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DECIDABILITY AND DEFINABILITY WITH CIRCUMSCRIPTION

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We consider McCarthy's notions of predicate circumscription and formula circumscription. We show that the decision problems "does θ have a countably infinite minimal model" and "does ϕ hold in every countably infinite minimal model of θ" are complete Σ_2^1 and complete Π_2^1 over the integers, for both forms of circumscription. The set of structures definable (up to isomorphism) as first order definable subsets of countably infinite minimal models is the set of structures which are Δ_2^1 over the integers, for both forms of circumscription. Thus, restricted to countably infinite structures, predicate and formula circumscription define the same sets and have equally difficult decision problems. With general formula circumscription we can define several infinite cardinals, so the decidability problems are dependent upon the axioms of set theory.

1. Introduction and definitions

In order to deal precisely with a form of non-monotonic inference, McCarthy ([8] and [9]) proposed two forms of a formalism called circumscription. He proposed it partly to express a sort of default logic: we have a default assumption which we apply whenever we cannot deduce the contrary; hence, if R represents the set of objects which violate the default, R should be as small as possible. The two forms are called predicate circumscription and formula circumscription.

For both forms we start with a finite language L and a symbol R not in L. (Normally R is a relation symbol.) For formula circumscription we have another symbol U not in L. The interpretation of R is to be minimized; the interpretation of U is allowed to vary freely as we try to minimize R; and the interpretations of the symbols in L stay fixed. We have a first-order formula $\theta[R]$ in $L[R]$, or $\theta[R, U]$ in $L[R, U]$, and we wish to consider only models of θ in which the interpretation of R is minimal. We shall use Fraktur $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{A}$ to denote L-structures, and the corresponding Latin M, N, A to denote their ground structures. So to denote an $L[R]$ structure we write $\langle \mathfrak{M}; R \rangle$. (We use R both for the symbol and for its interpretation in a structure; context should resolve any ambiguity.)

McCarthy has also proposed a third form, called prioritized circumscription. However, Lifschitz [5] has shown it to be no stronger than formula circumscription.

For predicate circumscription we have the following definitions: Given L, R, U, and θ as above, let MOD-P $= \{\langle \mathfrak{M}; R\rangle \models \theta[R]\}$. For $\langle \mathfrak{M}_1; R_1\rangle$ and $\langle \mathfrak{M}_2; R_2\rangle$ in Mod-P set $\langle \mathfrak{M}_1; R_1\rangle <_P \langle \mathfrak{M}_2; R_2\rangle$ if $\mathfrak{M}_1 = \mathfrak{M}_2$ and $R_1 \subsetneq R_2$. We restrict attention to $<_P$-minimal models in MOD-P. Note that $\langle \mathfrak{M}; R\rangle$ is a $<_P$-minimal model of $\theta[R]$ iff $\langle \mathfrak{M}; R\rangle \models \theta[R] \,\&\, \forall R' (R' \subsetneq R \rightarrow \neg\theta[R])$, iff $\langle \mathfrak{M}; R\rangle \models \forall R' (\theta[R] \,\&\, (R' \subsetneq R \rightarrow \neg\theta[R']))$. (Here "$R' \subsetneq R$" abbreviates the obvious first-order formula.) Thus, to say $\langle \mathfrak{M}; R\rangle$ is a $<_P$-minimal model of $\theta[R]$ is Π^1_1, i.e., it can be expressed by a formula consisting of a finite number of universal quantifiers $\forall R'$ over relations on the structure followed by a first-order formula in $L[R, R']$.

For formula circumscription we have analogous definitions: MOD-F $= \{\langle \mathfrak{M}; R, U\rangle \models \theta[R, U]\}$, and $\langle \mathfrak{M}_1; R_1, U_1\rangle <_F \langle \mathfrak{M}_2; R_2, U_2\rangle$ if $\mathfrak{M}_1 = \mathfrak{M}_2$ and $R_1 \subsetneq R_2$. (Note that *no* relationship is required between U_1 and U_2.) We restrict attention to $<_F$-minimal models in MOD-F. Being a $<_F$-minimal model of $\theta[R, U]$ is also Π^1_1:

$$\langle \mathfrak{M}; R, U\rangle \models \theta[R, U] \,\&\, \forall R', U' (R' \subsetneq R \rightarrow \neg\theta[R', U']).$$

Remark. We shall often want to have several relations $R_1 \cdots R_n$ rather than just one R, or several relations $U_1 \cdots U_k$. If we are allowed to assume every model of $\theta[R_1 \cdots R_n]$ contains at least two elements and to add constant symbols c_1, c_2 for two distinct elements, we can then combine all the R_i's into one R for our definition. For example, to combine $R_1(x)$, $R_2(x, y)$, and $R_3(x, y, z)$, we let R be 5-ary and set

$$\begin{aligned} R(u, v, x, y, z) \quad \text{iff} \quad & u = c_1 \,\&\, v = c_1 \,\&\, R_1(x) \,\&\, y = c_1 \,\&\, z = c_1 \\ \text{or} \quad & u = c_1 \,\&\, v = c_2 \,\&\, R_2(x, y) \,\&\, z = c_1 \\ \text{or} \quad & u = c_2 \,\&\, v = c_1 \,\&\, R_3(x, y, z). \end{aligned}$$

The notion of minimality resulting from this encoding is

$$\begin{aligned} & \langle \mathfrak{M}; R'_1, R'_2, R'_3\rangle <_P \langle \mathfrak{M}; R_1, R_2, R_3\rangle \quad \text{iff} \\ & \quad R'_1 \subseteq R_1 \,\&\, R'_2 \subseteq R_2 \,\&\, R_3 \subseteq R_3 \,\&\, (R'_1 \neq R_1 \vee R'_2 \neq R_2 \vee R'_3 \neq R_3). \end{aligned}$$

Convention. We shall always use R, R_i, R^*, etc., for the relations to be minimized, and U, U_i, U^*, etc., for the relations which are allowed to vary freely. Whenever we speak of a formula $\theta[R, U]$ in a first-order language L, we mean that R, U are not in L, and that θ is a formula in the language $L[R, U]$.

Proposition 1.1. *Let $\theta[R, U]$ and $\theta'[R', U']$ be first-order formulas in L.*

(A) *If $\langle \mathfrak{M}; R, U\rangle$ is a $<_P$ model of $\theta[R, U]$ and $\langle \mathfrak{M}; R', U'\rangle$ is a $<_P$-minimal model of $\theta'[R', U']$, then $\langle \mathfrak{M}; R, R', U, U'\rangle$ is a $<_P$-minimal model of $\theta[R, U] \,\&\, \theta'[R', U']$.*

(B) *If R, U, R', and U' are all distinct, the converse to* (A) *holds.*

(C) *If* $R = R'$ *or* $U = U'$ *the converse to* (A) *may fall.*
Also, all the analogous results for predicate circumscription also hold.

Proof. Note that R, R', U, U' are assumed not to be in L. Proofs of (A) and (B) are short and obvious. For (C) with $R = R'$: Let $\theta[R]$ say R has 1 or 3 elements; let $\theta'[R]$ say R has 2 or 3 elements. In a minimal model of θ, R has 1 element; in a minimal model of θ', R has 2 elements, and in a minimal model of $\theta \& \theta'$, R has 3 elements. (It is also easy to find such θ, θ' so that neither θ nor θ' has a minimal model but $\theta \& \theta'$ has minimal models.) For (C) with $U = U'$: let $\theta[R, U]$ say R has 1 or 3 elements and $R = U$; let θ' $[R', U]$ say R has 2 or 3 elements and $R' = U$; then proceed as before.

