Historia Mathematica **26** (1999), 125–160
Article ID hmat.1999.2233, available online at http://www.idealibrary.com on IDEAL®

Arthur Cayley as Sadleirian Professor: A Glimpse of Mathematics

Tony Crilly

Middlesex University Business School, The Burroughs, London NW4 4BT, UK

E-mail: t.crilly@mdx.ac.uk

This article contains the hitherto unpublished text of Arthur Cayley's inaugural professorial lecture given at Cambridge University on 3 November 1863. Cayley chose a historical treatment to explain the prevalent basic notions of analytical geometry, concentrating his attention in the period (1638–1750). Topics Cayley discussed include the geometric interpretation of complex numbers, the theory of pole and polar, points and lines at infinity, plane curves, the projective definition of distance, and Pascal's and Maclaurin's geometrical theorems. The paper provides a commentary on this lecture with reference to Cayley's work in geometry. The ambience of Cambridge mathematics as it existed after 1863 is briefly discussed. © 1999 Academic Press

Cet article contient le texte jusqu'ici inédit de la leçon inaugurale de Arthur Cayley donnée à l'Université de Cambridge le 3 novembre 1863. Cayley choisit une approche historique pour expliquer les notions fondamentales de la géométrie analytique, qui existaient alors, en concentrant son attention sur la période 1638–1750. Les sujets discutés incluent l'interpretation géométrique des nombres complexes, la théorie des pôles et des polaires, les points et les lignes à l'infini, les courbes planes, la définition projective de la distance, et les théorèmes géométriques de Pascal et de Maclaurin. L'article contient aussi un commentaire reliant cette leçon à l'oeuvre de Cayley en géométrie. L'atmosphère des mathématiques à Cambridge après 1863 est brièvement discutée. © 1999 Academic Press

MSC Classification: 01A55, 01A72, 01A73.
Key Words: Arthur Cayley; Sadleirian Chair of Pure Mathematics; geometry; English university education.

1. INTRODUCTION

Arthur Cayley (1821–1895) (see Fig. 1) was the first Sadleirian Professor of Pure Mathematics at the University of Cambridge, England [62].[1] Elected in 1863, Cayley felt it important to make a vigorous start in the role which was intended for him: to bring his knowledge of current mathematics to university teaching and research. Although Cayley was a prolific mathematician (his *Collected Mathematical Papers* occupy 13 large volumes) and he published virtually all his work, the notes he made in preparation for his first lecture at Cambridge have never been published. Their importance derives partially from the centrality of Cambridge to British mathematics and from Cayley's pivotal role there.

The text of Cayley's lecture (very likely the only manuscript of his lectures which has survived) also sheds light on his convictions about the nature of geometry itself. Nearly

[1] The holders of this chair have been Cayley (elected 1863); Andrew R. Forsyth (1895); Ernest W. Hobson (1910); G[odfrey] H[arold] Hardy (1931); Louis J. Mordell (1945); Philip Hall (1953); and John W. S. Cassels (1967). The present holder is John H. Coates (1986).

0315-0860/99 $30.00

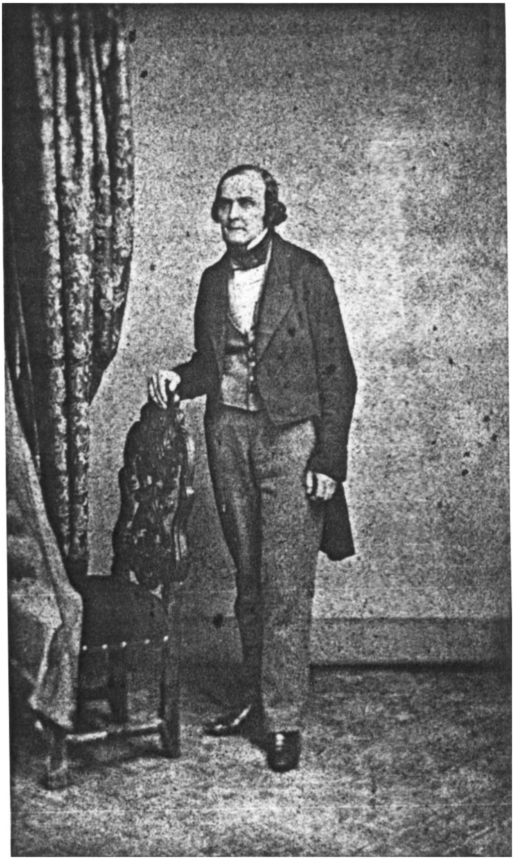

FIG. 1. Arthur Cayley (1821–1895). Photograph courtesy of St. John's College, Cambridge.

all of his mathematical writing was directed toward the research community, but in the lecture he addressed the needs of students. For Cayley, with his characteristically severe style of presenting mathematical papers with little accompanying exposition (unlike his friend, James Joseph Sylvester), the task of preparing a lecture for a broad audience would not have been effortless. By the time of his election to the chair, Cayley had secured a reputation as a research mathematician, but in preparation for his inaugural lecture, he was faced with the challenge of *explaining*.

2. THE INSIDER

Cayley's previous 14 years had been spent at the Bar where he specialized in drafting legal documents such as parliamentary drafts, family settlements, and wills. His mathematical career had continued in parallel with his legal work, but from the mid-1850s he indicated a preference for a position in academic life. Indeed, he had sought one on several occasions prior to 1863, but had been unsuccessful.[2] His competitors for the Sadleirian Chair were

[2] Cayley had applied for the chair of natural philosophy at Marischal College, Aberdeen in 1856, the "professorship" of mathematics at the aborted Western University of Great Britain at Neath in South Wales in 1857, and

all products of the Cambridge Mathematical Tripos and had been top students.[3] (Like Cayley, three of them—Todhunter, Ferrers, and Routh—had been the "Senior Wrangler" of their year, the champion student, and this was a strong point in their favor. Thirty years previously it would have been sufficient.) All of them were well-known Cambridge teachers of mathematics.

Cayley had little experience of teaching at any level, but his research record was startling: at this stage in his career he had published approximately 340 mathematical papers. None of his competitors could match this performance. Moreover, unlike his competitors, Cayley had played a significant part in the organization of science in London during the 1850s. He had been a Council member for both the Royal Society and the Royal Astronomical Society and had shown himself to be an efficient administrator. He was the obvious favorite for election. The university position at Cambridge would give him time to pursue research uninterrupted by the chore of drafting documents, though it probably meant a reduced income.[4] This did not deter him, and he was duly elected. On appointment to the Cambridge chair, Cayley achieved the position he most desired, whereupon a legal colleague wrote about him "now happily been transferred to a more congenial employment" [54, 1067].

The year 1863 was thus a turning point in Cayley's life. He was elected on 10 June and was married to Susan Moline on 8 September. Moreover, the first lecture on 3 November marked the University's public recognition of his work and, by founding a chair exclusive to "Pure Mathematics," its desire to promote his subject.

The Sadleirian Chair replaced lectureships in algebra established at Cambridge in the early eighteenth century. Lady Sadleir[5] died on 30 September 1706, and her will provided

was attracted to the Lowndean Chair of Geometry and Astronomy at Cambridge in 1858. In 1859, he expressed a strong interest in applying for the chair of astronomy at Glasgow University, a position that was coupled with the directorship of Glasgow Observatory.

[3] Cayley's competitors were Thomas Gaskin (c. 1809–1887), Percival Frost (1817–1898), James George Mould (1818–1902), Isaac Todhunter (1820–1884), Norman Macleod Ferrers (1829–1903), Edward John Routh (1831–1907), and John Clough Williams-Ellis (1833–1913) [12]. Todhunter, a scholar largely unappreciated today, was a college tutor and private tutor who produced a range of elementary textbooks which formed the backbone of English education in mathematics during the second half of the nineteenth century. Ferrers was editor of several British mathematical journals (*Cambridge and Dublin Mathematical Journal*, *Quarterly Journal of Pure and Applied Mathematics*) who became Master of his college (Gonville and Caius) and Vice-Chancellor of the university. Routh (like Todhunter) had been taught by Augustus De Morgan in London and became a private tutor. Ellis, the youngest applicant, was Third Wrangler in 1856. He was a Sadleirian lecturer but had little chance in the professorial contest. He became a celebrated mountaineer and father of Bertram Clough Williams-Ellis (1883–1978), the architect who modelled an entire village in North Wales on the Italian town of Portofino.

[4] In being elected to a Cambridge chair, Cayley enjoyed a high social standing if not a commensurate salary. For the first year, he received a stipend of £509 12s 9d, without the supplement of a college fellowship. Henry Fawcett (1883–1884), who was elected to the chair of political economy on 27 November 1863, received a similar sum for his professorship and fellowship combined. Charles Kingsley, as newly elected Regius Professor of Modern History in 1860, with a stipend of £370, soon found this did not allow him to keep up two houses, and he travelled to Cambridge only when necessary. Cayley did not follow the practice of being an "absentee professor," and he took up residence in Cambridge, purchasing Garden House, Little St. Mary's Lane, in 1864. By 1882, when new statutes of the university were enacted, the stipend for the Sadleirian Chair was £850. A Victorian's social standing was determined by birth, education, and professional status as well as salary. A professor at Cambridge would be comfortably off and could afford a large house and servants. The social class with salaries in the range £200–£1000 included well-to-do clergymen and the lesser gentry, while the upper middle class with incomes in excess of £1000 included the higher clergy and successful merchants [76].

[5] Cayley preferred the spelling convention "Sadlerian" but "Sadleirian" is used widely.

for the stipends of lecturers in algebra in different Cambridge colleges.[6] These lectureships had been generally held by college tutors whose prime duty was to teach various subjects which could be regarded as "algebraic." For example, in 1847, Samuel Blackall's lectures at St. John's College evidently took the form of setting 6 to 10 questions which included the elements of continued fractions and the theory of probability.[7] In 1852, the Report of the Royal Commissioners (one of whom was George Peacock, then Dean of Ely but still Lowndean Professor or Astronomy and Geometry) looking into the state of affairs at Oxford and Cambridge remarked on the unsatisfactory use of the Sadleir endowment. This Report found expression in the ensuing Cambridge University Act (1856).[8] The newly formed Council of the University Senate proposed that "a new direction should be given to the Endowment" by the establishment of the chair. This proposal was approved by the University Senate on 26 November 1857 and formally created by the Queen in Council in 1860. The restriction to algebra of the Sadleirian lectureships was removed to take into account modern developments in mathematics.

Cambridge University, which saw itself as a teaching institution, was slowly beginning to make moves to adopt the German ideal of education: that an educational institution was responsible not only for transmitting information but also for generating it. There was also a perceived gap between the twin activities of teaching and publishing new work. For the first half of the 19th century, the professors at Cambridge were separated from the teaching function of the colleges and, if their lectures were given at all, they rarely impinged on students' learning. This is understandable in the case of the professors of chemistry, botany, geology, natural philosophy, and mineralogy, for there were no undergraduates studying these subjects. Before 1824, the only route to a Cambridge honors degree was to study mathematics in the Mathematical Tripos while after that date, until midcentury, a student who wished to sit the Classical Tripos degree had first to appear on the Mathematical Tripos "order of merit" (that is, the published list of students who succeeded in the Mathematical Tripos examination and were awarded honors degrees).[9] Mathematical accomplishment was concentrated at the top end of the order of merit. Charles Kingsley, who graduated in the same year as Cayley (1842), but unlike him, intent on a Classical Tripos degree, was quite astonished to gain a second class honors degree in mathematics with "very little reading" [74, 1:38].

[6] The salary paid to the Sadleirian lectures was originally fixed at a minimum of £20/annum except at Emmanuel College (Lady Sadleir's first husband's college), where the sum was £30. The suitability of candidates was judged by the Lucasian Professor, the senior mathematics professor at Cambridge, and lecturers were appointed for 10 years. They were established at Emmanuel but spread to other colleges [47, 268–269; 52, 4:77]. By 1857, the stipends paid to Sadleirian lecturers had trebled and represented a useful supplement to income derived from a college fellowship.

[7] Samuel Blackall (1816–1899) from St. John's College was Fourth Wrangler in 1838 and was ordained an Anglican priest in 1842. He was a well-known teacher of mathematics to undergraduates but left Cambridge in 1847 to follow a church career.

[8] A reform contained in the modernising 1856 Act was the removal of the requirement for subscription to the 39 articles of the Anglican Church as a qualification for admission to most bachelor's degrees. The governance of the university was reformed, and one of the first acts of the Council of the Senate was to propose the establishment of the Sadleirian chair.

[9] Members of the aristocracy were excepted from this rule. Though Cayley himself was distantly related to the holder of the Cayley baronetcy whose seat was in Yorkshire, he had entered Trinity College in the normal mode of entry, as a *pensioner* (a student responsible for the payment of the full college fees).

It is bizarre that students of ability had no need to attend the mathematical professors' lectures and were actively dissuaded from doing so by their college tutors. The Mathematical Tripos underwent one of its many upheavals around 1850, and it was proposed that professors should have more influence in the education of undergraduates. In 1848, the Board of Mathematical Studies was established "with a view to encourage attendance at the Lectures of the Mathematical Professors, and to secure a correspondence between those Lectures and the Mathematical Examinations of the University; and also as a means of communicating to the Students themselves, from a body of experienced Examiners and Lectures, correct views of the nature and objects of our Mathematical Examinations" [11].[10]

By the time Cayley was elected 15 years later, this objective was still not realized.[11] At the age of 42, Cayley was one of the "new professors" (to use Sheldon Rothblatt's term [90])—men promoted on the strength of their research capability but with an understanding that their lectures should be made relevant to undergraduates. In terms of undergraduate teaching, Cayley was faced with the ingrained attitudes against professorial lectures from dons (the generic term for fellows and tutors of Oxford and Cambridge colleges) who aligned the study of mathematics with the aims of a "liberal education" (that is, mathematics, as a subject based on logic, was ideally suited for training the mind in preparation for future careers outside academe). The outward sign of the success of the university in mathematics was not new mathematics, but the publication year after year of the famed order of merit. In the 1860s, mathematical research was of little consequence, and to those engaged with daily teaching in the colleges, it was an irrelevance [72; 90; 91]. By this time, the Mathematical Tripos had lost its uniqueness as the only pathway to a Cambridge degree, but it still retained its immense prestige—it was still the "flagship" degree at Cambridge. In the *Student Guide* (1862), readers were advised that the existence of the Tripos was sufficient for the reputation of Cambridge: "The Mathematical Examination of Cambridge is widely celebrated, and has given to this University its character of the Mathematical University *par excellence*" [97, 15].

The study of mathematics for its own sake, Cayley's strongly held position, was thus not generally felt. The Mathematical Tripos remained the fulcrum on which all turned. In maintaining this *status quo*, William Whewell (1794–1866), the Master of Trinity College and a dominant Cambridge presence, was the guiding spirit. In recognition of the historic commitment of Cambridge colleges to teaching undergraduates and Whewell's desire to make professors' lectures relevant to undergraduates, the teaching function was included in the terms of the Sadleirian chair. The new impetus for research was partially recognised, and the terms of reference called upon Cayley to perform a dual function of teaching *and* research: "it shall be the duty of the Professor to explain and teach the principles of Pure Mathematics and to apply himself to the advancement of that Science" [62, xvi].[12] The

[10] The resolution to establish the Board of Mathematical Studies was adopted by Grace of the Senate 31 October 1848, and it met on 7 December 1848.

