (u^* - \frac{a}{b})\mathrm{e}^{-b(t-nT)}$, $nT < t \leq (n+1)T$, which is globally asymptotically stable, where $u^* = \frac{\frac{a}{b}(1-\theta)(1-\mathrm{e}^{-bT})}{1-(1-\theta)\mathrm{e}^{-bT}}$.

Proof. Integrating and solving the first equation of system (2.4) between pulses

$$u(t) = \frac{a}{b} + \left(u(nT) - \frac{a}{b}\right)\mathrm{e}^{-b(t-nT)}, \quad nT < t \leq (n+1)T,$$

where $u(nT)$ be the initial value at time nT, using the second equation of system (2.4), we deduce the stroboscopic map such that

$$u((n+1)T) = (1 - \theta)\left[\frac{a}{b} + \left(u(nT) - \frac{a}{b}\right)\mathrm{e}^{-bT}\right] = f(u(nT)), \tag{2.5}$$

where $f(u) = (1 - \theta)\left[\frac{a}{b} + (u - \frac{a}{b})\mathrm{e}^{-bT}\right]$. It is easy to know that system (2.5) has unique positive equilibrium $u^* = \frac{\frac{a}{b}(1-\theta)(1-\mathrm{e}^{-bT})}{1-(1-\theta)\mathrm{e}^{-bT}}$ which satisfies $u < f(u) < u^*$, if $0 < u < u^*$; $u^* < f(u) < u$ if $u > u^*$. From [5], we obtain that u^* is globally asymptotically stable for Eq. (2.5). It implies that the corresponding periodic solution of system (2.4)

$$\tilde{u}_e(t) = \frac{a}{b} + \left(u^* - \frac{a}{b}\right)\mathrm{e}^{-b(t-nT)}, \quad nT < t \leq (n+1)T$$

is globally asymptotically stable for system (2.4). Lemma 2.2 is proved. □

Lemma 2.3 ([14,26]). *Consider the following equation*

$$\dot{x}(t) = a_1 x(t - \tau) - a_2 x(t),$$

where $a_1, a_2, \tau > 0$; $x(t) > 0$, for $-\tau \leq t \leq 0$. We have :

(i) *If $a_1 < a_2$, then $\lim_{t \to \infty} x(t) = 0$;*

(ii) *If $a_1 > a_2$, then $\lim_{t \to \infty} x(t) = \infty$.*

The proof is given in [14,26].

3. Global attractivity of 'infection-free' periodic solution

In this section, we begin the analysis (2.2) by first demonstrating the existence of an 'infection-free' periodic solution, in which infectious individuals are entirely absent from the population permanently, i.e., $I(t) = 0$ for all $t \geq -\tau$. Under this condition, consider the following system:

$$\begin{cases} \left.\begin{aligned} \dot{S}(t) &= \mu - \mu S(t), \\ \dot{V}(t) &= -\gamma_1 V(t) - \mu V(t) \end{aligned}\right\}, & t \neq nT, n \in N \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ V(t^+) &= V(t) + \theta S(t) \end{aligned}\right\}, & t = nT, n \in N. \end{cases} \tag{3.1}$$

By Lemma 2.2., it is easy to obtain the periodic solution of system (3.1):

$$\begin{aligned} &\tilde{S}(t) = 1 + (S_0 - 1)\mathrm{e}^{-\mu(t-nT)}, \\ &\tilde{V}(t) = V_0 \mathrm{e}^{-(\mu+\gamma_1)(t-nT)} \end{aligned} \qquad t \in (nT, (n+1)T].$$

where

$$\begin{aligned} S_0 &= \frac{(1-\theta)(1-\mathrm{e}^{-\mu T})}{1-(1-\theta)\mathrm{e}^{-\mu T}}, \\ V_0 &= \frac{\theta[1+(S_0-1)\mathrm{e}^{-\mu T}]}{1-\mathrm{e}^{-(\mu+\gamma_1)T}} = \frac{\theta(1-\mathrm{e}^{-\mu T})}{(1-\mathrm{e}^{-(\mu+\gamma_1)T})[1-(1-\theta)\mathrm{e}^{-\mu T}]} \end{aligned}$$

which is globally asymptotically stable for system (3.1).

In the following theorem, we shall present the sufficient condition for the global attractivity of 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ of system (2.2).

Theorem 3.1. *If $R_1 < 1$, then 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ of system (2.2) is globally attractive, where* $R_1 = \frac{\mathrm{e}^{-\mu\tau}(1-\mathrm{e}^{-\mu T})}{(\gamma+\mu+\alpha)(1-(1-\theta)\mathrm{e}^{-\mu T})}\left(\beta + \frac{\beta_1\theta\mathrm{e}^{-(\mu+\gamma_1)T}}{1-\mathrm{e}^{-(\mu+\gamma_1)T}}\right)$.

