Total restrained domination in trees

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in $S$ and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the smallest cardinality of a total restrained dominating set of $G$. We show that if $T$ is a tree of order $n$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. Moreover, we show that if $T$ is a tree of order $n \equiv 0 \mod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We then constructively characterize the extremal trees $T$ of order $n$ achieving these lower bounds.

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1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Moreover, the notation $P_n$ will denote the path of order $n$. A set $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [3,4].

In this paper, we continue the study of a variation of the domination theme, namely that of total restrained domination. A set $S \subseteq V$ is a total restrained dominating set (denoted TRDS) if every vertex is adjacent to a vertex in $S$ and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. Every graph has a total restrained dominating set, since $S = V$ is such a set. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of $G$. A TRDS of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$-set.

The concept of restrained domination was introduced by Chen et al. [2], and further studied by Zelinka in [6]. We may note that the concept of total restrained domination was also introduced by Telle and Proskurowski [5], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set $S$, the complementary set $V - S$ and on edges between the sets $S$ and $V - S$. For example, if we require that every vertex in $V - S$ should be adjacent to some other vertex of $V - S$ (the condition on the set $V - S$) and to some vertex in $S$ (the condition on edges between...
Theorem 3. If $T$ is a tree of order $n$, then $\gamma_T(T) \geq \left\lceil \frac{n+2}{2} \right\rceil$. Moreover, we constructively characterize the extremal trees $T$ of order $n$ achieving this lower bound. Lastly, we show that if $T$ is a tree of order $n \equiv 0 \pmod{4}$, then $\gamma_T(T) \geq \left\lceil \frac{n+2}{2} \right\rceil + 1$, and also constructively characterize the extremal trees $T$ of order $n$ achieving this lower bound.

2. The lower bound

The following result was established in [2], using a more cumbersome proof. As we shall see, this result will be useful in establishing a sharp lower bound on the total restrained domination number of a tree.

Proposition 1. If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor$.

Proof. Suppose $S$ is a TRDS of $P_n$, whose vertex set is $V = \{v_1, \ldots, v_n\}$. Note that $v_1, v_2 \in S$. Moreover, any component of $V - S$ is of size exactly two. Each component is adjacent to a vertex of $S$, which, in turn, is adjacent to another vertex of $S$. Suppose there are $m$ such components. Then $2m + 2m + 2 \leq n$ and so $m \leq \left\lfloor \frac{n-2}{4} \right\rfloor$. Thus $|S| = n - 2m \geq n - 2\left\lfloor \frac{n-2}{4} \right\rfloor$. On the other hand, $V - \{v_i \mid i \in \{3, 4, 7, 8, \ldots, 4\left\lceil \frac{n-2}{2} \right\rceil - 1, 4\left\lfloor \frac{n-2}{4} \right\rfloor\}$ is a TRDS of $P_n$, whence $\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor$. \qed

Corollary 2. If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) \geq \left\lceil \frac{n+2}{2} \right\rceil$.

Proof. Since $n - 2\left\lfloor \frac{n-2}{4} \right\rfloor \geq \left\lceil \frac{n+2}{2} \right\rceil$, the result follows from Proposition 1. \qed

Let $T = (V, E)$ be a tree and $v, a, b \in V$ such that $\deg v \geq 3$ and $a, b \in N(v)$. Let $\ell_b$ be a leaf of the component of $T - v$ that contains $b$. Then the tree $T'$ which arises from $T$ by deleting the edge $va$ and joining $a$ to $\ell_b$ is called a $(v, a, b)$-pruning of $T$.

Theorem 3. If $T$ is a tree of order $n \geq 2$, then $\gamma_{tr}(T) \geq \left\lceil \frac{n+2}{2} \right\rceil$.

Proof. We use induction on $n$. It is easy to check that the result is true for all trees $T$ of order $n \leq 8$. Suppose, therefore, that the result is true for all trees of order less than $n$, where $n \geq 9$. Let $\gamma_{tr} = \min \{\gamma_{tr}(T) \mid T$ is a tree of order $n\}$. We will show that $\gamma_{tr} \geq \left\lceil \frac{n+2}{2} \right\rceil$.

Let $\mathcal{F} = \{T \mid T$ is a tree of order $n$ such that $\gamma_{tr}(T) = \gamma_{tr}\}$. Among all trees in $\mathcal{F}$, let $T$ be chosen so that the sum $s(T)$ of the degrees of its vertices of degree at least 3 is minimum. If $s(T) = 0$, then $T \cong P_n$, and so $\gamma_{tr} = \gamma_{tr}(P_n) \geq \left\lceil \frac{n+2}{2} \right\rceil$. Suppose, therefore, that $s(T) \geq 1$. Since $s(T) \geq 1$, there exists a vertex $v$ such that $\deg(v) \geq 3$. Let $S$ be a $\gamma_{tr}(T)$-set of $T$.