[11] With the creation of the Sadleirian Chair, Cayley joined the three other mathematical professors: Lucasian Professor of Mathematics (founded in 1663) held (from 1849) by George Gabriel Stokes (1819–1903); the Plumian Professor of Astronomy and Experimental Philosophy (founded in 1704) held (from 1836) by James Challis (1803–1882); and the Lowndean Professor of Astronomy and Geometry (founded in 1749) held (from 1859) by John Couch Adams (1819–1892).

[12] These terms became Cayley's mantra, and when challenged on his role as professor of pure mathematics or on curriculum reform in the Tripos during the 1860s and 1870s, he used it as a ready first line of defence.

German ideal manifested itself in this dual role and marked Cayley off from the traditional Cambridge professor.[13]

Cayley was best qualified to make a mark in the two roles of teaching and research through his expertise in analytical geometry. Following a French influence, geometry based on algebra had played a part in the Tripos since the beginning of the century, and, in the 1830s, several English texts had appeared [2, 19–20]. During the first 30 years of the century, analytical geometry was firmly established in England, and it had shaped Cayley's own mathematical education. Influential textbooks appeared including Henry Parr Hamilton's *Principles of Analytical Geometry* (1826) and John Hymers's *Analytical Geometry of Three Dimensions* (1830) and *Treatise on Conic Sections* (1837). However, promising students of this period were also introduced to French geometrical texts during their first term at Cambridge (through the supervision of their private tutors ("coaches"). For example, Francis Galton's tutor in 1839, Matthew O'Brien, introduced him to Jean-Louis Boucharlat's *Théorie des courbes et des surfaces du second ordre* (1810) during his first year at Trinity College [79, 1:144 f.n.]). Works such as Bourcharlat's treated the material from a mature standpoint and, by Cayley's undergraduate period (1838–1842), had helped to establish a prominent place for analytical geometry in the Tripos syllabus [64].

By the time Cayley was a young fellow of Trinity College, he was impatient to see the latest continental ideas becoming available to Cambridge undergraduates. Just after he made acquaintance with Sylvester around 1847, he wrote to William Thomson that "Sylvester is going to translate them [Michel Chasles's current lectures on geometry] so that at last one will have an English book on Modern Geometry: I hope it will get introduced into the Cambridge course" [44].[14] It was a relevant subject for undergraduates and a wise choice for his inaugural lecture in 1863. Geometry, both ancient and modern, was still central to the education of students at Cambridge. All students studied Euclid's *Elements*. Even the ordinary (poll) degree students had to satisfy the examiners with a (very) basic knowledge of its contents. An American student at Cambridge in the early 1860s, William Everett, identified Euclid as the basis of mathematical education in the university: "whereupon was raised the superstructure of Newtonian mathematics. Dear, indeed, to a Cambridge man is Euclid. His faith in it is truly sublime. It is to him not an author, but a system of demonstration, a science, a philosophy" [61, 41].

In 1863, the first-year course in mathematics in most Cambridge colleges comprised Euclid, algebra, plane trigonometry, and analytical geometry. Analytical geometry was regarded as so extensive that it should be commenced as soon as possible. It was even hoped (a pious hope for most Cambridge aspirants who came to take a poll degree and

[13] Although Charles Kingsley, the Regius Professor of Modern History, and Cayley were elected to their chairs only three years apart, Cayley was the "new professor" while Kingsley was locked into a former age, despite the fact that both graduated in the same year at Cambridge. According to Sheldon Rothblatt, Kingsley displayed all the hallmarks of *donnishness*—his brand of history was quaint and his behaviour bordered on the eccentric [90, 171]. By contrast, Cayley was dedicated to his subject in a serious-minded professional way, and he could hold his own against foreign competition.

[14] At this stage, Cayley did not know Sylvester well and evidently mistook his style in conducting mathematical research—which was to make his own investigations and not afford scholarly translations of existing work. In any event, Sylvester was too busy managing the Equity and Law Life Assurance Society, playing a leading part in the establishment of the Institute of Actuaries, examining for the College of Preceptors, and qualifying as a barrister to complete the task [78]. Chasles published his work in 1852 [46].

nothing more) that students would have studied some analytical geometry before arriving at the university: "[it] is now becoming so extensive that the Mathematical Student should commence his acquaintance with it as soon as possible, so as to be able to devote considerable time to it" [15, 88]. The *Student Guide* continued with the useful advice: "The student cannot be too often reminded that he must not expect to attain facility in the methods of Algebraic Geometry, without working a great number of examples" [15, 90].[15]

More texts were now available, and those written by Isaac Todhunter were regarded as basic. His books covering analytical geometry included *Treatise on Plane Coordinate Geometry as Applied to the Straight Line and the Conic Sections* (1855) and *Examples of Analytic Geometry of Three Dimensions* (1858). These introductory textbooks led to more advanced ones such as *Elementary Treatise of Trilinear Coordinates, the Method of Reciprocal Polars and the Theory of Projections* (1861), written by Norman Macleod Ferrers, and George Salmon's *Treatise on Conic Sections* (1848), *Treatise on the Higher Plane Curves* (1852), and *Treatise on the Analytical Geometry of Three Dimensions* (1862). John William Strutt's student booklist for the 1860s also included these geometrical texts [102, 29]. Though these texts introduced modern developments (such as the method of polars and projections) for students of the Tripos, they were elementary when compared with the level of mathematics Cayley could offer.

It is conjectural who attended Cayley's inaugural lecture. Almost certainly resident mathematical dons were present as perhaps were former competitors for the Sadleirian Chair such as Todhunter, Ferrers, Routh, and the geometer Frost.[16] Joseph Wolstenholme (1829–1891), a neglected Cambridge figure who collaborated with Frost in producing a *Treatise on Solid Geometry* (1863) and who wrote papers principally on analytical geometry, might have attended. He was the Third Wrangler in 1850 and was the Sadleirian lecturer at Christ's College. Also of interest are the impressionable undergraduates. The young William Kingdon Clifford (see Fig. 2) arrived at Trinity College (like Cayley from King's College, London) as an undergraduate just as Cayley arrived as professor. Frost was his tutor and possibly encouraged his attendance. Rawdon Levett, who was to become a schoolmaster and an ardent reformer of the teaching of geometry, was a third-year student in 1863, and William Peveril Turnbull, a finalist in 1863, published *An Introduction to Analytical Plane Geometry* in 1867 [111; 112]. I have been unable to confirm whether the listed "prominent" undergraduates (Table I) attended Cayley's inaugural lecture or the ensuing lecture course, but it seems that some would have appeared in the lecture room if only to enjoy a Cambridge mathematical occasion, when one of the university's more famed Senior

[15] The study of analytical geometry was not unique to Cambridge, but students would not have learned geometry with an analytical bias from Cayley's counterpart at Oxford. Henry John Stephen Smith (1826–1883) was elected to the Savilian Chair of Geometry in 1860 and from 1850 for a period of 20 years taught mathematics at Balliol College as a college lecturer. Smith saw some advantage in analytical methods but ultimately believed that they introduced a foreign element (namely, coordinate axes) into pure geometry. Charles Dodgson (otherwise known as the author Lewis Carroll) at Christ Church, Oxford taught a course on plane projective geometry in which he used homogeneous coordinates including trilinear coordinates [58]. A high proportion of Augustus De Morgan's Lower Senior Class at University College, London dealt with analytical geometry.

[16] The lecture was a Cambridge event. It would be interesting to know if auditors came from farther afield. The London-based mathematicians such as Thomas Archer Hirst (1830–1892) almost certainly did not attend, however, and it is likely that Augustus De Morgan was an absentee (being averse to leaving "town" for any reason). I do not know whether Sylvester attended or whether other London-based mathematicians (such as William Spottiswoode) were there.

FIG. 2. William Kingdon Clifford (1845–1879). Photograph courtesy of the London Mathematical Society.

Wranglers returned to his *alma mater* as professor. It would not be the grand event associated with inaugurals held in the Senate House, such as Charles Kingsley's in 1860, but the arrival of a new professor was nevertheless memorable.

In his lecture, Cayley chose to discuss the underlying tenets of analytical geometry, and the lecture, an overview, was by the way of an advertisement for the course of lectures to follow. A possible reason for Cayley not publishing his notes lay in the lecture's elementary character—a compelling reason for its interest today. The thoroughness and completeness of

TABLE I
Some Prominent Cambridge Wranglers (in residence November 1863) Who Went on to Careers in Mathematics, Science or Education[a]

		Freshmen (graduated 1867)	
Charles Niven	SW	1845–1923	Prof. Math., Queen's College, Cork, 1867–1880; Prof. Nat. Phil. Univ. Aberdeen, 1880–1922. FRS, 1880.
William Kingdon Clifford	2W	1845–1879	Prof. Applied Math., UCL, 1871–1879. FRS, 1874.
Carlton John Lambert	3W	1844–1921	Prof. R. N. Coll. Greenwich.
Osborne Reynolds	7W	1842–1912	Prof. Eng. Owens College, Manchester, 1868–1905. FRS, 1877.
Robert Kalley Miller	aeg.[b]	1842–1889	Prof. of Applied Math., R. N. Coll. Greenwich, 1873–1885.
		Second Year students (graduated 1866)	
Robert Morton	SW	c.1845–1872	Lecturer in Mathematics, Christ's College, Cambridge, 1867–1872.
Thomas Steadman Aldis	2W	1843–1908	Inspector of Schools, 1872; Chief Inspector of Schools, 1898.
William Davidson Niven	3W[c]	1842–1917	Prof. Math. R. E. Coll. Cooper's Hill, 1867; RMC Woolwich; Dir. R. N. Coll., Greenwich 1882–1903; Pres. LMS, 1908–1909; FRS, 1882.
James Stuart	3W[c]	1843–1913	Prof. Mechanism and Applied Math., Cambridge, 1875–1889. Member of Parliament.
George Pirie	5W	1843–1904	Prof. Math. Univ. Aberdeen 1878–1904.
		Third Year students (graduated 1865)	
John William Strutt	SW	1842–1919	3rd Baron Rayleigh, Cavendish Prof. Exp. Physics 1879–1884; FRS, 1873; Noble Prize, 1904.
Alfred Marshall	2W	1842–1924	Prof. Political Economy, Cambridge, 1884–1908; Founder of Cambridge School of Economics.
Henry Martyn Taylor	3W	1842–1927	Staff of Royal School of Naval Architecture, 1865–1869; Tutor, Trinity College Cambridge, 1869–1894; Innovator in Braille; FRS, 1898.
Rawdon Levett	11W	1843–1923	Schoolteacher in Birmingham. Active in the reform of the teaching of geometry in England.
Horatio Nelson Grimley	12W	c.1835–1919	BA (UCL, 1862); Prof. Math. and Nat. Phil. Univ. Coll. Aberystwyth, Wales, 1872–1879.
		Finalists[d] (graduated 1864)	
Henry John Purkiss	SW	1842–1865	Editor *Oxford and Cambridge and Dublin Messenger of Mathematics*; Principal R. S. Naval Architecture, 1865.
William Peveril Turnbull	2W	1841–1917	H. M. Inspectorate of Schools, 1871–1906.
Robert Pearce	3W	1841–1920	Prof. Math. Durham University, 1874–1895.

Note. The term "Wrangler" is derived from the process of wrangling, the debating method used to test undergraduates before the advent of written examinations. The Mathematical Tripos order of merit was subdivided into Wranglers, Senior Optimes, and Junior Optimes (corresponding to the First, Second, Third Class honours of the modern Cambridge degree). In 1863, for example, 100 students were listed in the order of merit, of which there were 33 Wranglers, 22 Senior Optimes, and 45 Junior Optimes. The total number of students appearing in the order of merit remained fairly stable until 1882 (when new regulations came into force), though the distribution between the three classes varied. The top student of the whole list was the Senior Wrangler (SW), followed by the Second Wrangler (2W) and so on. A student's position on the order of merit indelibly marked him for life, and, as a result, the Mathematical Tripos examination was extremely competitive.

[a] FRS = Fellow of the Royal Society; UCL = University College, London; LMS = London Mathematical Society.

[b] Aegrotat degree (a degree without classification).

[c] Students were placed equally (bracketed together).

[d] Honours degree students at Cambridge normally sat the Tripos examinations in January, after ten terms of residence.

his notes indicate that they were intended to be read verbatim.[17] As with his research papers, he adopted an outwardly proselike style, a way of expressing mathematics at variance with modern usage comprising concise definitions and theorems stated in relief. It was a natural form of expression for those who literally read their lectures, the way Cayley addressed scientific meetings of the Royal Society and the British Association. Although he was adept with solitary research pursued in the confines of his barrister's chambers, he had little experience in conducting day-to-day lectures, and Cambridge gossip indicates that his delivery before an audience was poor.[18]

When Cayley read his lecture in the Divinity Schools at noon on 3 November 1863, he was apprehensive. Sitting before his audience, he did not use the blackboard, a custom he only adopted in the late 1870s and then rarely. He was never at ease and acquired the habit of glancing at his watch before the end of the hour and, absorbed in his material, only managed occasional glances at the audience. The lecture was no doubt delivered at speed.[19]

3. THE LECTURE[20]

Analytical Geometry—Introductory Lecture
3rd November 1863[21]

In reference to the subject of my present course, it gave me some pleasure to remark that (as appears by the will of Dame Mary Sadleir dated the 25th September 1701) the Sadlerian lectures were instituted "for the full & clear explication and teaching [of] that part of mathematical knowledge commonly called Algebrare or the method and rule of contemplating quantities generall with particular application and use of it in Arithmetic and Geometry either according to the method of D[es]Cartes or any of those who have best improved it since."[22] The Professorship to which I have been appointed replaces to some extent these lectures and the duties of the Professorship regulated not by Lady Sadleir's will

[17] The manuscript of Cayley's preparatory notes is held at Columbia University, New York. It was given to W. W. Rouse Ball after Mrs. Cayley's death in 1923 and Rouse Ball passed it to David Eugene Smith as a memento of Cayley. Unlike Cayley's extant manuscripts, the notes are unpolished and contain his extensive revisions. Editorial changes have been added sparingly and are insubstantial (for instance, where Cayley wrote "thro," this has been replaced by "through"). Paragraphs have been introduced. Words or punctuation added to ease the flow have been inserted using [,]. When Cayley has used parentheses, round braces (,) have been used. The symbol // has been used to indicate a page-break in the manuscript. Cayley's many deletions have not been reinstated. All footnotes are mine.

[18] Before the election of the Sadleirian professor, Isaac Todhunter had written to George Boole: "I have of course no objection to Mr. Cayley; but it is obvious that he cannot teach or explain anything" [110, 234]. Cayley did not relish public speaking. In 1864, he was President of Section A (Mathematics and Physics) of the British Association meeting at Bath but declined to give the customary address.

[19] Writing to Cayley in 1883, on the occasion of his preparing his British Association address, Sylvester noted a previous occasion at which Cayley spoke: "You ought I think to read it just *half* as fast as you did on that memorable afternoon" [104, Sylvester's emphasis].

[20] This was the first lecture of a series delivered at noon on Tuesdays and Thursdays during the Michaelmas [Autumn] term, 1863 [8, 5].

[21] Cayley was sufficiently organized to submit a paper on differential equations to the *Philosophical Magazine* the day before [24].

[22] The next sentence of Lady Sadleir's will specified those qualified to perform this teaching duty: "The persons to read the said Lectures shall be at least of the Degree of Bachelor of Arts, of honest life and conversation, very well seeing and knowing in these parts of the Mathematics above mentioned" [47, v–vi; 82, 224].