Proof. Since $R_1 < 1$, we can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\frac{\mathrm{e}^{-\mu\tau}(1-\mathrm{e}^{-\mu T})}{1-(1-\theta)\mathrm{e}^{-\mu T}}\left(\beta + \frac{\beta_1\theta\mathrm{e}^{-(\mu+\gamma_1)T}}{1-\mathrm{e}^{-(\mu+\gamma_1)T}} + 2\varepsilon_1\right) < \gamma + \mu + \alpha. \tag{3.2}$$

From the first and second equations of system (2.2), we have $\dot{S}(t) < \mu - \mu S(t)$ and $\dot{V}(t) < -\gamma_1 V(t) - \mu V(t)$, then we consider the following comparison system with pulse

$$\begin{cases} \left.\begin{aligned} \dot{x}(t) &= \mu - \mu x(t), \\ \dot{y}(t) &= -\gamma_1 y(t) - \mu y(t) \end{aligned}\right\}, & t \neq nT, n \in N \\ \left.\begin{aligned} x(t^+) &= (1-\theta)x(t), \\ y(t^+) &= y(t) + \theta x(t) \end{aligned}\right\}, & t = nT, n \in N. \end{cases} \tag{3.3}$$

From Lemma 2.2, we obtain the periodic solution of system (3.3) i.e.,

$$\begin{aligned} \tilde{x}(t) &= 1 - \frac{\theta\mathrm{e}^{-\mu(t-nT)}}{1-(1-\theta)\mathrm{e}^{-\mu T}}, \\ \tilde{y}(t) &= \frac{\theta(1-\mathrm{e}^{-\mu T})\mathrm{e}^{-(\mu+\gamma_1)(t-nT)}}{(1-\mathrm{e}^{-(\mu+\gamma_1)T})[1-(1-\theta)\mathrm{e}^{-\mu T}]} \end{aligned} \qquad t \in (nT, (n+1)T].$$

By the comparison theorem [17, Theorem 3.1.1], there exists $m_1 \in N$ such that

$$\begin{aligned} &S(t) \leq x(t) < \tilde{x}(t) + \varepsilon_1, \\ &V(t) \leq y(t) < \tilde{y}(t) + \varepsilon_1, \end{aligned}$$

that is

$$\begin{aligned} S(t) &< \tilde{x}(t) + \varepsilon_1 \leq \frac{1-\mathrm{e}^{-\mu T}}{1-(1-\theta)\mathrm{e}^{-\mu T}} + \varepsilon_1 = S_\triangle, \\ V(t) &< \tilde{y}(t) + \varepsilon_1 \leq \frac{\theta(1-\mathrm{e}^{-\mu T})\mathrm{e}^{-(\mu+\gamma_1)T}}{(1-\mathrm{e}^{-(\mu+\gamma_1)T})[1-(1-\theta)\mathrm{e}^{-\mu T}]} + \varepsilon_1 = V_\triangle \end{aligned} \qquad t \in (nT, (n+1)T],\ n > m_1,\ n \in N. \tag{3.4}$$

Further, from the third equation of system (2.2), we have

$$\dot{I}(t) \leq (\beta\mathrm{e}^{-\mu\tau}S_\triangle + \beta_1\mathrm{e}^{-\mu\tau}V_\triangle)I(t-\tau) - (\gamma+\mu+\alpha)I(t)$$

Table 1
Critical values of some parameters of system (2.2) ($R_1 < 1$ must be satisfied).

	The conditions for global attractivity of $(\tilde{S}(t), \tilde{V}(t), 0)$
$\theta > \theta^*$	$\theta^* = \frac{(1-e^{-(\mu+\gamma_1)T})(1-e^{-\mu T})(\beta e^{-\mu\tau}-(\gamma+\mu+\alpha))}{(\gamma+\mu+\alpha)(1-e^{-(\mu+\gamma_1)T})e^{-\mu T}-\beta_1 e^{-(\mu+\gamma_1)T}e^{-\mu\tau}(1-e^{-\mu T})}$
$\tau > \tau^*$	$\tau^* = \frac{1}{\mu}\ln\frac{1-e^{-\mu T}}{(\gamma+\mu+\alpha)(1-(1-\theta)e^{-\mu T})}\left(\beta+\frac{\beta_1\theta e^{-(\mu+\gamma_1)T}}{1-e^{-(\mu+\gamma_1)T}}\right)$

then we consider the following comparison equation:

$$\dot{y}(t) = (\beta e^{-\mu\tau}S_\triangle + \beta_1 e^{-\mu\tau}V_\triangle)y(t-\tau) - (\gamma+\mu+\alpha)y(t).$$

From (3.2) and (3.4), by the Lemma 2.3, we have $\lim_{t\to\infty} y(t) = 0$. Therefore, $\lim_{t\to\infty} I(t) = 0$, i.e., for any sufficiently small $\varepsilon_2 > 0$, there exists an integer $m_2 > m_1$, such that $I(t) < \varepsilon_2$ for all $t > m_2T$. From the first and second equations of system (2.2), we have

$$\begin{aligned} &\dot{S}(t) > \mu - \mu S(t) - \beta\varepsilon_2 S(t), \\ &\dot{V}(t) > -\beta_1\varepsilon_2 V(t) - \gamma_1 V(t) - \mu V(t), \end{aligned} \qquad \text{for } t > m_2T.$$

Then we consider the following comparison system with pulse:

$$\begin{aligned} &\left.\begin{aligned} &\dot{\varphi}(t) = \mu - \mu\varphi(t) - \beta\varepsilon_2\varphi(t), \\ &\dot{g}(t) = -\beta_1\varepsilon_2 g(t) - \gamma_1 g(t) - \mu g(t) \end{aligned}\right\}, \quad t \neq nT, n \in N. \\ &\left.\begin{aligned} &\varphi(t^+) = (1-\theta)\varphi(t), \\ &g(t^+) = g(t) + \theta\varphi(t) \end{aligned}\right\}, \quad t = nT,\ n \in N. \end{aligned} \tag{3.5}$$

