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Realizing modules over the homology of a DGA

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Abstract

Let A be a DGA over a field and X a module over $H_*(A)$. Fix an A_∞-structure on $H_*(A)$ making it quasi-isomorphic to A. We construct an equivalence of categories between A_{n+1}-module structures on X and length n Postnikov systems in the derived category of A-modules based on the bar resolution of X. This implies that quasi-isomorphism classes of A_n-structures on X are in bijective correspondence with weak equivalence classes of rigidifications of the first n terms of the bar resolution of X to a complex of A-modules. The above equivalences of categories are compatible for different values of n. This implies that two obstruction theories for realizing X as the homology of an A-module coincide.

MSC: 55S35; 55U15; 16E45

1. Introduction

Let A be a differential graded algebra over a field k and let $R = H_*(A)$ be its homology. We say that an R-module X is *realizable* if there exists a differential graded module M over A with $H_*(M) \simeq X$. This paper deals with two obstruction theories for answering the question of whether or not a module is realizable.

One obstruction theory is based on the theory of A_n-structures. In [10], Stasheff introduced a hierarchy of higher homotopy associativity conditions for multiplications on chain complexes. An A_2-structure is just a bilinear multiplication m_2, while an A_3-structure is an A_2-structure together with a homotopy m_3 between the two ways of bracketing a 3-fold product. An A_∞-structure consists of a sequence of higher associating homotopies m_n satisfying certain conditions (see Section 2 for the definitions and also [6] for an excellent introduction to the theory of A_∞-algebras and modules).

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Kadeishvili proved [5] that there is a an A_∞-structure on $H_*(A)$ making it quasi-isomorphic to A as an A_∞-algebra. Such an equivalence induces an equivalence of derived categories of A_∞-modules and the derived category of (homologically unital) A_∞-modules over A is equivalent to the usual derived category of DG modules over A. This implies that a module X is realizable if and only if it admits the structure of an A_∞-module over $H_*(A)$. The $H_*(A)$-module structure on X makes it an A_2-module over the A_∞-algebra $H_*(A)$ and so the problem of realizability is naturally broken down into the problem of extending an A_n-module structure on X to an A_{n+1}-structure for successive n.

Given an A_n-structure on X, the obstruction to extending the underlying A_{n-1}-structure to an A_{n+1}-structure lies in $\mathrm{Ext}^{n,n-2}(X, X)$ (see Proposition 3.4). The original motivation for this paper was the observation that this first obstruction, i.e. the obstruction to extending the given A_2-structure to an A_4-structure, is the primary obstruction to realizability described by other means in a recent paper of Benson, Krause and Schwede [1].

In [1, Appendix A], the authors describe a general obstruction theory for realizability based on the notion of a Postnikov system (see Definition 5.1). This approach has its roots in stable homotopy theory (see [12] for example). The basic idea is the following. Since free modules and maps between them are clearly realizable, we can realize a free resolution for X in the derived category of A-modules. The problem is then whether this "chain complex of modules up to homotopy" can be rigidified to an actual chain complex of A-modules. This is explained in detail in Appendix A (see also [1]). The Postnikov system approach can be applied more generally [1, Appendix A] to the problem of realizing modules over endomorphism rings of compact objects in triangulated categories.

Benson, Krause and Schwede define a canonical Hochschild class[2] $\gamma_A \in HH^{3,1}(H_*(A))$ and show that the primary obstruction to building a Postnikov system for X is the cup product $\mathrm{id}_X \cup \gamma_A \in \mathrm{Ext}^{3,1}(X, X)$. Looking more closely one sees that the cocycles representing γ_A are (up to sign) precisely the A_3-structures on $H_*(A)$ which extend to an A_∞-structure quasi-isomorphic to A. It turns out that the primary obstruction $\mathrm{id}_X \cup \gamma_A$ to realizing X is the obstruction to putting an A_4-module structure on X (over any of these quasi-isomorphic A_∞-structures on $H_*(A)$) and so the primary obstructions to realizing a module coincide from the two points of view. A natural question is then whether the two obstruction theories described above coincide in general. We answer this question in the affirmative.

We construct a functor from A_n-modules over $H_*(A)$ to filtered differential graded A-modules. This filtration gives rise to an $(n-1)$-Postnikov system for X and our main result is that this functor induces an equivalence of categories.

Theorem 1.1. *Let A be a differential graded algebra over a field. There is an equivalence of categories between minimal A_n-modules over $H_*(A)$ and $(n-1)$-Postnikov systems based on the bar resolution.*[3]

This is proved below as Corollary 5.6 and Theorem 5.8. It implies in particular that for a fixed $H_*(A)$-module X, the moduli groupoid of A_n-structures on X is equivalent to the groupoid of $(n-1)$-Postnikov systems based on the bar resolution for X with isomorphisms which are the identity on the bar resolution (see Corollary 5.10). These equivalences of categories are compatible with the forgetful functors for varying n and hence yield an equivalence of the two obstruction theories (see Theorem 5.5).

It is a folk theorem in homotopy theory that given a chain complex $B_\bullet$ in a homotopy category $\mathrm{Ho}(\mathcal{C})$, n-Postnikov systems based on $B_\bullet$ correspond to rigidifications of the first n-terms of $B_\bullet$ to a chain complex in $\mathcal{C}$. We explain this in Appendix A. A consequence of this is the following result which is a direct consequence of Corollary 5.6 and Proposition A.6.

Corollary 1.2. *Isomorphism classes of A_n-module structures on X are in bijective correspondence with weak equivalence classes of rigidifications of the complex formed by the first n terms of the bar resolution for X.*

It would be interesting to know to what extent the previous results generalize to an abstract stable homotopy theoretic context.

The equivalence of Theorem 1.1 is reminiscent of the equivalence originally established by Stasheff [10] between the existence of an A_n-structure (defined as a certain diagram in the homotopy category) and an A_n-form on a topological space X. It would be interesting to understand the precise relation.

[2] The class in [1] is actually in $HH^{3,-1}(H^*(A))$. This is because of our convention (which follows [11]) that the kth shift $X[k]$ is the kth desuspension of X in the derived category, while in [1] it is the kth suspension.

[3] See Definition 5.1.

A related problem is to give a similar characterization of A_n-algebra structures on a graded algebra R. For such an A_n-structure, the obstruction to extending the underlying A_{n-1}-structure to an A_{n+1}-structure is a class in the Hochschild cohomology group $HH^{n+1,n-1}(R)$ (see Proposition 3.7). It would be nice to have a description of these obstructions in terms of rigidification of diagrams.

1.3. Conventions and notation

k denotes a ground field. We always deal with (not necessarily bounded) homological complexes of k-vector spaces (unlike [6] and [1] who consider cohomological complexes).

We use the Koszul sign conventions (as in [6]) so that if f, g are maps of graded modules, then

$$(f \otimes g)(x \otimes y) = (-1)^{|f||x|} f(x) \otimes g(y).$$

In particular, this implies the commutation rule

$$(f \otimes g) \circ (h \otimes j) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ j).$$

If C is a graded module we write $C[n]$ for the n-fold desuspension of C, i.e.

$$C[n]_k = C_{k+n}.$$

If C is a chain complex with differential $d\colon C[1] \to C$, then $C[n]$ is also a chain complex with differential given by $(-1)^n d$.

If $f : C \to D$ is a map of chain complexes, the standard model for the *homotopy fiber* of f is the map $F \xrightarrow{\pi} C$ where F is the complex (the desuspension of the mapping cone) defined by

$$F_k = D_{k+1} \oplus C_k$$

with differential given by the matrix

$$\begin{bmatrix} -d & f \\ 0 & d \end{bmatrix},$$

and π is the projection onto the second summand.

We will write $\lfloor x \rfloor$ for the greatest integer less than or equal to x.

Our differential graded algebras and modules are all unital.

1.4. Organization of the paper

In Section 2 we review the definition of A_n-algebra and module structures. In Section 3 we explain the obstruction to extending an A_n-module or algebra structure to an A_{n+1}-structure. In Section 4 we define the bar construction for an A_n-module. In Section 5 we recall the definition of Postnikov system and use the bar construction of the previous section to produce an equivalence of categories between A_n-modules over $H_*(A)$ and Postnikov systems based on the bar construction. There is one appendix where we explain the relation between Postnikov systems and rigidifying complexes in the homotopy category.

2. A_n-structures

In this section we recall some basic definitions and notation regarding A_n-structures [6,10] and point out some simplifications of the formulas that take place in our setting.

Definition 2.1. For $1 \le n \le \infty$, an A_n-*algebra structure* on a graded k-module R consists of maps

$$m_k : R^{\otimes k} \to R[k-2] \quad 1 \le k \le n$$

satisfying the following relations for $m \le n$

$$\sum_{r+s+t=m} (-1)^{r+st} m_{r+t+1} \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0. \tag{2.1}$$

An A_n-algebra is said to be *minimal* if $m_1 = 0$.

Definition 2.2. Let R be an A_n-algebra. A *right A_l-module X over R* with $l \leq n$ consists of a graded k-module X, together with maps

$$m_k^X : X \otimes R^{\otimes k-1} \to X[k-2] \quad 1 \leq k \leq l$$

satisfying, for each $m \leq l$,

$$\sum_{r+s+t=m} (-1)^{r+st} m_{r+t+1}^X (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0, \tag{2.2}$$

where m_s denotes m_s^X whenever $r = 0$ and m_s^R otherwise.

