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Fuzzy Intervals

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Abstract—This paper is devoted to the study of fuzzy intervals. Topological classification theorems on L-fuzzy intervals and H(λ)-intervals (both are generalizations of the ordinary intervals) are proved, and a series of properties of these fuzzy intervals are established. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

In this paper we study fuzzy intervals, precisely, L-fuzzy interval and their weakly induced modifications [1], for two considerations: first, this kind of L-fuzzy topological spaces have some thing to do with classical probability theory. For example, the L-fuzzy real line (for $L = [0,1]$) $R(L)$, a special L-fuzzy interval, exactly consists of all distribution functions on R (see [2]). Second, L-fuzzy intervals are of basic importance not only to topology, but probably also to many other fields. The L-fuzzy unit interval $I(L)$ (respectively, the L-fuzzy real line $R(L)$) was defined by Hutton [3] (respectively, by Höhle [4] and Gantner *et al.* [5]). There has been a great deal of interest (see, e.g., [1–13]) in $I(L)$ and $R(L)$ for their skillful definitions and their similarities with $I = [0,1]$ and $R = (-\infty, +\infty)$, the L-fuzzy unit interval and the L-fuzzy real line for the case of $L = \{0,1\}$. In addition, $I(L)$ has also been successfully applied to solving the compactification problem of weakly induced L-fuzzy topology spaces (see [12]). It is natural to regard L-fuzzy intervals (like ordinary intervals in a real line) to be the objects probably involved in many fields and used by researchers in these fields.

In this paper, we first give the definition of L-fuzzy intervals, among which are $I(L)$, $R(L)$, and all intervals of ordinary real line R. Then we prove the topological classification theorems of L-fuzzy intervals and their weakly induced modification. Finally, we present a series of topological properties of these fuzzy intervals.

Throughout this paper, L always stands for a fuzzy lattice, i.e., a completely distributive complete lattice with an order-reversing involution $'$ on it, and with a smallest element 0 and a largest element 1 ($0 \neq 1$). Obviously, for every nonempty set X, L^X is also a fuzzy lattice under the pointwise order. We denote the smallest element and the largest element of L^X by 0_X and

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1_X respectively, and the set of all nonzero coprime elements [14] of L (respectively, of L^X) by $M(L)$ (respectively, by $M(L^X)$). An L-fuzzy topology space is a pair (L^X, δ), where δ, called an L-fuzzy topology on L^X, is a subfamily of L^X closed under the operations of finite intersections and arbitrary unions.

Let (L^X, δ) be an L-fuzzy topology space. Then $\iota_{\alpha'}(\delta) = \{\iota_{\alpha'}(A) \mid A \in \delta\}$ is a topology on X, where $\iota_{\alpha'}(A) = \{x \in X \mid A(x) \not\leq \alpha'\}$ $(A \in \delta,\ \alpha \in M(L))$. Let $\iota_L(\delta)$ be the topology on X generated by the subbase $\cup\{\iota_{\alpha'}(\delta) \mid \alpha \in M(L)\}$ and $[\delta] = \{A \in \delta \mid A \text{ is a crisp set}\}$. Then $(X, [\delta])$ (respectively, $(X, \iota_L(\delta))$) is called the background space (respectively, underlying space) of (L^X, δ). Furthermore, let $\tilde{\delta}$ be the L-fuzzy topology on L^X generated by the subbase $\delta \cup \iota_L(\delta)$: we will make no distinction between a set E and its characteristic function χ_E. Then $(L^X, \tilde{\delta})$ is called the weakly induced modification [1] of (L^X, δ). (L^X, δ) is called induced if $\iota_L(\delta) \cup \{[\lambda] \mid \lambda \in L\} \subset \delta$, where $[\lambda] \in L^X$ is a constant mapping taking value λ.

A mapping $F : L^X \to L^Y$ is said to be an L-valued Zadeh function induced by an ordinary mapping $f : X \to Y$ if $F(A)(y) = \vee\{A(x) \mid f(x) = y\}$ for every $A \in L^X$ and every $y \in Y$. For the sake of convenience, we denote a mapping and an L-valued Zadeh function induced by it by the lower case and the upper case of the same letter, respectively. Given an L-valued Zadeh function $F : L^X \to L^Y$, a mapping $F^{-1} : L^Y \to L^X$ will be defined by $F^{-1}(B) = B \circ f$ for all $B \in L^Y$.

An L-valued Zadeh function F from an L-fuzzy topology space (L^X, δ) to another L-fuzzy topology space (L^Y, η) is said to be continuous if, for each $B \in \eta$, $F^{-1}(B) \in \delta$; F is said to be a homeomorphism if it is a one-to-one correspondence and both F and F^{-1} are continuous.