Claim 1. If $v$ is a vertex of degree at least 3, then

(i) $v \notin S$,
(ii) $v$ is adjacent to exactly one vertex of $S$,
(iii) $\deg(v) = 3$.

Proof. Suppose $v \in S$. Then there exist $a, b \in N(v)$ such that $b \in S$. Let $T'$ be a $(v, a, b)$-pruning of $T$. Then $S$ is a TRDS of $T'$, and so, by definition of $\gamma_{tr}$, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in \mathcal{F}$. However, as $s(T') < s(T)$, we obtain a contradiction.

Thus, assume $v \notin S$ and let $a, b \in N(v)$ such that $a \notin S$ and $b \in S$. If $c \in N(v) - \{a, b\}$ is in $S$, then, by considering the $(v, b, c)$-pruning of $T$, we obtain a contradiction as before. We therefore assume that $b$ is the only vertex in $S$ which is adjacent to $v$. 
Suppose \( \deg(v) \geq 4 \), let \( \{c_1, \ldots, c_{\deg(v)}\} = N(v) - \{a, b\} \), let \( c = c_1 \) and let \( \ell_b \) be a leaf of the component of \( T - v \) that contains \( b \). Let \( T' \) be the tree which arises from \( T \) by deleting the edges \( uv_i \) for \( i = 1, \ldots, \deg(v) - 2 \) and joining \( c \) to \( \ell_b, c_2, \ldots, c_{\deg(v)} \). Note that \( \deg_{T'}(v) = \deg_T(\ell_b) = 2 \), \( \deg_T(c) = \deg(c) + \deg(v) - 3 \geq \deg(c) + 1 \geq 3 \), while all other vertices have the same degree in \( T' \) as in \( T \). On the one hand, if \( \deg(c) = 2 \), then \( s(T') = s(T) - \deg(v) + \deg_T(c) = s(T) - 1 \). On the other hand, if \( \deg(c) \geq 3 \), then \( s(T') = s(T) - \deg(v) + \deg_T(c) - 3 = s(T) - 3 \). Then \( S \) is a TRDS of \( T' \). As \( T' \in \mathcal{F} \) and \( s(T') < s(T) \), we obtain a contradiction in both cases. Thus, \( \deg(v) = 3 \). \( \square \)

**Claim 2.** No two vertices of degree 3 are adjacent.

**Proof.** Using the notation employed in Claim 1, \( b \) is the only neighbor of \( v \) in \( S \). By Claim 1, \( \deg(b) \leq 2 \). If \( \deg(c) = 3 \), then, by Claim 1, \( c \) is adjacent to a vertex in \( V - S \) (other than \( v \)). Let \( T' \) be the \((v, c, b)\)-pruning of \( T \). Then \( S \) is a TRDS of \( T' \), and so, by definition of \( \gamma_T \), we have that \( \gamma_T \leq \gamma_T(T') \leq |S| = \gamma_T \). Hence, \( T' \in \mathcal{F} \). However, as \( s(T') < s(T) \), we obtain a contradiction. \( \square \)

Using the notation employed in the proof of Claim 1, the vertex \( b \in S \) and, as it must be adjacent to another vertex in \( S \), \( \deg(b) = 2 \) (cf. Claim 1). Let \( b' \in S \) be the vertex adjacent to \( b \) and suppose \( b' \) is not a leaf. Then, by Claim 1, \( \deg(b') = 2 \). Let \( b'' \) be the neighbor of \( b' \) different from \( b \). Then \( S \) is a TRDS of a tree \( T' \) obtained from \( T \) by deleting the edge \( b'b'' \) and joining the vertex \( b'' \) to some leaf of the component of \( T - v \) containing \( c \). Thus \( T' \in \mathcal{F} \) and \( b' \) is a leaf of \( T' \). Hence we may assume that \( b' \) is a leaf of \( T \).

By Claim 2, \( \deg(a) = \deg(c) = 2 \). Let \( a', c' \in S \) be the neighbor of \( a \) (\( c \), respectively) which is different from \( v \). Necessarily, \( a', c' \in S \). Then \( \deg(a') = \deg(c') = 2 \) (cf. Claim 1). As each vertex in \( S \) is adjacent to another vertex of \( S \), there exist vertices \( a'' \) and \( c'' \) in \( S \) which are adjacent to \( a' \) and \( c' \), respectively. We may assume, as we did for \( b' \), that \( a'' \) is a leaf of \( T \).