2. Examples of circumscription

Example 2.1 (default logic). Let $\theta[R]$ be $\phi \& (R(a) \& R(b))$, where ϕ does not involve R and a, b are constant symbols in L. Then for $\mathfrak{M} \models \phi$, $\langle \mathfrak{M}; R \rangle$ is a $<_P$-minimal model of $\theta[R]$ iff $R = \{a, b\}$.

Example 2.2 (default logic) McCarthy [8]. $\theta[R]$ is $\phi \& (R(a) \vee R(b))$ with ϕ, a, and b as above. Then for $\mathfrak{M} \models \phi$, $\langle \mathfrak{M}; R \rangle$ is a $<_P$-minimal model of $\theta[R]$ iff $R = \{a\}$ or $R = \{b\}$.

Example 2.3 (transitive closure) Davis [2]. Let $L = \{S, 0, +, *, <\}$. Let ϕ be

$$\begin{aligned}
&\forall x \neg(S(x) = 0) \& \forall xy\, (S(x) = S(y) \rightarrow x = y) \& \forall x\, (x = 0 \vee \exists y\, (x = S(y)))\\
&\& \forall x \neg(x < 0) \& \forall xy\, (x < S(y) \rightarrow x < y \vee x = y)\\
&\& \forall xy\, (x<y \vee x=y \vee y<x) \& \forall x \neg(x<x) \& \forall xyz\, (x<y \& y<z \rightarrow x<z)\\
&\& \forall x\, (x + 0 = x) \& \forall xy\, (x + S(y) = S(x + y))\\
&\& \forall x\, (x * 0 = 0) \& \forall xy\, (x * S(y) = x * y + x).
\end{aligned}$$

(This formula ϕ is a small modification of Davis' formula, since we need that in a later example. The difference is insignificant for this example.) Any model $\mathfrak{N}$ of ϕ will be linearly ordered by $<$, starting out with 0, $S(0)$, $S(S(0))$, ... Let us call those elements the *standard part* of $\mathfrak{N}$. The standard part of any such $\mathfrak{N}$ with the relations and functions S, $+$, $*$, and $<$ of $\mathfrak{N}$ restricted to the standard part can easily be shown to be isomorphic to the natural numbers (ω) under the intended operations. (Convention: we henceforth use the term 'integers' to be synonymous with 'natural numbers'—denoting only the non-negative natural numbers.) The structure $\langle \omega; S, 0, +, *, < \rangle$ is itself a model of ϕ; every other model has a *nonstandard part*—the part above (under the $<$ of the model) all elements in the standard part. There can be no least element in the nonstandard part—since if x is nonstandard, $x = S(y)$ for some y, and y must be nonstandard also.

Let $\theta_1[R_1]$ be $\phi \mathbin{\&} R_1(0) \mathbin{\&} \forall x\, (R_1(x) \rightarrow R_1(S(x)))$, and let $\langle \mathfrak{N}, R_1 \rangle \models \theta_1[R_1]$. It is easy to see that $\langle \mathfrak{N}, R_1 \rangle$ is minimal iff R_1 is the standard part of $\mathfrak{N}$.

More generally, we get transitive closures: Let L contain a unary function symbol S and a constant symbol c. Let $\theta[R]$ be $\phi \mathbin{\&} R(c) \mathbin{\&} \forall x\, (R(x) \rightarrow R(S(x)))$, where ϕ does not involve R. Then if $M \models \phi$, $\langle \mathfrak{M}; R \rangle$ is a $<_P$-minimal model of $\theta[R]$ iff $R = \{c, S(c), S(S(c)), \ldots\}$. R is a transitive closure, a special case of an inductively-definable set. Yet more generally, we can define inductively definable sets (see Moschovakis [11]). Transitive closures are an especially interesting sort of non-first-order definable sets in computer science since they correspond to very natural notions, such as ancestor and relative. They have been studied in several contexts, including circumscription. (See, e.g., Perlis–Minker [13] and Schlipf [14]).

Example 2.4. Let $L = \{\in, \mathrm{SET}\}$, the language of a theory of sets and classes. (A class is a collection of sets, which may be too large to be a set. Some classes are sets; some are not. Everything is a class.) Let GB denote the conjunction of the axioms of Gödel–Bernays set theory. (For those unfamiliar with Gödel–Bernays: the set part of a model of GB is always a model of ZF. Given a model of ZF we can always form a model of GB by taking as classes all first-order definable subsets of the model of ZF, and by taking the elements of that model to be sets. In general, there will also be lots of other collections of classes we could also take to form models of GB.) We use $\langle\ ,\ \rangle$ here to denote the ordered pair construction in set theory.

Let $\theta[R]$ be

$$\begin{aligned} &\mathrm{GB} \mathbin{\&} \exists z\, \forall x\, \forall y\, (R(x, y) \leftrightarrow \mathrm{SET}(x) \mathbin{\&} \mathrm{SET}(y) \mathbin{\&} \langle x, y \rangle \in z) \\ &\quad \mathbin{\&} \forall x\, \forall y\, (R(x, y) \rightarrow y \in x) \mathbin{\&} \forall x\, (x \neq 0 \mathbin{\&} \mathrm{SET}(x) \rightarrow \exists y\, R(x, y)). \end{aligned}$$

The first line of θ says R "is one of the classes of the model" — so we shall now use R both for the relation to be minimized and for the class. The second line says that R picks out at least one element from each nonempty set x. Now any model of GB contains some such class R, e.g., $\{\langle x, y \rangle : y \in x\}$. But not every model contains a minimal such R.

Claim. *If $\langle \mathfrak{M}; R \rangle$ satisfies $\theta[R]$, then it is $<_P$-minimal iff R picks out exactly one y from each nonempty x.*

Proof. The 'if' part is obvious. Suppose R picks out two elements, y and z, from some x. Then $R - \{\langle x, y \rangle\}$ also satisfies θ, so R is not minimal.

Such an R is called a global choice function. Its existence implies, among other things, that the underlying model of ZF satisfies the axiom of choice. So not every model of GB contains a global choice function.

Remark. Example 2.4 is interesting in that we are concerned not only with the minimal R but also with what the existence of a minimal R tells us about the structure. Other studies have dealt with circumstances which ensure that minimal R's exist (e.g. Etherington–Mercer–Reiter [3], Lifschitz [4]); we here exploit the reverse, making it difficult for minimal R's to exist and thus using their existence to conclude strong properties about the model $\mathfrak{M}$. In our next example we strengthen Davis' example in 2.3 in a similar way.

Example 2.5. Let ϕ and $\theta_1[R_1]$ be as in Example 2.3.
Let $\theta_2[R_1, R_2]$ be

$$\theta_1[R_1] \mathbin{\&} \forall x\, (R_1(x) \rightarrow R_2(x))$$
$$\mathbin{\&} (\forall x\, R_2(x) \vee \exists z\, \forall x\, (R_2(x) \leftrightarrow x < z)).$$

Note that every model $\mathfrak{N}$ of ϕ can be expanded to a model $\langle \mathfrak{N}; R_1, R_2 \rangle$ of $\theta_2[R_1, R_2]$: let $R_1 = R_2 =$ the entire model.