From Lemma 2.2, we obtain the periodic solution of system (3.5) i.e.,

$$\begin{aligned} &\tilde{\varphi}(t) = \frac{\mu}{\beta\varepsilon_2+\mu} - \frac{\frac{\theta\mu}{\beta\varepsilon_2+\mu}e^{-(\beta\varepsilon_2+\mu)(t-nT)}}{1-(1-\theta)e^{-(\beta\varepsilon_2+\mu)t}}, \\ &\tilde{g}(t) = \frac{\theta}{1-e^{-(\beta\varepsilon_2+\gamma_1+\mu)t}}\left[\frac{\mu}{\beta\varepsilon_2+\mu} - \frac{\frac{\theta\mu}{\beta\varepsilon_2+\mu}e^{-(\beta\varepsilon_2+\mu)(t-nT)}}{1-(1-\theta)e^{-(\beta\varepsilon_2+\mu)t}}\right], \end{aligned} \qquad t \in (nT, (n+1)T].$$

By the comparison theorem [17, Theorem 3.1.1], there exists an integer $m_3 > m_2$ such that

$$\begin{aligned} &S(t) \geq \varphi(t) > \tilde{\varphi}(t) - \varepsilon_2, \\ &V(t) \geq g(t) > \tilde{g}(t) - \varepsilon_2. \end{aligned} \qquad t \in (nT, (n+1)T],\ n > m_3,\ n \in N. \tag{3.6}$$

Since ε_1 and ε_2 are sufficiently small, from (3.4) and (3.6), we know that

$$\begin{aligned} &\tilde{S}(t) = \frac{1-e^{-\mu T}}{1-(1-\theta)e^{-\mu T}}, \\ &\tilde{V}(t) = \frac{\theta(1-e^{-\mu T})e^{-(\mu+\gamma_1)t}}{(1-e^{-(\mu+\gamma_1)T})[1-(1-\theta)e^{-\mu T}]}, \end{aligned} \qquad t \in (nT, (n+1)T]$$

is globally attractive. Hence, 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ of system (2.2) is globally attractive. The proof is completed. □

Corollary 3.1. *If $R_1 < 1$, then 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ is globally attractive, where $R_1 = \frac{\beta e^{-\mu\tau}}{\gamma+\mu+\alpha}$ as $\theta = 0$ or $T \to \infty$.*

Corollary 3.2. (i) *If $\beta + \frac{\beta_1\theta e^{-(\mu+\gamma_1)T}}{1-e^{-(\mu+\gamma_1)T}} \leq \gamma+\mu+\alpha$, then 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ is globally attractive.*

(ii) *If $\beta + \frac{\beta_1\theta e^{-(\mu+\gamma_1)T}}{1-e^{-(\mu+\gamma_1)T}} > \gamma+\mu+\alpha$ and $\theta > \theta^*$ or $\tau > \tau^*$, then 'infection-free' periodic solution $(\tilde{S}(t), \tilde{V}(t), 0)$ is globally attractive, where the critical values θ^* and τ^* are listed in Table 1 for system (2.2).*

4. Permanence

In this section, we say the disease is endemic if the infectious population persists above a certain threshold level for sufficiently large time. The endemic of the disease can be well captured and studied through the notion of uniform persistence and permanence.

Definition 4.1. System (2.2) is said to be uniformly persistent if there is an $\eta > 0$ (independent of the initial data) such that every solution $(S(t), V(t), I(t))$ with initial conditions (2.3) of system (2.2) satisfies

$$\liminf_{t\to\infty} S(t) \geq \eta, \qquad \liminf_{t\to\infty} V(t) \geq \eta, \qquad \liminf_{t\to\infty} I(t) \geq \eta.$$

Definition 4.2. System (2.2) is said to be permanent if there exists a compact region $\Omega_0 \in \operatorname{int}\Omega$ such that every solution of system (2.2) with initial data (2.3) will eventually enter and remain in region Ω_0.

Denote

$$R_2 = \frac{\beta e^{-\mu\tau}}{(\gamma+\mu+\alpha)}\left(1+\frac{\beta_1}{\beta}\frac{\theta e^{-(\beta_1+\mu+\gamma_1)T}}{1-e^{-(\beta_1+\mu+\gamma_1)T}}\right)\frac{(1-\theta)(1-e^{-\mu T})}{1-(1-\theta)e^{-\mu T}},$$
$$I^* = \frac{\mu}{\beta}(R_2-1).$$

Theorem 4.1. *Suppose $R_2 > 1$. Then there is a positive constant q such that each positive solution $(S(t), V(t), I(t))$ of system (2.2) satisfies $I(t) \geq q$, if t is large.*