If \( n = 9 \), then \( \gamma_T(T) = 6 = \left\lfloor \frac{n+2}{2} \right\rfloor \). Suppose, therefore, that \( n \geq 10 \). Let \( T' \) be the component of \( T - cc' \) containing \( c' \). Then \( S \cap V(T') \) is a TRDS of \( T' \), so that \( |S \cap V(T')| \geq \gamma_T(T') \). Hence, \( |S| \geq 4 + \gamma_T(T') \). Applying the inductive hypothesis to the tree \( T' \) of order \( n - 7 \), we have \( \gamma_T(T') \geq \left\lfloor \frac{n-5}{2} \right\rfloor \), and so \( \gamma_T(T) = |S| \geq \left\lceil \frac{n+3}{2} \right\rceil \geq \left\lceil \frac{n+2}{2} \right\rceil \). \( \square \)

**3. Extremal trees \( T \) with \( \gamma_T(T) = \left\lfloor \frac{n(T)+2}{2} \right\rfloor \)**

Let \( \mathcal{F} \) be the class of all trees \( T \) of order \( n(T) \) such that \( \gamma_T(T) = \left\lfloor \frac{n(T)+2}{2} \right\rfloor \). We will constructively characterize the trees in \( \mathcal{F} \). In order to state the characterization, we define four simple operations on a tree \( T \).

**O1.** Join a leaf or a remote vertex of \( T \) to a vertex of \( K_1 \), where \( n(T) \) is even.

**O2.** Join a vertex \( v \) of \( T \) which lies on an endpath \( vxz \) to a leaf of \( P_3 \), where \( n(T) \) is even.

**O3.** Join a vertex \( v \) of \( T \) which lies on an endpath \( vxy_1y_2z \) to a leaf of \( P_3 \), where \( n(T) \) is even.

**O4.** Join a remote vertex or a leaf of \( T \) to a leaf of each of \( \ell \) disjoint copies of \( P_4 \) for some \( \ell \geq 1 \).

Let \( \mathcal{G} \) be the class of all trees obtained from \( P_2 \) by a finite sequence of Operations O1–O4.

We will show that \( T \in \mathcal{F} \) if and only if \( T \in \mathcal{G} \).

**Lemma 4.** Let \( T' \in \mathcal{F} \) be a tree of even order \( n(T') \). If \( T \) is obtained from \( T' \) by one of the Operations O1–O3, then \( T \in \mathcal{F} \).

**Proof.** Let \( S \) be a \( \gamma_T(T') \)-set of \( T' \) throughout the proof of this result.

**Case 1:** \( T \) is obtained from \( T' \) by Operation O1.

Let \( u \) be a leaf or a remote vertex of \( T' \), and suppose \( T \) is formed by attaching the singleton \( v \) to \( u \). Then \( S \cup \{v\} \) is a TRDS set of \( T \), and so \( \left\lfloor \frac{n(T)+3}{2} \right\rfloor \leq \gamma_T(T) \leq \left\lfloor \frac{n(T)+2}{2} \right\rfloor + 1 \). Since \( n(T') \) is even, we have \( \gamma_T(T) = \left\lfloor \frac{n(T)+2}{2} \right\rfloor \). Thus, \( T \in \mathcal{F} \).

**Case 2:** \( T \) is obtained from \( T' \) by Operation O2 or Operation O3.
Suppose $v$ lies on the endpath $vzx$ or $v|x_1x_2z$ and $T$ is obtained from $T'$ by adding the path $y_1y_2z'$ to $T'$ and joining $y_1$ to $v$.

We show that $v \notin S$. First consider the case when $v$ lies on the endpath $vzx$. Suppose $v \in S$. Then $S' = S \setminus \{z\}$ is a TRDS of $T'' = T' \setminus \{z\}$, and so $\left\lceil \frac{n(T')+1}{2} \right\rceil \leq \gamma_{tr}(T'') \leq \left\lceil \frac{n(T')+2}{2} \right\rceil - 1$. However, as $n(T')$ is even, we have $\frac{n(T')+2}{2} < \gamma_{tr}(T'') \leq \frac{n(T')+2}{2} - 1$, which is a contradiction. Thus, $v \notin S$.

In the case when $v$ lies on the endpath $v|x_1x_2z$, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{y_2, z'\}$ is a TRDS of $T$, and so $\left\lceil \frac{n(T')}{2} + 1 \right\rceil \leq \gamma_{tr}(T) \leq \left\lceil \frac{n(T')+2}{2} \right\rceil + 2$. However, as $n(T')$ is even, we have $\gamma_{tr}(T) = \left\lceil \frac{n(T')+2}{2} \right\rceil$, and so $T \in \mathcal{F}$.

The proof is complete. \qed

**Lemma 5.** Let $T' \in \mathcal{F}$ be a tree of order $n(T')$. If $T$ is obtained from $T'$ by the Operation O4, then $T \in \mathcal{F}$.