Claim. *Suppose $\langle \mathfrak{N}; R_1, R_2 \rangle$ is a $<_P$-minimal model of $\theta_2[R_1, R_2]$. Then $\mathfrak{N}$ is the standard model, i.e., has empty nonstandard part.*

Proof. Suppose $\langle \mathfrak{N}; R_1, R_2 \rangle$ is such a minimal model. Let St denote its standard part and Nst its nonstandard part, and suppose Nst is not empty. As with θ_1, clearly St is the minimal possible R_1, so $\mathrm{St} = R_1$. Since Nst is nonempty, the minimal R_2 must be $\{x : x < z\}$ for the minimal z in Nst — which doesn't exist.

Thus $\theta_2[R_1, R_2]$ has a unique $<_P$-minimal model — the standard integers.

Example 2.6. Let the formula ϕ be as in Example 2.5, and let $\theta[R, U]$ be

$$\phi \mathbin{\&} (F \text{ maps } \{x : x \leqslant a\} \text{ 1–1 and onto } \{x : x \leqslant b\} \mathbin{\&} a < b)$$
$$\mathbin{\&} (\forall x\, R(x) \vee \exists z\, \forall x\, (R(x) \leftrightarrow x \leqslant z))$$
$$\mathbin{\&} (U \text{ maps } R \text{ 1–1 onto the entire model}).$$

We show ϕ has $<_F$-minimal models, but every $<_F$-minimal model is uncountable. First, we show there is a $<_F$-minimal model: We know there are 'ω_1-like' models of Peano Arithmetic, that is, models such that each element has only countably many predecessors, but where the entire model has cardinality $\aleph_1$. Furthermore, ϕ is provable in Peano Arithmetic. So let $\mathfrak{N}$ be an ω_1-like model of Peano Arithmetic, and let a, b be interpreted by any two elements with infinitely many predecessors. Since U maps R 1–1 onto the entire model, R must be uncountable; it follows that R must be the entire model. So with R equal to the entire model and U the identity function, we have a $<_F$-minimal model. Second, we show that there is no countable $<_F$-minimal model. For by the axiom about F, the standard integers cannot be a model. Suppose we had a countably infinite model. The axioms concerning R and U tell us that R would have to be $\{x : x \leqslant z\}$ for some z where the set is infinite — and for R to be minimal that z would have to be the first z with infinitely many predecessors. As in Example 2.5, such a z cannot exist.

3. Definability and decidability

We are interested in three questions:

(1) How complex is the question "Does $\theta[R]$ (resp. $\theta[R, U]$) have a $<_P$-minimal (resp. $<_F$-minimal) model?"

(2) How complex is the question "Does $\phi[R]$ (resp. $\phi[R, U]$) hold in every $<_P$-minimal (resp. $<_F$-minimal) model of $\theta[R]$ (resp. $\theta[R, U]$)?"

(3) What structures are definable using circumscription? (We shall define precisely below what we mean by 'definable'.)

We could ask these three questions in three contexts:

(A) Limited to finite $\mathfrak{M}$.

(B) Limited to infinite $\mathfrak{M}$.

(C) Limited to countably infinite $\mathfrak{M}$.

A good deal of work in circumscription has dealt with context (B). We deal primarily with context (C), avoiding problems of changing cardinalities and allowing us to build all our structures over the integers. We shall answer the three questions in that case. We shall also show that when we move to context (B), with formula circumscription far more is definable, and decidability problems are dependent upon our axioms for set theory.

Remark. Since predicate circumscription is a special case of formula circumscription, the two decision problems are at least as complicated for formula circumscription as for predicate circumscription, and anything definable using predicate circumscription is also definable using formula circumscription (under any plausible definition of definable, including the one we give below).

A formula is said to be Σ^1_2 if its is of the form $\exists S_1 \cdots \exists S_n\ \Phi$ where Φ is Π^1_1. A formula is Π^1_2 if it is of the form $\forall S_1 \cdots \forall S_n\ \Phi$ where Φ is Σ^1_1.

Theorem 3.1. (A) *The predicate "$\theta[R]$ (resp. $\theta[R, U]$) has a countably infinite $<_P$-minimal (resp. $<_F$-minimal) model" is Σ^1_2 over the integers.*

(B) *The predicate "$\phi[R]$ (resp. $\phi[R, U]$) holds in every countably infinite $<_P$-minimal (resp. $<_F$-minimal) model of $\theta[R]$ (resp. $\theta[R, U]$)" is Π^1_2 over the integers.*

Proof. (A) Since the model is to be countably infinite, we may assume it to have universe the integers. For simplicity, assume that θ has only relation symbols $S_1 \cdots S_k$ in addition to R (and possibly U). Then the formula saying θ has a minimal model with universe the integers is

$$\exists S_1 \cdots \exists S_k\, \exists R\, \exists U\, (\theta[S_1, \ldots, S_k, R, U]\ \&\ \forall R' \subsetneqq R\, \forall U'$$
$$(\neg\theta[S_1, \ldots, S_k, R', U']))$$

(or the analogous formula without the U), which is clearly Σ^1_2.

The proof of part (B) is analogous.

In our next definitions we define the notion of defining a structure. For notational convenience only (to avoid awful subscripting) we shall state the definitions for structures with two binary relations; generalizations to structures for any finite language are obvious. Similarly, we speak in the following theorems only about structures with one or two binary relations. We shall give several analogous definitions, indicating only the changes made from one definition to the next.

Definition. We say that a formula $\theta[R]$ (resp. $\theta[R, U]$) *defines* a structure $\langle N; S_1, S_2\rangle$ *using predicate* (resp. *formula*) *circumscription* if for some first-order formulas ϕ_0, ϕ_1, ϕ_2 (possibly involving R and U):

(1) $\theta[R]$ has a $<_P$-minimal model (resp., $\theta[R, U]$ has a $<_F$-minimal model);

(2) in every minimal model $\langle \mathfrak{M}; R\rangle$ (or $\langle \mathfrak{M}; R, U\rangle$) of θ,

if $N' = \{x: \langle \mathfrak{M}; R, U\rangle \models \phi_0(x)\}$ and $S_i' = \{\langle x, y\rangle; \langle \mathfrak{M}; R, U\rangle \models \phi_i(x, y)\}$, then

$$\langle N', S_1', S_2'\rangle \cong \langle N, S_1, S_2\rangle.$$

We say that the formula defines the structure using (predicate or formula) circumscription *over countably infinite models* if we require that the minimal model be countably infinite in (1) and (2) above—or, equivalently, if we redefine MOD-P and MOD-F to include only the countably infinite models of the sentences.

We say that the formula defines the structure using (predicate or formula) circumscription *by restriction* (possibly over countably infinite models) if the defining formulas ϕ_i do not involve the symbols R and U. (The idea is that we do not explicitly use R or U to define the structure; rather, we use circumscription only to restrict the class of models we consider, to those models $\mathfrak{M}$ for which a minimal R exists. Recall Examples 2.4, 2.5, and 2.6, where we used circumscription to restrict the class of models.)

Remark. Example 2.3 shows we can define the integers using predicate circumscription, and Example 2.5 shows us we can define the integers using predicate circumscription by restriction.