[but] by the Statutes for the Mathematical Professorship sanctioned by an Order in Council of the 7th March 1860—it is thereby provided that it shall be the duty of the Professor to explain & teach the principles of Pure Mathematics and to apply himself to the advancement of that Science.

I have selected Analytical geometry—the creation of Descartes—for the subject of my course—and I propose to consider it, indeed also theoretically, but in a great measure // historically, for the present attending chiefly to the period beginning with the *Géométrie* of Descartes [56], and in order to fix a termination ending with Cramer's *Traité des lignes Courbes* [53], the included period being 1638 to 1750.[23] I have said *Analytical* Geometry—not that I intend to exclude from consideration any of the methods and theories which belong to Modern Geometry—but partly because I do intend to dwell chiefly on the analytical view of the subject—partly because I am not able to satisfy myself that the Modern Geometry[24] does not rest essentially on an analytical basis.[25]

The fundamental notion [of analytical geometry] is that of a locus *of a given order*, viz. considering (as, in the absence of any explanation I shall always do) [in] the case of plane geometry, a curve which meets every line whatever in a given number of points. A conic is a curve of the second order, but this is not in Ancient Geometry the definition of it; so far from being so, an ancient geometer (not recognising the existence of imaginary points) would be unable to admit [the] truth of the proposition that a conic does meet every line whatever in two points.[26] Before going further it may be right to remark that in this Introductory Lecture I am of necessity assuming in the whole of my audience an acquaintance with principles and theories // which in subsequent lectures, I shall of course not give them credit for, but which it will be a chief object to me to explain and familiarize them with; I wish to make this apology beforehand for anything which may appear difficult in the present lecture; I shall throughout endeavor to make my lectures intelligible to those who have commenced to occupy themselves with the subject.

[23] Of the two aspects of modern geometry, analytical and pure (synthetic), Cayley championed analytical geometry. Cayley used "analytical geometry" as the subject in which algebra is applied to geometry. (The wholly different meaning of "Analytic" and "Synthetic" as a facet of the methodology of mathematics has been briefly discussed by Ivor Grattan-Guinness [68, 1:135].) Cayley wrote only on analytical geometry for the *Encyclopaedia Britannica* article "Geometry" (the half on synthetic geometry was written by Olaus Henrici (1840–1918)) [36]. Of the two approaches, pure geometry was a fairly neglected area of research in the second half of the 19th century in England. Henrici put the case for its study: "One of the great advantages of the purely geometrical methods is that all operations are performed by constructions mostly in three dimensions. Thus the student learns to realise figures in space, whilst in Coordinate Geometry the geometrical meaning of the algebraic operations is too easily lost sight of" [89, 140]. While Cayley and Henrici were leading exponents of these distinct geometrical styles, both valued the making and use of physical geometrical models for illustration and understanding of geometrical ideas. In Britain, H. J. S. Smith and T. A. Hirst were leading synthetic geometers, while George Salmon (1819–1904) and Cayley were analytical geometers.

[24] Here, Cayley struck out "Higher Geometry" as an equivalent of "Modern Geometry." Cayley used "Modern Geometry" in distinction to the geometry of Euclid. Modern geometry encompassed both the analytical approach of Descartes and the synthetic approach espoused by Jean-Victor Poncelet (1788–1867) and Jakob Steiner (1796–1863). Leading analytical geometers of the day included August Ferdinand Möbius (1790–1868) and Julius Plücker (1801–1868).

[25] This is the central tenet of Cayley's view of geometry and partially explains his attraction to algebraic invariant theory as a means of studying geometry.

[26] Cayley was an ardent admirer of the Greek tradition: "the achievements of Euclid, Archimedes and Apollonius are as admirable now as they were in their own days" [37, 459] (see note 74).

Resuming now the foregoing question [of lines intersecting curves]; it may be noticed that in the case of a conic there are a whole set of lines each of which does meet the curve in two real points, but even admitting that this is suggestive of the theorem that every line meets the curve in two points, we are not in the same way led to the theorem for a curve of any order [every line meets the curve in n points]: a curve of the fourth order, taken at random[27] will it is true be met in four real points by each of a whole set of lines; but it is possible to find curves of the fourth order which are not met by any line in more than two real points. (For example the curve [an oval][28] $x^4 + y^4 = c^4$; each of the lines $x = \pm c$, $y = \pm c$ meets however this curve in *four* coincident real points.) And there is this further geometrical difficulty, that if we admit the existence of imaginary points, and for example allow that a line not visibly intersecting a conic may nevertheless intersect [it] in two points // then the question arises, how do we know that any line whatever does not intersect a conic in *more* than two points, the number of real intersections being two at most.

For all these difficulties & inquiries we have an analytical solution; a curve of the nth order is a curve such that the coordinates of any point are connected by an equation of the nth order; the coordinates, for instance the x-coordinates, of the points of intersection by an arbitrary line are then given by an equation of the same order, and since this equation has n roots, the number of points of intersection is equal to n: the only assumption is that an equation of the nth order has n roots.

I stop to enquire what this means:—the analytical conception, no matter how arrived [at] of magnitude$_{[,]}$ is that it is of the form $\alpha + \beta i_{[,]}$ α and β being positive or negative real magnitudes, i being used as [a] literal to denote $\sqrt{-1}$, what the signification of i or $\sqrt{-1}$ is, I do not undertake to explain; all that needs to be known is that magnitudes defined as above may be combined in the way of addition$_{[,]}$ subtraction$_{[,]}$ multiplication$_{[,]}$ and division; this includes of course involution [exponentiation], the exponent being a positive integer number.[29]

This being so the theorem is that$_{[,]}$ consider an equation for instance the quadric equation $x^2 + ax + b = 0$ where a and b may be either real magnitudes, or magnitudes as above defined [complex numbers], to fix the ideas let them be real magnitudes. Then for any assumed value $x = \alpha + \beta i$ of x, the // expression $x^2 + ax + b$ (the characteristic or nilfactum of the equation) is a determinate calculable magnitude, and the general theorem applied to the case in hand, is that there are precisely two such values of x (that is, two sets of real values of α and β) for which the magnitude $x^2 + ax + b$ is $= 0$. And so for an equation of the nth order—there are n such values of x (that is n sets of real values of α and β) for

[27] The phrase "taken at random" is equivalent to Cayley assuming that the curve is a *general* curve, thus ensuring that its points of intersection with the line are *distinct*. The phrase conveys the belief that special curves, those without this convenient property, are rare. As an algebraic example, a general polynomial equation of degree n "taken at random" would be assumed to have n distinct roots. For Cayley, this was the *normal* case, and, only considering this case, he would conclude results were true "in general," i.e., normally true but admitting exceptions [35, 463]. In restricting himself to the normal case, polynomials with coincident roots would be ignored—Cayley would regard these as special cases and hardly worth considering. Similarly, matrices were assumed to be invertible "in general." An analysis of Cayley's work in terms of the general case (generic reasoning) has been made by Thomas Hawkins [70].

[28] According to one of Cayley's definitions made in the context of quartic curves (1864), an *oval* is a closed curve without nodes or cusps [28, 468].

[29] As often happened at the time, Cayley used the term *imaginary* magnitude to apply to the *entire* expression $\alpha + \beta i$ in which α and β are real magnitudes.

which the magnitude represented by the characteristic or nilfactum of the equation is $= 0$. So exhibited the analytical theory presents no difficulty whatever if you will only be content to remain ignorant of the meaning of $\sqrt{-1}$. But while thus deriving from analysis or at any rate supporting by analysis the notion of a curve of the nth order as one which is met by a line in n points, and the theorem of the existence of such curves I mean only that we are led to the notion by analysis and are unable so far [as] I can perceive, to arrive at it otherwise; I do not in the least mean to imply that the notion is not essentially a geometrical one; on the contrary I fully believe that it is so.[30]

And we have therefore now to consider the theory geometrically.[31] And first I must say something of a current theory of the geometrical interpretation of the imaginary quantities. It is frequently very convenient in dealing with the analytical magnitude $\alpha + \beta i$ to take α and β as the rectangular coordinates of a point in a plane:[32] for instance, as with regard to the real roots of an equation we may enquire how many real roots lie between real limits $x = a$, $x = b$, // so with regard to the roots real or imaginary we may enquire how many roots lie within the limits given by the contour of a certain curve: if to fix the ideas the curve be a circle radius unity described about the origin as centre, this is merely asking how many are the roots $\alpha + \beta i$ which are such that $\alpha^2 + \beta^2$ (or say the norm or squared modulus of the root) is not greater than unity: in the last mentioned case the question would naturally be so asked, but you will readily understand how [in] more complicated cases, the introduction of the curve facilitates the comprehension of the question. To this mode of representation of the analytical magnitude $\alpha + \beta i$ there is no objection whatever; and it is in fact the basis of a series of most refined and interesting investigations relating in particular to the theory of multiple integrals.[33]

[30] Cayley expressed similar thoughts about quaternions being essentially a geometrical conception once he obtained an algebraic quaternion expression that represented a rotation in space [17; 69].

[31] The drive to explain complex numbers, algebraic triples and quaternions was made through geometry. The quest for interpreting geometrical meaning for complex numbers was far from dead around the period of Cayley's lecture [88].

[32] Here Cayley describes the geometric representation of a complex number in the way shown in Gauss (1831), as a point in the plane. Other representations were suggested in the 1860s, and one commented on by Cayley was the theory due to Alexander J. Ellis (1814–1890). Ellis was Sixth Wrangler at Cambridge in 1837 (to J. J. Sylvester's Second Wrangler) and became an eminent philologist. He worked with Isaac Pitman (1813–1897) on a phonetic system, published a large work on pronunciation and proposed "Glossic" as a new system in spelling. Ellis's interests ranged widely, and he counted amongst his publications *Algebra Identified with Geometry* and *Horse Taming*. After graduating at Cambridge, he translated *Geist der mathematischen Analysis* by Martin Ohm (1792–1872). With his interest in foundational work, he suggested an interpretation for imaginaries in geometry, in which a point is represented as $xOM + yON$ from an origin O where x, y, possibly complex numbers, satisfy an equation $f(x, y) = 0$ (without being explicit, Ellis considered "vectors" with complex components). Cayley rejected Ellis's attempt: "Some of the views in regard to imaginaries in algebra and geometry—which I in no-wise agree to—are perhaps as orthodox as my own, but it does not much signify either way" [86; 59]. Ellis also suggested "scalar and clinant" geometry which Cayley also dismissed: "The memoir may be valuable to those who adopt the views of Peacock, Gregory and Walton in interpreting imaginary quantities in geometry. But in the wholly different point of view from which I look at imaginary quantities in geometry, the speculation appears to be a very unprofitable one" [87; 60] (see note 35).

[33] Cayley had shown a limited attention to Cauchy's theory of complex-variable integration up to this time. In 1845, he referred to the calculus of residues in Cauchy's *Exercices de mathématiques* in connection with the expansion of reciprocal functions as the summation of series [18, 174–181]. In 1847, he again referred to the *Exercices de mathématiques* when he outlined a theory of the beta and gamma function, a theory of his own to which he attached "some value" [20].

But it is quite another thing to assume connexion of $\sqrt{-1}$ with the notion of perpendicularity, or to suppose that the analytical magnitude $\alpha+\beta i$ is in anywise necessarily represented by the point the rectangular coordinates wh[ere]of are α and β; or to state the erroneous assumption in its crudest form, to assume that if in seeking for a point on the axis of x which shall satisfy a certain condition, we obtain as the result of the investigation $x=\alpha+\beta i$, to assume I say that the point so obtained is the point the coordinates whereof are α, β. So stated // the assumption is its own confutation; you obtain as a result what is in direct contradiction with your hypothesis, a point *off* the axis of x as the result of an investigation which assumes that the point is *on* the axis of x.[34] And, to merely allude to deductions which have been made from such a theory, I may remark that by parity of reason, and *a fortiori* if that were possible, I discard altogether the notion of branches out of the plane of reference, of a curve which is by hypothesis in the plane of reference.[35] The last mentioned theory being removed out of the way, we return to the inquiry as to the geometrical theory of imaginary points. And I may say at once that I have no sensuous representation to give you; I cannot make cognisable by the senses that which while I believe it to exist, is certainly incognisable by the senses.

The way in which geometrically we are led to imaginary points is as follows; we have for instance a conic, and a point which may be either outside or inside the conic; in the former case there are two real tangents from the point to the conic in the latter case there is not any real tangent; but without looking to see whether the point is outside or inside we may in either case by a precisely similar construction obtain a line possessed of geometrical properties the same in each of the two cases: for instance, and using for convenience the // ordinary terms pole and polar to denote the point and the line respectively, if the pole moves in a line the polar revolves about a point; and so if the pole moves in a conic the polar envelopes a conic, etc. Now in the case where the pole is outside the conic the polar does in fact pass through the points of contact of the two real tangents drawn through the pole; in other words the polar in this case might be defined to be, and at any rate is, the line joining the points of contact [of] the two tangents drawn through the pole. It is at any rate natural to extend this definition to the other case where the point is inside the conic and to say generally that the polar is the line joining the points of contact of the two tangents through

[34] Here, Cayley rejects the notion of an "imaginary" axis at right angles to the x-axis in seeking the solution to $x^2=-1$ to arrive with roots $x=\pm i$ which are "off the axis of x" (see note 35).

[35] Cayley's interpretation of the equation of a circle $x^2+y^2=1$ is the standard one which implicitly assumes that $|x|\leq 1$. Cayley emphatically rejected the notion of the equation representing a hyperbola in a plane at right angles to the original plane, a construction made possible by considering values of x outside this range. The geometrical interpretation of complex numbers through branches of a curve out of the plane of reference is the so-called "English" theory. It was supported by George Peacock (1791–1858) and Duncan F. Gregory (1813–1844) and promulgated by William Walton (c. 1813–1901) in the 1840s and 1850s [89; 93]. Writing to George Boole in 1847, Cayley had remarked: "I wonder we should never have stumbled in our previous correspondence on the subject of my *utter disbelief* of the received "English" theory of the geometrical interpretation of $\sqrt{-1}$. I would much more easily admit witchcraft on the philosophers stone" [38; 4, 191–197]. Cayley gave no reason for his "utter disbelief," but we may infer he found the interpretation peculiar. H. J. S. Smith could find no fault in it *analytically* but rejected it as "barren of results," the principal criticism of a geometer [100, 3–4]. The "English" view is traceable to John Playfair (1778) who noted that propositions in geometry were free of controversy but that "the doctrine of negative quantities and its consequences have often perplexed the analyst, and involved him in the most intricate disputations" [83, 318].

the pole. But such definition assumes that in the case of a pole inside the conic, the polar, which is here a line not visibly meeting the conic, does in fact meet it in two points; or that a line in general meets the conic in two points, or that there are imaginary points. And by way of confirmation it is to be noticed that taking the definition of the polar to be the line joining the two points of contact we find analytically an equation which in fact belongs (as well in the case of an inside as of an outside point) to the real line given by the before mentioned geometrical construction. As another & a different but equally interesting example I might take the common chord of two circles which do or do not intersect in a pair of real points, but the single // example of the polar was sufficient for my present purpose.