**Proof.** Let $S$ be a $\gamma_{tr}(T')$-set of $T'$, and suppose $v$ is a remote vertex or a leaf of $T'$. Then $v \in S$. Let $T$ be the tree which is obtained from $T'$ by adding the paths $u_iy_i\overline{y}_iz_i$ to $T'$ and joining $u_i$ to $v$ for $i = 1, \ldots, \ell$. Then $S' = S \setminus \{y_2, z'\}$ is a TRDS of $T$, and so $\left\lceil \frac{n(T')}{2} + 4\ell + 2 \right\rceil \leq \gamma_{tr}(T) \leq \left\lceil \frac{n(T')+2}{2} \right\rceil + 2\ell$. Consequently, $\gamma_{tr}(T) = \left\lceil \frac{n(T')+2}{2} \right\rceil$, and so $T \in \mathcal{F}$. \qed

We are now in a position to prove the main result of this section.

**Theorem 6.** $T$ is in $\mathcal{G}$ if and only if $T$ is in $\mathcal{F}$.

**Proof.** Assume $T \in \mathcal{G}$. We show that $T \in \mathcal{F}$, by using induction on $c(T)$, the number of operations required to construct the tree $T$. If $c(T) = 0$, then $T = P_2$, which is in $\mathcal{F}$. Assume, then, for all trees $T' \in \mathcal{G}$ with $c(T') < k$, where $k \geq 1$ is an integer, that $T'$ is in $\mathcal{F}$. Let $T \in \mathcal{G}$ be a tree with $c(T) = k$. Then $T$ is obtained from some tree $T'$ by one of the Operations O1–O4. But then $T' \in \mathcal{G}$ and $c(T') < k$. Applying the inductive hypothesis to $T'$, $T'$ is in $\mathcal{F}$. Hence, by Lemma 4 or Lemma 5, $T$ is in $\mathcal{F}$.

To show that $T \in \mathcal{G}$ for a nontrivial $T \in \mathcal{F}$, we use induction on $n$, the order of the tree $T$. If $n = 2$, then $T = P_2 \in \mathcal{G}$. Let $T \in \mathcal{F}$ be a tree of order $n \geq 3$, and assume for all trees $T' \in \mathcal{F}$ of order $2 \leq n(T') < n$, that $T' \in \mathcal{G}$. Since $n(T) \geq 3$, $\text{diam}(T) \geq 2$.

If $\text{diam}(T) = 2$, then $T$ is a star with exactly two leaves, which can be constructed from $P_2$ by applying Operation O1. Thus, $T \in \mathcal{G}$.

Since no double star is in $\mathcal{F}$, we may assume $\text{diam}(T) \geq 4$. Throughout $S$ will be used to denote a $\gamma_{tr}(T)$-set of $T$.

**Claim 3.** Let $z$ be a leaf of $T$. If $S \setminus \{z\}$ is a TRDS of $T' = T - z$, then $T \in \mathcal{G}$.

**Proof.** Assume $S \setminus \{z\}$ is a TRDS of $T'$. Then $\left\lceil \frac{n(T')}{2} + 2 \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n(T')+2}{2} \right\rceil - 1$. This yields a contradiction when $n$ is even. Hence, $n$ is odd, and $\gamma_{tr}(T') = \frac{n+1}{2} = \left\lceil \frac{n(T')+2}{2} \right\rceil$. Thus, $T' \in \mathcal{F}$, with $n(T') = n - 1$ even. By the induction assumption, $T' \in \mathcal{G}$. The tree $T$ can now be constructed from $T'$ by applying Operation O1, whence $T \in \mathcal{G}$. \qed

Claim 3 implies that if $vzx$ is an endpath of $T$, then we may assume $v \notin S$, since otherwise the tree is constructible.

Claim 3 also implies that every remote vertex of $T$ is adjacent to exactly one leaf, since otherwise it is constructible.

**Claim 4.** If $u$ is a leaf of $T$ and $v$ is either another leaf of $T$ or the remote vertex adjacent to $u$, then $S' = S \setminus \{u, v\}$ is not a TRDS of $T' = T - u - v$.

**Proof.** Suppose, to the contrary, that $S'$ is a TRDS of $T'$. Then $\left\lceil \frac{n(T')}{2} + 2 \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n(T')+2}{2} \right\rceil - 2$. Thus, $\left\lceil \frac{n}{2} \right\rceil + 2 \leq \left\lceil \frac{n(T')+2}{2} \right\rceil$, which yields a contradiction. \qed
Let $T$ be rooted at a leaf $r$ of a longest path.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 2$ from $r$. Suppose $v$ lies on the endpath $vyz'$. Then, by the remark above, $v \notin S$, which implies that $v$ is not adjacent to a leaf. If $v$ also lies on the endpath $vzx$, then $S - \{x, z\}$ is a TRDS of $T - x - z$, which is a contradiction by Claim 4.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from $r$ has degree 2.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$. Let $vy_1y_2z'$ be an endpath of $T$. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of $v$ have degree at least 2.