Remark. For countably infinite structures $\mathfrak{N}$, $\mathfrak{N}$ is definable using circumscription over countably infinite models iff for some formula θ', $\mathfrak{N}$ is isomorphic to the reduct of every minimal model of θ' to the language $\{S1, S2\}$—if θ defines $\mathfrak{N}$ in the sense of our definition, we need merely additionally assert in θ' that F is an isomorphism from the entire structure to $\{x: \phi_0(x)\}$ carrying $\{\langle x, y\rangle: \phi_i(x, y)\}$ to S_i.

The importance of definition by restriction is shown in the following proposition, which allows us to compose definitions using circumscription. (Since the result holds for both predicate and formula circumscription, we shall avoid saying so explicitely.)

Proposition 3.2. *Suppose $X = \{x : \phi_X(x)\}$ in every minimal model of $\theta_X[R_X]$. And suppose, for a formula $\theta_Y[P_X, R_Y]$ in the language extended with a relation P_X intended to denote X, in every minimal model of $\theta_Y[P_X, R_Y]$ in which P_X happens to be X, $Y = \{y : \phi_Y(y)\}$. Let $\theta^*[R_X, R_Y]$ be $\theta_X[R_X]$ & $\theta_Y[\phi_X, R_Y]$, where that last formula denotes the result of substituting the definition ϕ_X of X in for P_X (using an alphabetic variant if needed to avoid variable scope problems). Assume θ_X does not involve R_Y, and θ_Y does not involve R_X. If ϕ_X does not involve R_X—that is, if θ_X defines X by restriction—then in every minimal model of θ^*, $Y = \{y : \phi_Y(y)\}$. That need not hold if ϕ_X involves R_X.*

Proof. Suppose ϕ_X does not involve R_X. Then the first conjunct of θ^* does not involve R_Y, and the second conjunct of θ^* does not involve R_X, so, by Proposition 1.1(B), a minimal model of θ^* is minimal for each conjunct. So $\{x : \phi_X(x)\}$ is X in the model, and, as a result, $\{y : \phi_Y(y)\}$ is Y in the model.

To construct a counterexample when ϕ_X does involve R_X one need only modify the example of Proposition 1.1(C).

Proposition 3.3. *If a structure is definable using predicate (resp. formula) circumscription over countably infinite models, it is definable using predicate (resp. formula) circumscription. If a structure is definable using predicate (resp. formula) circumscription over countably infinite models by restriction, it is definable using predicate (resp. formula) circumscription by restriction.*

Proof. Suppose the structure is definable by $\theta[R]$ using circumscription over countably infinite models. Let $\theta'[R']$ be the formula of Example 2.5, where all symbols are chosen to be distinct from those of $\theta[R]$. Let $\theta^*[R, R']$ be $\theta[R]$ & $\theta'[R']$. Then, by Theorem 1.1(B), the minimal models of $\theta^*[R, R']$ are the models which are simultaneously minimal models of $\theta[R]$ and $\theta'[R']$. By Example 2.5, those are the countably infinite minimal models of $\theta[R]$, which, as noted above, are exactly the minimal models in the class of countably infinite models.

Remark. We immediately see that far more is definable using circumscription than is definable in first-order logic, or in first-order logic restricted to countably infinite models.

Definition. A relation S on the integers is Δ^1_2-definable over the integers if it is definable by both Σ^1_2 and Π^1_2 formulas over the integers. A structure $\langle N; S, T\rangle$ is Δ^1_2-definable over the integers if N, S, and T are all Δ^1_2-definable over the integers.

Theorem 3.4. *Suppose a structure $\langle N; S, T\rangle$ is definable using predicate or formula circumscription over countably infinite models. Then some isomorphic copy of it is Δ^1_2-definable over the integers.*

Proof. Our problem here is dealing with isomorphic models. Every countably infinite structure is isomorphic to a structure with universe the integers. Now if one of those models, together with the structure of the integers, were definable using predicate circumscription over countably infinite models, we could finish as in the proof of Theorem 3.1, but we don't know that any single such model is definable. But the result is a corollary of wellknown generalized recursion-theoretic results: Suppose $\theta[R]$ defines $\langle N; S, T\rangle$, and formulas ϕ_0, ϕ_1, and ϕ_2 are as in the definition of defining using circumscription. For $\mathfrak{M}$ a structure with universe the integers, to say that $\langle\mathfrak{M}; R\rangle$ is minimal is Π^1_1 over the integers. So for $\langle N; S, T\rangle$ a structure over the integers, to say that there is some structure $\langle\mathfrak{M}; R\rangle$ over the integers which is a minimal model of $\theta[R]$ and in which $\langle N; S, T\rangle$ is definable by the ϕ_i's is Σ^1_2. Now the Basis Theorem for Σ^1_2 (see Moschovakis [12, p. 236]) implies that at least one such structure $\langle N; S, T\rangle$ is Δ^1_2 definable over the integers, which is what we want. (The Basis Theorem for Δ^1_2 is a corollary of the Novikov–Kondo–Addison Theorem of Kondo [7]; it appears in Moschovakis [12, p. 235].)

Definition. A Σ^1_2 relation R on the integers is complete Σ^1_2 if for each Σ^1_2 relation S on the integers there is a recursive function F such that, for all integers n, $S(n)$ iff $R(F(n))$. (Note that we have defined only completeness only for unary relations, since that is all we need here; the extension to other relations gives no important extensions since there is a recursive function mapping the n-tuples of integers 1–1 onto the integers. When we speak of the pair of integers n and m below, we mean the pair as encoded by an integer using some such function.)

The term 'complete Π^1_2' is defined analogously.

Theorem 3.5. (A) *The relation "there exists a countably infinite $<_P$-minimal model of $\theta[R]$", considered as a relation on the Gödel number of θ, is complete Σ^1_2.*

(B) *The relation "ϕ holds in every countably infinite $<_P$-minimal model of $\theta[R]$", considered as a relation on the pair of the Gödel number of ϕ and the Gödel number of θ, is complete Π^1_2.*

Proof. Deferred to Section 4.

Although the formulas expressing notions of circumscription are of these familiar categories, they are still quite restrictive. Saying that R is the minimal relation on the integers satisfying some formula is far more restrictive than an arbitrary Π^1_1 relation on the integers. For example, the $\forall R'$ quantifies (in effect) only over subsets of the relation R. At first glance it appears that circumscription is not much more complicated than a logic that allows construction of a finite number of transitive closures (or more generally, a finite number of inductively definable sets). (Circumscription also allows a sort of non-determinacy, as shown in Example 2.2. However, this only partially enters into our three questions.) Thus, Theorems 3.5 and 3.7 are rather surprising.

Corollary 3.6. *The complexities of our two decision problems* (1) *and* (2) *from the top of this section, restricted to countably infinite models, are the same for predicate circumscription as they are for formula circumscription.*

This corollary, and Corollary 3.8, are particularly surprising since formula circumscription seems such a significant extension of predicate circumscription.

Theorem 3.7. *Suppose a structure* $\langle N, S, T\rangle$ *is* Δ^1_2 *definable over the integers. Then* $\langle N, S, T\rangle$ *is definable using predicate circumscription by restriction over countably infinite structures.*

Proof. Deferred to Section 4.