A *left A_l-module X over R* consists of maps $m_k^X : R^{\otimes k-1} \otimes X \to X[k-2]$ satisfying (2.2) with m_s now denoting m_s^X whenever $t = 0$ and m_s^R otherwise.

An A_l-module is said to be *minimal* if $m_1^X = 0$.

Note that $m_1 m_1 = 0$ so that an A_l-module is a complex.

Definition 2.3. Let R be an A_n-algebra and $l \leq n$. A *morphism of A_l-modules* over R, $f : X \to Y$, consists of maps of k-modules

$$X \otimes R^{\otimes k-1} \xrightarrow{f_k} Y[k-1] \quad 1 \leq k \leq l$$

satisfying the following equation in $\mathrm{Hom}(X \otimes R^{\otimes m-1}, Y[m-2])$ for each $m \leq l$:

$$\sum_{r+s+t=m} (-1)^{r+st} f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{i+j=m} (-1)^{(i+1)j} m_{j+1}(f_i \otimes 1^{\otimes j}), \tag{2.3}$$

where again m_s denotes m_s^X whenever $r = 0$ and m_s^R otherwise.

A morphism is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism.

Definition 2.4. A *morphism of A_n-algebras* $f : A \to B$ consists of maps

$$A^{\otimes k} \xrightarrow{f_k} B[k-1] \quad k \leq n$$

satisfying the following equation in $\mathrm{Hom}(A^{\otimes m}, B[m-2])$ for each $1 \leq m \leq n$:

$$\sum_{r+s+t=m} (-1)^{r+st} f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum (-1)^v m_u (f_{i_1} \otimes \cdots \otimes f_{i_u}), \tag{2.4}$$

where, on the right-hand side, the sum is over all decompositions $i_1 + i_2 + \cdots + i_u = m$ and $v = (u-1)(i_1 - 1) + (u-2)(i_2 - 1) + \cdots + 2(i_{u-2} - 1) + (i_{u-1} - 1)$.

A morphism is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism.

The A_∞-algebras we will be dealing with arise from the following result.

Theorem 2.5 (*Kadeishvili*). *Let A be a differential graded algebra over a field k. There is an A_∞ structure on $H_*(A)$ together with a quasi-isomorphism of A_∞-algebras*

$$f : H_*(A) \to A.$$

Moreover $m_1^{H_*(A)} = 0$ and $m_2^{H_*(A)}$ is the associative multiplication induced by the multiplication on A.

The A_∞-structure on $H_*(A)$ and the quasi-isomorphism f are constructed inductively. The induction is started by picking a splitting f_1 for the projection $Z(A) \to H_*(A)$ and then making use of the following simplification of (2.4).

Lemma 2.6. *Let B be a minimal A_∞-algebra and A a differential graded algebra. Then an A_∞-morphism $f : B \to A$ consists of maps $f_n : B \to A[n-1]$ satisfying the following formulas for all n:*

$$\begin{aligned} m_1 f_n = {} & f_1(m_n) + \sum_{0<u+t<n-1} (-1)^{u+t(n-u-t)} f_{u+t+1}(1^{\otimes u} \otimes m_{n-u-t} \otimes 1^{\otimes t}) \\ & + \sum_{i=1}^{n-1} (-1)^i m_2 (f_i \otimes f_{n-i}). \end{aligned}$$

Remark 2.7. In the situation of Theorem 2.5, writing d for the differential and μ for the multiplication on A, the maps

$$m_n^A = (-1)^n \mu \circ (f_{n-1} \otimes 1) \colon H_*(A)^{\otimes n-1} \otimes A \to A[n-1] \quad \text{for } n \geq 2$$

together with $m_1 = d$ make A a left A_∞-module over $H_*(A)$.

Given a differential graded algebra A, **we fix an A_∞-algebra structure on $H_*(A)$ and a quasi-isomorphism** $f \colon H_*(A) \to A$ given by Theorem 2.5 for the rest of the paper.

A graded module X over the associative graded algebra $H_*(A)$ has a natural A_2-structure with $m_1^X = 0$ and m_2^X given by the action of $H_*(A)$. Moreover, an arbitrary map $m_3^X \colon X \otimes H_*(A)^{\otimes 2} \to X[1]$ gives X the structure of an A_3-module over $H_*(A)$.

The formulas (2.2) defining an A_n-module simplify for modules over $H_*(A)$ whose underlying A_2-structure arises from the situation described in the previous paragraph. For example, an A_2-module structure consists of a map $m_2^X \colon X \otimes H_*(A) \to X$ satisfying no hypothesis and an A_3-structure consists of an associative action m_2^X together with a map $m_3^X \colon X \otimes H_*(A)^{\otimes 2} \to X[1]$ satisfying no hypothesis. More generally, an A_n-structure puts no restriction on the map m_n^X. We state the simplified formulas here as they will be used repeatedly.

Lemma 2.8. *Let X be a graded vector space. An A_n-module structure on X over $H_*(A)$ with $m_1^X = 0$ consists of maps*

$$m_k^X \colon X \otimes H_*(A)^{\otimes (k-1)} \to X[k-2] \quad 2 \leq k \leq n$$

satisfying the following equations[4] *for $2 \leq k < n$:*

$$\sum_{\substack{2 \leq r+t+1 \leq k \\ r+s+t=k+1}} (-1)^{r+st+1} m_{r+t+1}^X \circ (1^r \otimes m_s \otimes 1^t) = 0.$$

3. The obstruction to extending an A_n-structure to an A_{n+1}-structure

Let R be a minimal A_∞-algebra over the field k (so m_2 makes R an associative algebra) and let X be a minimal right A_n-module over R (so if $n \geq 3$, X is in particular a module over the associative algebra R in the usual sense). In this section we describe the set of A_{n+1}-structures on X extending the given A_n-structure. We show that the obstruction to the existence of an A_{n+1}-structure extending the underlying A_{n-1}-structure is an element in $\mathrm{Ext}_R^{n,n-2}(X, X)$.

The exact same computation shows that if S is a graded algebra and one considers A_n-algebra structures on S extending the underlying A_2-structure then the obstructions to extension are classes in the Hochschild cohomology $HH^{n,n-2}(S)$.

This obstruction theory for minimal A_n-algebras is also described in [7, Appendix B.4]. Lefèvre also discusses obstruction theory for non-minimal A_n-algebras and modules in [7, Appendix B] but this does not seem relevant to the minimal case we consider here.

Recall the bar resolution of a module M over a graded algebra R (see for instance [11, 8.6.12]):

$$\cdots \to M \otimes_k R \otimes_k R \to M \otimes_k R \to M.$$

This is a free resolution of M as a right R-module. If N is another right R-module we write

$$\left(\mathrm{Bar}^{p,q}(M, N), \partial\right) = \mathrm{Hom}^q(M \otimes R^{\otimes (p+1)}, N)$$

for the induced cochain complex of graded k-modules. The cohomology of this complex is $\mathrm{Ext}_R^{p,q}(M, N)$.

If R is a graded k-algebra, M a graded k-module, N a right R-module, and $f \colon M \to N$ a map of graded k-modules, we write

$$f * 1 \colon M \otimes R \to N$$

for the canonical extension of f to a map of R-modules.

[4] Note the difference from the usual formula in the range of the summation.

Given R and X satisfying our standing assumptions, the map

$$m_n^X : X \otimes R^{\otimes(n-1)} \to X[n-2]$$

yields a map

$$m_n^X * 1 \in \mathrm{Bar}^{n-1,n-2}(X, X).$$

In this section we will write

$$\phi_n = -m_2^X(1 \otimes m_n) + \sum_{\substack{2<r+t+1<n \\ r+s+t=n+1}} (-1)^{r+st} m_{r+t+1}^X (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}). \tag{3.1}$$

Note that the condition $\phi_n = 0$ is precisely the $(n+1)$th condition for $(m_2^X, \ldots, m_{n-1}^X, 0, 0)$ to be an A_{n+1}-structure on X (see Definition 2.2).

Lemma 3.1. *Let ∂ be the coboundary operator in the complex $\mathrm{Bar}^{\star,*}(X, X)$. There is an A_{n+1}-structure on X extending a given A_n-structure if and only if the following equation holds:*

$$\partial(m_n^X * 1) + \phi_n * 1 = 0. \tag{3.2}$$

Furthermore, when an extension exists, the set of extensions is in 1-1 correspondence with the set of k-module maps $m_{n+1}^X : X \otimes R^{\otimes n} \to X[n-1]$.