Assume $v$ also lies on the path $vzx$, where $z$ is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, $vzx$ is an endpath. If $v$ is dominated by a vertex other than $x$, then $S - \{x, z\}$ is a TRDS of $T' = T - x - z$, which is a contradiction (cf. Claim 4). Hence, $v$ is dominated only by $x$. Then $S' = S - \{y_2, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - z'$ and so $\frac{n+3+2}{2} \leq \gamma_{tv}(T') \leq \frac{n}{2} - 2$. This yields a contradiction when $n$ is even. Hence, $n$ is odd and $\gamma_{tv}(T') = \frac{n-1}{2} = \left\lceil\frac{n(T') + 2}{2}\right\rceil$. Thus, $T' \in \mathcal{F}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{G}$.

The tree $T$ can now be constructed from $T'$ by applying Operation O2, whence $T \in \mathcal{G}$.

Assume $v$ lies on the path $vx_1x_2z$. Since $x_1$ ($x_2$, respectively) is on a longest path at distance $\text{diam}(T) - 2$ (diam $(T) - 1$, respectively) from $r$, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that $vx_1x_2$ is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a TRDS of $T' = T - x_1 - x_2 - z$. Thus, $\frac{n-3+2}{2} \leq \gamma_{tv}(T') \leq \frac{n+2}{2} - 2$. This yields a contradiction when $n$ is even. Hence, $n$ is odd and $\gamma_{tv}(T') = \frac{n(T') + 2}{2}$. Thus, $T' \in \mathcal{F}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{G}$ and $T$ can now be constructed from $T'$ by applying Operation O3, whence $T \in \mathcal{G}$.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$ has degree 2.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 4$ from $r$. As $P_5 \notin \mathcal{F}$, $v \notin r$ and $\text{diam}(T) \geq 5$.

Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of $T$. Then, as $y_2y_3z'$ is an endpath of $T$, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\frac{n-4+2}{2} \leq \gamma_{tv}(T') \leq \frac{n+2}{2} - 2$, whence $\gamma_{tv}(T') = \frac{n(T') + 2}{2}$. We conclude that $T' \in \mathcal{F}$, and by the induction assumption, $T' \in \mathcal{G}$. If $\deg_T(v) = 2$ or when $v$ is a remote vertex, then $T$ can be constructed from $T'$ by applying Operation O4.

We therefore assume that $\deg_T(v) \geq 3$ and that $v$ is not adjacent to a leaf.

If $v$ also lies on the path $vzx$, where $z$ is a leaf, then $v \notin S$, which is a contradiction.

We now suppose $v$ lies on the path $vx_1x_2z$, where $z$ is a leaf. Then, since $x_2$ is a remote vertex, we have $\deg(x_2) = 2$. As $x_1x_2$ is an endpath of $T$, it follows that $x_1 \notin S$. As $x_1$ must be adjacent to another vertex in $V - S$, vertex $x_1$ lies on a path $x_1, u_1, u_2, z''$. But then $x_1$, with $\deg(x_1) \geq 3$, is a vertex at distance $\text{diam}(T) - 3$ on a longest path from $r$, which is a contradiction.

Let $e$ be the edge that joins $v$ with its parent, and let $T(v)$ be the component of $T - e$ that contains $v$. Then $T(v)$ consists of $\ell$ disjoint paths $u_i x_i y_i z_i (i = 1, \ldots, \ell)$ with $v$ joined to $u_i$ for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_i y_i z_i$ is an endpath of $T$, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \bigcup_{i=1}^{\ell}(y_i, z_i)$ is a TRDS of $T' = T - (T(v) - v)$, and so $\frac{n-4\ell+2}{2} \leq \gamma_{tv}(T') \leq \frac{n+2}{2} - \ell$, whence $\gamma_{tv}(T') = \frac{n(T') + 2}{2}$. Thus, $T' \in \mathcal{F}$, and by the induction assumption, $T' \in \mathcal{G}$. Note that $v$ is a leaf of $T'$. The tree $T$ can now be constructed from $T'$ by applying Operation O4, whence $T \in \mathcal{G}$. \qed

**Theorem 7.** Let $T$ be a tree of order $n(T)$. If $n(T) \equiv 0 \mod 4$, then $\gamma_{tv}(T) \geq \left\lceil\frac{n(T) + 2}{2}\right\rceil + 1$.