Proof. Note that the third equation of (2.2) can be rewritten as

$$\begin{aligned}\dot{I}(t) &= \beta e^{-\mu\tau}S(t)I(t) + \beta_1 e^{-\mu\tau}V(t)I(t) - (\gamma+\mu+\alpha)I(t)\\ &\quad - \beta e^{-\mu\tau}(S(t)I(t) - S(t-\tau)I(t-\tau)) - \beta_1 e^{-\mu\tau}(V(t)I(t) - V(t-\tau)I(t-\tau))\\ &= \beta e^{-\mu\tau}S(t)I(t) + \beta_1 e^{-\mu\tau}V(t)I(t) - (\gamma+\mu+\alpha)I(t)\\ &\quad - \beta e^{-\mu\tau}\frac{d}{dt}\int_{t-\tau}^{t} S(\sigma)I(\sigma)d\sigma - \beta_1 e^{-\mu\tau}\frac{d}{dt}\int_{t-\tau}^{t} V(\sigma)I(\sigma)d\sigma. \end{aligned} \tag{4.1}$$

Let us consider any positive solution $(S(t), V(t), I(t))$ of system (2.2). According to this solution, we define

$$F(t) = I(t) + \beta e^{-\mu\tau}\int_{t-\tau}^{t} S(\sigma)I(\sigma)d\sigma + \beta_1 e^{-\mu\tau}\int_{t-\tau}^{t} V(\sigma)I(\sigma)d\sigma.$$

In view of (4.1), we calculate the derivative of F along the solution of (2.2)

$$\begin{aligned}\dot{F}(t) &= (\beta e^{-\mu\tau}S(t) + \beta_1 e^{-\mu\tau}V(t))I(t) - (\gamma+\mu+\alpha)I(t)\\ &= (\gamma+\mu+\alpha)I(t)\left(\frac{\beta e^{-\mu\tau}}{\gamma+\mu+\alpha}S(t) + \frac{\beta_1 e^{-\mu\tau}}{\gamma+\mu+\alpha}V(t) - 1\right).\end{aligned} \tag{4.2}$$

Since $R_2 > 1$, we easily see that $0 < I^* \leq 1$, and there exists sufficiently small $\varepsilon_3 > 0$ such that

$$\begin{aligned}&\frac{\beta e^{-\mu\tau}}{\gamma+\mu+\alpha}\left(\frac{\mu}{\mu+\beta I^*}\frac{(1-\theta)(1-e^{-(\mu+\beta I^*)T})}{1-(1-\theta)e^{-(\mu+\beta I^*)T}} - \varepsilon_3\right)\\ &+ \frac{\beta_1 e^{-\mu\tau}}{\gamma+\mu+\alpha}\left(\frac{\theta e^{-(\beta_1 I^*+\gamma_1+\mu)T}}{1-e^{-(\beta_1 I^*+\gamma_1+\mu)T}}\frac{\mu}{\mu+\beta I^*}\frac{(1-\theta)(1-e^{-(\mu+\beta I^*)T})}{1-(1-\theta)e^{-(\mu+\beta I^*)T}} - \varepsilon_3\right) > 1.\end{aligned} \tag{4.3}$$

Suppose that there is a $t_0 > 0$ such that $I(t) < I^*$ for all $t \geq t_0$. It follows from the first and second equations of system (2.2), that for $t \geq t_0$,

$$\begin{aligned}\dot{S}(t) &> -\beta I^* S(t) + \mu - \mu S(t)\\ &= \mu - (\mu+\beta I^*)S(t)\\ \dot{V}(t) &> -(\beta_1 I^* + \gamma_1 + \mu)V(t).\end{aligned}$$

Consider the following comparison impulsive system for $t \geq t_0$,

$$\begin{cases}\left.\begin{aligned}\dot{u}(t) &= \mu - (\mu+\beta I^*)u(t),\\ \dot{w}(t) &= -(\beta_1 I^* + \gamma_1 + \mu)w(t)\end{aligned}\right\}, & t \neq nT,\ n \in N\\ \left.\begin{aligned}u(t^+) &= (1-\theta)u(t),\\ w(t^+) &= w(t) + \theta u(t)\end{aligned}\right\}, & t = nT,\ n \in N.\end{cases} \tag{4.4}$$

According to Lemma 2.2, we obtain that

$$\begin{cases} \tilde{u}_e(t) = \dfrac{\mu}{\mu+\beta I^*} + \left(u^* - \dfrac{\mu}{\mu+\beta I^*}\right) \mathrm{e}^{-(\mu+\beta I^*)(t-nT)}, \\ \tilde{w}_e(t) = \dfrac{\theta}{1-\mathrm{e}^{-(\beta_1 I^*+\gamma_1+\mu)T}} \dfrac{\mu}{\mu+\beta I^*} \dfrac{(1-\theta)(1-\mathrm{e}^{-(\mu+\beta I^*)T})}{1-(1-\theta)\mathrm{e}^{-(\mu+\beta I^*)T}} \mathrm{e}^{-(\beta_1 I^*+\gamma_1+\mu)(t-nT)} \end{cases} \quad t \in (nT, (n+1)T]$$

is the unique positive periodic solutions of (4.4), which is globally asymptotically stable, where

$$u^* = \frac{\mu}{\mu+\beta I^*} \frac{(1-\theta)(1-\mathrm{e}^{-(\mu+\beta I^*)T})}{1-(1-\theta)\mathrm{e}^{-(\mu+\beta I^*)T}}.$$