In what precedes I have been concerned with the intersections real or imaginary of a curve by a line; but the theory must be looked at in a more general point of view; the notion of an imaginary point must be considered not as opposed to but as including that of a real point; space is the locus in quo not of real points merely but of points real or imaginary; a real curve means merely a curve which is in part real, but it is nevertheless the locus of a series of points imaginary or real and comprises points as well imaginary as real; and we have lines and other curves & figures which are altogether imaginary.[36]

We have in the first instance to ignore altogether the distinction while retaining for convenience, and as opposed to each other the terms *real* and *imaginary* we do in fact in Modern Geometry, ignore for the most part the distinction between the so called real and the so called imaginary: our space is the locus in quo not of real points only but of points real or imaginary, and we are concerned and have to deal with the curves and other figures which belong to this generalised conception of space and of points existing therein: a real curve is merely a curve comprising // (among the points whereof it is the locus) real points but it comprises also imaginary points; and a curve or other figure may be altogether imaginary. We never in the first instance inquire whether a point curve or figure is or is not real, but say for example that two curves of the order m & n respectively intersect in mn points,[37] that a curve of the third order has 9 inflexions etc. knowing that all or some of these may be imaginary, or perhaps knowing that they are all of them necessarily imaginary, and not caring whether this is so or not: the inquiry may doubtless be made, whether the figure or any particular part of it is or is not real; and such inquiry is frequently an interesting and difficult one, but it is merely supplementary to the general theory of the figure.

Of course in an actual construction we cannot directly make use of imaginary points or lines; a given construction frequently presupposes subsidiary constructions for effecting as regards the imaginary parts of a figure that which as // regards the real points is effected directly: Suppose for instance that as part of the construction of any problem you are told to join the points of intersection of two given circles: if the points of intersection are real you at once join them; if they are imaginary you require to know the solution of the subsidiary

[36] For Cayley, this would be a *mathematical* or *ideal* space, an extension of the so-called ordinary space wherein reside real curves and real surfaces. While Cayley rejected some notions of imaginaries in geometry (see note 35), he accepted the notion of a circle with imaginary radius without qualm. Any mention of such ideas as imaginary lengths caused some practical scientists exasperation with pure mathematics (for example, the astronomer, George Biddell Airy).

[37] Bezout's theorem, that a curve of order m intersects a curve of order n in mn points, is named after Étienne Bezout (1730–1783) [6, 163]. This is another example of a statement which is true "in general," but may be false in particular cases (see note 27).

problem to construct the line through the points of intersection of two circles which do not meet.[38]

The imaginary space of Modern Geometry exists for the purposes of the science—but does it exist, as an "$o\upsilon\tau\omega\varsigma\ o\nu\tau\alpha$"—I use the Greek expression merely because the word real as above remarked is appropriated as the contradictory of imaginary.[39] In support of the opinion that it does so exist it may be argued, according to the theory we adopt of ordinary space, that on the one hand that if space is only a form of the perceiving mind, then, since in the development by geometrical science of the notion of space imaginary space has presented itself to the mind, it has as good a right as ordinary space to be considered as a form of the perceiving mind—on the other hand if ordinary space be an existence exterior to the mind, then there [is] no reason why imaginary space should // also exist exterior to the mind, incognisable to the senses, and only arrived at as a necessary substratum for the truths of geometry.[40] But the question is [one] which hardly admits of any profitable discussion.[41]

[38] Cayley was driven by analysis to interpret complex numbers in geometry: "The notion of imaginary intersections, thus presenting itself, through algebra, must be accepted in geometry—and in fact plays an all-important part in modern geometry" [35, 463]. Cayley's orthodox interpretation appealed to Poncelet's *principle of continuity*. This principle, which was widely accepted by geometers in the nineteenth century, is traceable to Kepler and Desargues. The principle assumes that properties of a construction, which are true for a particular figure, remain true when the method of construction remains the same but the figure changes form. In Cayley's example, two circles meet (in imaginary points) even when the circles do not visibly meet. The principle allows general statements to be made as in: "all circles drawn in the plane intersect each other." For a *construction* of imaginary points, which the principle of continuity does not provide (and is not concerned with), Felix Klein guided him to von Staudt's work in 1889. Then Cayley noted: "I have looked at Staudt's *Beiträge zur Geometrie der Lage*—the sections on the Addition and Multiplication of "Würfe" [throws] which leaves nothing to be desired and the point of view is a better one than mine" [42].

[39] Cayley used the Greek expression "$o\upsilon\tau\omega\varsigma\ o\nu\tau\alpha$" [really real], instead of "real" since this word was reserved for use with complex numbers. Cayley wrote to Boole in 1847: "My own theory as far as I can express it is that a distance x, *whether real or imaginary* is an "$o\upsilon\tau\omega\varsigma\ o\nu\tau\alpha$" capable of being *measured* (that word won't do I admit, but capable of existing) in any direction real or imaginary: somewhat as if beings whose space was of two dimensions only (which I think is conceivable) had by their science of geometry arrived at the notion of a third dimension." Finally, he came to his interpretation of imaginary: "In fact I should admit no distinction between real and imaginary; it is only when you draw the figures that the difficulty arises. This is rather idiosyncratical I am afraid, and I do not expect or wish to convert you" [38, Cayley's emphasis; 4, 191–197].

[40] In 1865 Cayley published a "Note on Lobatschewsky's Imaginary Geometry" [29]. He felt Lobachevsky's interpretation of certain trigonometrical equations was "hard to be understood." He reworked the equations but concluded: "I do not understand this; but it would be very interesting to find a *real* [his italics] geometrical interpretation of the system of equations" which defined Lobachewsky's geometry in which the sum of angles of a triangle is strictly less than π. Cayley wanted to interpret Lobachewsky's geometry in terms of the *really real* space of ordinary Euclidean space in the same sense that spherical geometry could be. For a discussion of the way the new non-Euclidean geometry impinged on the commonly accepted view of geometry as "measuring" ordinary three-dimensional space, see [89, 61–114]. As a separate development, Cayley's analytical geometry concerned figures in "mathematical space" [complex n-space].

[41] Cayley was certainly interested in the philosophical questions associated with the foundation of mathematics, as is shown by the content of this lecture and more extensively by the text of his presidential lecture to the British Association in 1883 [37]. After Cayley's death, J. W. L. Glaisher noted that philosophy of mathematics and logic were the only aspects of mathematics which did not claim Cayley's attention, but Glaisher was perhaps thinking in specific terms [66, 174]. Cayley did become impatient with the probing of foundational issues when he saw no likelihood of mathematical advance, but his sensitivity to the "metaphysics of mathematics" dated from the 1840s. It is true that, compared with his substantial contributions to algebra and geometry, his papers

I have said that a curve of the *n*th order is intersected by any line whatever in *n* points—it will be easily understood—and that as well geometrically as analytically that two or more of the points of intersection may coincide together; the case is that of a line touching the curve or passing through a singular point. It may also happen that a point or two or more points lie at infinity on the cutting line. But there is another case to be considered viz. in plane geometry infinity is a line, and as such, like any other line, meets the curve in *n* points. To show how this is, you are all familiar with the notion that two parallel lines meet at infinity, or say, in the point at infinity on each line, or in the infinity of each line. Two planes meet in a line, and two parallel planes meet in the line at infinity on each plane, or in the infinity of each plane; that is, in the same way as the infinity of a line is a point, so the infinity of a plane is a line; and for the several lines in a plane, the points at infinity are those in which these lines meet respectively the line infinity. The notion of the line infinity although not really difficult, // does at first [appear] a strange one, it requires consideration and time to familiarise oneself with it. I remark as regards the analytical theory that for ordinary Cartesian coordinates the equation of the line infinity itself [results] in the paradoxical form $1 = 0$, but that there is no such difficulty for trilinear coordinates.[42] And by way of illustration of the geometrical application of the notion—you will at once see how it leads to a classification of curves of any order—e.g. a curve of the second order (conic) may meet the line infinity in two distinct points which may be real (giving the hyperbola) or imaginary (giving the ellipse) or again it may touch the conic, meet it (that is) in two coincident points, giving the parabola.[43]

And as a further instance take the cubic curve with the theory of projections; we may in fact so project a figure that a given line in the original figure shall become the line infinity in the projected figure. This is frequently used in order to obtain, as the projection, a simplified figure and to derive therefrom properties of the original figure; but the converse use is far more important, viz. we under[stand] the nature at infinity [and?] of the projection by the consideration of the original more general figure. Thus it is easier to found the notion of a conic touched by a line, than that of the Parabola considered as a conic touched by the line infinity. //

Among the theories of Modern Geometry may be mentioned one which presents itself under various forms; the theory of related figures. The most simple case but a very important one, is the relation of Projections; viz. two plane figures may be the projections each of the other, by lines meeting in a point; this assumes in the first instance that the figures are in different planes: but the planes may be made to coincide together, or by other means the

which directly impinge on the philosophy of mathematics and logic are few. Examples of Cayley's papers which include a "philosophical" component are "On the Notion and Boundaries of Algebra" [27], "A Memoir on Abstract Geometry" [32], and "Note on the Calculus of Logic" [33].

[42] The lines $y = x$ and $y = x + 1$ expressed in ordinary Cartesian coordinates pass through the same "point at infinity." In homogeneous coordinates (x, y, z), the line at infinity is given by the equation $z = 0$. The "line infinity" was Cayley's basis for classifying the geometry of the plane. George Salmon's *Higher Plane Curves,* 1st ed. (1852), began with "natural" points of view [93, 1], but Cayley replaced Salmon's introductory remarks in the 2nd ed. (1873) by a passage which delineated metrical and projective geometry: "We have in the plane a special line, the line infinity; and on this line two special (imaginary) points, the circular points at infinity. A geometrical theorem has either no relation to the special line and points, and it is then *descriptive*; or it has a relation to them, and it is then *metrical* [94, 1, Cayley's emphasis] (see note 49).

[43] The intersection of the line at infinity $z = 0$ with the conics represented by $x^2 - y^2 = z^2$, $x^2 + y^2 = z^2$, $x^2 = yz$ (respectively).

figures may be conceived as in the same plane. It is hardly necessary to remark that the very notion of a conic as the section of a right cone is a notion of projection—a conic of any kind whatever, ellipse, parabola or hyperbola is in fact the projection of a circle. I shall have a great deal to say to you about Newton's analogous theorem for curves of the third order.[44] But what I wish particularly to remark as to the principle of Projection is that it may be used in two opposite ways;

1°. to simplify a figure, so as for instance, to obtain the circle instead of the ellipse—or parallel lines instead of lines meeting in a point—or what includes this, to project some line of the original figure into the line infinity; we have then in the simplified figure properties which are frequently // self evident or nearly so—and we thence arrive at the existence of corresponding properties in the more complicated original figure. This is in fact the way in which the principle has been most used—in particular in Poncelet's Classic work the "Traité des Propriété's projectives" (1817) [84]; and I may in passing refer to the very beautiful theory of the in and circumscribed polygon.[45]

But the principle of projections may be used in the opposite way—

[2°] we really understand the simplified figure better—and we escape from a very great number of special forms, which would otherwise need to be examined—by considering the simplified figure merely as a case of the original more general one. Thus for instance the notion of a conic touched by a line is an easier one than that of the parabola considered as a conic touched by the line infinity.

Two figures, the projection one of the other, and situate[d] in the same plane may lie perspectively in regard to each other—as for instance if they are two different projections of one and the same figure in another plane, by lines proceeding from two different points. But if we displace one of the two figures then in general they will not any longer lie perspectively in regard to each other; and we arrive at a more // general relation of the two figures; capable indeed of being derived from the perspective relation, but this, although perhaps historically the origin, is not the natural origin of the more general relation—the relation referred to is the homographic or collinear relation; the characteristic feature is that to points in a line or

[44] On his return to Cambridge in 1863, Cayley began publishing papers in the *Transactions of the Cambridge Philosophical Society* again. (Prior to this his only paper in this journal was published in 1843.) In February 1864, he read papers to the *Cambridge Philosophical Society* on the theory of involution and its application to cubic curves. In April, he read two expository papers on the classification of cubic curves [30; 31]. A plane cubic curve expressed in Cartesian coordinates x, y is of the form $A(x, y) + B(x, y) + C(x, y) + d = 0$ in which $A(x, y)$, $B(x, y)$, $C(x, y)$ are the homogeneous cubic, quadratic, and linear terms, respectively. Newton's "analogous theorem" stated that any cubic curve was the projection of one of five "divergent parabolas." Cayley's first expository paper compared the Newtonian classification of cubic curves (into 78 species under 14 genera) with Julius Plücker's classification (of 219 species in 61 groups) [30]. By combining Newton's theorem with an observation of Michel Chasles, Cayley thought that the true classification method for cubic curves was obtained by taking plane sections through five different cubic cones (which he called *simplex, complex, acnodal, crunodal,* and *cuspidal*). In this way, the theorem for cubic curves bears direct comparison with the classical theorem for conics (that any conic can be obtained as a plane section through a right cone) [31].

[45] The theory involves finding conditions which must be satisfied by two conics so that a polygon may simultaneously inscribe one and circumscribe the other. The simplest case of a triangle was the case Cayley first considered in 1853, but he elaborated more general cases in many publications between 1853 and 1871 [21; 34]. Evidently, Cayley was unaware of the little-known English topographer, philologist, and scholar, William Chapple (1717?–1781), who intuitively arrived at the correct result for the triangle and published the result in *Miscellanea Curiosa Mathematica* (1746). Jean-Victor Poncelet and C. G. J. Jacobi (1804–1851) proved deep results about this problem, Poncelet in 1813–1814 while a prisoner of war during Napoleon's campaign in Russia. For an historical analysis of the problem with a prehistory and modern developments, see [5].

lines through a point in the one figure, there correspond points in a line or lines through a point in the other figure.

A different and less easily intelligible relation is that of reciprocity; arising historically out of the theory of poles and polars, but which in its general form is better obtained independently. [T]he characteristic feature is that to lines through a point or points in a line in the one figure there correspond in the other figure points in a line or lines through a point. This principle of Reciprocity or say the Principle of *Duality* has perhaps more than anything else contributed to the progress of Modern Geometry.[46] But beautiful and valuable as it is[,] it seems in Analytical Geometry to sink along with the theory of homography into a particular case of a theory of the geometrical interpretation of the Analytical Symbols.[47]

I will merely allude to another question, but shall not much go into [it] either now or indeed in the present course. It would at first sight appear that a broad separation exists between metrical and descriptive geometry; where by metrical geometry is to be understood all that relates to magnitude—including therein equality and // perpendicularity: by descriptive geometry all that is wholly independent of magnitude. As an instance of a purely descriptive theorem (take Pascal's theorem for two lines). But in fact this in not so; it has gradually been becoming evident that descriptive geometry includes metrical geometry, as for instance, the circle instead of being considered as the curve all the points whereof are equidistant from a centre, is considered as a conic passing through two fixed points, the circular points at infinity; lines at right angles to each other—are in fact lines which are harmonics in regard to the lines drawn through the two circular points at infinity etc. And the ultimate conclusion appears to be that descriptive geometry is in fact *all* geometry, the difference being that in descriptive geometry the figure is considered apart by itself, in metrical geometry in connexion with a certain conic wherein the notions of linear and angular distance have respectively their origin.[48] But so far as the distinction is recognised at all I shall be chiefly concerned with descriptive geometry.[49]

[46] It is significant that the principle of duality was taught in the Mathematical Tripos in the early 1850s. In the official report on the conduct of the Tripos examinations of 1851 (with Cayley as Senior Moderator and Thomas Gaskin as Junior Moderator), it was noted that the "Theory of Reciprocal Polars" was given greater prominence in 1851 as compared with the previous year, and the report of 8 April 1851 stressed its importance: "the subject is one the knowledge of which cannot fail to be of great service in the solution of many questions [in modern geometry] which naturally arise in the course of a Senate House Examination" [11]. The author was Gaskin, but Cayley's influence is clear. Treating the coefficients of equations as coordinates, the essence of geometric duality was recognised and discussed by Cayley during the 1840s. During the 1860s (the decade of Cayley's "geometrical period"), he introduced the "six coordinates of a line" and identified the importance of the one-to-one correspondence between the space of lines and the points of a five-dimensional space in which the elements are conics. A leading light in these developments was Julius Plücker, and Cayley's geometrical period coincided with Plücker's return to pure mathematics in the mid-1860s. Plücker published his work in the *Philosophical Transactions of the Royal Society*, and his standing in English scientific society was high. He attended the British Association Birmingham meeting in 1865, and in the following year, at the Nottingham meeting, he displayed physical geometrical models for the Association and spoke about the quadratic line complex (a complex of lines in which the line coordinates satisfy an equation of the second degree [92, 214–217]). In November of that year, he was awarded the Royal Society of London's Copley Medal.