**Proof.** We will show that every tree $T$ in $\mathcal{F} = C$ has $n(T) \not\equiv 0 \mod 4$, by using induction on $s(T)$, the number of operations required to construct the tree $T$. If $s(T) = 0$, then $T = P_2$, and $2 \not\equiv 0 \mod 4$. Assume, then, for all trees $T' \in \mathcal{G}$ with $s(T') < k$, where $k \geq 1$ is an integer, that $n(T') \not\equiv 0 \mod 4$. Let $T \in \mathcal{G}$ be a tree with $s(T) = k$. Then $T$ is obtained from some tree $T'$ by one of the Operations O1–O4. Then $T' \in \mathcal{G}$, and by the induction hypothesis, $n(T') \not\equiv 0 \mod 4$. If $T$ is obtained from $T'$ by one of the Operations O1–O3, then $n(T') \equiv 2 \mod 4$, and, since either a path of order 1 or a path of order 3 is attached to $T'$ to form $T$, $n(T) \not\equiv 0 \mod 4$. Moreover, $n(T) = n(T') + 4$ if $T$ is obtained from $T'$ by Operation O4, whence $n(T) \not\equiv 0 \mod 4$. The result now follows. \qed
4. Extremal trees $T$ of order $n(T) \equiv 0 \mod 4$ with $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$

Let $\mathcal{F}^* = \{ T \mid T$ is a tree of order $n(T) \equiv 0 \mod 4$ such that $\gamma_{tr}(T) = \left\lceil \frac{n+2}{2} \right\rceil + 1 \}$. In order to constructively characterize the trees in $\mathcal{F}^*$, we define the following operations on a tree $T$:

O5. Join a leaf or a remote vertex $v$ of $T$ to a vertex of $K_1$, where $n(T) \equiv 3 \mod 4$.
O6. Join a vertex $v$ of $T$ which lies on an endpath $vxz$ to a vertex of $K_2$, where $n(T) \equiv 2 \mod 4$.
O7. Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a vertex of $K_2$, where $n(T) \equiv 2 \mod 4$.
O8. Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a leaf of $P_3$, where $n(T) \equiv 1 \mod 4$.
O9. Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a leaf of $P_3$, where $n(T) \equiv 1 \mod 4$.

Let $\mathcal{F} = \{ T \mid T$ is a tree obtained by applying one of the Operations O5–O9 to a tree $T' \in \mathcal{C}$ exactly once $\}$. Let $\mathcal{C}^* = \{ T \mid T$ is a tree obtained from a tree $T' \in \mathcal{F}$ by applying Operation O4 to $T'$ zero or more times $\}$. We will show that $\mathcal{F}^* = \mathcal{C}^*$.

Lemma 8. Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 3 \mod 4$. If $T$ is obtained from $T'$ by Operation O5, then $T \in \mathcal{F}^*$.

Proof. Let $u$ be a leaf or a remote vertex of $T'$, and suppose $T$ is formed by attaching the singleton $v$ to $u$. Let $S$ be a $\gamma_{tr}(T')$-set of $T'$. Then $S \cup \{v\}$ is a TRDS set of $T$, and so, since $n(T) \equiv 0 \mod 4$, $\left\lceil \frac{n(T)+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 1 = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1 = \left\lceil \frac{n(T)+1}{2} \right\rceil + 1$. Hence, $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$, and so $T \in \mathcal{F}^*$. □

Lemma 9. Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 2 \mod 4$. If $T$ is obtained from $T'$ by either Operation O6 or Operation O7, then $T \in \mathcal{F}^*$.

Proof. Let $\{u, v\}$ be the vertex set of $K_2$ and let $S$ be a $\gamma_{tr}(T')$-set. The set $S \cup \{u, v\}$ is a TRDS of $T$, and so, since $n(T) \equiv 0 \mod 4$, $\left\lceil \frac{n(T)+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \left\lceil \frac{n(T)+2}{2} \right\rceil + 2 = \left\lceil \frac{n(T)+1}{2} \right\rceil + 2$. Hence, $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$, and so $T \in \mathcal{F}^*$. □

Lemma 10. Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 1 \mod 4$. If $T$ is obtained from $T'$ by either Operation O8 or Operation O9, then $T \in \mathcal{F}^*$.

Proof. Let $S$ be a $\gamma_{tr}(T')$-set of $T'$. Assume $v$ lies on the endpath $vxz$ or $vx_1x_2z$ and $T$ is obtained from $T'$ by adding the path $y_1y_2z'$ to $T'$ and joining $y_1$ to $v$. We show that $v \notin S$.

First consider the case when $v$ lies on the endpath $vxz$. Suppose $v \in S$. Then $x, z \in S$, and $S - \{z\}$ is TRDS of $T'' = T' - z$. Since $n(T'') \equiv 0 \mod 4$, $\left\lceil \frac{n(T'+2)}{2} \right\rceil + 1 \leq \gamma_{tr}(T'') \leq |S| - 1 = \left\lceil \frac{n(T)+2}{2} \right\rceil - 1 = \left\lceil \frac{n(T)+1}{2} \right\rceil - 1$, and so $n(T') + 4 \leq n(T'') + 2$, which is a contradiction. Thus, $v \notin S$.