Corollary 3.8. *Exactly the same structures are definable using predicate circumscription over countably infinite models as are definable using formula circumscription over countably infinite models.*

Corollary 3.9. *Exactly the same structures are definable using circumscription over countably infinite models as are definable using circumscription by restriction over countably infinite models.*

It is instructive to use this analysis to compare the expressive power of circumscription to the expressive poser of dynamic logic. Dynamic logic is a logic of programs, expressing the idea of state changing over time. It was shown by Meyer and Parikh [10] to be equivalent to the fragment of infinitary logic in the admissible set $L(\omega_1^{ck})$, where ω_1^{ck} is the least ordinal not the order type of a recursive well-ordering of a subset of the integers, L is Gödel's constructible universe, and for α any ordinal, $L(\alpha)$ denotes the sets contructable in the first α levels of the construction. In infinitary logic we extend finitary logic by allowing, among our formula building techniques, taking conjunctions and disjunctions of infinite sets of formulas. (In the cases we are interested in, these will be countably infinite sets of formulas.) For further information on infinitary logic see, e.g., Barwise [1].

Let σ_0 denote the least ordinal not the order type of a Δ^1_2 wellordering of the integers. More is said about σ_0 in Section 4; for further information there are many standard references, including Barwise [1], Moschovakis [12], and Hinman [6]. We merely wish to note here that, although it is countable, it is *far* larger than ω_1^{ck}, and hence that the set of infinitary sentences in the admissible set $L(\sigma_0)$ is far larger, and hence more expressive, than the set of infinitary sentences in the admissible set $L(\omega_1^{ck})$. These ordinals are both admissible ordinals (see, e.g., Barwise [1]). (The admissible ordinals were originally identified as nice ordinals on which to do generalized recursion theory.) Wellknown results about the size of σ_0 include: Let τ_α denote the αth admissible ordinal. So $\tau_0 = \omega$. And $\tau_1 = \omega_1^{ck}$. An ordinal α is said to be recursively inaccessible if $\alpha = \tau_\alpha$. σ_0 is recursively inaccessible. Moreover, σ_0 is the σ_0th recursively inaccessible ordinal.

Theorem 3.10. (A) *For every (infinitary) sentence Φ in $L(\sigma_0)$ in any finite language K there are a finite language K′ extending K and a first-order sentence $\theta'[R]$ in K′ such that the models of Φ are just the reducts of the $<_P$-minimal models of $\theta'[R]$ to the language K.*

(B) *For every (infinitary) sentence Φ in $L(\sigma_0)$ in any finite language K and for every first-order sentence $\theta[R]$ (resp. $\theta[R, U]$) in K, there are a finite language K′ extending K and a first-order sentence $\theta'[R]$ (resp. $\theta'[R, U]$) in K′ such that the $<_P$-minimal (resp. $<_F$-minimal) models of Φ & θ are just the reducts of the $<_P$-minimal (resp. $<_F$-minimal) models of θ & θ' to K.*

Proof. Deferred to Section 4.

Accordingly, circumscription is far more powerful than dynamic logic—and hence far more difficult, in general, to prove theorems about.

In the remainder of this section we drop our limitation to countably infinite models.

Definition. A formula $\theta[R]$ (resp. $\theta[R, U]$) *defines* a cardinal κ *using predicate* (resp. *formula*) *circumscription* if it defines a structure $\langle K \rangle$ for the empty language using predicate (resp. formula) circumscription, where the cardinality of K is κ.

We define $\theta[R]$ (resp. $\theta[R, U]$) *defining* a cardinal κ *by restriction using predicate* (resp. *formula*) *circumscription* analogously.

We say $\theta[R]$ (resp. $\theta[R, U]$) *defines* an ordinal α *using circumscription*—in any of the possible variations—if $\theta[R]$ (resp. $\theta[R, U]$) defines the structure $\langle \alpha, < \rangle$.

Remark. Example 2.5 shows we can define $\aleph_0$ using predicate circumscription by restriction.

Our final main theorem is proved by using a variant of the trick of Example 2.6 over and over again. We defer the proof to Section 4 since it is long. It is interesting to note that our proof fails if we do not include in the hypothesis below that the ordinal is defined using circumscription by restriction.

Theorem 3.11. *Suppose we can define an ordinal α using formula circumscription by restriction. Then we can define the cardinal $\aleph_\alpha$ using formula circumscription by restriction.*

Corollary 3.12. *For each ordinal $\alpha < \sigma_0$, we can define the cardinal $\aleph_\alpha$ using formula circumscription by restriction.*

Corollary 3.13. *There is a formula $\theta[R, U]$ which has a $<_F$-minimal model iff the continuum hypothesis fails (in the real world). In fact, for each ordinal $\alpha < \sigma_0$, there is a formula $\theta_\alpha[R, U]$ which has a $<_F$-minimal model iff there are at least $\aleph_\alpha$ subsets of the integers.*

Proof Sketch. We can define the model in three parts: the first part is used to define the integers (e.g., as in Example 2.5), the second part defines the cardinal $\aleph_\alpha$ by restriction, and the third part is used to code up $\aleph_\alpha$ distinct subsets of the integers from the first part. We know we correctly define the integers and the cardinal $\aleph_\alpha$ by Proposition 3.2.

Questions. (1) If a countable structure is definable using circumscription, is it definably using circumscription over countably infinite models?

(2) Are any uncountable structures definable using predicate circumscription?

(3) Are any cardinals and ordinals, other than those mentioned above, definable using circumscription?

4. Proofs of main results

In this section we use a good deal of the theory of admissable sets and infinitary languages. All needed background material can be found in Chapters I–V of Barwise [1]. We shall include here a little intuitive description of the theory for the reader who is unfamiliar with it but who is familiar with basic ZF set theory and basic model theory.

KP (Kripke–Platek) set theory is a much weaker theory than ZF. It has no axiom of infinity and no powerset axiom. It has variants of the separation and replacement axiom schemes; most notably, they are restricted to Δ_0 *formulas*, i.e., formulas where all quantifiers ae *bounded*: $\forall x \in a$ or $\exists x \in a$. (We shall usually be interested in models of KP + the axiom of infinity.)

For any infinite cardinal κ, let $H(\kappa)$ be $\{x: \text{card}(\text{trans-clos}(x)) < \kappa\}$, where trans-clos$(x)$, the transitive closure of a set x, is the smallest y such that $x \subseteq y$ and $\forall z\,(z \in y \rightarrow z \subseteq y)$. Then $\langle H(k), \in \rangle \models$ KP. (Notational change: from now on, when we write a structure $\langle A; \ldots, \in, \ldots \rangle$ we mean the real $\in$ relation. If we have some other relation E interpreting the symbol $\in$, we shall write $\langle A; \ldots, E, \ldots \rangle$.) Let $L(\beta)$ denote the sets constructible in Gödel's constructible universe by stage β. For X a transitive set, let $L(X, \beta)$ denote the sets constructible by stage β starting from X. Then for any infinite cardinal κ, $\langle L(\kappa), \in \rangle \models$ KP. (Here we must formulate the construction of L using some variant of Gödel's F functions, not iterated Δ_0 definability.) Also $\langle L(\omega_1^{ck}), \in \rangle \models$

KP and $\langle L(\sigma_0), \in\rangle \vDash$ KP. We use L to denote the collection of sets constructible at any stage, and $L(X)$, for transitive X, to denote the collection of sets constructible from X at any stage.