Proof. By Lemma 2.8 we need to check that

$$\partial(m_n^X * 1) = (-1)^n m_2^X(m_n^X \otimes 1) * 1 + \sum_{i=0}^{n-1} (-1)^i m_n^X(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes(n-i-1)}) * 1. \tag{3.3}$$

The map,

$$\partial(m_n^X * 1) : X \otimes R^{\otimes(n+1)} \to X[n-2]$$

is given by the formula

$$\begin{aligned}\partial(m_n^X * 1)(x, \zeta_1, \ldots, \zeta_{n+1}) = {} & m_n^X(x\zeta_1, \ldots, \zeta_n)\zeta_{n+1} - m_n^X(x, \zeta_1\zeta_2, \ldots, \zeta_n)\zeta_{n+1} + \cdots \\ & + (-1)^{n-1} m_n^X(x, \zeta_1, \ldots, \zeta_{n-1}\zeta_n)\zeta_{n+1} + (-1)^n m_n^X(x, \zeta_1, \ldots, \zeta_{n-1})\zeta_n\zeta_{n+1}.\end{aligned}$$

On the other hand, applying the right-hand side of (3.3) to $(x, \zeta_1, \ldots, \zeta_{n+1})$ we get

$$\begin{aligned}& (-1)^n m_n^X(x, \zeta_1, \ldots, \zeta_{n-1})\zeta_n\zeta_{n+1} + m_n^X(x\zeta_1, \ldots, \zeta_n)\zeta_{n+1} \\ & \quad + \cdots + (-1)^{n-1} m_n^X(x, \zeta_1, \ldots, \zeta_{n-1}\zeta_n)\zeta_{n+1}. \quad \square\end{aligned}$$

Lemma 3.2. *If X is an A_n-module over R then $\phi_n * 1 \in \mathrm{Bar}^{n,n-2}(X, X)$ is a cocycle.*

Proof. It suffices to check the condition $(\phi_n * 1) \circ \partial = 0$ on module generators. Thus we need to check that

$$\sum_{i=0}^{n} (-1)^i \phi_n \left(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes n-i}\right) + (-1)^{n+1} \phi_n * 1 = 0. \tag{3.4}$$

In the course of this proof, through (3.10), we omit

$$\sum_{\substack{r+s+t=n+1 \\ 2\le r+t+1<n \\ (r,s,t)\neq(0,n,1)}}$$

which is understood to precede each sum.

Expanding the left summand in expression (3.4) we obtain

$$\sum_{0\le i<r} (-1)^{r+st+i} m^X_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r-i-1} \otimes m_s \otimes 1^{\otimes t}) \tag{3.5}$$

$$+ \sum_{r\le i<r+s} (-1)^{r+st+i} m^X_{r+t+1}(1^{\otimes r} \otimes m_s(1^{\otimes i-r} \otimes m_2 \otimes 1^{\otimes r+s-i-1}) \otimes 1^{\otimes t}) \tag{3.6}$$

$$+ \sum_{r+s\le i\le n} (-1)^{r+st+i} m^X_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes i-r-s} \otimes m_2 \otimes 1^{\otimes r+t+s-i-1}). \tag{3.7}$$

Using the A_{s+1}-structure on the middle term (3.6) we find that it is equal to

$$\sum_{\substack{j+k+l=s+1\\ 2\le j+l+1<s}} (-1)^{st+j+kl+1} m^X_{r+t+1}(1^{\otimes r} \otimes m_{j+l+1}(1^{\otimes j} \otimes m_k \otimes 1^{\otimes l}) \otimes 1^{\otimes t}). \tag{3.8}$$

Separating out the term where $(j, k, l) = (1, s, 0)$ and combining it with the term (3.5) we obtain

$$\sum_{0\le i\le r} (-1)^{r+st+i} m^X_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r+t-i})(1^{\otimes r+1} \otimes m_s \otimes 1^{\otimes t}).$$

Similarly combining the term of (3.8) where $(j, k, l) = (0, s, 1)$ with the term (3.7) we obtain

$$\sum_{r\le i\le r+t} (-1)^{r+st+s+i-1} m^X_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r+t-i})(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}).$$

Together the last two expressions yield

$$\sum_{0\le i\le r+t} (-1)^{r+st+s+i-1} m^X_{r+t+1}(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes r+t-i})(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}). \tag{3.9}$$

Using the $(r + t + 2)$-module structure in (3.9) yields

$$\sum_{\substack{a+b+c=r+t+2\\ b>2}} (-1)^{r+st+s+a+bc} m^X_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c}) \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}). \tag{3.10}$$

The terms in (3.10) corresponding to $(a, b, c) = (1, r + t + 1, 0)$ cancel with those terms in (3.8) that remain (i.e. $(j, k, l) \notin \{(1, s, 0), (0, s, 1)\}$) and satisfy $(r, s) = (1, n)$. On the other hand, the terms in (3.10) corresponding to $(a, b, c) = (0, r + t + 1, 1)$ cancel with the term $(-1)^{n+1}\phi_n * 1$ in (3.4).

We are left with showing that the terms

$$\sum_{\substack{r+s+t=n+1\\ 2<s<n}} \sum_{\substack{a+b+c=r+t+2\\ 2<b\le r+t}} (-1)^{r+st+s+a+bc} m^X_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c}) \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t+1}). \tag{3.11}$$

left from (3.10) and

$$\sum_{\substack{r+s+t=n+1\\ 2<s<n}} \sum_{\substack{j+k+l=s+1\\ 2<k<s}} (-1)^{st+1+j+kl} m^X_{r+t+1}(1^{\otimes r} \otimes m_{j+l+1}(1^{\otimes j} \otimes m_k \otimes 1^{\otimes l}) \otimes 1^{\otimes t}). \tag{3.12}$$

left from (3.8) add up to zero. In (3.11), the terms of the form $m^X_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes r-a-b} \otimes m_s \otimes 1^{\otimes t+1})$ appear twice with opposite signs. Finally, the remaining terms of (3.8), which are of the form $m^X_{a+c+1}(1^{\otimes a} \otimes m_b(1^{\otimes r-a} \otimes m_s \otimes 1^{\otimes t+1-c}) \otimes 1^{\otimes c})$, cancel with (3.12). □

Remark 3.3. We have observed that, by definition, $(m^X_2, \dots, m^X_{n-1}, 0)$ extends to an A_{n+1}-structure if and only if $\phi_n = 0$. Writing $\tilde{\phi}_n = 0$ for the A_{n+1}-structure equation that $(m^X_2, \dots, m^X_{n-1}, m^X_n)$ must satisfy in order to extend, it is easy to check that $(\tilde{\phi}_n - \phi_n) * 1$ is a cocycle in the bar complex. Thus, given an A_n-structure on X, the previous lemma implies that $\tilde{\phi}_n * 1$ is also a cocycle.

We summarize the previous arguments in the following statement.

Proposition 3.4. *Let X be an A_n-module over R.*

(a) *The underlying A_{n-1}-structure on X can be extended to an A_{n+1}-structure iff the class $[\phi_n * 1] \in \mathrm{Ext}_R^{n,n-2}(X, X)$ vanishes.*

(b) *If $[\phi_n * 1] = 0$, the set of A_{n+1}-structures on X extending the underlying A_{n-1}-structure is in bijective correspondence with pairs of R-module maps*

$$\psi : X \otimes R^{\otimes n} \to X[n-2], \quad \xi : X \otimes R^{\otimes(n+1)} \to X[n-1]$$

such that

$$\partial(\psi) = \phi_n * 1.$$

Proof. Statement (a) is the content of Lemmas 3.1 and 3.2. Statement (b) follows immediately from the fact that Eq. (3.2) is the only equation involving m_n^X among the equations defining an A_{n+1}-structure on X. □

Remark 3.5. Let A be a differential graded algebra. In [1] the authors consider the problem of deciding whether an $H_*(A)$-module X is the homology of an A-module. They define a Hochschild cohomology class $\gamma_A \in HH^{3,1}(H_*(A))$ and show that the first obstruction is

$$1_X \cup \gamma_A \in \mathrm{Ext}^{3,1}_{H_*(A)}(X, X)$$

(see [1, Corollary 6.3]). The choice of a cocycle representing γ_A precisely corresponds to the choice of $m_3^{H_*(A)}$ in the inductive proof of Kadeishvili's theorem (compare Lemma 2.6 with [1, Construction 5.1 and Remark 5.8]).

The special case of Proposition 3.4 when $n = 3$ says that an R-module X has an A_4-structure if and only if the map

$$(m_2^X(1 \otimes m_3)) * 1$$

is a coboundary in $\mathrm{Bar}^{3,1}(X, X)$.Thus the obstruction described in [1] is exactly the obstruction to the existence of an A_4-structure on X.

Example 3.6. This example amplifies on the example considered in [1, 7.3,7.4,7.6]. Let $L = k[z]/z^n$ be the truncated polynomial algebra of height n over a field k. Let A be the endomorphism DGA of the complete resolution $\hat{P}$ of the trivial L-module k. $\hat{P}$ is defined by $\hat{P}_i = L$ for each $i \in \mathbb{Z}$ with differentials $d_i : \hat{P}_i \to \hat{P}_{i-1}$ given by the formulas

$$d_i = \begin{cases} \text{multiplication by } -z^{n-1} & \text{if } i \text{ is even,} \\ \text{multiplication by } z & \text{otherwise.} \end{cases}$$

Note that if k has characteristic p and n is a power of p, L is isomorphic to the group algebra of the cyclic group C_n and then the homology of A is the Tate cohomology of C_n.