In the case when $v$ lies on the endpath $vx_1x_2z$, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{y_2, z'\}$ forms a TRDS of $T$, so that $\left\lceil \frac{n(T)+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \left\lceil \frac{n(T)+2}{2} \right\rceil + 2 = \left\lceil \frac{n(T)+1}{2} \right\rceil + 2$. Hence, $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$, and so $T \in \mathcal{F}^*$. □

The proof of the following result is similar to that of Lemma 5.

Lemma 11. If $T$ is obtained from $T' \in \mathcal{F}^*$ by Operation O4, then $T \in \mathcal{F}^*$.

Lemma 12. If $T$ is in $\mathcal{F}$, then $T$ is in $\mathcal{F}^*$.

Proof. Assume $T \in \mathcal{F}$. Then $T$ is obtained from $T' \in \mathcal{C}$ by applying one of the Operations O5–O9 exactly once. Then, by Lemmas 8–10, $T \in \mathcal{F}^*$. □
Theorem 13. T is in $\mathcal{G}^*$ if and only if T is in $\mathcal{F}^*$.

Proof. Assume $T \in \mathcal{G}^*$. We show that $T \in \mathcal{F}^*$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T \in \mathcal{A}$, and the result follows from Lemma 12. Assume, then, for all trees $T' \in \mathcal{G}^*$ with $c(T') < k$, where $k \geq 1$ is an integer, that $T'$ is in $\mathcal{F}^*$. Let $T \in \mathcal{G}^*$ be a tree with $c(T) = k$. Then T is obtained from some tree $T'$ by applying Operation O4. But then $T' \in \mathcal{G}^*$ and $c(T') < k$. Applying the inductive hypothesis to $T'$, $T'$ is in $\mathcal{F}^*$. Hence, by Lemma 11, T is in $\mathcal{F}^*$.

To show that $T \in \mathcal{G}^*$ for a nontrivial $T \in \mathcal{F}^*$, we employ induction on $4n$, the order of the tree T. Suppose $n = 1$. Then $T \cong K_{1,3}$ or $T \cong P_4$, and T can be constructed from $P_3 \in \mathcal{G}$ by applying Operation O5.

Let $T \in \mathcal{F}^*$ be a tree of order $4n$, where $n \geq 2$, and suppose $T' \in \mathcal{G}^*$ for all trees $T' \in \mathcal{F}^*$ of order $4n'$ where $n' < n$.

The only trees T with $diam(T) \leq 3$ which are in $\mathcal{F}^*$ are $K_{1,3}$ and $P_4$. As $4n \geq 8$, it follows that $diam(T) \geq 4$. Throughout $S$ will be used to denote a $\gamma_{tr}$-set of T, i.e. $|S| = \left\lceil \frac{n+2}{2} \right\rceil + 1$.

Claim 5. If $u$ and $v$ are vertices of $T$ such that $T' = T - u - v$ is a tree and $S' = S - \{u, v\}$ is a TRDS of $T'$, then $n(T') \equiv 2 \mod 4$ and $T' \in \mathcal{G}$.

Proof. As $\left\lceil \frac{n+2}{2} \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil + 1$, we have $\gamma_{tr}(T') = \left\lceil \frac{n-2+2}{2} \right\rceil = \left\lceil \frac{n(T)+2}{2} \right\rceil$, and so $T' \in \mathcal{G}$. □

Claim 6. Let z be a leaf of T. If $S - \{z\}$ is a TRDS of $T' = T - z$, then $T \in \mathcal{G}^*$.

Proof. Assume $S - \{z\}$ is a TRDS of $T'$. Then $\left\lceil \frac{n-1+2}{2} \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil + 1 - 1 = \left\lceil \frac{n+2}{2} \right\rceil$. Hence, $n - 1 \equiv 3 \mod 4$ and $\gamma_{tr}(T') = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{n(T)+2}{2} \right\rceil$. Thus, $T' \in \mathcal{G}$. The tree T can now be constructed from $T'$ by applying Operation O5, whence $T \in \mathcal{G}^*$. □

Claim 6 implies that if $vzx$ is an endpath of T, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 6 also implies that every remote vertex of T is adjacent to exactly one leaf, since otherwise it is constructible.

Let $T$ be rooted at a leaf $r$ of a longest path.

Let $v$ be any vertex on a longest path at distance $diam(T) - 2$ from $r$. Suppose $v$ lies on the endpath $vyy'z'$. Then, by the remark above, $v \notin S$, which implies $v$ is not adjacent to a leaf. If $v$ also lies on the endpath $vzx$, then $S - \{x, z\}$ is a TRDS of $T - x - z$ and so $T' \in \mathcal{G}$ (cf. Claim 5), whence $T \in \mathcal{G}^*$ (as it can be constructed from $T'$ by applying Operation O6).