Denote

$$w^* = \tilde{w}_e(t)_{\min} = \frac{\theta \mathrm{e}^{-(\beta_1 I^*+\gamma_1+\mu)T}}{1-\mathrm{e}^{-(\beta_1 I^*+\gamma_1+\mu)T}} \frac{\mu}{\mu+\beta I^*} \frac{(1-\theta)(1-\mathrm{e}^{-(\mu+\beta I^*)T})}{1-(1-\theta)\mathrm{e}^{-(\mu+\beta I^*)T}}.$$

Let $(S(t), V(t), I(t))$ be the solution of system (2.2) with initial values (2.3) and $S(0^+) = S_0,\ S_0 > 0$, and $V(0^+) = V_0,\ V_0 > 0$, $u(t)$ and $w(t)$ be the solution of system (4.4) with initial value $u(0^+) = S_0$, $w(0^+) = V_0$. By comparison theorem in impulsive differential equation, we know that, there exists $t_1 (> t_0 + \tau)$ and sufficiently small ξ such that the following inequality holds true for $t \geq t_1$,

$$\begin{cases} S(t) > \tilde{u}_e(t) - \xi, \\ V(t) > \tilde{w}_e(t) - \xi. \end{cases}$$

Thus,

$$\begin{aligned} &S(t) > u^* - \xi = \delta, \\ &V(t) > w^* - \xi = \Lambda, \end{aligned} \quad \text{for } t \geq t_1. \tag{4.5}$$

From (4.3), we have $\frac{\beta \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\delta + \frac{\beta_1 \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\Lambda > 1$. By (4.2) and (4.5), we have

$$\dot{F}(t) > (\gamma+\mu+\alpha) I(t) \left(\frac{\beta \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\delta + \frac{\beta_1 \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\Lambda - 1 \right) \quad \text{for } t \geq t_1. \tag{4.6}$$

Set

$$I_L = \min_{t\in[t_1, t_1+\tau]} I(t)$$

we will show that $I(t) \geq I_L$ for all $t \geq t_1$. Suppose the contrary. Then there is a $T_0 \geq 0$ such that $I(t) \geq I_L$ for $t_1 \leq t \leq t_1 + \tau + T_0$, $I(t_1 + \tau + T_0) = I_L$ and $\dot{I}(t_1 + \tau + T_0) \leq 0$. However, the third equation of system (2.2) and (4.5) imply that

$$\begin{aligned} \dot{I}(t_1+\tau+T_0) &\geq (\beta \mathrm{e}^{-\mu\tau} S(t_1+T_0) + \beta_1 \mathrm{e}^{-\mu\tau} V(t_1+T_0)) - (\gamma+\mu+\alpha) I_L \\ &> (\gamma+\mu+\alpha) \frac{\mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha} (\beta\delta + \beta_1\Lambda - 1) I_L \\ &> 0. \end{aligned}$$

This is a contradiction. Thus, $I(t) \geq I_L$ for all $t \geq t_1$. As a consequence, (4.6) leads to

$$\dot{F}(t) > (\gamma+\mu+\alpha) \left(\frac{\beta \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\delta + \frac{\beta_1 \mathrm{e}^{-\mu\tau}}{\gamma+\mu+\alpha}\Lambda - 1 \right) I_L \quad \text{for } t \geq t_1.$$

Which implies that as $t \to \infty$, $F(t) \to \infty$. But by Lemma 2.1, we can obtain

$$\begin{aligned} F(t) &= I(t) + \beta \mathrm{e}^{-\mu\tau} \int_{t-\tau}^{t} S(\sigma) I(\sigma) \mathrm{d}\sigma + \beta_1 \mathrm{e}^{-\mu\tau} \int_{t-\tau}^{t} V(\sigma) I(\sigma) \mathrm{d}\sigma \\ &\leq 1 + \beta \mathrm{e}^{-\mu\tau} \int_{t-\tau}^{t} \mathrm{d}\sigma + \beta_1 \mathrm{e}^{-\mu\tau} \int_{t-\tau}^{t} \mathrm{d}\sigma \\ &= 1 + \tau\beta \mathrm{e}^{-\mu\tau} + \tau\beta_1 \mathrm{e}^{-\mu\tau}. \end{aligned}$$

This is a contradiction. Hence, for any $t_0 > 0$, it is impossible that $I(t) < I^*$ for all $t \geq t_0$.

Following, we are left to consider two cases. First, $I(t) \geq I^*$ for all large t. Second, $I(t)$ oscillates about I^* for all large t.

Define

$$L = \min\left\{\frac{I^*}{2}, \quad L_1\right\} \quad \text{and} \quad L_1 = I^* e^{-(\gamma+\mu+\alpha)\tau}.$$

We hope to show that $I(t) \geq L$ for all large t. The conclusion is evident in the first case. For the second case, Let $t^* > 0$ and $\gamma > 0$ satisfy

$$I(t^*) = I(t^* + \gamma) = I^*,$$

and

$$I(t) < I^* \quad \text{for } t^* < t < t^* + \gamma,$$

where t^* is sufficiently large such that

$$\begin{cases} S(t) > \delta, \\ V(t) > \Lambda \end{cases} \quad \text{for } t^* < t < t^* + \gamma.$$