The homology algebra of A is [1, Theorem 7.3]:

$$H_*(A) = \begin{cases} k[x^{\pm 1}] & \text{if } n = 2, \\ \Lambda(x) \otimes k[y^{\pm 1}] & \text{if } n > 2, \end{cases}$$

where $\Lambda(x)$ denotes the exterior algebra on x, and $|x| = -1$ and $|y| = -2$. In the proof the authors define the first two maps $f_1 : H_*(A) \to A$ and $f_2 : H_*(A) \to A[1]$ in a quasi-isomorphism of A_∞-algebras $f : H_*(A) \to A$ (cf. Theorem 2.5) and use this to find the A_3-structure on $H_*(A)$, $m_3 : H_*(A)^{\otimes 3} \to H_*(A)[1]$ (in their terminology this is the Hochschild cocycle m representing the canonical class as in the previous remark).

For $n \neq 3$, m_3 vanishes, while for $n = 3$, it is given by the formula

$$m_3(a, b, c) = \begin{cases} 0 & \text{if } |a|, |b|, \text{ or } |c| \text{ is even,} \\ y^{i+j+k+1} & \text{if } a = xy^i, b = xy^j, c = xy^k. \end{cases}$$

Proceeding as in the proof of [1, Theorem 7.3] we can inductively find formulas for the remaining maps $f_i : H_*(A) \to A[i-1]$. From this we see that, in general, the A_∞-structure on $H_*(A)$ consists of only m_2 and m_n with all other m_k's vanishing and m_n given by the formula

$$m_n(a_1, \ldots, a_n) = \begin{cases} 0 & \text{if one of the } |a_i|\text{'s is even,} \\ y^{j_1+\cdots+j_n+1} & \text{if } a_i = xy^{j_i}. \end{cases}$$

In the case when $n = 3$ the authors show in [1, Example 7.6] that the realizable $H_*(A)$-modules are precisely the free ones. For $n > 3$, any $H_*(A)$-module X admits a trivial A_n-structure with $m_k^X = 0$ for $2 < k \leq n$. The argument of [1, Example 7.6] shows more generally that for this trivial A_n-structure to extend to an A_{n+1}-structure, X must be a free module. However, it is no longer true that only free modules are realizable. In fact, for $n > 3$, all modules are direct summands of realizable modules (by the previous calculation together with the main result of [1]) but they are certainly not all direct summands of free modules.

For example, for $n > 3$,

$$H_*(\mathrm{Hom}(\hat{P}, k[z]/z^2)) = k[y^{\pm 1}] \oplus k[y^{\pm 1}][1]$$

(with x acting trivially) is obviously realizable.

On the other hand, the $H_*(A)$-module

$$X = k[y^{\pm 1}] = H_*(A)/xH_*(A)$$

is not realizable. Indeed, for any choice of $m_k^X \colon X \otimes H_*(A)^{\otimes(k-1)} \to X[k-2]$ and any $a \in X$ we have

$$m_k^X(a, x, \ldots, x) = 0$$

since X is concentrated in even degrees. It follows that X cannot be given an A_{n+1}-structure: when we evaluate

$$\sum_{\substack{2 \leq r+t+1 \leq n \\ r+s+t=n+1}} (-1)^{r+st} m_{r+t+1}^X (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})(a, x, \ldots, x)$$

all terms except $am_n(x, \ldots, x) = ay$ vanish (either because x acts trivially on X, $x^2 = 0$, $m_k^X(a, x, \ldots, x) = 0$, or because $m_k = 0$ for $k < n$).

The algebra $\mathrm{Ext}^{\star,*}(X, X)$ is a polynomial algebra on $k[y^{\pm 1}]$ on a generator in bidegree $(1, -1)$ (y has degree $(0, -2)$). The obstruction to extending the trivial A_n-module structure on X to an A_{n+1}-structure must therefore be a generator of $\mathrm{Ext}^{n,n-2}_{H_*(A)}(X, X)$.

It is somewhat surprising that for the realizable module $Y = k[y^{\pm 1}] \oplus k[y^{\pm 1}][1]$ we cannot choose m_k^Y to vanish for $k < n$. One can check that an A_∞-structure on Y can be defined in the following way. Let a and b be module generators for Y in degrees 0 and -1 respectively.

If n is even, set (for $k \geq 3$)

$$m_k^Y(m, xy^{i_2}, \ldots, xy^{i_k}) = \begin{cases} by^{i_1+\cdots+i_k} & \text{if } m = ay^{i_1} \text{ and } k = \dfrac{n}{2} - 1, \\ ay^{i_1+\cdots+i_k} & \text{if } m = by^{i_1} \text{ and } k = \dfrac{n}{2} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, set

$$m_k^Y(m, xy^{i_2}, \ldots, xy^{i_k}) = \begin{cases} by^{i_1+\cdots+i_k} & \text{if } m = ay^{i_1}, k \in \left\{\dfrac{n+1}{2}, \dfrac{n+3}{2}\right\}, \text{ and } k \text{ is even}, \\ ay^{i_1+\cdots+i_k} & \text{if } m = by^{i_1}, k \in \left\{\dfrac{n+1}{2}, \dfrac{n+3}{2}\right\}, \text{ and } k \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

The analog for algebras of Proposition 3.4 is the following. Consider the Hochschild complex

$$(C^{n,m}(S) = \mathrm{Hom}^m_{S \otimes S^{op}}(S^{\otimes(n+2)}, S), \partial_H).$$

An element in $C^{n,m}(S)$ is represented by a map of vector spaces $f \colon S^{\otimes n} \to S$ of degree m and, in these terms, the differential is given by the formula

$$\partial_H(f) = m_2(1 \otimes f) - \sum_{j=0}^{n-1} (-1)^j f \circ (1^{\otimes j} \otimes m_2 \otimes 1^{\otimes n-j-1}) + (-1)^n m_2(f \otimes 1).$$

The analog of (3.2) is that an A_n-algebra structure on S can be extended to an A_{n+1}-algebra structure iff the following equation is satisfied:

$$\partial_H(m_n) = \sum_{\substack{2<r+t+1<n \\ r+s+t=n+1}} (-1)^{r+st} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})$$

and the same computations as in the proofs of Lemmas 3.1 and 3.2 give the following analog of Proposition 3.4 which can also be found in [7, Lemma B.4.1].

Proposition 3.7. *Let S be a graded algebra. Given an A_n-structure on S extending the given A_2-structure, the underlying A_{n-1}-structure can be extended to an A_{n+1}-structure iff the Hochschild cocycle*

$$\sum_{\substack{2<r+t+1<n \\ r+s+t=n+1}} (-1)^{r+st} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})$$

represents the trivial class in $HH^{n+1,n-2}(S)$.

If S is an A_∞-algebra with $m_1 = m_3 = \cdots = m_{n-1} = 0$ then $m_n : S^{\otimes n} \to S$ is a Hochschild cocycle. The primary obstruction to realizing a module X is then the obstruction to giving X an A_{n+1}-structure, namely the class $1_X \cup [m_n] \in \mathrm{Ext}_S^{n,n-2}(X, X)$. In fact, any S-module X can be given an A_{n-1}-module structure with $m_3^X = \cdots = m_{n-1}^X = 0$ and one can extend this to an A_{n+1}-module structure on X if and only if

$$(m_2^X(1 \otimes m_n)) * 1,$$

which is a cocycle representing $1_X \cup [m_n]$, is a coboundary in $\mathrm{Bar}^{n,n-2}(X, X)$. This is exactly the situation for the non-realizable module X in Example 3.6.

Example 3.8. Let $S = R[\epsilon]/\epsilon^2$ where R is a k-algebra concentrated in degree 0 and $|\epsilon| = n-2$. If $\{m_n\}$ is an A_∞-structure on S then for degree reasons $m_i = 0$ for $i \neq 2, n$ and m_n is determined by a k-linear map $R^{\otimes n} \to R$ which must be a Hochschild cocycle. One can check that two A_∞-structures on S are quasi-isomorphic iff the corresponding cocycles represent the same cohomology class (cf. [6, 3.2]).

4. The bar construction

Recall that we have fixed an A_∞-structure on $H_*(A)$ and a quasi-isomorphism $f : H_*(A) \to A$, which in turn gives A the structure of an A_∞-$H_*(A)$-module (see Remark 2.7). The goal of this section is to construct a functor, denoted $B(-, H_*(A), A)$, from A_∞-$H_*(A)$-modules to differential graded A-modules. The functor can be written as a directed colimit of functors $B_{n-1}(-, H_*(A), A)$, from A_n-$H_*(A)$-modules to differential graded A-modules.

Given a (minimal) A_n-module structure on an $H_*(A)$-module X

$$m_k^X : X \otimes H_*(A)^{\otimes(k-1)} \to X, \quad 2 \le k \le n,$$

let R_k denote the free differential graded A-module defined by

$$R_k = X \otimes H_*(A)^{\otimes k} \otimes A.$$

For $1 \le l \le k+1$, let

$$M_{k,l} : R_k \to R_{k-l+1}[l-2]$$

be defined as

$$M_{k,l} = \sum_{i=0}^{k+2-l} (-1)^{i(l-1)} 1^{\otimes i} \otimes m_l \otimes 1^{\otimes k-l-i+2},$$

where, in the first term of the sum, m_l stands for m_l^X and, in the last term, m_l stands for $m_l^A = (-1)^l f_{l-1} * 1$ if $l > 1$ and for the differential d on A if $l = 1$ (see Remark 2.7). We will sometimes write D for $M_{k,2}$ and d for $M_{k,1}$.