The support of an L-fuzzy set $A \in L^X$ (respectively, an L-fuzzy topology space (L^X, δ)) is defined as the set $\operatorname{supp} A = \{x \in X \mid A(x) > 0\}$ (respectively, X). The definitions of $I(L)$ and $R(L)$ can be found in [5]. For other undefined notions and symbols, such as H(λ)-unit interval (the weakly induced modification of $I(L)$), H(λ)-real line (the weakly induced modification of $R(L)$), completely regular L-fuzzy topology spaces, H(λ)-completely regular L-fuzzy topology spaces, $W(L^X, \delta)$, etc., please refer to [7].

2. DEFINITIONS AND CLASSIFICATION THEOREMS

DEFINITION 2.1. *Let $J \subset R$ be an ordinary interval with left-hand member a and right-hand member b, $(a, b \in R \cup \{-\infty, +\infty\},\ a < b)$, and $H(J)$ be the set of all monotonic decreasing mappings $\lambda : R \longrightarrow L$ satisfying the following conditions:*

(i) *if $a \in J$, $\lambda(t) = 1$ for all $t \in (-\infty, a)$;*
(ii) *if $a \notin J$, $\vee\{\lambda(t) \mid t > a,\ t \in R\} = 1$;*
(iii) *if $b \in J$, $\lambda(t) = 0$ for all $t \in (b, +\infty)$;*
(iv) *if $a \notin J$, $\wedge\{\lambda(t) \mid t < b,\ t \in R\} = 0$.*

For each $t \in R$ and each $\lambda \in H(J)$, write $\lambda(t+) = \vee\{\lambda(r) \mid r \in R,\ r > t\}$ and $\lambda(t-) = \wedge\{\lambda(l) \mid l \in R,\ l < t\}$. We define an equivalence relation $\sim$ on $H(J)$ as follows:

$$\lambda_1 \sim \lambda_2 \text{ iff } \lambda_1(t+) = \lambda_2(t+) \text{ and } \lambda_1(t-) = \lambda_2(t-), \qquad (\forall t \in R).$$

Let $H(J)/\sim$ be the quotient set, and δ be the L-fuzzy topology on $L^{H(J)/\sim}$, which has $\{L_t, R_t \mid t \in R\}$ as a subbase, where $L_t([\lambda]) = (\lambda(t-))'$ and $R_t([\lambda]) = \lambda(t+)$ $(t \in R, \lambda \in H(J))$, $[\lambda]$ is the equivalence class containing λ. The L-fuzzy topology space $(L^{H(J)/\sim}, \delta)$ is called L-fuzzy interval, and is briefly denoted by $J(L)$. The weakly induced modification of $J(L)$, denoted by $\tilde{J}(L)$, is called an H(λ)*-interval.*

Similar to Theorem 3.10 in [7], we may show the following.

LEMMA 2.2. *Neither $R(L)$ nor $(0, 1](L)$ is* N*-compact [15].*

THEOREM 2.3. *For a given L, there are exactly three nonhomeomorphic L-fuzzy intervals: $I(L)$, $R(L)$, and $(0, 1](L)$.*

PROOF. Analogous to Proposition 3.1 in [7], we may show that every L-fuzzy interval $J(L)$ is homeomorphic with $I(L)$, $R(L)$, or $(0,1](L)$. Since $I(L)$ is N-compact [1] and N-compactness is an invariant of homeomorphic L-valued Zadeh functions, by Lemma 2.2, $I(L)$ is homeomorphic with neither $R(L)$ nor $(0,1]$. Therefore, we only need to show that $R(L)$ is not homeomorphic with $(0,1](L)$.

Suppose that $F : (L^X, \delta) \longrightarrow (L^Y, \eta)$ is a homeomorphism, where $(L^X, \delta) = R(L)$ and $(L^Y, \eta) = (0,1](L)$. We say that $[\lambda] \in X$ is a crisp element of X if $\lambda(R) \subset \{0,1\}$. We may show that both f and f^{-1} preserve crisp elements. Indeed, if $[\lambda] \in X$ is a crisp element, then $R_t([\lambda]) \in \{0,1\}$ and $L_t([\lambda]) \in \{0,1\}$ $(\forall t \in R)$, and so $A([\lambda]) \in \{0,1\}$ $(\forall A \in \delta)$. Since f is a one-to-one correspondence, we have $F(A)(f([\lambda])) = A([\lambda]) \in \{0,1\}$ $(\forall A \in \delta)$, and thus, $B(f([\lambda])) \in \{0,1\}$ $(\forall B \in \eta)$, particularly, $R_t(f([\lambda])) \in \{0,1\}$ and $L_t(f([\lambda])) \in \{0,1\}$ $(\forall t \in R)$, which implies that $f([\lambda]) \in Y$ is a crisp element. Similarly, f^{-1} preserves crisp elements. It follows that R is homeomorphic with its subspace $(0,1]$. This is a contradiction.

Analogously, we may show the following theorem.