$I(t)$ is uniformly continuous since the positive solutions of (2.2) are ultimately bounded and $I(t)$ is not effected by impulses. Hence, there is a $T(0 < T < \tau$, and T is independent of the choice of $t^*)$ such that $I(t) > \frac{I^*}{2}$ for $t^* \leq t < t^* + T$. If $\gamma \leq T$, there is nothing to prove. Let us consider the case where $T < \gamma \leq \tau$, since $\dot{I(t)} > -(\gamma + \mu + \alpha)I(t)$, and $I(t^*) = I^*$, it is obvious that $I(t) \geq L_1$ for $t^* < t < t^* + \gamma$. If $\gamma > \tau$, by the second equation of (2.2), we obtain $I(t) \geq L$ for $t \in [t^*, t^* + \tau]$. Then, proceeding exactly as the proof for above claim, we see that $I(t) \geq L$ for $t \in [t^* + \tau, t^* + \gamma]$. Since this kind of interval $[t^*, t^* + \gamma]$ is chosen in an arbitrary way (we only need t^* to be large), we conclude that $I(t) \geq L$ for all large t in the second case. In view of our above discussions, the choice of L is independent of the positive solution, and we have proved that any positive solution of (2.2) satisfies $I(t) \geq L$ for all large t. The proof of Theorem 4.1 is completed. □

Theorem 4.2. *System (2.2) is permanent provided $R_2 > 1$.*

Proof. Denote $(S(t), V(t), I(t))$ be any solution of system (2.2). From the first and second equations of system (2.2). We have

$$\begin{cases} \dot{S}(t) \geq \mu - (\beta + \mu)S(t), \\ \dot{V}(t) \geq -\beta_1 V(t) - \gamma_1 V(t) - \mu V(t). \end{cases}$$

By the similar arguments as those in the proof of Theorem 3.1, we have that

$$\lim_{t\to\infty} S(t) \geq M_1, \qquad \lim_{t\to\infty} V(t) \geq M_2 \tag{4.7}$$

where

$$M_1 = \frac{\mu}{\beta+\mu}\frac{(1-\theta)(1-e^{-(\beta+\mu)T})}{1-(1-\theta)e^{-(\beta+\mu)T}} - \varepsilon_4,$$
$$M_2 = \frac{\theta}{1-e^{-(\beta_1+\mu+\gamma_1)T}}\frac{\mu}{\beta+\mu}\frac{(1-\theta)(1-e^{-(\beta+\mu)T})}{1-(1-\theta)e^{-(\beta+\mu)T}} - \varepsilon_4 \qquad \text{for } (\varepsilon_4 \text{ is sufficiently small}).$$

We let $\Omega_0 = \{(S(t), V(t), I(t)) : M_1 \leq S(t), M_2 \leq V(t), L \leq I(t), S(t) + V(t) + I(t) \leq 1\}$. From Theorem 4.1 and inequality (4.7), we know that the set Ω_0 is global attractor in Ω, and of course, every solution of system (2.2) with initial conditions (2.3) will eventually enter and remain in region Ω_0. Therefore, system (2.2) is permanent. The proof of Theorem 4.2 is completed. □

5. Numerical analysis

In this section, we test the correctness of our conclusions by numerical analysis. We can clearly see they are in agreement with our conclusions. Let $\mu = 0.1$, $\beta = 0.5$, $\beta_1 = 0.05$, $\gamma_1 = 0.06$, $\gamma = 0.06$, $\alpha = 0.05$, $\theta = 0.8$, $\tau = 1$, $T = 4$, then $R_1 = 0.8934 < 1$. According to Theorem 3.1, we know the 'infection-free' periodic solution of system (2.2) is globally attractive for this case. Its epidemiological implication is that the infectious population vanishes, i.e., the disease dies out (see Fig. 1). If we let $\mu = 0.1$, $\beta = 0.7$, $\beta_1 = 0.5$, $\gamma_1 = 0.06$, $\gamma = 0.06$, $\alpha = 0.05$, $\theta = 0.4$, $\tau = 1$, $T = 4$, then $R_2 = 1.0199 > 1$. According to Theorem 4.1, the disease will be permanent and there is a positive constant q such that each positive solution $(S(t), V(t), I(t))$ of system (2.2) satisfies $I(t) \geq q$, if t is large (see Fig. 2). If we let $\mu = 0.1$, $\beta = 0.6$, $\beta_1 = 0.5$, $\gamma_1 = 0.06$, $\gamma = 0.06$, $\alpha = 0.05$, $\theta = 0.6$, $\tau = 1$, $T = 4$, then $R_1 = 1.8141 > 1$, $R_2 = 0.4837 < 1$. Computer observation shows that the disease is still permanent (see Fig. 3).

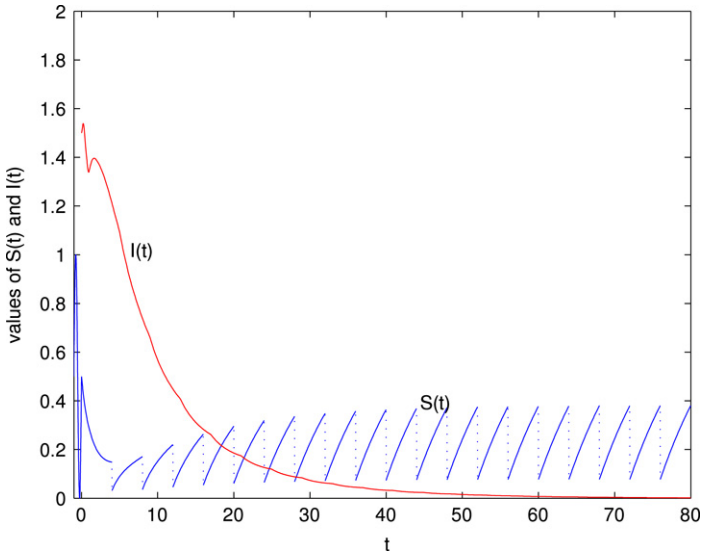

Fig. 1. This figure shows that movement paths of S and I as functions of time t. $R_1 = 0.8934 < 1$. The disease dies out.