The formulas in the following definition were obtained when attempting to construct a Postnikov system associated to an A_{n+1}-module (see Theorem 5.3). They are very reminiscent of Stasheff's tilde bar construction [10, II.(2.4)].

Definition 4.1. Given an A_{n+1}-module X over $H_*(A)$ (with $1 \leq n \leq \infty$), the *bar construction on* X is the right A-module $B_n(X, H_*(A), A)$ defined by

$$\oplus_{i=0}^{n}(X \otimes H_*(A)^{\otimes i} \otimes A)[-i] = \oplus_{i=0}^{n} R_i[-i].$$

The differential on $B_n(X, H_*(A), A)$ is defined on the summand R_l by the following formula

$$\partial_{|R_l} = \sum_{i+j+k=l+2} (-1)^{k+j+ij+\left\lfloor \frac{j-1}{2} \right\rfloor} 1^{\otimes i} \otimes m_j \otimes 1^{\otimes k} = \sum_{j=1}^{l+1} (-1)^{l+\left\lfloor \frac{j-1}{2} \right\rfloor} M_{l,j}. \tag{4.1}$$

We use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. The following easily checked formula will be used constantly in computations.

Lemma 4.2. *For any integers i and j*

$$\left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor \equiv \left\lfloor \frac{j-i}{2} \right\rfloor + ij \bmod 2.$$

Lemma 4.3. *The formula (4.1) gives $B_n(X, H_*(A), A)$ the structure of a differential graded A-module.*

Proof. It is easy to check that the Leibniz rule holds so it is enough to check that (4.1) defines a differential on $B_n(X, H_*(A), A)$. The projection to R_m of $\partial^2_{|R_l}$ is given by the formula

$$\sum_{j=1}^{l-m+1} (-1)^{1-j+\left\lfloor \frac{j-1}{2} \right\rfloor + \left\lfloor \frac{l-j-m+1}{2} \right\rfloor} M_{l-j+1,l-j-m+2} M_{l,j}. \tag{4.2}$$

By Lemma 4.2 the sign in the previous expression is equal to

$$(-1)^{(l-m)(j-1)+\left\lfloor \frac{l-m}{2} \right\rfloor}.$$

Since $(-1)^{\left\lfloor \frac{l-m}{2} \right\rfloor}$ is independent of j, this factor can be eliminated and the equation $\partial^2 = 0$ then follows from the relations that must be satisfied because $H_*(A)$ is an A_{n+1}-algebra and, X and A are A_{n+1}-modules over $H_*(A)$. □

We also need to explain the functoriality of the bar construction.

Proposition 4.4. *Let $g : X \to Y$ be a map of A_{n+1}-modules (with $1 \leq n \leq \infty$). The map*

$$B_n(g) \colon B_n(X, H_*(A), A) \to B_n(Y, H_*(A), A)$$

defined by the matrix with entries

$$B_n(g)_{i,j} = (-1)^{\left\lfloor \frac{j-i+1}{2} \right\rfloor} g_{j-i+1} \otimes 1^{\otimes i} \tag{4.3}$$

for $1 \leq i \leq j \leq n+1$, or

$$\begin{bmatrix} g_1 \otimes 1 & -g_2 \otimes 1 & -g_3 \otimes 1 & g_4 \otimes 1 & \cdots \\ 0 & g_1 \otimes 1^{\otimes 2} & -g_2 \otimes 1^{\otimes 2} & -g_3 \otimes 1^{\otimes 2} & \cdots \\ 0 & 0 & g_1 \otimes 1^{\otimes 3} & -g_2 \otimes 1^{\otimes 3} & \cdots \\ 0 & 0 & 0 & g_1 \otimes 1^{\otimes 4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is a map of differential graded A-modules.

Proof. This computation is similar to the one above and hence is omitted. □

We also write $B(g) = B_\infty(g)$.

Proposition 4.5. *For $1 \leq n \leq \infty$, the assignments*

$$X \mapsto B_n(X, H_*(A), A) \quad \left(X \xrightarrow{g} Y\right) \mapsto B_n(g)$$

define a functor from $A_{n+1} - H_*(A)$-modules to differential graded A-modules.

Proof. Matrix multiplication precisely corresponds to the composition of A_{n+1}-module maps as defined on [6, p. 15]. □

Remark 4.6. The quasi-isomorphism of A_∞-algebras $f\colon H_*(A) \xrightarrow{\sim} A$ makes A and A_∞-$H_*(A)$-A-bimodule. Although the formula for the differential (4.1) is different, it seems likely that $B_\infty(-, H_*(A), A)$ is equivalent to the functor $-\overset{\infty}{\otimes}_{H_*(A)} A$ considered in [7, Section 4.1, p.114].

5. A_n-structures and Postnikov systems

In this section we describe the obstruction theory to realizing a module based on the notion of a Postnikov system [1] and show that the bar construction of the previous section gives us a functor from A_{n+1}-module structures to n-Postnikov systems. We then show that the obstructions to extending an A_{n+1}-structure or its associated n-Postnikov system agree. It follows by induction that any Postnikov system arising from the bar resolution of X comes from an A_{n+1}-structure. Finally we prove that this assignment is fully faithful in completing the proof of Theorem 1.1.

In this section, we will often use the following simple formula for the maps in the derived category of A-modules when the source is free: if V is a k-module and N is a differential graded module over A then

$$[V \otimes A, N] = \mathrm{Hom}_{H_*(A)}(V \otimes H_*(A), H_*(N)).$$

Definition 5.1. Let A be a differential graded algebra and X be an $H_*(A)$-module. An *n-Postnikov system* for X is a commutative diagram in the derived category of A-modules

$$\begin{tikzcd}
Y_n \arrow[d, "j_n"'] & Y_{n-1} \arrow[d, "j_{n-1}"'] & Y_{n-2} \arrow[d, "j_{n-2}"] & & Y_1 \arrow[d, "j_1"'] & \\
C_n \arrow[r, "d_n"'] \arrow[ur, "i_n"] & C_{n-1} \arrow[r, "d_{n-1}"'] \arrow[ur, "i_{n-1}"] & C_{n-2} & \cdots & C_1 \arrow[r, "i_1"'] & C_0
\end{tikzcd}$$

satisfying

(i) j_k is the homotopy fiber of i_k (i.e. $Y_k \to C_k \to Y_{k-1}$ is part of a triangle),
(ii) C_k is a free A-module,
(iii) there is a map $H_*(C_0) \to X$ such that the following is an exact sequence $H_*(C_n) \to \cdots \to H_*(C_0) \to X \to 0$.

Maps of n-Postnikov systems are maps of diagrams in the derived category which restrict to maps of triangles.

We say that an n-Postnikov system is *based on the bar resolution* if $H_*(C_\star)$ is isomorphic to the bar resolution for X. A map is *based on the bar resolution* if the maps $H_*(C_k) \to H_*(C'_k)$ are of the form $g \otimes 1^{\otimes(k+1)}$ with $g\colon X \to X'$ a map of $H_*(A)$-modules.

Remark 5.2. The previous definition differs from the definition of n-Postnikov system in [1, Definition A.6] in that the homotopy fiber of i_n is included in the diagram. This distinction is only relevant when considering maps of Postnikov systems.

A simple diagram chase shows (see [1, Lemma A.12]) that an n-Postnikov system yields an exact sequence

$$0 \to X[n-1] \to H_*(Y_{n-1}) \to H_*(C_{n-1}) \to \cdots \to H_*(C_0) \to X \to 0.$$

It follows from the proof of this result that the following diagram commutes

$$\begin{tikzcd} & H_*(C_0)[n-1] \arrow[d] \arrow[dr, "j"] & \\ 0 \arrow[r] & X[n-1] \arrow[r] \arrow[d] & H_*(Y_{n-1}) \\ & 0 & \end{tikzcd} \tag{5.1}$$

where j denotes the composite

$$H_*(C_0)[n-1] \to H_*(Y_1)[n-2] \to \cdots \to H_*(Y_{n-2})[1] \to H_*(Y_{n-1}).$$

Note that when C_* is the bar resolution, the nontrivial vertical map in (5.1) is the multiplication map $X \otimes H_*(A) \to X$.

Theorem 5.3. *Let X be an A_{n+1}-module over $H_*(A)$, $R_k = X \otimes H_*(A)^{\otimes k} \otimes A$ and $Y_k = B_k(X, H_*(A), A)[k]$. Then the following diagram of A-modules projects to an n-Postnikov system for X:*

$$\begin{tikzcd} Y_n \arrow[d, "\pi_n"'] & Y_{n-1} \arrow[d, "\pi_{n-1}"'] & Y_{n-2} \arrow[d, "\pi_{n-2}"'] & \cdots & Y_1 \arrow[d, "\pi_1"'] & \\ R_n \arrow[ur, "i_n"] \arrow[r, "M_{n,2}"'] & R_{n-1} \arrow[ur, "i_{n-1}"] \arrow[r, "M_{n-1,2}"'] & R_{n-2} & \cdots & R_1 \arrow[r, "M_{1,2}"'] & R_0. \end{tikzcd}$$

Here π_k denotes the projection onto the last summand and

$$i_k = \begin{bmatrix} (-1)^{\lfloor \frac{k}{2} \rfloor} M_{k,k+1} \\ \vdots \\ (-1)^{\lfloor \frac{j-1}{2} \rfloor} M_{k,j} \\ \vdots \\ M_{k,2} \end{bmatrix}.$$

This assignment is functorial with respect to maps of A_{n+1}-modules.

