On the Planarity of Hanoi Graphs

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Abstract. Hanoi graphs are the state graphs for Tower of Hanoi problems with three or more pegs. We prove hamiltonicity and present a complete analysis of planarity of these graphs.

1 Hanoi graphs

The Tower of Hanoi puzzle (cf., e.g., [5]) consists of n discs of different size distributed among 3 + m vertical pegs (m, n ∈ No) in such a way that only the topmost disc on a peg can be moved to the top position on another peg with the additional restriction that no larger disc may be placed on a smaller one (divine rule). Starting from a perfect state, where all discs are on one and the same peg in natural order with the smallest on top, this means that any reachable state is regular and can be represented by a unique element s of V^m n := \{0, \ldots, 2 + m\}^n; here s_d is the peg where disc d, numbered according to size, is lying. With this set as the vertex set, the Hanoi graph \( H^m_n \) is defined if an edge is a pair of vertices obtained from each other by a legal move of a single disc. See Figure 1 for the example of \( H^3_5 \) (with the vertex s labeled as s1s2s3).

A lot is known for the graphs \( H^m_n \). They first appeared in [12], where the proof of (2-)connectedness and planarity is sketched. They were named Hanoi graphs by Lu [8], who showed that they are hamiltonian. (Hanoi graphs with n > 1 cannot be eulerian, since there are always vertices whose degrees are odd; cf. [13, Theorem 6B].) For more detailed discussions of properties of \( H^m_0 \), see [10, 9].

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Fig. 1 Hanoi graph $H^3_0$

Obviously, Hanoi graphs have been employed to visualize and analyze tasks in the Tower of Hanoi puzzle, the classical one being to find a shortest path from one perfect state to another. For $m = 0$, the length of this path, $2^n - 1$, is also the diameter of $H^n_0$. But while $H^n_m$ is still simple and $(2+m)$-connected for $m \geq 1$, its diameter, apart from some trivial cases, is not known, and it is even not clear, if it is assumed by the distance of two distinct perfect states! Among the many nice results about the graphs $H^n_0$, let us mention their isomorphy to Sierpiński triangles and (finite cuts of) Pascal’s arithmetical triangle (cf. [4, 10]). This has been used to determine the average distance of two points on the Sierpiński gasket (cf. [6]) by starting from the corresponding value on $H^n_0$, namely

$$\frac{466}{885} \cdot 2^n - \frac{1}{3} \left( \frac{12}{59} + \frac{18}{1003} \sqrt{17} \right) \left( \frac{5 + \sqrt{17}}{18} \right)^n - \frac{3}{5} \left( \frac{1}{3} \right)^n + \left( \frac{12}{59} - \frac{18}{1003} \sqrt{17} \right) \left( \frac{5 - \sqrt{17}}{18} \right)^n$$

(cf. [3, Proposition 7]), rescaling, i.e. dividing by the diameter $2^n - 1$, and letting $n$ tend to infinity. Another interesting result is that the graphs $H^n_0$ support perfect error-correcting codes (cf. [2]).

As mentioned before, very little is known about the general case of $H^n_m$. Let us start by proving hamiltonicity.

**Theorem 1.** *Every Hanoi graph is hamiltonian.*

**Proof.** a) For $n = 0$, this is trivial.
i) By induction on $n \in \mathbb{N}$ we show that there is a hamiltonian path starting and ending in distinct perfect states. This is trivial for $n = 1$. Now let disc $n + 1$ move stepwise from peg 0 to 1 and so on to $2 + m$, say. Before each such step, a hamiltonian path in $H^m_n$ can be performed to transfer the tower consisting of the $n$ smaller discs to a peg allowing disc $n + 1$ to move. Finally, after the last move of disc $n + 1$, the $n$-tower has to be transferred to peg 2, again on a hamiltonian path through $H^m_n$.

ii) A hamiltonian circuit on $H^{n+1}_m$, $n \in \mathbb{N}_0$, can now be constructed by starting in state $(1, \ldots, 1, 0) \in V^{n+1}_m$, say, and transferring the tower of $n$ smaller discs on hamiltonian paths according to (i) in a cyclic fashion from peg to peg, each complete transfer being followed by a single move of disc $n + 1$ to the next peg in the same direction. 

Obviously, the number of vertices in $H^n_m$ is $|V^n_m| = (3 + m)^n$ and, less obviously, the number of edges is $|E^n_m| = \frac{(3 + m)(2 + m)}{4} \left((3 + m)^n - (1 + m)^n\right)$ (cf. [7, Corollary 3.3], where this result has been used to obtain some combinatorial identities). A notorious open question is to find the minimum number of moves in the classical Tower of Hanoi problem, i.e. perfect to perfect, for more than three pegs. For instance, in the case $m = 1$, this has been resolved so far only for $n \leq 17$ by an exhaustive inspection of the corresponding graphs with the aid of a computer (cf. [1]). It would therefore be desirable to have more, both qualitative and quantitative, information on Hanoi graphs. From some graphical representations by A. Rukhin [11], we were inspired to settle the question of planarity.