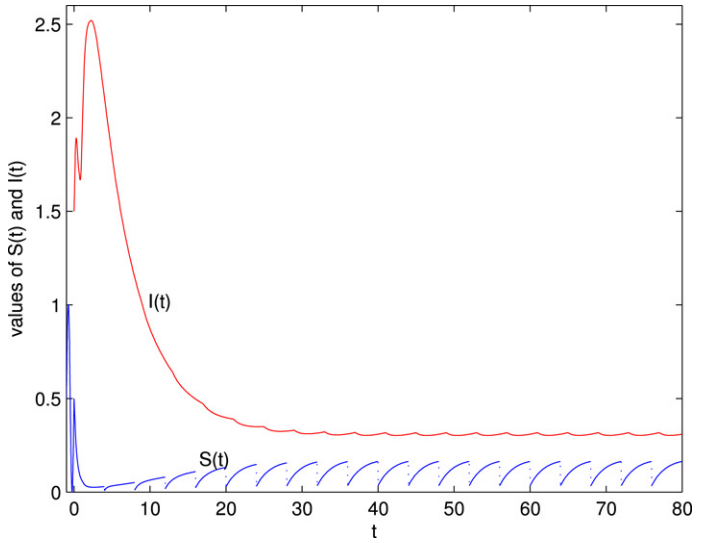

Fig. 2. This figure shows that movement paths of S and I as functions of time t. $R_2 = 1.1697 > 1$. The disease is permanent.

6. Conclusion

The strategy of pulse vaccination (PVS) consists of periodical repetitions of impulsive vaccinations in a population. Some theoretical considerations, practical advantages, and examples of the PVS are presented in [8,18,21,22]. For example, some successes against poliomyelitis and measles have been attributed to repeated PVS [19]. As indicated in [11], models have clearly shown the advantages of a mass campaign approach in rapidly achieving high measles population immunity and interrupting measles virus circulation.

In this study, we have studied the dynamical behavior of a delayed SVEIR epidemic model with pulse vaccination and time delay. We introduced two threshold values R_1 and R_2 (see Theorems 3.1 and 4.1) and further obtain: if $R_1 < 1$ then the disease will be extinct, if $R_2 > 1$ then the disease will be permanent which means that after some period of time the disease will become endemic. Corollary 3.2 show that $\theta > \theta^*$ or $\tau > \tau^*$ implies the disease will fade out. Our results indicate that a long latent period of the disease or a large pulse vaccination rate will lead to eradication of the disease. So PVS is effective.

In this work, we have discussed two cases: (1) $R_1 < 1$, (2) $R_2 > 1$. Obviously, $R_1 > R_2$. When $R_1 > 1$ and $R_2 < 1$, the dynamical behavior of model (2.2) has not been clear. The extinction of the disease and uniformly persistence has not been obtained. By numerical simulation, we see that the disease is uniformly persistent between R_1 and R_2 (see Fig. 3). Hence,

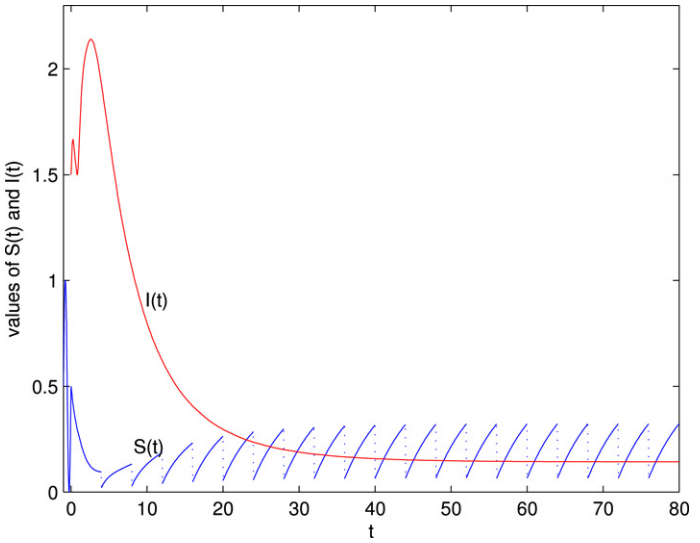

Fig. 3. This figure shows that movement paths of S and I as functions of time t. $R_1 = 1.479 > 1$ and $R_2 = 0.5419 < 1$. The disease is still permanent.

we conjecture that the system (2.2) is permanent when $R_1 > 1$, i.e., R_1 is the threshold value whether the disease will go to extinction or not. These works will be left as our future consideration.