[47] Cayley referred here to the theory of projective transformations [98, 52]. Cayley made a note here, "Explain this" and also a marginal note, "Remodel this."

[48] In 1853, the young Edmond Laguerre (1834–1886) defined *angle* in a projective space in terms of the circular points at infinity [3; 75].

[49] The general relationship between descriptive geometry and metrical geometry is set out in Cayley's "Sixth Memoir on Quantics" [22]. It was arguably Cayley's most influential paper. C. S. Peirce described it as an "immortal

For some time I have been speaking rather of Modern Geometry than of Analytical Geometry—of the theories and principles which in the brilliant career of the Science in the last fifty years, have been established and developed partly geometrically partly analytically, rather than of the analytical method of coordinates which is applicable to, and is I think the appropriate instrument for the development of these theories.[50] In using the expression method of // coordinates it must be understood that I refer not exclusively to the Cartesian Coordinates, but to the more general system of trilinear coordinates—or rather (since the coordinates made use of are not of necessity point coordinates) homogeneous coordinates;—the analysis thus in a great measure becomes a theory of homogeneous functions.[51] But this generalisation of the system of coordinates, and consequent transformation and further development of the analytical theory are circumstances to which I shall have to direct your attention in considering the History of the subject, and I need not at present speak of them further.[52]

With regard to the Analytical Method generally, it is to be // noticed, that given a geometrical theorem of any kind, or say a supposed theorem only, you may always—potentially at least—by a direct application of the method of coordinates demonstrate the truth or falsehood of such [a] theorem: this is of course a perfection in the method, but it is the worst way of using it, so to apply it to the demonstration of an isolated theorem, considered apart from the geometrical theory to which such theorem belongs and the various other theorems

memoir" and judged it to be one of the greatest mathematical influences on his life [81]. Cayley's memoir contains his projective definition of distance, the distance $d(P_1, P_2)$ (defined as a mathematical function of the coordinates of the two points P_1, P_2) which satisfies the additive property $d(P_1, P_3) = d(P_1, P_2) + d(P_2, P_3)$ for three points P_1, P_2, P_3 lying on a line. Cayley based his definition of distance on a bilinear form $F(x, y, z) = f(x, y, z; x, y, z)$ which defines the Absolute conic $f(x, y, z; x, y, z) = 0$ and showed that it reduces to the Pythagorean distance $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ as a limiting case when the Absolute conic is degenerate (consists of two points). Cayley had defined distance in terms of bilinear forms for spherical geometry in the 1840s [19, 218–219]. With distance defined projectively and Euclidean distance obtained as a special case, a different light was cast on the relationship of Euclidean and projective geometry. Projective geometry had usually been thought of as part of Euclidean geometry in which geometrical constructions were based on metric notions of length and angle. In the new viewpoint, Cayley's best-known quotation arises: "Metrical geometry is thus part of the descriptive geometry, and descriptive geometry is *all* geometry" [22, 592]. At this stage, Cayley did not connect the projective distance with non-Euclidean geometry, as Felix Klein and William K. Clifford soon would [101; 92, 228–230].

[50] The method of coordinates was the hallmark of Cayley's geometry, a view amplified in his 1883 Presidential Address to the British Association: "Descartes' method of coordinates is a possession for ever" [37, 37].

[51] Here, Cayley parenthetically alludes to trilinear *line* coordinates. By referring to the general homogeneous coordinates and a theory of homogeneous functions, he means a treatment of geometrical theory as a branch of invariant theory. For example, the study of plane curves becomes a study of homogeneous functions of three variables, or *ternary quantics*, as Cayley designated them. The relevance of invariant theory to geometry arose initially in the study of cubic curves by Otto Hesse (1811–1874). The "Hessian," so named by Sylvester, is an example of a covariant which defines the Hessian cubic curve, a new curve which cuts the original curve in its points of inflection. The geometric side of invariant theory seems to have been overwhelmed by the algebraic side during the course of the nineteenth century, but the exact historical relationship between the two aspects of invariant theory has yet to be told [77; 1:157–206].

[52] Cayley had an extensive knowledge of the history of mathematics. Andrew R. Forsyth noted how Cayley used exact references to the mathematical literature (by 19th-century standards) "to give indications of the history of the subject" [62, xxviii]. Cayley's view of mathematical research was to build on the past, as his general articles contributed to the *Encyclopaedia Britannica* (all developed historically) indicate. To the *Encyclopaedia*, he contributed several biographical sketches of leading mathematicians (Galois, Gauss, John Landen (1719–1790), Monge, and John Wallis (1616–1703)). When W. W. R. Ball's *A Short Account of the History of Mathematics* appeared in 1888, he warmly praised it and supported its aims.

with which it is connected.[53] And the analysis required for the demonstration of an isolated theorem would in many cases be far more difficult and complicated than would be necessary in order to [give] the demonstration of some more general theorem of which the other is a mere Corollary.[54]

I am not prepared to say that when a theorem is considered from a sufficiently general point of view the analysis is always easy; on the contrary I rather believe that while by applying it to the right questions you give greater simplicity to the analysis, so the difficulty of the questions to which it becomes applicable and the consequent complexity of the analysis increase // also. But still the analysis employed in analytical geometry is on the whole of a simple character, sometimes indeed fascinatingly so; the difficulty of the subject is not in the analysis but in developing and grasping in the mind the vast number of geometrical theorems, often apparently unconnected together, and relating to quite different figures, which are really included in an analytical investigation, or it may be, in what is simple enough to be termed a mere analytical observation: a good illustration both of the kind of analysis [made] use of, and of the series of geometrical theorems founded upon it is the following. The [theorem] which I give now, though I shall have to return to it in my subsequent lectures.

The equation of a curve of the third order or cubic curve contains nine arbitrary constants;[55] hence the equation of a cubic curve passing through eight given points should contain a single arbitrary constant—or what is the same thing, given a cubic equation in the coordinates (x, y, z), containing a single arbitrary constant and belonging to a cubic curve through eight given points—any cubic curve whatever through the eight points may be represented by the equation in question. Let the eight points be any eight of the nine points of intersection of the two cubic curves $U = 0$, $V = 0$; then the equation $U + \lambda V = 0$ where 1 [one] is arbitrary is satisfied by the // coordinates of the eight points respectively, or it belongs to a cubic curve passing through the eight points. But the equation is satisfied also by the coordinates of the ninth point—this is all the analysis—and we have the theorem,

> Every cubic curve whatever which passes through eight of the nine points of intersection of two cubic curves passes also through the ninth point of intersection.[56]

[53] The setting down of geometrical theorems without recourse to background theory was of little interest to Cayley. He rejected a paper submitted to the *Cambridge and Dublin Mathematical Journal* which treated geometry in "that way without any reference to general geometrical theories or without any attempt to make a 'Zusammensetzung' [composition] of the whole mass of theorems one obtains, it is very uninteresting work" [45].

[54] Cayley was not solely motivated by the need for proof. He was often satisfied by, and had a positive liking for, the method of "verification" whereby a result would be tested in a few simple cases and proclaimed true in general. Alternatively, the "proof" of an algebraic result would be made to depend on an unproven (but intuitively plausible) geometric "fact." Cayley worked in a fairly primitive mathematical environment, as his strong motivation for designing suitable notation indicates. Looking back, we now see that he was hampered by his active dislike for both subscripted and superscripted notation (in his later work), dislikes which barred his way to providing adequate proof in questions of n-dimensional geometry, and, of necessity, he confined his examples to 2 and 3 dimensions.

[55] The plane cubic of the form $A(x, y) + B(x, y) + C(x, y) + d = 0$ contains $4 + 3 + 2 + 1 = 10$ constants and, hence (on division by one of them), 9 *arbitrary* constants (see note 44).

[56] Cayley attributes this theorem to Chasles. A point of interest is Cayley's acceptance of a proof which assumes that *all* cubic curves passing through the points of intersection of $U = 0$ and $V = 0$ are expressible in the form $\lambda U + \mu V = 0$. Cayley showed how the theorem may be used to prove Pascal's Theorem and Maclaurin's Theorem in 1843 [16]. He returned to the problem in 1862 [23]. For a discussion of the Cramer paradox, see [6, 246–248].

Now this includes in itself the two very different theorems.

> 1^0. *Pascal's Theorem*. If a hexagon be inscribed in a conic the three intersections of pairs of opposite sides lie in a line.
>
> 2^0. *Maclaurin's Theorem*. The line joining two inflexions of a cubic curve, again meets it in a third inflexion.

This is perhaps somewhat of a puzzle wherewith to conclude the present lecture, but to some of you it would be really a good exercise to think out for yourselves the mode in which the lastmentioned two theorems are really included in the theorem relating to the nine points of intersection of the two cubic curves.

4. THE OUTSIDER

Understandably, Cayley submitted little to journals while he settled into Cambridge, and the period from June to November 1863 was a rare fallow period in his mathematical life.[57] Evidently, he encouraged the young Clifford and, within three weeks of the inaugural lecture, Clifford had written a paper on analogues of Pascal's Theorem [50].[58] The material of Cayley's lecture on the intersections of cubic curves had had an effect on the 18-year-old, in the same vein as Cayley himself, who had proved a general theorem on the intersection of curves as a young man of 22 in 1843 [16]. This material, which Cayley touched upon in the inaugural lecture, was covered in the ensuing course. In the following August 1864, Clifford wrote "Analytical Metrics," a topic inspired by Cayley's "Sixth Memoir on Quantics" [51, 84–85; 22].[59]

Geometry dominated Cayley's research interests for the remainder of the 1860s, a decade in which he published his most extensive memoirs (not unusually, they exceeded a 100 pages in length). Besides analytical geometry, other aspects of the subject he selected for lecture topics were graphical geometry (after Möbius) and solid geometry.[60] In each academic year up to 1886, he was required to give one lecture course of a term's duration. From 1887 to 1894, two courses of lectures each year were required. Sadly, the examination system effectively sidelined Cayley, and from being the new insider at Cambridge in June 1863, he gradually began to appear as an outsider when his teaching duties are considered. He

[57] Nevertheless, by the end of November 1863, Cayley submitted some observations on an analytical theorem relating to conics for publication [25; 26].

[58] In his first year at Cambridge, Clifford turned challenger and set questions for the newly established *Mathematical Questions with Their Solutions from the Educational Times*. In a reversal of roles, two of them were solved by the new professor. Both problems drew on the material in Cayley's course of lectures. The first was a problem in pure geometry (on conics) and solved by an application of the theory of pole and polar [48]. The second was a problem on the projective cross-ratio which Cayley treated as a problem in analytical geometry [49]. Cayley kept a watchful eye on Clifford's progress, particularly during the Tripos examinations in 1867, in which Clifford was placed Second Wrangler in the order of merit [39].

[59] Cayley did not explore this topic in his lecture course, perhaps thinking it had not been sufficiently developed to make it suitable for presentation to undergraduates. That Clifford should write a paper on the subject within such a short time indicates his mathematical talent. It is notable that he was willing to grapple with problems at the forefront of mathematics at such a young age.

[60] Cayley lectured on analytical geometry in 1863–1865, 1869, 1870, 1888, 1889, and 1891–1894; on graphical geometry in 1871 and 1882; and on solid geometry in 1878 and 1889.

was not the only mathematical professor left in limbo. His friend, the astronomer John Couch Adams (1819–1892), shared the same dismal experience. Less than six months after Cayley's opening lecture, Whewell noted on a flysheet, dated 8 March 1864, "that a spontaneous attendance upon lectures, even admirable lectures, often dwindles down to a half-dozen or even less" [14].[61]

Attendance at the lectures of other Cambridge professors was kept artificially buoyant by the university regulation which compelled ordinary degree students to attend at least one term's lectures from a professor selected from a "starred" list—a list which did *not* include the mathematical professors. During the 1860s, this compulsion did much to swell attendance in the courses of the nonmathematical professoriate. Men such as Henry Fawcett, the professor of political economy, had lecture rooms brimful with students whose main concern was to have his signature on their certificates of attendance.[62] By contrast, for honors degree students of mathematics in 1863, the informal position was that the professor's lectures were "of service" but not required. In practice, the recommended study regime for success in the Tripos was to attend College lectures with the necessary element of supervision from private tutors on alternate days.[63] The private tutors drilled their students in "Tripos technique." The effective ones knew the correct balance to place on each well-defined topic. In their pupil-rooms students received a strictly administered diet of learning, practising, and endless testing so that by the end of this meticulous preparation they could produce examination answers on demand.[64]

Cayley did not help his own cause. His lecture titles were only faintly indicative of the intended content, and he did not feel bound by them. He would often treat the occasion as an opportunity to give a spontaneous account of one of his latest discoveries before embarking on the advertised lecture topic. Andrew R. Forsyth (1858–1942), his successor at Cambridge, reported that "old notes were never used a second time" [62, xvii]. It was all too obvious that his state-of-the-art lectures would have little relevance to students whose principal object was to score well in the Mathematical Tripos examination. Cayley was concerned with mathematics *per se*, while most students wanted competence in a circumscribed syllabus. Any deviation from it was regarded as a waste of time and effort. James Wilson, who attended Cambridge a few years earlier, noted that one of the functions of the private tutor

[61] The small numbers attending professorial lectures in mathematics at Cambridge could be compared with more than a hundred undergraduates who, in his inaugural year, of 1860, crowded Kingsley "out of room after room, till he had to have the largest of all the schools, and we crowded that—crammed it" [74, 2:118].

[62] Kingsley also benefited from this regulation. When it was abandoned in 1876, Henry Fawcett fulfilled his professorial obligations and gave his lectures to empty lecture rooms.

[63] The individual standard charge for coaching in 1860 was about £7 per term and £12 for the "Long," the long summer vacation [72, 449]. In pecuniary terms being a coach was a far better proposition than being a professor, but in availability of free time, the opposite was the case. The private tutor taught for the whole year including the long vacation, but the professors' lecturing duties amounted to a few hours per week during term time only. They lived an unfettered existence. Jointly, these two categories were not bound by college rules, unlike college fellows who were dependent on the College Chest (college fellows were not allowed to marry, a rule which remained in place for most colleges until the 1880s).