An admissible set is a transitive model $\langle A, \in\rangle$ of KP. An admissible ordinal is an ordinal α such that $\langle L(\alpha), \in\rangle \vDash$ KP. Every wellfounded $\langle A, E\rangle \vDash$ KP is isomorphic to an admissible set $\langle A', \in\rangle$ (by Mostowski collapsing). Of course, there are also non-wellfounded models of KP.

Inside KP we can formalize the construction of L, and whenever transitive $\langle A, \in\rangle \vDash$ KP and β is an ordinal of A, the $L(\beta)$ constructed inside A is the real $L(\beta)$ (absoluteness), and if α is the least ordinal not in A, the L of $\langle A, \in\rangle$ is $L(\alpha)$ and $\langle L(\alpha), \in\rangle \vDash$ KP. Similarly, if transitive $X \in A$, then the analogous results hold for $L(X, \beta)$ and $L(X, \alpha)$. (See Barwise [1, §2.5], for details.) We shall be most interested in the theory KP + $V = L(X)$.

We need a finitely axiomatizable theory T in a language extending the language $\{\in\}$ of KP such that the reducts of models of T are exactly the models of KP + $V = L(X)$ for some transitive X. We shall call the conjunction of that theory by the awful acronym GKPLX. Its construction is routine, and we shall merely sketch it. We add to our language new predicates Form and Δ_0-form to represent the sets of formulas and Δ_0-formulas. We add closure axioms appropriate to these classes, but we do not assert any axioms that would force them to contain nonstandard formulas (or nonstandard integers as Gödel numbers of formulas). We add a satisfaction predicate Sat and axioms saying it obeys the inductive definition of a satisfaction predicate (for formulas in the sets above). The infinite axiom schemes of KP can be replaced by single formulas which quantify over Form and Δ_0-form. Then we add Gödel F functions plus axioms saying they obey their definitions and that the model is $L(X)$.

We need the following facts about models of KP. We phrase the first as a result about models of GKPLX merely because that is where we use it.

Property 4.1 (Variant of Truncation Lemma). *Let* $\mathfrak{A} = \langle A, E, X, \ldots\rangle$ *be a model of* GKPLX. *Let* $O =$ *the set of standard ordinals of the model, i.e., the set of ordinals whose predecessors really are wellordered by E. Assume that the formulas in* Form *and* Δ_0-form *are all standard.*

Suppose X is in the wellfounded part of $\mathfrak{A}$.

Let $A^* = \bigcup_{\alpha\in O}\{a \in A: \langle A, \in\rangle \vDash a \in L(X, \alpha)\}$.

Then $\langle A^*, E, X, \ldots\rangle \vDash$ GKPLX, *and* $\langle A^*, E, X, \ldots\rangle$ *is wellfounded* (*and thus isomorphic to some* $\langle A', \in, X', \ldots\rangle \vDash$ GKPLX).

(If our formulas are not all standard, we can take the subsets of Form and Δ_0-form whose formulas are standard, and the resultant $\mathfrak{A}$ is still a model of GKPLX. But in our examples, they will be standard.)

Property 4.2 (Special Case of Barwise Completeness). *Suppose* $\{A, \in\rangle \vDash$ KP;

suppose a structure $\mathfrak{M} \in A$. *Then for any* Π_1^1 *formula* $\Phi = \forall R_1 \cdots \forall R_n \; \phi[R_1 \cdots R_n]$ *in the language of* $\mathfrak{M}$, $\mathfrak{M} \models \Phi$ *iff there is an infinitary proof in A of* $\phi[R_1 \cdots R_n]$ *from the infinitary diagram of* $\mathfrak{M}$. (See Barwise [1, Chapters III–IV], for details.)

Note that this result fails for non-wellfounded $\mathfrak{A} \models \mathrm{KP}$ since the proof may be in the non-wellfounded part of $\mathfrak{A}$ and may contain an infinite regression of falsehoods.

The ordinal σ_0 (also called δ_2^1) is the least ordinal which is not the order type of a Δ_2^1 wellordering of a subset of the integers. It is also the least ordinal α such that whenever a formula $\exists x_1 \cdots \exists x_n \; \phi$ holds in the universe of sets, for ϕ Δ_0, the formula also holds in $L(\alpha)$. And $\langle L(\alpha), \in \rangle \models \mathrm{KP}$.

In the following proofs, when we write a Greek letter α or β, we imply it varies over the ordinals of the model. Recall that when we use the symbol $\in$ in a formula, we mean the formal symbol $\in$, but when we speak of a structure $\langle A, \in \rangle$, we mean that the symbol $\in$ is interpreted by the actual $\in$ relation.

Lemma 4.3. *Suppose a structure* $\langle B; \in \rangle$ *is definable using formula (resp. predicate) circumscription by restriction, where B is a transitive set. Then there is a formula* $\Theta_B[R_0, R_1, R_2, R_3, R']$ (*resp.* $\Theta_B[R_0, R_1, R_2, R_3, R', U']$) *such that*:

(1) *For any* $C \subseteq B$, *if* α *is the least ordinal such that* $L(B \cup \{C\}, \alpha)$ *is admissible, then* $\langle L(B \cup \{C\}, \alpha); \in \rangle$ *can be expanded to a* $<_{\mathrm{P}}$*-minimal (resp.* $<_{\mathrm{F}}$*-minimal) model of* Θ_B.

(2) *If a model* $\langle A; E, Y, \ldots \rangle$ *of* Θ_B *is* $<_{\mathrm{P}}$*-minimal (resp.* $<_{\mathrm{F}}$*-minimal), then for some subset C of B,* $\langle A; E, Y \rangle \cong \langle L(B \cup \{C\}, \alpha); \in, B \cup \{C\} \rangle$ *where* α *is the least ordinal such that* $\langle L(B \cup \{C\}, \alpha); \in \rangle$ *is admissible.*

(3) *Let* ϕ *be a sentence in the language* $\{\in, X\}$ *— the language of set theory plus the symbol X for our* $B \cup \{C\}$. *Then* $\Theta_B \,\&\, \phi$ *has a* $<_{\mathrm{P}}$*-minimal (resp.* $<_{\mathrm{F}}$*-minimal) model iff* ϕ *holds in some* $\langle L(X, \alpha); \in, X \rangle$ *where* $X = B \cup \{C\}$ *for some* $C \subseteq B$ *and* α *is the least ordinal such that the structure is an admissible set.*

Proof. We prove this only for predicate circumscription; the other proof is virtually identical. Suppose that $\theta'[R']$ defines the structure $\langle B; \in \rangle$ using circumscription by restriction. So there are formulas ϕ_1, ϕ_2 such that in any minimal model $\langle \mathfrak{M}; R' \rangle$ of θ', $\langle \{x : \phi_0(x)\}, \{\langle x, y \rangle : \phi_1(x, y)\} \rangle \cong \langle B, \in \rangle$. (Note that for any $C \subseteq B$, $B \cup \{C\}$ is also transitive.)