Proof. By definition

$$Y_k = R_0[k] \oplus \cdots \oplus R_{k-1}[1] \oplus R_k$$

as a graded A-module, and the ijth entry ($1 \leq i \leq j \leq k+1$) of the matrix ∂_{Y_k} is

$$(-1)^{k-j+1+\lfloor \frac{j-i}{2} \rfloor} M_{j-1,j-i+1}.$$

Therefore the differential on Y_k satisfies the following inductive formula

$$\partial_{Y_k} = \begin{bmatrix} -\partial_{Y_{k-1}} & i_k \\ 0 & d \end{bmatrix}.$$

It follows that i_k is a map of differential graded modules because this condition is precisely the condition that the upper right-hand vector in the matrix $\partial^2_{Y_k}$ vanishes. Clearly Y_k is the homotopy fiber of $i_k\colon R_k \to Y_{k-1}$. Finally, functoriality follows from Proposition 4.5. $\square$

Definition 5.4. The *canonical n-Postnikov system* associated to an A_{n+1}-structure on X is the Postnikov system defined in Theorem 5.3.

Theorem 5.5. *Let X be an A_{n+1}-module over $H_*(A)$. There is a bijective correspondence between the sets of*

(i) *A_{n+2}-structures $(m_2^X, \ldots, m_{n+1}^X, \phi)$ on X,*

(ii) *lifts in the homotopy category*

$$\begin{tikzcd} & Y_n \arrow[d, "\pi_n"] \\ R_{n+1} \arrow[ur, dashed, "j"] \arrow[r, "D"'] & R_n . \end{tikzcd}$$

The assignment sends $(m_2^X, \ldots, m_{n+1}^X, \phi)$ *to the homotopy class of the map* i_{n+1} *defined in Theorem 5.3 from the* A_{n+2}*-structure.*

In other words, an A_{n+1}*-structure on* X *extends one stage iff its associated canonical* n*-Postnikov system extends one stage and, in that case, the extensions are in bijective correspondence.*

Proof. The canonical n-Postnikov system associated to the A_{n+1}-structure on X extends if and only if the map $i_n D$ is null. Since R_{n+1} is a free A-module, this is equivalent to $H_*(i_n D)$ being the zero map. As $X[n-1] \to H_*(Y_{n+1})$ is an inclusion, this amounts to the vanishing of the map $\overline{i_n D}$ in the commutative diagram

$$\begin{tikzcd} {} \arrow[r, "H_*(D)"] & H_*(R_{n+1}) \arrow[r, "H_*(D)"] \arrow[d, dashed, "\overline{i_n D}"] & H_*(R_n) \arrow[d, "H_*(i_n)"'] \arrow[dr, "H_*(D)"] & & \\ 0 \arrow[r] & X[n-1] \arrow[r] & H_*(Y_{n-1}) \arrow[r] & H_*(R_{n-1}) \arrow[r] & H_*(R_{n-2}). \end{tikzcd}$$

We will show that

$$\overline{i_n D} = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor + n+1} \left(\sum_{\substack{r+s+t=n+2 \\ 2 \le r+t+1 \le n+1}} (-1)^{r+st} m_{r+t+1}^X (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \right) * 1. \tag{5.2}$$

Lemma 3.1 then implies that the canonical $(n+1)$-Postnikov system extends if and only if the A_{n+1}-structure extends to an A_{n+2}-structure.

To prove (5.2), we need to compute $H_*(i_n D)$. We will add a null homotopic map to $i_n D$ in order to perform the computation. For $n \ge 2$, let $H_n \colon R_{n+1} \to Y_{n-1}$ be the map defined by the column vector

$$\begin{bmatrix} (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor + n+1} 1 \otimes m_{n+2}^A \\ (-1)^{\left\lfloor \frac{n-2}{2} \right\rfloor} M_{n+1,n+1} \\ \vdots \\ (-1)^{\left\lfloor \frac{n-j}{2} \right\rfloor} M_{n+1,n+3-j} \\ \vdots \\ M_{n+1,3} \end{bmatrix}.$$

We now compute the effect of the map

$$i_n D + (\partial_{Y_{n-1}} H_n + H_n d)$$

on homology. We will first show that $i_n D + (\partial_{Y_{n-1}} H_n + H_n d)$ factors through $R_0[n-1]$: For $i \ge 2$ the ith component of this map is

$$(-1)^{\left\lfloor \frac{n-i+1}{2} \right\rfloor} M_{n,n+2-i} M_{n+1,2} + \sum_{j=i}^{n} (-1)^{n-j+\lfloor (j-i)/2 \rfloor + \lfloor (n-j)/2 \rfloor} M_{j-1,j+1-i} M_{n+1,n+3-j}$$
$$+ (-1)^{\lfloor (n-i)/2 \rfloor} M_{n+1,n+3-i} M_{n+1,1}$$

and this simplifies to

$$(-1)^{n+\left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=i}^{n+2} (-1)^{j(n+i)} M_{j-1,j+1-i} M_{n+1,n+3-j}$$

which up to sign is exactly the sum (4.2) and therefore vanishes (only the A_{n+1}-structure is used).

The first component of $i_n D + (\partial_{Y_{n-1}} H_n + H_n d)$ is

$$(-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(\sum_{j=2}^{n+1} (-1)^{j(n+1)} M_{j-1,j} M_{n+1,n+3-j} + d(1 \otimes m^A_{n+2}) + (-1)^{n+1} 1 \otimes (m^A_{n+2} d) \right). \tag{5.3}$$

Using the A_∞-$H_*(A)$-module structure on A, a computation similar to the proof of Lemma 4.3 shows that this formula simplifies to

$$(-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor + n+1} \left(\sum_{\substack{2 \le r+t+1 \le n+1 \\ r+s+t=n+2}} (-1)^{r+st} m^X_{r+t+1} (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \right) \otimes 1. \tag{5.4}$$

By the commutativity of diagram (5.1), the map $H_*(R_{n+1}) \to X[n-1]$ is obtained by composing (5.4) with the multiplication map $X \otimes H_*(A) \to X$. This proves (5.2).

It remains to prove the bijection between A_{n+2}-structures extending the given A_{n+1}-structure and the extensions of the canonical n-Postnikov system when an extension exists. In that case, the A_{n+2}-structures are arbitrary k-module maps

$$\phi \colon X \otimes H_*(A)^{\otimes(n+1)} \to X[n].$$

On the other hand, a homotopy class of maps $j \colon R_{n+1} \to Y_n$ lifting D is the same as an $H_*(A)$-module map $H_*(R_{n+1}) \to X[n]$. Writing $i_{n+1}(\phi)$ for the lift associated to a k-module map ϕ, the formula for $i_{n+1}(\phi)$ shows that $i_{n+1}(\phi) - i_{n+1}(0)$ factors through $R_0[n]$ and hence (see diagram (5.1)) the $H_*(A)$-module map associated to $i_{n+1}(\phi) - i_{n+1}(0)$ is

$$\phi * 1 \colon H_*(R_{n+1}) \to X[n].$$

This shows that homotopy classes of lifts of D are in bijective correspondence with k-module maps $X \otimes H_*(A)^{\otimes(n+1)} \to X[n]$ and completes the proof. $\square$

Corollary 5.6. *Any n-Postnikov system based on the bar resolution for X is isomorphic to the canonical n-Postnikov system associated to an A_{n+1}-structure on X.*

Proof. For $n = 1$ the statement is clearly true. The result follows by induction from Theorem 5.5. $\square$

Lemma 5.7. *Let $(g_1, \ldots, g_k)$ be an A_k-map between two A_{k+1}-modules X and X'. Then the square*

$$\begin{tikzcd} R_k \arrow[r, "i_k"] \arrow[d, "g_1 \otimes 1^{\otimes(k+1)}"'] & Y_{k-1} \arrow[d, "B_{k-1}(g)"] \\ R'_k \arrow[r, "i'_k"] & Y'_{k-1} \end{tikzcd}$$

commutes up to homotopy if and only if $(g_1, \ldots, g_k, 0)$ is an A_{k+1}-map.