In this article, we have proposed a SVEIR epidemic model with time delay and pulse vaccination. Using the stroboscopic maps and comparison theorem for impulsive ODE, we have also established the sufficient conditions for the global attractivity of the disease-free periodic solution and the permanence of the epidemic model. Our results indicate: (I) If $R_1 < 1$, then 'infection-free' periodic solution of system (2.2) is globally attractive. (II) System (2.2) is permanent provided $R_2 > 1$. We can see that a smaller pulse vaccination rate or a shorter latent period of the disease or a shorter immunity period of the recovered could cause global attractive 'infection-free' periodic solution to lose and cause epidemic disease to be permanent. We can also see that when the latent period of disease and the temporary immunity period of the recovered are very small, the very larger pulse vaccination rate is need in order to eradicate the epidemic disease.

Acknowledgments

This work is supported by program for the National Natural Science Foundation of China (10675096) and the New Century Excellent Talents in University (NCET-06-0837).

References

[1] M.E. Alexander, C. Bowman, S.M. Moghadas, R. Summors, A.B. Gumel, B.M. Sahai, A vaccination model for transmission dynamics of influenza, SIAM J. Appl. Dyn. Syst. 3 (2004) 503–524.
[2] J. Arino, C.C. McCluskey, P. van den Driessche, Global results for an epidemic model with vaccination that exhibits backward bifurcation, SIAM J. Appl. Math. 64 (2003) 260–276.
[3] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman Scientific and Technical, New York, 1993.
[4] K.L. Cooke, P. van den Driessche, Analysis of an SEIRS epidemic model with two delays, J. Math. Biol. 35 (1996) 240–260.
[5] P. Cull, Global stability for population models, Bull. Math. Biol. 43 (1981) 47–58.
[6] A. D'Onofrio, Pulse vaccination strategy in the SIR epidemic model: Global asymptotic stable eradication in presence of vaccine failures, Math. Comput. Modelling 36 (2002) 473–489.
[7] A. D'Onofrio, Vaccination policies and nonlinear force of infection: Generalization of an observation by Alexander and Moghadas (2004), Appl. Math. Comput. 168 (2005) 613–622.
[8] A. D'Onofrio, Mixed pulse vaccination strategy in epidemic model with realistic distributed infectious and latent times, Appl. Math. Comput. 151 (2004) 181–187.
[9] A. Gabbuti, L. Romanò, P. Blanc, et al., Long-term immunogenicity of hepatitis B vaccination in a cohort of Italian healthy adolescents, Vaccine 25 (2007) 3129–3132.
[10] J. Hui, L. Chen, Impulsive vaccination of SIR epidemic models with nonlinear incidence rates, Discrete Contin. Dyn. Syst. Ser. B 4 (2004) 595–605.
[11] B.S. Hersh, G. Tambini, A.C. Norgueira, P. Carrasco, C.A. de Quadros, Review of regional measles surveillance in the Americas, 1996–1999, Lancet 355 (2000) 1943–1948.
[12] A. Korobeinikov, G.C. Wake, Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models, Appl. Math. Lett. 15 (2002) 955–960.
[13] C.M. Kribs-Zaleta, M. Martcheva, Vaccination strategies and backward bifurcation in an age-since-infection structured model, Math. Biosci 177–178 317–332.
[14] Y. Kuang, Delay Differential Equation with Application in Population Dynamics, Academic Press, NewYork, 1993, pp. 67–70.
[15] M. Li, H. Smith, L. Wang, Global dynamics of an SEIR epidemic model with vertical transmission, SIAM J. Appl. Math. 62 (1) (2001) 58–69.
[16] M. Langlais, C. Suppo, A remark on a generic SEIRS model and application to cat retroviruses and fox rabies, Math. Comput. Modelling 31 (2000) 117–124.
[17] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[18] D.J. Nokes, J. Swinton, The control of a childhood viral infection by pulse vaccination, IMA. J. Math. Appl. Med. Biol. 12 (1995) 29–53.

[19] D.J. Nokes, J. Swinton, Vaccination in pulses: A strategy for global eradication of measles and polio?, Trends Microbiol 5 (1997) 14–19.
[20] A.G. Richard, et al., Immunology, W.H. Freeman, New York, 2002.
[21] B. Shulgin, L. Stone, Z. Agur, Theoretical examinations of pulse vaccination policy in the SIR epidemic model, Math. Comput. Modelling 30 (2000) 207–215.
[22] B. Shulgin, L. Stone, Z. Agur, Pulse vaccination strategy in the SIR epidemic model, Bull Math. Biol. 60 (1998) 1123–1148.
[23] X. Song, L. Chen, A predator–prey system with stage-structured and harvestion for predator, Ann. Differential Equations 18 (3) (2002) 264–277.
[24] K. Van Herck, G. Leroux-Roels, P. Van Damme, K. Srinivasa, B. Hoet, Ten-year antibody persistence induced by hepatitis A and B vaccine ($Twinrix^{TM}$) in adults, Travel. Med. Infect. Dis. 5 (3) 171–175.
[25] H.M. Wei, X.Z. Li, M. Martcheva, An epidemic model of a vector-borne disease with direct transmission and time delay, J. Math. Anal. Appl. 342 (2) (2008) 895–908.
[26] Y. Xiao, L. Chen, Modelling and analysis of a predator–prey model with disease in the prey, Math. Biosci. 171 (2001) 59–82.