Let $\theta^*[R_0, R_1]$ be the conjunction of the universal quantifications of

$$R_0(\emptyset), \qquad R_0(x) \rightarrow R_0(\mathrm{Succ}(x)), \qquad R_0(x) \rightarrow R_1(x),$$
$$\forall x\, (R_1(x) \leftrightarrow x \text{ is a finite ordinal}) \vee \exists m \in \omega\, \forall x\, (R_1(x) \leftrightarrow x \leq m)$$

and let $\theta^{**}[R_2, R_3]$ be the conjunction of the universal quantifications of

$$R_2(a) \rightarrow \forall x \in a\, R_2(x), \qquad \forall \beta\, (R_2(\beta) \leftrightarrow R_2(L(X, \beta)),$$
$$\forall \beta\, (R_2(\beta) \rightarrow R_2(\mathrm{Succ}(b))),$$
$$\langle R_2, \in, X, \ldots \rangle \models \mathrm{GKPLX}, \qquad R_2(x) \rightarrow R_3(x),$$
$$\forall x\, R_3(x) \vee \exists \beta\, \forall x\, (R_3(x) \leftrightarrow x \in L(X, \beta)).$$

Let Θ_B denote the formula that defines a structure $\langle N; \ldots \rangle$ which consists of 2 disjoint pieces, a set-theory part and an $\mathfrak{M}$-part, which consists of

$$\mathrm{GKPLX}\ \&\ \forall \beta \neg(\langle L(X, \beta), \in, X, \ldots \rangle \models \mathrm{GKPLX})\ \&\ \theta^*[R_0, R_1]\ \&\ \theta^{**}[R_2, R_3]$$
all restricted to the set-theory part of the model
& $\theta[R']$ restricted to the $\mathfrak{M}$ part of the model
& $X = B \cup \{C\}$ for some transitive set B and some $C \subseteq B$
& F maps $\langle \{x : \phi_0(c)\}, \{\langle x, y \rangle : \phi_1(x, y)\} \rangle$ (defined in the $\mathfrak{M}$-part) isomorphically onto that $\langle B, \in \rangle$.

(Note that we have an innocuous triple use of the symbol B: as the transitive set B, as the ground structure of any isomorphic copy of $\langle B; \in \rangle$, and as the formal symbol to denote the set B in our model. Which we intend should always be clear from context.)

Proof of part (1). We already remarked that any such $\langle L(B\{C\}, \alpha); \in, B \cup \{C\} \rangle$ can be expanded to a model of GKPLX. That it is then also a model of the second conjunct of θ_B is trivial from a being the least such ordinal. Now let R_0 and R_1 both be interpreted by ω, and let R_2 and R_3 be interpreted by the entire model. Let $\langle \mathfrak{M}; R' \rangle$ be any minimal model of $\theta'[R']$; inside it we define an isomorphic copy of $\langle B, \in \rangle$, so let F be the isomorphism between B and the isomorphic copy.

We claim the resultant structure is minimal. As in Example 2.5, R_0 is obviously minimal, so R_1 is also minimal. To show that R_2 is minimal: our axioms guarantee that R_2 is transitive and a model of $V = L$; thus R_2 must be some $L(B \cup \{C\}, \beta)$. But the least such admissible $L(B \cup \{C\}, \beta)$ is $L(B \cup \{C\}, \alpha)$. Since $R_2 \subseteq R_3$, R_3 is also minimal. So, by Proposition 1.1(A), the entire structure is minimal.

Proof of part (2). Let $\langle A; E, Y, \ldots, R_0, R_1, R_2, R_3, R' \rangle$ be a $<_P$-minimal model of Θ_B. By Proposition 1.1(B), it is minimal as a model of $\theta^*[R_0, R_1]$, as a model of $\theta^{**}[R_2, R_3]$, and as a model of $\theta'[R']$ restricted to B. By minimality as a model of θ^*, as in Example 2.5, it is ω-standard. Hence the Form and Δ_0-form of the model contain only standard formulas (or Gödel numbers), so we can apply our Truncation Lemma (4.1). By minimality as a model of θ', we have an isomorphic copy of $\langle B, \in \rangle$. Since $\langle B, \in \rangle$ is wellfounded, B, and hence Y, is in the wellfounded part of $\langle A; E, \ldots \rangle$.

Now, so far, we don't know whether $\langle A; E, \ldots\rangle$ is standard. Let A^* be as in 4.1. So $\langle A^*; E, \ldots\rangle \models \text{GKPLX}$. For simplicity of language we shall identify $\langle A^*; E, \ldots\rangle$ with its transitive (Mostowski) collapse. Let α be the least ordinal such that $\langle L(Y, \alpha), \in\rangle$ is admissible. Then α is not in A^* since otherwise $\langle A; E, Y, \ldots\rangle$ would not be a model of the second conjunct of θ_B (which says "$\langle A; E, Y, \ldots\rangle$ thinks it is the minimal mode of GKPLX for this Y"). (Here we use heavily the absolutness of the construction of $L(Y, \beta)$'s — the construction inside $\langle A; E, Y, \ldots\rangle$ is the same as the construction in the real world.) So $A^* = L(Y, \alpha)$.

Suppose $A^* \neq A$. Then the minimal R_2 is obviously A^*. And no minimal R_3 could exist since it would have to be $L(Y, \beta)$ for th minimal β in $A - A^*$, and if such a β existed it would have to be in the wellfounded part of A also.

Proof of part (3). Immediate from part (2) and Proposition 1.1(B).

Remark. We showed in Example 2.5 that we can define ω by restriction. Thus we have a formula Θ_ω as above.

4.4. Proof of Theorem 3.5. (A) Suppose U is a Σ^1_2 unary relation on the integers. So $n \in X$ iff (the integers are a model of) $\exists Y\, \forall Z\, \phi(Y, Z, n)$ for some first order ϕ. So $n \in U$ iff for some $C \subseteq \omega$, some transitive set A, and some p

$$(\langle A, \in\rangle \models \text{KP})\ \&\ (\omega \in A)\ \&\ (C \in A)\ \&\ (p \in A)$$
$$\&\ (p \text{ is an infinitary proof of } \theta(C, D, n)).$$

Furthermore, such a p exists in every such admissible set, in particular, the least one. So $n \in U$ iff for some $C \subseteq \omega$, if $X = \omega \cup \{C\}$,

$$\langle L(X, \alpha); \in\rangle \models \exists p\ (p \text{ is an infinitary proof of } \phi(C, D, n)),$$

where α is the least ordinal for which that structure is admissible. Let ϕ_n be the formula

$$\exists C, p\ (C \subseteq \omega\ \&\ X = \omega \cup \{c\}$$
$$\&\ p \text{ is an infinitary proof of } \phi(C, D, n) \text{ (over the integers)}),$$

where n appears in ϕ_n as a numeral for n — e.g. $\emptyset$ for 0, Succ($\emptyset$) for 1, etc. Then, by 4.3, n is in U iff $\Theta_\omega\ \&\ \phi_n$ has a $<_P$-minimal model.

(B) Part (B) follows from part (A), since $\exists x\,(x \neq x)$ holds in all countably infinite minimal models iff there are no countably infinite minimal models.