Proof. Because g_1 is a map of $H_*(A)$-modules, the square

$$\begin{tikzcd} R_k \arrow[r] \arrow[d] & R_{k-1} \arrow[d] \\ R'_k \arrow[r] & R'_{k-1} \end{tikzcd}$$

commutes strictly and so the difference on homology lies in the kernel of $Y'_{k-1} \to R'_{k-1}$. This kernel is a desuspension of X. We want to compute the map

$$B_{k-1}(g) i_k - i'_k (g_1 \otimes 1^{\otimes(k+1)}) : H_*(R_k) \to X'[k-1].$$

We will add a nulhomotopic map so as to make the factorization of this map through $R'_0[k-1]$ apparent. The homotopy is given by the formula

$$H_k = \begin{bmatrix} 0 \\ (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} g_k \otimes 1^{\otimes 2} \\ \vdots \\ (-1)^{\left\lfloor \frac{k-i+2}{2} \right\rfloor} g_{k-i+2} \otimes 1^{\otimes i} \\ \vdots \\ -g_2 \otimes 1^{\otimes k} \end{bmatrix}.$$

One computes that

$$B_{k-1}(g)i_k - i'_k(g_1 \otimes 1^{\otimes(k+1)}) + (\partial_{Y'_{k-1}} H_k + H_k d)$$

has all components zero except the first one because $(g_1, \dots, g_k)$ is an A_k-map. When composed with the multiplication $m_2^{X'}: R'_0 \to X$, the first component yields the $(k+1)$-ary map whose vanishing is synonymous with $(g_1, \dots, g_k, 0)$ being an A_{k+1}-map. $\square$

Using the functor $B_n(-, H_*(A), A)$ from the last section, we can now complete the proof of Theorem 1.1.

Theorem 5.8. *Let X and X' be (minimal) A_{n+1}-$H_*(A)$-modules. There is a bijective correspondence between A_{n+1}-maps $g: X \to X'$, and maps between the associated canonical n-Postnikov systems based on the bar resolution.*

Proof. Given g, the desuspensions of the maps $B_k(g)$, $1 \le k \le n$, described in Proposition 4.4 give the desired map of Postnikov systems. It is easy to check that this assignment is injective (if two A_n-maps first differ on g_k, the induced maps $Y_{k-1} \to Y'_{k-1}$ will not be homotopic).

The converse is proved by induction. For $n = 1$, a map of Postnikov systems of the sort described above is determined by a map of $H_*(A)$-modules $g_1: X \to X'$ and a map

$$f_1: Y_1 \to Y'_1$$

such that

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, "f_1"] & R_1 \arrow[r] \arrow[d, "g_1 \otimes 1 \otimes 1"] & R_0 \arrow[d, "g_1 \otimes 1"] \\ Y'_1 \arrow[r] & R'_1 \arrow[r] & R'_0 \end{tikzcd}$$

is a map of triangles. Thus f_1 can be represented by a matrix

$$\begin{bmatrix} g_1 \otimes 1 & \tilde{g}_2 \\ 0 & g_1 \otimes 1 \otimes 1 \end{bmatrix}.$$

Since g_1 is a map of $H_*(A)$-modules, the matrix above with $\tilde{g}_2 = 0$ also defines a map of triangles. The difference between these two matrices factors as

$$Y_1 \to R_1 \to R'_0[1] \to Y'_1.$$

There is a unique representative for the homotopy class of the middle map of the form $g_2 \otimes 1$ and therefore f_1 has a unique representative of the form

$$\begin{bmatrix} g_1 \otimes 1 & -g_2 \otimes 1 \\ 0 & g_1 \otimes 1 \otimes 1 \end{bmatrix}.$$

The only requirement for (g_1, g_2) to be a map of A_2-modules is that g_1 commutes with the multiplication. This completes the proof for $n = 1$.

Suppose given a map of k-Postnikov systems based on the bar construction. By induction we know that there is a unique map $g = (g_1, \dots, g_k)$ of A_k-modules such that $Y_j \to Y'_j$ is $B_j(g)$ for $j \le k - 1$.

There is a commutative square

$$\begin{tikzcd} Y_k \arrow[r] \arrow[d, "f_k"] & R_k \arrow[r, "i_k"] \arrow[d, "g_1\otimes 1^{\otimes(k+1)}"] & Y_{k-1} \arrow[d, "B_{k-1}(g)"] \\ Y'_k \arrow[r] & R'_k \arrow[r, "i'_k"'] & Y'_{k-1} \end{tikzcd}$$

By Lemma 5.7, $(g_1, \ldots, g_k, 0)$ is an A_{k+1}-module map. Let $d = f_k - B_{k+1}(g_1, \ldots, g_k, 0)$. This difference factors as

$$Y_k \to R_k \to Y'_{k-1}[1] \to Y'_k.$$

The homotopy class of a map from R_k is determined by its effect on homology. Since $R_k \to Y'_k$ factors through $Y'_{k-1}[1]$, it is 0 along R'_k and hence its image lies in the kernel of the map $Y'_k \to R'_k$ which is $X'[k]$.

Therefore it factors through $R'_0[k]$ up to homotopy and the homotopy class is therefore represented uniquely by a map of the form $(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} g_{k+1} \otimes 1$. We conclude that the homotopy class of f_k is equal to that of $B_{k+1}(g_1, \ldots, g_{k+1})$ (note that any choice of g_{k+1} will give an A_{k+1}-map). $\square$

The previous Theorem shows that the functor sending an A_{n+1}-structure to its canonical n-Postnikov system is full and faithful. Corollary 5.6 asserts that this functor is essentially surjective hence it is an equivalence of categories. This completes the proof of Theorem 1.1.

Remark 5.9. It follows from Theorem 5.8 that if $g : X \to X'$ is an A_k-map such that g_1 is an isomorphism then g is also an isomorphism.

Let X be an $H_*(A)$-module. The *moduli groupoid of A_{n+1}-structures on X* is the groupoid with objects A_{n+1}-module structures on X and quasi-isomorphisms g between them with $g_1 = \mathrm{id}$. Note that this is equivalent to the groupoid of A_{n+1}-modules X' together with an isomorphism of $H_*(A)$-modules $X' \to X$.

Corollary 5.10. *The moduli groupoid of A_{n+1}-structures on X is equivalent to the groupoid of n-Postnikov systems for X based on the bar resolution and isomorphisms which are the identity on the bar resolution.*

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Appendix A. Relation between realization of Postnikov systems and chain complexes

In this appendix we explain the relation between Postnikov systems and rigidifying complexes in a homotopy category (see [3] for the general theory of realizing diagrams). We explain this in the setting of model categories (see [4]). The model category $\mathcal{C}$ which is relevant for this paper is the category of differential graded modules over a DGA A with the standard projective model structure (see for example [9]).

Definition A.1. Let $\mathcal{C}$ be a pointed category. A *chain complex* in $\mathcal{C}$ is a sequence of maps in $\mathcal{C}$

$$\cdots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_0$$

such that $dd = *$.

Definition A.2. If $\mathcal{C}$ is a pointed model category, a *Postnikov system* is a commutative diagram in $\mathrm{Ho}(\mathcal{C})$

$$\begin{tikzcd} & Y_n \arrow[d, "j_n"'] & Y_{n-1} \arrow[d, "j_{n-1}"] & & Y_1 \arrow[d, "j_1"'] & \\ \cdots \arrow[r] & C_n \arrow[r] \arrow[ur, "i_{n-1}"] & C_{n-1} \arrow[r] & \cdots & C_1 \arrow[r, "i_0"'] & C_0 \end{tikzcd}$$

where for each k, the sequence

$$Y_k \xrightarrow{j_k} C_k \xrightarrow{i_{k-1}} Y_{k-1}$$

is a homotopy fiber sequence (we set $Y_0 = C_0$).

An *m-Postnikov system* is a diagram as above but with objects only those Y_i and C_i where $i \geq m+1$.

Note that in a Postnikov system, $C_\bullet$ is a chain complex in $\mathrm{Ho}(\mathcal{C})$.

Definition A.3. Let $\mathcal{C}$ be a model category, $\pi : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the canonical functor and I be a small category. A diagram $F : I \to \mathrm{Ho}(\mathcal{C})$ is *realizable* if there exists a diagram $\tilde{F} : I \to \mathcal{C}$ together with a natural isomorphism $\phi : \pi \tilde{F} \to F$. The diagram $\tilde{F}$ is then called a *realization* of F.

If $\mathcal{C}$ is pointed we say that a diagram is *strictly realizable* if $F(\alpha) = *$ implies that $\tilde{F}(\alpha) = *$ and $\tilde{F}$ is then called a strict realization of F.

Proposition A.4. *Let $\mathcal{C}$ be a pointed model category. Let $C_\bullet$ be a chain complex in* $\mathrm{Ho}(\mathcal{C})$. *Then the following are equivalent:*

(i) *$C_\bullet$ is strictly realizable,*
(ii) *$C_\bullet$ extends to a Postnikov system,*

Proof. (i) $\Rightarrow$ (ii): Replacing $C_\bullet$ if necessary by an isomorphic complex we may assume that

$$\cdots \to C_n \xrightarrow{d_{n-1}} C_{n-1} \to \cdots$$

is a chain complex in $\mathcal{C}$ projecting to $C_\bullet$.

Replacing the map $C_1 \xrightarrow{d_0} C_0$ by a fibration we obtain a diagram

$$\begin{tikzcd} C_3 \arrow[r, "d_2"] & C_2 \arrow[r, "d_1"] \arrow[d, dashed, "i_1"'] & C_1 \arrow[r, "d_0"] \arrow[d, "\sim"'] & C_0 \\ & Y_1 \arrow[r] \arrow[rr, bend right] & C_1' \arrow[ur] & * \arrow[u] \end{tikzcd}$$

where Y_1 is the homotopy fiber of $C_1' \to C_0$. Since $d_0 d_1 = *$, there is a canonical factorization $C_2 \xrightarrow{i_1} Y_1$. Furthermore, the composite

$$C_3 \xrightarrow{d_2} C_2 \xrightarrow{i_1} Y_1$$

is the zero map since its composite with the map $Y_1 \to C_1'$ is zero by the construction of i_1.