THEOREM 2.4. *For a given L, there are exactly three nonhomeomorphic* H(λ)*-intervals: $\tilde{I}(L)$, $\tilde{R}(L)$, and $\widetilde{(0,1]}(L)$.*

THEOREM 2.5. *For a given L ($L \neq \{0,1\}$), there are exactly six nonhomeomorphic L-fuzzy intervals and H(λ)-intervals: $I(L)$, $R(L)$, $(0,1](L)$, $\tilde{I}(L)$, $\tilde{R}(L)$, and $\widetilde{(0,1]}(L)$.*

PROOF. Suppose that $L \neq \{0,1\}$. Then, the background space of $I(L)$, as shown in [13], is antidiscrete [16]. This is also true for every L-fuzzy interval $J(L)$. On the other hand, for every L-fuzzy interval $J(L)$, the background space of $\tilde{J}(L)$ is Hausdorff (see [12, Corollary 1]), which implies that there is no pair $(J_1(L), J_2(L))$ of L-fuzzy intervals such that $J_1(L)$ is homeomorphic with $\tilde{J}_2(L)$. Thus, Theorem 2.5 follows from Theorem 2.3 and Theorem 2.4.

COROLLARY 2.6. *For any intervals $J_1, J_2 \subset R$, the following statements are equivalent:*

(1) *$J_1(L)$ is homeomorphic with $J_2(L)$;*
(2) *$\tilde{J}_1(L)$ is homeomorphic with $\tilde{J}_2(L)$;*
(3) *J_1 is homeomorphic with J_2.*

3. PROPERTIES OF FUZZY INTERVALS

An L-fuzzy topology space (L^X, δ) is said to be T_k if $(X, [\delta])$ is a T_k topological space ($k = 0, 1, 2, 3, 3\ 1/2, 4$). Obviously, $T_4 \Longrightarrow T_{3\ 1/2} \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$.

The main results of this section are as follows.

THEOREM 3.1. *$J(L)$ and its weakly induced modification $\tilde{J}(L)$ have the following properties:*

(1) *$J(L)$ is completely regular; it is not T_0 when $L \neq \{0,1\}$.*
(2) *$\tilde{J}(L)$ is* H(λ)*-completely regular and $T_{3\ 1/2}$; it is T_4 when $W(L) \leq \aleph_0$, where $W(L)$ is the weight of L (see [14, Definition 4.5, p. 170]).*
(3) *Neither $J(L)$ nor $\tilde{J}(L)$ is induced [1] when $L \neq \{0,1\}$.*
(4) *The closure of a crisp set in $\tilde{J}(L)$ is still a crisp set.*
(5) *$W(J(L)) = \aleph_0$, $W(\tilde{J}(L)) = W(L) \cdot \aleph_0$.*
(6) *$\tilde{J}(L)$ is* N*-compact $\Longleftrightarrow$ $J(L)$ is* N*-compact $\Longleftrightarrow$ J is compact.*
(7) *$\tilde{J}(L)$ is II-paracompact (see [7]) if L satisfies*
    (i) *$W(L) \leq \aleph_0$;*
    (ii) *L is a chain or $L - \{1\} = \cup\{\downarrow e \mid e \in P\}$ for some finite subset $P \subset L$.*

PROOF.

(1) Since $I(L)$ is completely regular, complete regularity is hereditary (see [1]) and the background space of $I(L)$ is antidiscrete [13], (1) follows from Theorem 2.3.

(2) As $\tilde{I}(L)$ is H(λ)-completely regular, H(λ)-complete regularity is hereditary and the underlying space of $I(L)$ is Hausdorff (see [12, Theorem 9 and Corollary 2]), the first half of (2) follows from Theorem 2.3; the second half of (2) follows from (5).

(3) As $\tilde{I}(L)$ is not induced when $L \neq \{0,1\}$ (see [12, Theorem 3]), (3) follows from Theorems 2.3–2.5. (4) follows from Theorem 2.4 and [12, Theorem 3], (5) follows from Theorems 2.3–2.5 and [7, Proposition 3.2, Theorem 3.3], (6) follows from Corollary 2.6, and (7) follows from Theorem 2.4 and [7, Theorem 3.14]. This completes the proof of Theorem 3.1.

REMARK 3.2. Connectedness and local connectedness of fuzzy intervals will be studied in a different paper.

Finally, let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the topologies on X_L (the support of $R(L)$) generated by the bases $\{\text{supp}(a,b)(L) \mid a,b \in R, a < b\} \cup \{X_L\}$ and $\{\text{supp}[a,b)(L) \mid a,b \in R, a < b\} \cup \{X_L\}$ respectively, $R_L = (X_L, \mathcal{J}_1)$ and $S_L = (X_L, \mathcal{J}_2)$. Obviously, R_L (respectively, S_L) is homeomorphic with R (respectively, Sorgenfrey line [16]) when $L = \{0,1\}$. It is easy to verify the following proposition (where $d(S_L)$, $\chi(S_L)$ and $W(S_L)$ are the density, character and weight of S_L, respectively, see [16]).

PROPOSITION 3.3. *$d(S_L) = \chi(S_L) = W(R_L) = \aleph_0$, $W(S_L) = \aleph$. Therefore, S_L is not homeomorphic with R_L.*

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