4.5. Proof of Theorem 3.7. Suppose $\langle N; S, T\rangle$ is Δ^1_2-definable over the integers. Say that n is in N iff $\exists Y\, \forall Z\, \phi_1(Y, Z, n)$, and n is not in N iff $\exists Y\, \forall Z\, \phi_2(Y, Z, n)$. Similarly, $\langle n, m\rangle$ is in S iff $\exists Y\, \forall Z\, \phi_3(Y, Z, \langle n, m\rangle)$, and $\langle n, m\rangle$ is not in S iff $\exists Y\, \forall Z\, \phi_4(Y, Z, \langle n, m\rangle)$, and analogously for T and ϕ_5 and ϕ_6, where inside a formula $\langle\ ,\ \rangle$ denotes some recursive function mapping ω 1–1 onto ω. Let ϕ say

$X = \omega \cup \{C\}$, and that for each integer n,

(i) if C_{1n} denotes $\{m: \langle n, \langle 1, m\rangle\rangle \in C\}$ either

(a) there is an infinitary proof of $\phi_0(C_1, Z, n)$ or

(b) there is an infinitary proof of $\phi_1(C_1, Z, n)$, and

(ii) if C_{2n} denotes $\{\langle m_1, m_2\rangle: \langle n, \langle 2, \langle m_1, m_2\rangle\rangle\rangle \in C\}$ either

(a) there is an infinitary proof of $\phi_2(C_2, Z, n)$ or

(b) there is an infinitary proof of $\phi_3(C_2, Z, n)$, and

(iii) similarly for C_{3n}, ϕ_4, and ϕ_5.

Let $\theta = \Theta_\omega \;\&\; \phi$.

Now we can easily construct a minimal model for this θ: for each n in N pick a Y such that $\forall Z\, \phi_0(Y, Z, n)$, and for each m in Y put $\langle n, \langle 1, m\rangle\rangle$ into C. For each n not in N pick a Y such that $\forall Z\, \phi_1(Y, Z, n)$, and for each m in Y put $\langle n, \langle 1, m\rangle\rangle$ into C. Similarly, add elements $\langle n, \langle i, \langle m_1, m_2\rangle\rangle\rangle$ for $i = 2$ to 3 using the definitions of S and T as in (ii)–(iii) above. Let $X = \omega \cup \{C\}$ as usual, and our model, by 4.3(i), is an expansion of $L(X, \alpha)$ for the minimal α where that is admissible. By Barwise completeness, the required proofs all exist in that model.

On the other hand, any minimal model of θ contains sets C_{1n}–C_{3n} as described. Since our admissible set is standard, so the infinitary proofs are valid, it follows that $\{n$: there is a proof of $\forall Z\, \phi_0(C_{1n}, Z, n)\} \subseteq N$, and that $\{n$: there is a proof of $\forall Z\, \phi_1(C_{1n}, Z, n)\} \subseteq \omega - N$. Since for each n either (ia) or (ib) holds, those two sets are complements of each other, so the former set is N. The same argument shows that S and T are defined correctly also, so $\langle N; S, T\rangle$ is a definable subset of every minimal model. (It is actually definable only up to isomorphism since we may get only an isomorphic copy of the admissible set.)

4.6. Proof of Theorem 3.10. (A) Our formula is in some $L(\alpha)$ for $\alpha < \sigma_0$, so we can define that $\langle L(\alpha), \in\rangle$ and the formula Φ using predicate circumscription. (To show that Φ is definable: every element of $L(\alpha)$ is definable via F functions from a finite number of ordinals $<\alpha$, and each such ordinal is definable.) We now define a model in three parts: (1) $L(\alpha)$ with Φ distinguished; (2) a K structure $\mathfrak{M}$ plus the admissible set $HF(\mathfrak{M})$ — $\mathfrak{M}$ together with all hereditarily finite subsets of $\mathfrak{M}$; (3) a satisfaction predicate for formulas in $L(\alpha)$ and sequences of elements of $\mathfrak{M}$. Finally, we assert that Φ is in the satisfaction predicate.

(B) Immediate from part (A) and Proposition 1.1(B).

Our final proof to fill in is the proof of Theorem 3.11. As the previous proofs extended the trick of Example 2.5, so this proof extends the trick of Example 2.6.

Lemma 4.7. *Suppose we can define an infinite cardinal $\aleph_\beta$ using formula circumscription by restriction. Then we can define the cardinal $\aleph_{\beta+1}$ using formula circumscription by restriction.*

Proof. Suppose that there is a formula $\theta_\beta[R, U]$ such that in every $<_F$-minimal model of θ_β, $\{x: \phi_\beta(x)\}$ has cardinality $\aleph_\beta$.

We define a model in 3 parts:

Part I: a model of $\theta_\beta[R, U]$.

Part II: a model of the formula ϕ from Examples 2.3 and 2.5–2.6. (Recall that that formula was a weak axiom for number theory.)

Part III: a model of GB set theory–powerset.

In addition, we assert the existence of the following functions:

F_1 maps $\{x: \phi_\beta(x)\}$ in Part I 1–1 onto Part II.

F_2 maps $\{x: \phi_\beta(x)\}$ in Part I 1–1 onto (the ordinals which are predecessors of) some infinite cardinal κ in the model of GB-powerset, and that κ is not the largest cardinal of the model.

F_3 maps Part II 1–1 onto the predecessors of some c in Part II. (Thus the predecessors of c have the same cardinality as all of Part II.)

Finally, we assert the following:

$<$ is the linear ordering consisting of Part I of the model followed by the ordinals of Part III $<\kappa^+$, the least cardinal of the model of GB $>\kappa$.

O is the field of $<$, i.e., $\{x: \exists y\,(x<y \vee y<x)\}$.

Either $R=O$ or for some d in O, $R=\{x \in O: x<d\}$.

U is a 1–1 function from R onto O.

First we show that in any minimal model of the theory, the cardinality of the κ^+ of the model is $\aleph_{\beta+1}$. By Proposition 1.1(B), in any minimal model the cardinality of $\{x: x<\kappa\}$ is $\aleph_\beta$, and any minimal model is minimal with respect to the last two clauses above (regarding R and U). Clearly the cardinality of κ^+ is $\leqslant\aleph_{\beta+1}$ since each predecessor of κ^+ has at most $\aleph_\beta$ predecessors. Now suppose that the cardinality of κ^+ is $\aleph_\beta$. Then the cardinality of the κ^+ is the same as the cardinality of $\{x$ in Part II: $x<c\}$. So for R to be minimal, R must be $\{x \in O: x<d\}$ for some $d \leqslant c$. And d must be the first element of Part II with $\aleph_\beta$ predecessors. But then the predecessor of d would also have $\aleph_\beta$ predecessors, giving us a contradiction.

Second, we wish to show that our axioms have a minimal model. This is easy. As the sets of our model of GB-powerset choose $H(\aleph_{\beta+2})$, and choose its powerset as the classes. Interpret κ as $\aleph_\beta$, κ^+ as $\aleph_{\beta+1}$, and the rest of the model in obvious ways.

4.8. Proof of 3.11. As usual, our model will be built in several parts, and we shall use the trick of composing definitions that definition by restriction allows. One part will define, by restriction, the ordering $\langle \alpha+1, <\rangle$. A second will be a model of GB-powerset, and we shall have a function F mapping the ordinals $<\alpha+1$ 1–1 and increasing onto an initial segment of the cardinals of that model. We shall want to prove by transfinite induction on $\beta \leqslant \alpha$ that the cardinality of the set of ordinals of the model of GB-powerset $<F(\beta)$ actually is $\aleph_\beta$. For $\beta=0$

we shall use the trick of Example 2.5. For each successor ordinal β we shall use the trick of Lemma 4.7 — so we shall have a family of models of ϕ indexed by ordinals $\beta \leqslant \alpha$, a family of R's and U's indexed by ordinals $\beta \leqslant \alpha$, etc. The step for limit ordinals β is trivial.

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