[64] Being trained by a coach would have seemed quite natural for students entering Cambridge from public schools (in England these are fee-paying schools such as Eton and Harrow), a principal route of entry. In these schools, masters would give their pupils extra lessons for extra payment. It was another instance when the cross-over from school to Oxbridge to the Inns of Court could be seamless for a Victorian young man with the right connections.

was to deflect students away from professorial lectures, which, while of interest, did not "pay" in the Tripos. Wilson was fascinated by trilinear coordinates and the topical "abridged notation," but his coach advised that a better use of time would be Tripos practice [116, 42–43]. The paying work was what students wanted and what their parents expected them to receive. Once over the Tripos hurdle, they were free to study whatever they wished and were regarded as having gained the all important prerequisite—for it was surely a mark of a Cambridge education in mathematics that after admission to an honors degree, one could "get-up" any subject whether it was law, medicine, or any other intellectual challenge met with in later professional life.

The system of teaching mathematics at Cambridge was so deeply imbedded that it proved relatively immune to the occasional sideswipes from mathematicians. The Mathematical Tripos was one of Augustus De Morgan's favourite targets: "The Cambridge examination is nothing but a hard trial of what we must call problems—since they call them so—of the Senior Wrangler that is to be of this present January [1865], and the Senior Wranglers of some three or four years ago. The whole object seems to be to produce problems, or, as I should prefer to call them, hard 10-minute conundrums" [55, 283]. A more stinging attack came from Leslie Stephen—not withstanding his disenchantment with pure mathematics as a suitable instrument for education. De Morgan had been a student at Cambridge in the 1820s, but Stephen had more recent experience. He was Twentieth Wrangler in 1854 and had been a mathematics tutor at Cambridge for 7 years. In 1865, he published the popularly received *Sketches from Cambridge by a Don*, by way of a parting shot when he resigned his fellowship in 1862. Stephen purloined the well-known image of the Tripos as a horse race, when he described the Senior Wrangler as the winner of the Derby and the pupil-rooms of the different coaches as the stables in which the horses were groomed.

Cayley was not drawn into sweeping criticisms of the Tripos, but he did become embroiled in one heated debate which took place during 1866 and 1867. Its occurrence must have caused him to think of the peace of 2, Stone Buildings, his erstwhile chambers in Lincoln's Inn. In the dispute, Cayley met a direct challenge from Sir George Biddell Airy (1801–1892), a man of wide scholarship and intellectual interests.[65] Airy occupied a pivotal position in British science and astronomy and was Astronomer Royal from 1836 to 1881. During his own student days at Cambridge, Airy had been guided by George Peacock and was the Senior Wrangler in 1823. Newton apart, Whewell bracketed him with Cayley as the two finest mathematicians Trinity College had ever produced. Superficially, the disagreement between these two respected men of science appeared to hinge on the balance between pure and applied mathematics but, as the debate unfolded, Cayley's attitudes to the teaching of mathematics became apparent. He was not opposed to the ideals of a liberal education, but neither did he see the *raison d' être* of mathematics as grist for the Tripos mill nor his field as a servant of science to be rudely beckoned for any ill-considered scientific task. Airy, for his part, was not satisfied by mathematical sophistication alone, and he directed a vigorous campaign aimed at the introduction of more applicable mathematics into the Tripos course.

[65] In 1857 Airy bemoaned sending 22-year-old Cambridge graduates into the world without the slightest knowledge of physical problems. Amongst them was the important question relating to the shape of the earth viewed as a rotating body. To indicate the vibrancy of Airy's campaign, if not its limited vision, Airy suggested to the Vice Chancellor that a university capable of this conduct "has already sunk to the position of a second-rate Academy" [13].

His views counted in the mathematical fraternity. He was a mathematician and practising scientist but one with a strong interest in the physical applications of mathematics.

On 24 May 1866, the Board of Mathematical Studies with Cayley as President met at Garden House to consider Airy's proposed scheme. After undue equivocation, the Board produced a Report a year later on 8 May 1867 that set the stage for addressing whether the Mathematical Tripos as the capstone of the existing Cambridge syllabus was an appropriate kernel for a mathematical education. A collateral concern was the increasing specialisation and diversity of mathematical subjects and the need to reconcile this with the traditional comprehensive nature of the Tripos course of study. The Report proposed, among other things, reinstatement of parts of the syllabus in applications of mathematics (heat, electricity and magnetism, elastic solids) lost in the earlier reform when the Board was founded in 1848. In their Report, the Board stressed the importance of some of Cayley's subjects. The introduction of elliptic integrals was suggested, and the Board

> have already recommended the re-introduction of Laplace's Co-efficient and the Figure of the Earth considered as heterogeneous, in doing which they are partly influenced by the publication of a recent work on these subjects. While curtailed in some directions, the [Mathematical Tripos] course has however extended itself in others, especially in those of analytical geometry and higher algebra, so that it is really as extensive as ever, while yet there are some important subjects which are entirely omitted. [11]

The Report noted that since "the progress of science is continually enlarging the field of Mathematical knowledge by extending the range of pure mathematics, and reducing fresh branches of physics to mathematical laws," there was a need continually to evaluate the course of mathematics and to maintain the comprehensive nature of the examination without making it too onerous [117, 223]. Following publication of the Report, the mathematical professors met to consider ways in which the Tripos could be modified and the Smith's prizes distributed. Ahead of any conclusions, Cayley was involved in a short, sharp correspondence with Airy in November and December of 1867.[66]

It is difficult to imagine men of such different temperaments. Cayley was a retiring scholar with the manners of a gentleman from an earlier generation, but one who stood his ground, while Airy had a sure gift for pedantry, tenacity, and a reputation for forthright attack. Airy had already suggested that the large part of "useless algebra" should be reduced and applications of mathematics be introduced. Cayley, the appointed standardbearer of pure mathematics was pitted against a man with diametrically opposed views of what actually constituted useful mathematics.

The opening salvo came from Airy on 8 November 1867 in the guise of a discussion on partial differential equations. In his letter, Airy derided the usefulness of such subjects as analytical geometry for the majority of students. In reply, Cayley neatly deflected the attack by making direct appeal to his statutory duty of the Sadleirian professor "to explain and teach the principles of pure mathematics." This response did not satisfy the Astronomer Royal.

[66] In 1857, Airy had a personal axe to grind. His two eldest sons, Wilfrid and Hubert, had been in residence at Trinity College, and he had become concerned about the mathematics they were *not* studying. By 1867, his elder sons had graduated, but his third son, Osmund, had been admitted to Trinity College in the year Cayley was appointed. Of his three sons, Osmund showed the surest sign of mathematical talent (he graduated 27th in the order of merit). Airy was now a concerned parent who was attuned to the teaching offered at Cambridge. Moreover, he had become an "insider" at Trinity College and was among the first batch of Honorary Fellows elected in June 1867.

Cayley argued that applications frequently gave rise to mathematical problems in which students had no hope of initiating their own mathematical thought and were obliged to adopt the methods found in textbooks. He was on surer ground when he countered Airy's views on the teaching of modern geometry. It was in this subject, Cayley maintained, that there was room for experimentation and through it, the opportunity of developing geometrical intuition:

> I cannot but differ from you *in toto* as to the educational value of Analytical Geometry, or I would rather say of Modern Geometry generally.... Whereas Geometry (of course to an intelligent student) is a real inductive and deductive science of inexhaustible extent, in which he can experiment for himself—the very tracing of a curve from its equation (and still more the consideration of the cases belonging to different values of the parameters) is the construction of a theory to bind together the facts—and the selection of a curve or surface proper for the verification of any general theorem is the selection of an experiment in proof or disproof of a theory. [1, 275–276]

The empirical approach was precisely the way in which Cayley had carried out his own geometrical investigations, but Airy was difficult to convince. Cayley bravely asserted that geometry was a developing theory just as much as physics. Moreover, he intimated that geometry developed all the faculties that Airy claimed could only be achieved by studying physics. By seeing mathematics as the servant of science, Airy argued for the teaching of those parts of mathematics that could help students understand *real* science. He balked at any hint of artificiality as might be found, he suggested, in a "game of billiards with novel islands on a newly shaped billiard table" [1, 277]. Cayley, his accommodating nature sorely tried, insisted that pure mathematics was a science in its own right with its own intrinsically interesting problems. To Cayley, these were as real as any in Airy's physical world. In his reply Cayley argued, perhaps unwisely, that Airy's proposed extension of ordinary billiards, "*if it* were found susceptible of interesting mathematical developments, would be a fit subject of study" [1, 278; Cayley's emphasis].[67]

As the debate proceeded, Cayley began to lose sight of the average undergraduate:

> But admitting (as I do not) that Pure Mathematics are only to be studied with a view to Natural and Physical Science, the question still arises how are they best to be studied in that view.... Now taking for instance the problem of three bodies—unless this is to be gone on with by the mere improvement in detail of the present approximate methods—it is at least conceivable that the future treatment of it will be in the direction of the problem of two fixed centres, by means of elliptic functions, etc.; and that the discovery will be made not by searching for it directly with the mathematical resources now at our command, but by "prospecting" for it in the field of these functions. Even improvements in the existing methods are more likely to arise from a study of differential equations in general than from a special one of the equations of the particular problem. [1, 278–279]

Cayley explained that "my idea of a University is that of a place for the cultivation of all science" [1, 278]. For Cayley, this included pure mathematics, and this was to be studied for its own sake. Cayley's ultimate position was that mathematics was a thing of beauty and did not need an immediate purpose. The way of progress was best achieved by allowing mathematicians to go "prospecting" in mathematics unconstrained by immediate practicalities. It was Cayley's ill-advised sentence: "I do not think everything should be subordinated to

[67] Billiards played on rectangular tables was a popular game in the 1860s and had been analyzed mathematically. The game has a long history and had already been played on triangular and octagonal shaped tables. The mathematics related to more esoteric tables have also been treated since Cayley's time. For example, see [73].

the educational element" that caused Airy the greatest consternation. "I cannot conceal my surprise at this sentiment," wrote Airy. "Assuredly the founders of the Colleges intended them for education (so far as they apply to persons in *statu pupillari*), the statutes of the University and the Colleges are framed for education, and the fathers send their sons to the University for education. If I had not had your words before me, I should have said that it is impossible to doubt this" [1, 279]. When Airy was a student at Cambridge in the 1820s, the University had been primarily a teaching institution where the education of undergraduates was paramount, but, with the new professors appointed in the 1860s, the seeds of change were planted. Research became a watchword, though it was little pursued in the 1860s.

In the Cambridge environment of the 1860s, Airy's argument won the day. It was decided that the advanced part of the Tripos should contain a thorough study of such applied topics as heat, electricity, and magnetism. W. P. Turnbull, who gained a Trinity College fellowship in 1865, noted that his own *Analytical Geometry* "was produced at a time when abstract thought was rather at a discount, for physical research was in the ascendant" [112, 14]. On 2 June 1868, new regulations for the Tripos were narrowly approved by the Senate and came into operation for the first time in January 1873.

For those students who did attend Cayley's classes in the aftermath of this debate, the lectures evoked fond memories [63; 80]. Cayley was the mentor of several distinguished Victorian scientists. The brilliant Clifford was the first, but the next, James Whitbread Lee Glaisher (1848–1928) (see Fig. 3) was, in several ways, Cayley's inheritor, the one to remain in Cambridge and pursue Cayley's subjects (in particular, elliptic functions and astronomy). The story of Cayley's "somewhat remarkable" lectures attended by the young J. J. Thomson (1856–1940) is perhaps well known. Thomson, an undergraduate from 1876 to 1880, was looked upon by his Cambridge friends as an outstanding student and perhaps for this reason was permitted to attend Cayley's lectures. Thomson with two Masters of Arts made up the total audience. Cayley did not use the blackboard, but sat at the end of a long narrow table and wrote with a quill pen on sheets of large foolscap paper. Thomson sat opposite and only saw Cayley's lecture upside down [109, 47].[68]

Thus sidelined, Cayley had little contact with potential disciples, what Cambridge termed the "professed mathematicians," who would naturally be found among the putative top

[68] This proved only a minor setback for Thomson, and the future physicist's early papers consisted of pure mathematics and the theory of electricity interspersed. While an undergraduate, he submitted three notes to the *Messenger of Mathematics* on the eight-squares problem, the calculus of operations, and elliptic functions, all subjects on which Cayley was expert [106, 107, 108]. The advanced topics in the Mathematical Tripos during Thomson's time were skewed in the direction of mathematical physics (astronomy, dynamics, thermodynamics, hydrodynamics, sound, optics, elasticity, electricity, and magnetism were prominent). These had been traditional Cambridge "mixed mathematics," and while they occupied a place in the Tripos of the 1840s, they were taught differently in the 1870s after the opening of the Cavendish Laboratory in 1874 with James Clerk Maxwell as first Professor of Experimental Physics [114]. The rapid rise of physics from a zero-baseline to an international reputation was the success story of Cambridge from 1870. Pure mathematics lived in the shadow of these developments. In the upper echelons of the Mathematical Tripos, only a limited palette of pure mathematics was available (higher algebra, higher analytical geometry, elliptic functions), and Cayley argued vigorously that the theory of numbers should be included. Consequently, it was more likely that graduates would take up mathematical physics for research than subjects in pure mathematics. Forsyth himself toyed with the kinetic theory of gases before becoming Cayley's apostle. William Burnside (1852–1927), who was taught by the well-known coach William H. Besant (a close rival to Routh), wrote papers on the kinetic theory of gases in the late 1880s, and only in the 1890s did he turn to group theory in pure mathematics.

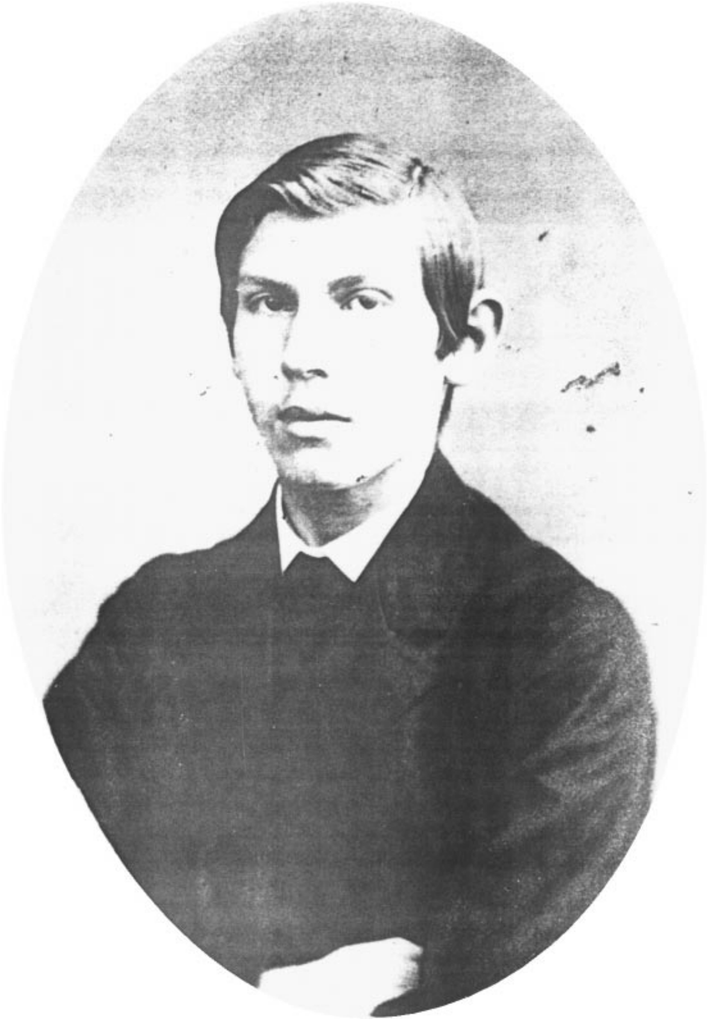

FIG. 3. James Whitbread Lee Glaisher (1848–1928). Photograph courtesy of the London Mathematical Society.

