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The Ginzburg-Landau Equation

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Abstract—The decomposition method is applied to the Ginzburg-Landau equation.

Keywords—Decomposition method, Real G-L equation, Complex G-L equation, A_n polynomials.

The general equation is

$$u_t = (v + i\alpha)u_{xx} - (k + i\beta)|u|^2u + \gamma u, \qquad \begin{array}{l} u(x,0) = f(x), \quad u(0,t) = g(t), \\ 0 \le x < \infty, \quad 0 \le t < \infty, \end{array}$$

where $\alpha, \beta, \gamma, v, k$ are real, $v > 0$, $k > 0$. Let's consider first the real G-L equation where $\alpha = \beta = 0$. Then

$$u_t + vu_{xx} - k|u|^2u + \gamma u = 0.$$

Let $L_t = \frac{\partial}{\partial t}, L_t^{-1} = \int_0^1 (\cdot)\, dt, u = \sum_{n=0}^{\infty} u_n,\ f(u) = |u|^2 u = \sum_{n=0}^{\infty} A_n,$

$$\begin{aligned}
u &= u(0) - L_t^{-1}\gamma \sum_{n=0}^{\infty} u_n - L_t^{-1} v \left(\frac{\partial^2}{\partial x^2}\right) \sum_{n=0}^{\infty} u_n + k \sum_{n=0}^{\infty} A_n,\\
u_0 &= u(t=0) = f(x),\\
u_1 &= -L_t^{-1}\gamma u_0 - L_t^{-1} v \left(\frac{\partial^2}{\partial x^2}\right) u_0 + kA_0,\\
u_2 &= -L_t^{-1}\gamma u_1 - L_t^{-1} v \left(\frac{\partial^2}{\partial x^2}\right) u_1 + kA_1,\\
&\ \vdots
\end{aligned}$$

When the A_n polynomials [1, 2] are evaluated, we can determine u as closely as necessary by computing an m-term approximant $\varphi_m = \sum_{n=0}^{m-1} u_n$ which converges to u.

For the general case, proceeding in the same manner,

$$u = u(t=0) + L_t^{-1}(v + i\alpha)\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} u_n - L_t^{-1}(k + i\beta) \sum_{n=0}^{\infty} A_n + L_t^{-1}\gamma \sum_{n=0}^{\infty} u_n.$$

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Hence, the decomposition components are: [1]

$$
\begin{aligned}
u_0 &= u(t=0),\\
u_1 &= L_t^{-1}(v+i\alpha)\frac{\partial^2}{\partial x^2}\,u_0 - L_t^{-1}(k+i\beta)A_0 + L_t^{-1}\gamma\,u_0,\\
u_2 &= L_t^{-1}(v+i\alpha)\frac{\partial^2}{\partial x^2}\,u_1 - L_t^{-1}(k+i\beta)A_1 + L_t^{-1}\gamma\,u_1,\\
&\vdots\\
u_n &= L_t^{-1}(v+i\alpha)\frac{\partial^2}{\partial x^2}\,u_{n-1} - L_t^{-1}(k+i\beta)A_{n-1} + L_t^{-1}\gamma\,u_{n-1}.
\end{aligned}
$$

To evaluate the $f(u)$ in terms of the A_n polynomials, we write

$$
\begin{aligned}
f(u) &= u|u|^2,\\
|u| &= u\eta(u); \qquad \eta(u) = H(u) - H(-u),
\end{aligned}
$$

where H is the Heaviside (step) function of the first kind and η is the Heaviside (step) function of the second kind

$$
\begin{aligned}
H(u) &= +1, && \text{for } u>0 \text{ and } 0 \text{ for } u<0,\\
\eta(u) &= +1, && \text{for } u>0 \text{ and } -1 \text{ for } u<0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|u|^2 &= u^2\eta^2(u),\\
&= u^2[H^2(u) - 2H(u)H(-u) + H^2(-u)],\\
f(u) &= (u)(u^2\eta^2 u) = u^3\eta^2(u).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f(u) &= \sum_{n=0}^{\infty}\eta^2(u)A_n\{u^3\},\\
A_0 &= u_0^3,\\
A_1 &= 3u_0^2u_1,\\
A_2 &= 3u_0^2u_2 + 3u_1^2u_0,\\
A_3 &= u_1^3 + 3u_0^2u_3 + 6u_0u_1u_2,\\
A_4 &= 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3,\\
A_5 &= 3u_0^2u_5 + 3u_1^2u_3 + 3u_2^2u_1 + 6u_0u_1u_4 + 6u_0u_2u_3,\\
&\vdots
\end{aligned}
$$

The $f(u)$ is singular at the origin since the function is piecewise-differentiable there which is due to the modelling. We can use $\sum_{n=0}^{\infty} A_n$ for $f(u)$ as long as we avoid the origin. Thus, one can now write the m-term approximant $\varphi_m = \sum_{n=0}^{m-1} u_n$ using the above A_n. We can also retain this $f(u)$ but replace it with a smooth approximation as in [3] which avoids any problem at the origin.

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