We may apply the same procedure to the sequence of maps

$$\cdots \to C_3 \xrightarrow{d_2} C_2 \xrightarrow{i_1} Y_1$$

and continuing inductively we obtain a Postnikov system which we denote by $P(C_\bullet)$.

This construction is clearly functorial so we have defined a functor

$$P : \mathcal{CC} \longrightarrow \mathcal{PS} \tag{A.1}$$

from the category of chain complexes in $\mathcal{C}$ to the category of Posnikov systems in $\mathrm{Ho}(\mathcal{C})$ which sends weak equivalences to isomorphisms.

(ii) ⇒ (i): Let

$$\begin{tikzcd}
 & Y_n \arrow[d, "j_n"'] & Y_{n-1} \arrow[d, "j_{n-1}"] & & Y_1 \arrow[d, "j_1"'] & \\
\cdots \arrow[r] & C_n \arrow[r] \arrow[ur, "i_{n-1}"] & C_{n-1} \arrow[r] & \cdots & C_1 \arrow[r, "i_0"'] & C_0
\end{tikzcd}$$

be a Postnikov system in $\mathrm{Ho}(\mathcal{C})$.

We will write $\overline{f}$ for an arbitrary representative of the map $f \in \mathrm{Ho}(\mathcal{C})$ and $[\psi]$ for the homotopy class of $\psi \in \mathcal{C}$.

First note that we can assume that all the objects Y_k and C_k are fibrant and cofibrant. We will construct a chain complex $\tilde{C}_\bullet$ in $\mathcal{C}$ lifting $C_\bullet$ inductively.

Let $\tilde{C}_0 = C_0$. Let

$$\begin{tikzcd}
C_1 \arrow[r, "\phi_1"] \arrow[d, "\overline{d}_0"'] & \tilde{C}_1 \arrow[dl, two heads, "\tilde{d}_0"] \\
C_0 &
\end{tikzcd}$$

be a factorization of $\overline{d}_0$ into a trivial cofibration followed by a fibration and

$$\tilde{Y}_1 \xrightarrow{\tilde{j}_1} \tilde{C}_1$$

be the inclusion of the fiber of $\tilde{d}_0$. Since Y_1 is the homotopy fiber of d_0, there is an isomorphism $\psi_1 \colon Y_1 \to \tilde{Y}_1$ such that

$$\begin{tikzcd}
Y_1 \arrow[r, "\psi_1"] \arrow[d, "j_1"'] & \tilde{Y}_1 \arrow[d, "{[\tilde{j}_1]}"] \\
C_1 \arrow[r, "{[\phi_1]}"] & \tilde{C}_1
\end{tikzcd}$$

commutes.

Now factor $\overline{\psi_1}\ \overline{i}_1 \colon C_2 \to \tilde{Y}_1$ (which exists because C_2 is cofibrant and $\tilde{Y}_1$ is fibrant) as a trivial cofibration ϕ_2 followed by a fibration $\tilde{i}_1$. We get a commutative diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\phi_2"] \arrow[d, "\overline{i_1}"'] & \tilde{C}_2 \arrow[d, two heads, "\tilde{i}_1"'] \\
Y_1 \arrow[r, "\overline{\psi_1}"'] & \tilde{Y}_1
\end{tikzcd}$$

Let $\tilde{d}_1 = \tilde{j}_1\tilde{i}_1$. Since $\tilde{Y}_1$ is the fiber of $\tilde{d}_0$ it follows that the composite $\tilde{d}_0\tilde{d}_1$ is the zero map.

Let $\tilde{j}_2 \colon \tilde{Y}_2 \to C_2$ denote the inclusion of the fiber of $\tilde{i}_1$. Since $Y_2 \to C_2 \to Y_1$ is a fiber sequence, there is an isomorphism $\psi_2 \colon Y_2 \to \tilde{Y}_2$ in $\mathrm{Ho}(\mathcal{C})$ such that

$$\begin{tikzcd}
Y_2 \arrow[r, "\psi_2"] \arrow[d, "j_2"'] & \tilde{Y}_2 \arrow[d, "{[\tilde{j}_2]}"] \\
C_2 \arrow[r, "{[\phi_2]}"] & \tilde{C}_2
\end{tikzcd}$$

commutes and we can proceed inductively to obtain a realization $\tilde{C}_\bullet$ of $C_\bullet$. □

Remark A.5. The statements in Proposition A.4 are equivalent to the vanishing of the *Toda brackets* $\langle d_0, \ldots, d_n\rangle$ for all $n \geq 2$. The Toda bracket can be defined in several different ways. We use the following definition: $\langle d_0, \ldots, d_n\rangle$ is a subset (possibly empty) of $\mathrm{Ho}(\mathcal{C})(C_{n+1}, \Omega^{n-1}C_0)$ consisting of all possible lifts ϕ in the diagrams of the form (A.2), for all choices of n-Postnikov systems extending $C_n \to \cdots \to C_0$.

$$\begin{tikzcd}
 & & \Omega^{n-1}C_0 \arrow[d] & & & \\
 & & \Omega^{n-2}Y_1 \arrow[d] & & & \\
 & & \vdots \arrow[d] & & & \\
 & & \Omega Y^{n-2} \arrow[d] & & & \\
 & & Y_{n-1} \arrow[d, "j_{n-1}"] & & Y_1 \arrow[d] & \\
C_{n+1} \arrow[r, "d_n"'] \arrow[uuuuurr, dashed, "\phi"] & C_n \arrow[r, "d_{n-1}"'] \arrow[ur, "i_{n-1}"] & C_{n-1} \arrow[r, "d_{n-2}"'] & \cdots \arrow[r] & C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd} \tag{A.2}$$

Ω^j denotes the jth iterate of the loop functor and the vertical maps belong to the homotopy fiber sequences which end in $Y_k \to C_k \to Y_{k-1}$ (see [4, Chapter 6]).

We say that a Toda bracket vanishes if it contains the zero map. It is clear that the n-Postnikov system in (A.2) extends one stage if and only if ϕ can be chosen to be zero. Thus, an n-Postnikov system encodes the vanishing of the Toda bracket of the maps in the underlying chain complex.

The higher order cohomology operations in [8, 16.3] are defined as Toda brackets with the above definition. The definition of Toda bracket in [12, IV.1] is very similar. Whitehead works in a stable setting where cofiber and fiber sequences are equivalent. To define the Toda bracket of $\langle d_0, \ldots, d_n\rangle$ he considers all possible diagrams[5]

$$\begin{tikzcd}
 & X_{n-1} \arrow[dr] & X_{n-2} & & X_1 \arrow[dr, "i_1"] & \\
X_n = C_{n+1} \arrow[r, "d_n"'] & C_n \arrow[r, "d_{n-1}"'] \arrow[u] & C_{n-1} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_2 \arrow[r, "d_1"'] \arrow[u] & C_1
\end{tikzcd}$$

where the sequences $X_i \to C_i \to X_{i-1}$ are cofiber sequences and defines the Toda bracket to be the set of all possible extensions of $d_0 i_1$ along

$$X_1 \to \Sigma X_2 \to \cdots \to \Sigma^{n-1}C_n,$$

where Σ denotes the suspension functor.

It is possible to check that our definition and Whitehead's agree by exhibiting both sets as certain choices of $(n-1)$-spheres $\partial\Delta^n \subset \mathrm{Hom}(C_n, C_0)$ in the homotopy function complex from C_n to C_0. For more on this perspective, see [2, Examples 3.10,3.20].

Proposition A.6. *Let $\mathcal{C}$ be a pointed model category, $\mathcal{CC}$ be the category of length n chain complexes in $\mathcal{C}$ (with $n \le \infty$) and $\mathcal{PS}$ be the category of n-Postnikov systems in* $\mathrm{Ho}(\mathcal{C})$. *Then the functor*

$$P : \mathcal{CC} \to \mathcal{PS}$$

(see (A.1)) induces a bijection from weak equivalence classes in $\mathcal{CC}$ to isomorphism classes of objects in $\mathcal{PS}$.

[5] It is easy to check using the limited naturality of triangles that in the definition of Toda bracket in [12] we may assume that either the map i_0 or j_n is the identity and we are taking j_n to be the identity.

Proof. A chain complex realizing a Postnikov system will, by definition, realize any equivalent Postnikov system so we have already proved in Proposition A.4 that the functor P is essentially surjective.

On the other hand using the homotopy lifting property for fibrations, the construction of the chain complex from the Postnikov system in Proposition A.4 will also yield lifts of isomorphisms between Postnikov systems to weak equivalences between chain complexes in $\mathcal{C}$ (because the $\tilde{C}_i$ are fibrant and cofibrant, the $\tilde{Y}_i$ are the actual fibers of maps and the maps $\tilde{C}_{i+1} \to \tilde{Y}_i$ are fibrations). $\square$

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