wranglers of each year. Had he been a university politician (like William Whewell, for instance) rather than a reticent scholar, a more lively atmosphere might have resulted. He defended the position of pure mathematics during the 1860s and 1870s, but when the time was ripe for establishing a school of mathematics during the 1880s, nothing even remotely like the one Felix Klein created in Germany was forthcoming. Glaisher was the sharpest critic of the prevailing atmosphere: "In Cambridge so much Mathematics was taught, and yet so little original mathematical work was done by Cambridge men. The difference between foreign Universities and Cambridge in respect of amount of original work in Mathematics was appalling" [9, 510]. Cayley felt himself powerless to change the undergraduate regime, and, moreover, there was no mechanism that permitted a student who did not graduate at Cambridge to transfer to the university for more advanced studies. Coupled with this undeveloped postgraduate system, Cayley's inspiration was limited to only a few undergraduates who passed through Cambridge.[69]

Thus in May of 1881 Sylvester could tempt Cayley to visit Johns Hopkins University in Baltimore by promising "a class of Glaishers for your auditors and the seed you might sow would fall upon a fertile soil" [103]. Cayley's classes were always small, and later in the year, when he had 12 students at Cambridge, he remarked on the "exceptional number" [40]. While interest in advanced pure mathematics at Cambridge was minimal, it was healthy in comparison with Oxford. In 1883, advising Felix Klein on whether to apply for the Savilian chair, Cayley informed him that "there is I am afraid the serious objection of the want of interest in mathematics at Oxford" [41]. (This would change when Sylvester took up the post of Savilian professor the following year.) Five new university mathematical lectureships at Cambridge were created in 1882, and this move would provide seedcorn to aid the revival of British mathematics in the 20th century.[70]

Cayley was able to offer help and advice to a few students, but invariably this was on an individual basis. He adjudicated Trinity College fellowship dissertations from 1872, and though professors were normally cut off from examining for the Tripos, he regularly adjudicated in the Smith's Prize contest. This had been known as the "Professors' Examination" but the two annual prizes, small in monetary terms but high in prestige, were awarded for essays in the 1880s. If he saw exceptional work, he encouraged young mathematicians to submit work for publication in the Royal Society's *Philosophical Transactions* with his support (such as Glaisher's, Forsyth's, R. C. Rowe's, and G. T. Bennett's juvenalia). By the end of the 1880s, Cayley (then seriously ill) recommended ambitious young mathematicians to study with Felix Klein in Germany. He was still working, but his principal interest, invariant theory, was idiosyncratic work, and his approach to it had already been superseded.[71] The younger students did not appear to have the same sense of reverence towards the old man as the students of the previous generation. Grace Chisholm, who later gained a doctorate at Göttingen, complained of the "stifling" Cambridge atmosphere and criticized Cayley who, she wrote, "sat, like a figure of Buddha on its pedestal, dead-weight on the mathematical school of Cambridge" [67, 115].

At school level, the vexed question of Euclid in English education arose in the 1860s and for 30 years continued to simmer. In the previous decade, Cambridge had widened its influence by conducting examinations for school-aged children through examinations

[69] The most prominent included Clifford, James Stuart (1843–1913), Glaisher, Alfred B. Kempe (1849–1922), Karl Pearson (1857–1936), Thomson, Forsyth, Richard C. Rowe (1853–1884), Arthur Berry (1862–1929), Henry F. Baker (1866–1956), and Geoffrey T. Bennett (1868–1943). In the 1890s, the situation regarding research was gradually improving. Unfortunately, it was too late for Cayley himself (though he helped set up the Isaac Newton postgraduate studentships).

[70] For each lectureship there was a stipened of £50. Forsyth, Hobson, Thomson, Richard T. Glazebrook (1854–1935), and William H. Macaulay (1853–1936) were appointed [105, 118].

[71] In the 1880s, David Hilbert effectively transformed the subject to an "abstract" one, whereas Cayley's approach was primarily algorithmic (and hence "concrete"). Henry F. Baker, one of Cayley's last students at Cambridge, initially studied invariant theory with him, but when he studied at Göttingen later, Felix Klein had a greater influence. Baker came to regard the once great subject of invariant theory as merely a branch of group theory which was then in the ascendant.

held at local examination centres. They and other examining bodies in the United Kingdom conducted the geometrical component of their examinations by insisting on Euclid's order of presentation and his methods of proof. Opposition to this philosophy was gathering amongst the schoolmasters who saw their pupils not learning geometry, but attempting to "parrot-learn" his chains of reasoning. In 1869, a high-level British Association Committee which included Cayley, Sylvester, and H. J. S. Smith was appointed to study the question. While others (including Sylvester) supported the reformers, Cayley argued the case for the retention of Euclid and fought the reformers every inch of the way. He doubted whether the schoolmasters' intention of supplying a text to take the place of the *Elements* would be any improvement on the way geometry had been studied by the Greeks, and, moreover, the form of the *Elements* (especially Book V) was to be regarded as a thing of beauty and an intrinsic part of a liberal education [85, 28–33; 115].

With other prominent reformers, Rawdon Levett, a schoolmaster and an ardent supporter of the Association for the Improvement of Geometrical Teaching (he was its secretary from 1871 to 1884), journeyed to Cambridge in the 1880s to continue the argument for reform.[72] He had been a third-year student at the time of Cayley's inaugural lecture and had proposed an Anti-Euclid Association in *Nature* in 1870. At the meeting at Cambridge on 23 April 1887, Levett recalled that Cayley dominated the meeting and effectively blocked any opposition to his point of view [99, 18]. Cayley argued that Euclid should not be tampered with, despite the archaic appearance of the text and the difficulty this put in the way of school pupils. He was unwilling to compromise, and though the Mathematical Board admitted that the rigid adherence to Euclid's text was not in the best interests of education, only Cayley opposed this view and chose not to sign the Board's recommendation [7, 29; 10, 759–761].[73]

Cayley held unrealistic expectations of students. Rawdon Levett also recalled Cayley's preposterous view that "the proper way to learn geometry is to start with the geometry of n dimensions and then come down to the particular cases of 2 and 3 dimensions" [99, 18]. On another occasion, Cayley recommended that "a complete knowledge of invariants and covariants of ternary forms ought to be presupposed in the teaching of Higher Plane Curves" [96, 141]. In one important way Cayley and Levett were similar: both were deeply affected by the beauty of mathematics. As a teacher, Levett naturally focused on the plight of his pupils, while Cayley's experience caused him to think first of his subject.

In his defence, Cayley always admitted that he had little contact with the teaching of elementary mathematics [95, 483]. When he was once asked what parts of elementary

[72] The insistence of the Examination Boards on paying continued respect to Euclid's authority was advantageous to those who adopted the "cram" method of teaching. Though it led to an improvement of the "portative memory," it did little for the acquisition of geometrical understanding in the school population. In the Cambridge Local Examinations Syndicate Report on the examinations for 1883, for example, the question on Euclid was answered poorly by the Junior candidates (those up to the age of 16): "A very large percentage of the candidates failed in the proposition 'draw a straight line perpendicular to a given straight straight line from a given point without it,' many of them betraying great ignorance of Geometry in their attempts to solve this problem" [9, 496].

[73] The schoolmasters' argument was gradually accepted. Shortly afterwards, in 1887 and 1888, both Oxford and Cambridge passed regulations that allowed proofs of geometrical theorems other than the ones found in Euclid. Unlike Cayley, Forsyth was progressive in educational reform, and, when he came to occupy the Sadleirian chair in 1895, he would have nothing to do with any report that supported the retention of Euclid's order of presentation. One long chapter was coming to an end, and by the turn of the new century, Euclid had ceased to dominate English education.

mathematics the advanced students were most deficient in, he answered with plain honesty: "My classes are very small and as I give lectures only I naturally assume in my hearers a sufficient acquaintance with the lower subjects" [43, 255]. His views on what should be taught in the "lower subjects" were conservative and reflected his own education shaped at King's College, London, and Cambridge. Euclid was a firm fixture of the King's curriculum and a mainstay of the undergraduate course at the university. His first-year Cambridge tutor George Peacock was also of the view that Euclid's *Elements* approached perfection.

The lines of deduction of Euclid's propositions, memorized by generations of school pupils, were poor preparation for the geometric investigations that lay outside the formalised network of Euclid's interconnected definitions and propositions. Cayley believed Euclid was where school pupils should start, but in his view of geometry as progressive, Euclid did not represent the whole subject. While Cayley admired the achievement of Euclid and placed the *Elements* on a pedestal, it is worth noting that his own approach to geometry was at variance with it. Behind the formality of his memoirs was an intuitive mind. He learned by making models, drawing curves, and preparing multicolored diagrams in water color, suggesting his was a tactile approach to geometry. The geometrical investigations of the type Cayley carried out required a spirit of discovery and investigation, what Hobson, a later successor to the Sadleirian chair, viewed as essentially inductive: "The actual evolution of mathematical theories proceeds by a process of induction strictly analogous to the method of induction employed in building up the physical sciences; observations, comparison, classification, trials, and generalisation are essential in both cases" [71, 520]. Cayley's championing of Euclid, with its inherent formality, actually camouflaged his own methods of conducting research.

With Cayley's concern for mathematical investigation at variance with the main day-to-day business of the university, that of teaching undergraduates, it was inevitable that he would be sidelined. In the teaching of mathematics, Edward Routh, another "outsider" of the official system, enjoyed tangible success. While Routh unsuccessfully applied for the Sadleirian Chair in 1863, this erudite man enjoyed a subsequent career as a renowned mathematical coach, and when he retired in 1888, he did so with an unequalled teaching reputation.[74] His professional life was in stark contrast to Cayley's, officially elected in 1863 "to explain and teach the principles of Pure Mathematics and to apply himself to the advancement of that Science" [62, xvi].

At Cambridge, Cayley ultimately failed to establish a school of mathematics. Writing in 1890, this time with regret, his outstanding student J. W. L. Glaisher could only remark: "I am afraid that the old saying that we have generals without armies is as true as ever" [65, 724]. For his part, Cayley was faithful to the terms of the Sadleirian chair. He pursued his mathematical interests relentlessly and quietly transmitted his knowledge to the most talented students who passed through Cambridge, the ones who were influential in the mathematical investigations of the 20th century.[75]

[74] His career spanned the period from 1856 until 1888, and he became the "Senior Wrangler Maker" of Cambridge University. During his career, he taught something like half the wranglers, and for 20 consecutive years (from 1862), *all* the Senior Wranglers. He failed to obtain a professorship on later occasions [57].

[75] Cayley was of the opinion that a good piece of mathematics was never wasted. While much of his work is now rarely heeded, his papers in such areas as division algebras, group theory, matrix algebra, invariant theory, and graph theory have influenced 20th-century mathematicians (e.g. [113; 114]).

ACKNOWLEDGMENTS

Many years ago, Ivor Grattan-Guinness alerted me to the existence of Cayley material at Columbia University. For this and general help over the years I thank him. For comments on various drafts of this article I would like to thank Adrian Rice, Maria Panteki, Roy and Monty Chisholm, Frank Smithies, Andrew Warwick, and Clive Kilmister. I acknowledge help received from John Coates, the current Sadleirian professor, who gave me information on the Sadleir foundation. A grant from the British Academy has enabled me to carry out this work. It is a recurring pleasure to thank the librarians and authorities at Cambridge for their hospitality and for permission to quote from the collections under their care. In particular, I thank the Master and Fellows of Trinity College, the Master and Fellows of St. John's College, and the librarian of the Cambridge University Library. I am grateful to the Royal Society of London and the Niedersächsische Staats- und Universitätsbibliothek, Göttingen, for the same reason, and to the London Mathematical Society for permission to reproduce the portraits of W. K. Clifford and J. W. L. Glaisher, and to the Master and Fellows of St. John's College for permission to reproduce the portrait of Cayley. I thank Jean Ashton (Director) and Bernard R. Crystal (Curator of Manuscripts) for their help in obtaining permission to publish Arthur Cayley's inaugural lecture held in the David Eugene Smith Historical Papers, Rare Book and Manuscript Library, Columbia University, New York. Finally, special thanks are due to Karen Hunger Parshall in her capacity as Editor and for her suggestion and encouragement in widening the scope of an earlier draft of this article.

REFERENCES

Note. Nearly all of Cayley's mathematical papers were reprinted in *The Collected Mathematical Papers of Arthur Cayley*, ed. Arthur Cayley and A. R. Forsyth, 14 vols., Cambridge: University Press, 1889–1898. In the following list of references, these volumes will be referred to in the style: CP[volume number]([year of publication]), [pagination]. Pagination of Cayley's work in the body of this paper refers to *The Collected Mathematical Papers*.

1. George Biddell Airy, *Autobiography of Sir George Biddell Airy*, ed. Wilfrid Airy, Cambridge, UK: Cambridge University Press, 1896.
2. Harvey W. Becher, William Whewell and Cambridge Mathematics, *Historical Studies in the Physical Sciences* **11** (1980), 1–48.
3. Michael Bernkopf, Edmond Nicolas Laguerre, *Dictionary of Scientific Biography*, New York: Charles Scribner & Sons, 1973, 7:573–576.
4. George Boole, *George Boole: Selected Manuscripts on Logic and Its Philosophy*, ed. Ivor Grattan-Guinness and Gérard Bornet, Boston/Berlin: Birkhäuser, 1997.
5. Henk J. M. Bos, Cees Kers, Frans Oort, and Diederik W. Raven, Poncelet's Closure Theorem, *Expositiones Mathematicae* **5** (1987), 289–364.
6. Carl B. Boyer, *History of Analytic Geometry*, Scripta Mathematica, 1956.
7. William H. Brock, Geometry and the Universities: Euclid and His Modern Rivals 1860–1901, *History of Education* **4**(2) (1975), 21–35.
8. *Cambridge Chronicle*, 24 October 1863.
9. *Cambridge University Reporter*, 1882–1883 (No. 468), 14 March 1883.
10. *Cambridge University Reporter*, 1886–1887 (No. 698), 31 March 1887.
11. Cambridge University Library (Manuscripts): Min. V.7.
12. Cambridge University Library (Manuscripts): O.XIV.52.127.
13. Cambridge University Library (Rare Books): G.F.109.
14. Cambridge University Library (Rare Books): H.C.1.
15. William M. Campion, On the Course of Reading for the Mathematical Tripos, in *The Students Guide to the University of Cambridge*, ed. Sir J. R. Seeley, Cambridge, UK: Deighton Bell, 1862, pp. 82–102.
16. Arthur Cayley, On the Intersection of Curves, *Cambridge Mathematical Journal* **3** (1843), 211–213, or CP1(1843), 25–27.