\section{Planarity}

The main result of this note is the following.

**Theorem 2.** The only planar Hanoi graphs are $H^n_m$, $H^n_0$, $H^1_1$ and $H^1_2$ ($m, n \in \mathbb{N}_0$).

**Proof.** The case of $H^n_0$ is trivial.

Planarity of $H^n_0$ is shown by proving the following statement by induction on $n \in \mathbb{N}$: $H^n_0$ can be represented by a plane graph, the infinite face (cf. [13, p. 65]) of which is the complement of an equilateral triangle (of side length $2^n - 1$, say) whose corners are the perfect states. This is obvious for $n = 1$. Assume that the statement has been proved for $n \geq 1$. Now take three copies of these plane graphs isomorphic to $H^n_0$, one for each different position of disc $n + 1$. Then they can be arranged and joined by three new edges in the plane in an appropriate way to form the desired plane graph which is isomorphic to $H^{n+1}_0$. The example in Figure 1 shows how to get from $n = 1$ to $n = 2$ and from $n = 2$ to $n = 3$. 

$H_1^3$ is isomorphic to the complete graph $K_4$ and therefore planar. The proof of planarity of $H_2^7$ is by constructing the plane representation in Figure 2.

![Figure 2 Hanoi graph $H_1^3$](image)

We now show that $H_1^3$ is not planar. This will complete the proof, since all $H_1^n$ with $n > 3$ contain (an isomorphic copy of) $H_1^3$ as a subgraph and every $H_m^n$ with $m > 1$ and $n > 0$ contains some $H_2^2$ as a subgraph (in any regular state, the smallest disc can choose among at least five pegs), which is isomorphic to the non-planar $K_5$ (cf. [13, Theorem 12A]).

By Kuratowski's Theorem (cf. [13, Theorem 12B]), we can use the same kind of argument to prove non-planarity of $H_1^3$ by devising a subgraph which is homeomorphic to $K_5$ or $K_{3,3}$. However, we want to present two alternative and more elegant arguments.

Assume that we have a plane representation of $H_1^3$, consisting of four copies of $H_1^2$ (one for each position of disc 3) which are interconnected by altogether 24 extra edges, as can be seen by a combinatorial analysis or by recourse to the above formula for the numbers of edges $|E_2^2| = 36$ and $|E_1^3| = 168$. The faces of $H_1^3$ cannot have more than four vertices each, as can be seen by isomorphy to the graph depicted in Figure 2. In particular, the infinite face has either three or four vertices. In the former case, these three vertices have at least degree 3 in $H_1^2$ and since there are 12 edges going out from $H_1^2$ (moves of disc 3), at least one of the vertices gets another 4 degrees. However, all vertices in $H_1^3$ have degrees strictly less than 7. (In any state of the Tower of Hanoi with four pegs, disc 1 can move to exactly three different goals, disc 2 to at most two, whereas disc 3 can move to
two different goals only if disc 2 can not move at all.) Similarly, if the infinite face of one
of the copies of $H^2_1$ has four vertices, then another look at Figure 2 reveals that they all
have degree 5 in $H^2_1$ and at least one of them has to get at least three more edges incident
to it, again leading to a degree of more than 6.

Another alternative for the disproof of planarity of $H^3_1$ can be based on Euler's formula
for plane graphs. By inspection of Figure 2 one realizes that each face of $H^3_1$ has either
two or three (adjacent) vertices (if at all) allowing the largest disc to move. By dismissing
the other vertices, we arrive at a subgraph of $H^3_1$ consisting of $l \in \{0, 1, 2, 3, 4\}$ copies
of $K_2$ and $4 - l$ copies of $K_3$ which are interconnected by 24 edges. That is, for this
graph, the number of vertices is $v = 2l + 3(4 - l) = 12 - l$, and the number of edges is
e$ = l + 3(4 - l) + 24 = 36 - 2l$. But then $3v - 6 = 30 - 3l < 36 - 2l = e$, such that this
graph, and consequently $H^3_1$, cannot be planar by [13, Corollary 13d].

For those Hanoi graphs which are not planar, an interesting problem is to find out their
genera (cf. [13, p. 70]). It took almost 80 years from the conjecture of Heawood (1890) to
the proof of Ringel and Youngs (1968) to settle the question for complete graphs, i.e. for
Hanoi graphs with $n = 1$, with the result that
\[
g \left( H^1_m \right) = \left\lceil \frac{m(m - 1)}{12} \right\rceil
\]
(cf. [13, Theorem 14d]). The question for other non-planar Hanoi graphs is open.

Acknowledgement. We thank Christian Clason (TU München) and Wolfgang Sobetzki (EADS
München) for technical support.

References

139(1999), 113–122.

208/209(1999), 157–175.


538–544.


Received: 18.02.2002