Thus, we assume each vertex on a longest path at distance $diam(T) - 2$ or $diam(T) - 1$ from $r$ has degree 2.

Let $v$ be any vertex on a longest path at distance $diam(T) - 3$ from $r$. Let $vy_1y_2y_3'$ be an endpath of $T$. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of $v$ have degree at least 2.

Assume $v$ also lies on the path $vxz$, where $z$ is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, $vzx$ is an endpath. If $v$ is dominated by a vertex other than $x$, then $S - \{x, z\}$ is a TRDS of $T' = T - x - z$ and so $T' \in \mathcal{G}$ (cf. Claim 5), whence $T \in \mathcal{G}^*$ (as it can be constructed from $T'$ by applying Operation O7). Hence, $v$ is dominated only by $x$. Then $S' = S - \{y_2, z\}$ is a TRDS of $T' = T - y_1 - y_2 - z'$ and so $\left\lceil \frac{n-3+2}{2} \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil - 1$.

But then $\gamma_{tr}(T') = \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n(T)+2}{2} \right\rceil$. Thus, $T' \in \mathcal{G}$. The tree T can now be constructed from $T'$ by applying Operation O8.

Assume $v$ lies on the path $vx_1x_2x_3z$. Since $x_1$ ($x_2$, respectively) is on a longest path at distance $diam(T) - 2$ ($diam(T) - 1$, respectively) from $r$, we have $deg(x_1) = 2$ ($deg(x_2) = 2$, respectively). This implies that $vx_1x_2x_3z$ is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a TRDS of $T' = T - x_1 - x_2 - z$. Thus, $\left\lceil \frac{n-3+2}{2} \right\rceil \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil - 1$. But then $\gamma_{tr}(T') = \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n(T)+2}{2} \right\rceil$. Thus, $T' \in \mathcal{G}$ and so $T$ can now be constructed from $T'$ by applying Operation O9.

Thus, we assume each vertex on a longest path at distance $diam(T) - 3$ from $r$ has degree 2.

Let $v$ be any vertex on a longest path at distance $diam(T) - 4$ from $r$. As $P_3 \notin \mathcal{F}^*$, $v \neq r$ and $diam(T) \geq 5$.

Assume $deg_T(v) \geq 3$. Let $y_1y_2y_3z'$ be an endpath of $T$. But then, as $y_2y_3z'$ is an endpath of $T$, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z\}$ is a TRDS of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\left\lceil \frac{n-4+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil - 1$, whence $\gamma_{tr}(T') = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$. We conclude that $T' \in \mathcal{F}^*$, and by the induction
assumption, $T' \in \mathcal{C}^*$. If $\deg_T(v) = 2$ or when $v$ is a remote vertex, then $T$ can be constructed from $T'$ by applying Operation O4, whence $T \in \mathcal{C}^*$.

We therefore assume that $\deg_T(v) \geq 3$ and that $v$ is not adjacent to a leaf.

If $v$ also lies on the path $vzx$, where $z$ is a leaf, then $v \notin S$, which is a contradiction.

We now suppose $v$ lies on the path $vx_1x_2z$, where $z$ is a leaf. Then, since $x_2$ is a remote vertex, we have $\deg(x_2) = 2$. As $x_1x_2z$ is an endpath of $T$, it follows that $x_1 \notin S$. As $x_1$ must be adjacent to another vertex in $V - S$, vertex $x_1$ lies on a path $x_1, u_1, u_2, z''$. But then $x_1$, with $\deg(x_1) \geq 3$, is a vertex at distance $\text{diam}(T) - 3$ on a longest path from $r$, which is a contradiction.

Let $e$ be the edge that joins $v$ with its parent, and let $T(v)$ be the component of $T - e$ that contains $v$. Then $T(v)$ consists of $\ell$ disjoint paths $u_ix_iy_iz_i$ ($i = 1, \ldots, \ell$) with $v$ joined to $u_i$ for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of $T$, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \bigcup_{i=1}^{\ell} \{y_i, z_i\}$ is a TRDS of $T' = T - (T(v) - v)$, and so $\left\lceil \frac{n - 4\ell + 2}{2} \right\rceil + 1 \leq \gamma_{tr}(T') \leq \left\lceil \frac{n + 2}{2} \right\rceil - 2\ell + 1$, whence $\gamma_{tr}(T') = \left\lceil \frac{n(T') + 2}{2} \right\rceil + 1$. Thus, $T' \in \mathcal{F}^*$, and by the induction assumption, $T' \in \mathcal{C}^*$. Note that $v$ is a leaf of $T'$. The tree $T$ can now be constructed from $T'$ by applying Operation O4, whence $T \in \mathcal{C}^*$. □

References