17. Arthur Cayley, On Certain Results Relating to Quaternions, *Philosophical Magazine* **26** (1845), 141–145, or CP1(1845), 123–124.
18. Arthur Cayley, Mémoire sur les fonctions doublement périodiques, *Journal des mathématiques pures et appliquées* **10** (1845), 385–420, or CP1(1845), 156–182.
19. Arthur Cayley, On Some Analytical Formulae, and Their Application to the Theory of Spherical Coordinates, *Cambridge and Dublin Mathematical Journal* **1** (1846), 22–33, or CP1(1846), 213–223.
20. Arthur Cayley, Sur quelques formules du calcul intégral, *Journal des mathématiques pures et appliquées* **12** (1847), 231–240, or CP1(1847), 309–316.
21. Arthur Cayley, Note on the Porism of the In-and-Circumscribed Polygon, *Philosophical Magazine* **6** (1853), 99–102, or CP2(1853), 87–90.
22. Arthur Cayley, A Sixth Memoir on Quantics, *Philosophical Transactions of the Royal Society of London* **149** (1859), 61–90, or CP2(1859), 561–592.
23. Arthur Cayley, On the Construction of the Ninth Point of Intersection of the Cubics Which Pass through Eight Given Points, *Quarterly Journal of Pure and Applied Mathematics* **5** (1862), 222–233, or CP4(1862), 495–504.
24. Arthur Cayley, On Differential Equations and Umbilici, *Philosophical Magazine* **26** (1863), 373–379, 441–452, or CP5(1863), 115–130.
25. Arthur Cayley, Analytical Theorem Relating to the Four Conics Inscribed in the Same Conic and Passing through the Same Three Points, *Philosophical Magazine* **27** (1864), 42–43, or CP5(1864), 131–132.
26. Arthur Cayley, Analytical Theorem Relating to the Sections of a Quadric Surface, *Philosophical Magazine* **27** (1864), 43–44, or CP5(1864), 133–134.
27. Arthur Cayley, On the Notion and Boundaries of Algebra, *Quarterly Journal of Pure and Applied Mathematics* **6** (1864), 382–384, or CP5(1864), 292–294.
28. Arthur Cayley, On Quartic Curves, *Philosophical Magazine* **29** (1865), 105–108, or CP5(1865), 468–470.
29. Arthur Cayley, Note on Lobatchewsky's Imaginary Geometry, *Philosophical Magazine* **29** (1865), 231–233, or CP5(1865), 471–472.
30. Arthur Cayley, On the Classification of Cubic Curves, *Cambridge Philosophical Society Transactions* **11**(Part 1) (1866), 81–128, or CP5(1866), 354–400.
31. Arthur Cayley, On Cubic Cones and Curves, *Cambridge Philosophical Society Transactions* **11**(Part 1) (1866), 129–144, or CP5(1866), 401–415.
32. Arthur Cayley, A Memoir on Abstract Geometry, *Philosophical Transactions of the Royal Society of London* **160** (1870), 51–63, or CP6(1870), 456–459.
33. Arthur Cayley, Note on the Calculus of Logic, *Quarterly Journal of Pure and Applied Mathematics* **11** (1871), 282–283, or CP8(1871), 65–66.
34. Arthur Cayley, On the Problem of the In-and-Circumscribed Triangle, *Philosophical Transactions of the Royal Society of London* **161** (1871), 369–412, or CP8(1871), 212–257.
35. Arthur Cayley, Curve, *Encyclopaedia Britannica*, 9th ed., 1877, 6:716–728, or CP11(1877), 460–489.
36. Arthur Cayley, Geometry, Part 2—Analytical Geometry, *Encyclopaedia Britannica*, 9th ed., 1879, 10:408–420, or CP11(1879), 546–582.
37. Arthur Cayley, [Presidential] Address by Arthur Cayley, *Report of the Fifty-Third Meeting of the British Association for the Advancement of Science 1883*, London: John Murray, 1884, pp. 3–37, or CP11(1883), 429–459.
38. Arthur Cayley to George Boole, 7 December 1847, Royal Society of London, Mss782, E-13. [Copy made by Boole]
39. Arthur Cayley to T. A. Hirst, 21 January 1867, 25 January 1867, London Mathematical Society, University College, London.
40. Arthur Cayley to T. A. Hirst, 31 March 1882, London Mathematical Society, University College, London.
41. Arthur Cayley to Felix Klein, 16 April 1883, Niedersächsische Staats- und Universitätsbibliothek, Göttingen, Letter 375.

42. Arthur Cayley to Felix Klein, 23 July 1889, Niedersächsische Staats- und Universitätsbibliothek, Göttingen, Cod. Ms. F. Klein 8-386.

43. Arthur Cayley to David Eugene Smith, 9 October 1893, in Lao G. Simons, Among the Autograph Letters in the David Eugene Smith Collection, *Scripta Mathematica* **11** (1945), 247–262.

44. Arthur Cayley to William Thomson, 5 November 1847, Cambridge University Library, Add 7342.C56.

45. Arthur Cayley to William Thomson, 8 February 1847, Cambridge University Library, Add 7342.C45.

46. Michel Chasles, *Traité de géométrie supérieure*, Paris: Bachelier, 1852.

47. John Willis Clark, *Endowments of the University of Cambridge*, Cambridge, UK: Cambridge University Press, 1904.

48. William Kingdon Clifford, Problem 1387, *Mathematical Questions with Their Solutions from the Educational Times* **1** (1863), 43–45, or CP5(1863), 561–562.

49. William Kingdon Clifford, Problem 1409, *Mathematical Questions with Their Solutions from the Educational Times* **1** (1863), 77–79, or CP5(1863), 562–563.

50. William Kingdon Clifford, Analogues of Pascal's Theorem, *Quarterly Journal of Pure and Applied Mathematics* **6** (1864), 216–222, or *Mathematical Papers by William Kingdon Clifford*, ed. Robert Tucker, London: Macmillan, 1882, 72–79.

51. William Kingdon Clifford, Analytical Metrics, *Quarterly Journal of Pure and Applied Mathematics* **7** (1865), 54–67; **8** (1866), 16–21, 119–126, or *Mathematical Papers by William Kingdon Clifford*, ed. Robert Tucker, London: Macmillan, 1882, pp. 80–109.

52. Charles Henry Cooper, *Annals of Cambridge*, 5 vols., Cambridge, UK: Warwick & Co., 1842–1853.

53. Gabriel Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750.

54. Charles Davidson, Jacob Waley, and Thomas Key, eds., *Davidson's Precedents and Forms*, 3rd ed., vol. 3, pt. 2, London: William Maxwell, 1873.

55. Sophia De Morgan, *Memoir of Augustus de Morgan*, London: Longmans Green, 1882.

56. René Descartes, *La géométrie*, Leyden: Elzevir, 1637.

57. James Douglas Hamilton Dickson, Routh, Edward John, *Dictionary of National Biography*, 2nd suppl. London: Smith, Elder & Co., 1912, 3:233–235.

58. Charles L. Dodgson, *A Syllabus of Plane Algebraical Geometry*, pt. 1, Oxford: J. H. and P. Parker, 1860.

59. Alexander J. Ellis, On the Laws of Operation, and the Systematization of Mathematics, *Proceedings of the Royal Society of London* **10** (1859–1860), 85–94.

60. Alexander, J. Ellis, On Scalar and Clinant Algebraical Coordinate Geometry, Introducing a New and More General Theory of Analytical Geometry, Including the Received as a Particular Case, and Explaining "Imaginary Points," "Intersections," and "Lines," *Proceedings of the Royal Society of London* **10** (1859–1860), 415–426.

61. William Everett, *On the Cam: Lectures on the University of Cambridge*, London: Beeton, 1866.

62. Andrew R. Forsyth, Arthur Cayley, *Proceedings of the Royal Society of London* **58** (1895), ix–xliv, or CP8(1895), ix–xliv.

63. Andrew R. Forsyth, Old Tripos Days at Cambridge, *Mathematical Gazette* **19** (1935), 162–179.

64. James W. L. Glaisher, The Mathematical Tripos, *Proceedings of the London Mathematical Society* **18** (1886), 4–38.

65. James W. L. Glaisher, 1891, Presidential Address—Section A, *Report of the Sixtieth Meeting of the British Association for the Advancement of Science*, London: John Murray, 1890, pp. 719–727.

66. James W. L. Glaisher, Arthur Cayley, *The Cambridge Review* **16** (7 February 1895), 174–176.

67. Ivor Grattan-Guinness, A Mathematical Union: William Henry and Grace Chisholm Young, *Annals of Science* **29** (1972), 105–186.

68. Ivor Grattan-Guinness, *Convolutions in French Mathematics*, 3 vols., Basel/Boston/Berlin: Birkhäuser, 1990.

69. Jeremy Gray, Olinde Rodrigues' Paper of 1840 on the Transformation Groups, *Archive for History of Exact Sciences* **21** (1980), 375–385.

70. Thomas Hawkins, Another Look at Cayley and the Theory of Matrices, *Archives internationales d'histoire des sciences* **27** (1977), 82–112.

71. Ernest W. Hobson, Presidential Address—Section A, in *Report of the Eightieth Meeting of the British Association for the Advancement of Science, 1910*, London: John Murray, 1911, pp. 509–522.

72. Henry Jackson, Cambridge Fifty Years Ago, *The Cambridge Review* **31** (2 June 1910), 449–451.

73. Jonathan L. King, Billiards Inside a Cusp, *The Mathematical Intelligencer* **17**(1) (1995), 8–16.

74. Frances E. Kingsley, ed., *Charles Kingsley: His Letters and Memories of His Life*, 2 vols., 8th abridged ed., London: C. Kegan Paul & Co., 1880.

75. Edmond Laguerre-Verly, Sur la théorie des foyers [foci], *Nouvelles annales de mathématiques* **12** (1853), 57–66, or *Oeuvres de Laguerre*, ed. Charles Hermite, Henri Poincaré, Eugène Rouché, 2 vols., Paris: Gauthier-Villars, 1898, 2:6–15.

76. Frank Musgrove, Middle-Class Education and Employment in the Nineteenth Century, *Economic History Review* **12** (1959), 99–111.

77. Karen Hunger Parshall, Toward a History of Nineteenth-Century Invariant Theory, in *The History of Modern Mathematics*, 2 vols., ed. David E. Rowe and John McCleary. San Diego: Academic Press, 1989, 1:157–206.

78. Karen Hunger Parshall, *James Joseph Sylvester: Life and Work in Letters*, Oxford: Clarendon Press, 1998.

79. Karl Pearson, *The Life, Letters and Labours of Francis Galton*, 4 vols., Cambridge, UK: Cambridge University Press, 1914–1930.

80. Karl Pearson, Old Tripos Days at Cambridge, as Seen from Another Viewpoint, *Mathematical Gazette* **20** (1936), 27–36.

81. Charles S. Peirce to Bertrand Russell, 24 April 1909, in *The New Elements of Mathematics by C. S. Peirce*, 5 vols., ed. Carolyn Eisele, The Hague/Paris: Mouton. 1976, 3(2):983–984.

82. Henry Philpott, *Trusts, Statutes and Directions Affecting the Professorships of the University*, Cambridge, UK: Deighton Bell, 1876.

83. John Playfair, On the Arithmetic of Impossible Quantities, *Philosophical Transactions of the Royal Society of London* **68** (1778), 318–343.

84. Jean-Victor Poncelet, *Traité des propriétés projectives des figures*, Paris: Metz, 1822.

85. Michael H. Price, *Mathematics for the Multitude?: A History of the Mathematical Association*, The Mathematical Association, 1994.

86. Ref. Rep., 1 June 1859, Royal Society of London, RR.4.72.

87. Ref. Rep., 25 April 1860, Royal Society of London, RR.4.79.

88. Adrian Rice, Inexplicable? The Status of Complex Numbers in Britain, 1750–1850, preprint.

89. Joan L. Richards, *Mathematical Visions: The Pursuit of Geometry in Victorian England*, San Diego: Academic Press, 1988.

90. Sheldon Rothblatt, *The Revolution of the Dons*, London: Faber and Faber, 1967.

91. Walter W. Rouse Ball, *A History of the Study of Mathematics at Cambridge*, Cambridge, UK: Cambridge University Press, 1889.

92. David. E. Rowe, The Early Geometrical Works of Sophus Lie and Felix Klein, in *The History of Modern Mathematics*, 2 vols., ed. David E. Rowe and John McCleary, San Diego: Academic Press, 1989, 1:209–273.

93. George Salmon, *A Treatise on the Higher Plane Curves*, Dublin: Hodges and Smith, 1852.

94. George Salmon, *A Treatise on the Higher Plane Curves*, 2nd, ed., Dublin: Hodges and Smith, 1873.

95. George Salmon, Science Worthies—Arthur Cayley, *Nature* **28** (20 September 1883), 481–485.

96. Charlotte Angas Scott, Arthur Cayley, *Bulletin of the American of the Mathematical Society* **2** (1895), 133–141.

97. [Sir] J. R. Seeley, ed., *The Student's Guide to the University of Cambridge*, Cambridge, UK: Deighton Bell, 1862.

98. Abe Shenitzer, The Cinderella Career of Projective Geometry, *The Mathematical Intelligencer* **13**(2) (1991), 50–55.

99. Arthur W. Siddons, Progress, *Mathematical Gazette* **20** (1936), 7–26.

100. Henry J. S. Smith, On Some of the Methods at Present in Use in Pure Geometry, *Transactions of the Ashmolean Society* **2**(35) (1851), or *The Collected Mathematical Papers of H. J. S. Smith*, ed. James W. L. Glaisher, 2 vols., Oxford: Clarendon, 1894, 1:1–23.

101. John Stillwell, *Sources of Hyperbolic Geometry*, History of Mathematics, vol. 10, Providence: American Mathematical Society/London: London Mathematical Society, 1996.

102. Robert John Strutt, Baron Rayleigh, *John William Strutt, Third Baron Rayleigh*, London: Edward Arnold, 1924.

103. James Joseph Sylvester to Arthur Cayley, 12 May 1881, St. John's College Library, Cambridge.

104. James Joseph Sylvester to Arthur Cayley, 3 August 1883, St. John's College Library, Cambridge.

105. Joseph R. Tanner, ed., *The Historical Register of the University of Cambridge Being a Supplement to the Calendar with a Record of University Offices Honours and Distinctions to the Year 1910*, Cambridge, UK: Cambridge University Press, 1917.

106. Joseph John Thomson, On the Resolution of the Product of Two Sums of Eight Squares into the Sum of Eight Squares, *Messenger of Mathematics* **7** (1878), 73–74.

107. Joseph John Thompson, An Extension of Arbogast's Method of Derivations, *Messenger of Mathematics* **7** (1878) 142–143.

108. Joseph John Thomson, Note on the Addition Equation in Elliptic Functions, *Messenger of Mathematics* **9** (1880), 52–53.

109. Joseph John Thomson, *Recollections and Reflections*, London: G. Bell and Sons, 1936.

110. Isaac Todhunter to George Boole, 23 May 1863, in *George Boole*, Desmond MacHale, Dublin: Boole Press, 1985, pp. 233–234. [=Papers of G. I. Taylor, Trinity College, Cambridge]

111. William Peveril Turnbull, *An Introduction to Analytical Plane Geometry*, Cambridge, UK: Deighton Bell & Co., 1867.

112. Herbert W. Turnbull, *Some Memories of William Peveril Turnbull*, London: George Bell, 1919.

113. William T. Tutte, *Graph Theory As I Have Known It*, Oxford: Clarendon Press, 1998.

114. David B. Wilson, The Educational Matrix: Physics Education at Early-Victorian Cambridge, Edinburgh and Glasgow Universities, in *Wranglers and Physicists*, ed. P. M. Harman, Manchester: Manchester University Press, 1985, pp. 12–48.

115. James M. Wilson, The Early History of the [Mathematical] Association, *Mathematical Gazette* **10** (1920–21), 239–244.

116. James M. Wilson, *James Maurice Wilson: An Autobiography 1836–1931*, London: Sidgwick and Jackson, 1932.

117. Denys A. Winstanley, *Later Victorian Cambridge*, Cambridge, UK: Cambridge University Press, 1947.