Complete Balanced Howell Rotations for 16k + 12 Partnerships

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A complete balanced Howell rotation for 4n partnerships is an arrangement of 4n elements in a square array (also known as a balanced Room square) of side 4n - 1 such that:

(i) Each of the (4n - 1)^2 cells is either empty or contains an ordered pair of distinct elements.

(ii) Each of the 4n elements appears precisely once in each row and each column.

(iii) Each unordered pair of distinct elements occurs in exactly one cell of the array.

(iv) Each pair of distinct elements appears together in a block exactly 2n - 1 times.

In this paper we show that such a rotation exists for 16k + 12 partnerships if 8k + 5 is a prime power. Our method is to construct skew balanced starters using cyclotomy theory.

1. INTRODUCTION

A complete balanced Howell rotation, abbreviated as CBHR, for n partnerships is an arrangement of n elements in a square array (also known as a balanced Room square) of side s, s = n - 1 for n even and s = n for n odd, satisfying the following four properties.

(i) Each of the s^2 cells is either empty or contains an ordered pair of distinct elements.
(ii) Each of the \( n \) elements appears precisely once in each row and each column. (If \( n \) is odd, then one row and one column is excepted.)

(iii) Each unordered pair of distinct elements occurs in exactly one cell of the array.

To describe condition (iv), we have to define blocks in a CBHR. Each row of a CBHR generates a pair of complementary blocks; one block consisting of those elements which are first elements of the ordered pairs of the row and the other block consisting of the second elements. Therefore the \( s \) rows together generate a total of \( 2s \) blocks.

(iv) Each pair of distinct elements appears together in a block exactly \( \lfloor n/2 \rfloor - 1 \) times where \( \lfloor x \rfloor \) denotes the integral part of \( x \).

When a CBHR is used for a bridge tournament, each element represents a partnership, each row a board and each column a round. If cell \((i,j)\) contains the ordered pair \((x,y)\), then partnership \( x \) opposes partnership \( y \) on board \( i \) in round \( j \), with \( x \) playing the NS direction of the board and \( y \) the EW direction. Two partnerships are said to compete with each other on a given board if they play the same direction. Property (i) assures that each NS partnership has an EW opponent and vice versa. Property (ii) assures that each partnership plays each board once and plays at each round once. Property (iii) assures that each partnership opposes every other partnership once. Property (iv) assures that each partnership competes with every other partnership an equal number of times.

CBHR\((4n)\)'s have been constructed for the following cases:

(i) \( 4n - 1 \) is a prime power [1].

(ii) \( 2n - 1 \) is a prime power and \( n \) is even [5] (a generalization given in [3] contains an incomplete proof, as pointed out by Schellenberg [6]).

(iii) \( 2n - 1 = 2^kt + 1 \) is a prime power where \( t > 1 \) is odd and

\[
a = \frac{1 + x^{d+1}}{1 + x^{d-1}}, \quad b = \frac{1 - x^{d+1}}{1 - x^{d-1}}
\]

are quadratic residues in \( GF(2n - 1) \), with \( d = 2^{k-1} \) and \( x \) a generator of \( GF(2n - 1) \) [5].

In case (iii), it is not known for what values of \( n \) the conditions on \( a \) and \( b \) are satisfied. In this paper we show explicitly that when \( 2n - 1 = 8k + 5 \) (but not 5) is a prime power, then a CBHR\((4n)\) exists.

2. Skew Balanced Starter

Let \( S_1, S_2, \ldots, S_m \) be a family of subsets of the elements in \( GF(2n - 1) \) where \( 2n - 1 \) is a prime power. Let \( D_i \) denote the set of symmetric
differences generated by $S_i$, i.e., $D_i = \{x - x' \text{ for all } x \text{ and } x' \text{ in } S_i, x \neq x'\}$. Then $S_1, S_2, \ldots, S_m$ are called supplementary difference sets if $D_1, D_2, \ldots, D_m$ together contain each nonzero element of $GF(2n - 1)$ an equal number of times.

The set of $n - 1$ pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$ is called a strong balanced starter if:

(i) The $n - 1$ pairs contain each nonzero element of $GF(2n - 1)$ exactly one.

(ii) The $n - 1$ pairs are supplementary difference sets.

(iii) The two sets $\{x_1, x_2, \ldots, x_{n-1}\}$ and $\{y_1, y_2, \ldots, y_{n-1}\}$ are supplementary difference sets.

(iv) $x_1 + y_1, x_2 + y_2, \ldots, x_{n-1} + y_{n-1}$ are all distinct mod $(2n - 1)$.

It is well known [4] that a CBHR$(2n - 1)$ can be constructed from a strong balanced starter on $GF(2n - 1)$ by assigning the pair $(x_i, y_i + j)$ to cell $(j, x_i + y_i + j)$ for $j = 0, 1, \ldots, 2n - 2$. For $n$ even, then by further adding the pair $(\infty, j)$ to cell $(j, j)$ for $j = 0, 1, \ldots, 2n - 2$, we obtain a CBHR$(2n)$ [5]. We will refer to this method as the cyclic method.

The transpose of a strong balanced starter $S$ is defined as the first column of the CBHR obtained from $S$ by the cyclic method. Therefore the transpose of $S$ consists of the pairs $(-x_1, -x_1), (-x_2, -x_2), \ldots, (-x_{n-1}, -x_{n-1})$ which clearly satisfy all conditions of a strong balanced starter. A strong balanced starter is symmetric if $\{x_1, x_2, \ldots, x_{n-1}\} = \{-x_1, -x_2, \ldots, -x_{n-1}\}$.

A strong balanced starter is called a skew balanced starter if $\pm(x_i, y_i), \pm(x_2 + y_2), \ldots, \pm(x_{n-1} + y_{n-1})$ are all distinct mod$(2n - 1)$. This skew property implies that cell $(1, j)$ contains a pair if and only if cell $(j, 1)$ is empty. Furthermore, due to the cyclic method, it also implies that each cell $(i, j)$ contains a pair if and only if the cell $(j, i)$ is empty.

Let $R$ and $Q$ be two CBHR's of order $n$. Then $R$ and $Q$ are called a latin pair if on superimposing $R$ and $Q$ and placing the pair $(i, i)$ in each cell $(i, i)$, we obtain a pair of superimposed orthogonal latin squares $R \odot Q$ of side $n - 1$. If $n$ is even then we have to remove all pairs containing the element $\infty$ in both $R$ and $Q$. A common transversal of $R \odot Q$ is a set of $n - 1$ cells, one from each row and one from each column, whose $n - 1$ ordered pairs have $n - 1$ distinct first elements and $n - 1$ distinct second elements. Schellenberg [5] gave the following theorem:

**Theorem 1.** Suppose the CBHR$(2n)$'s, $R$ and $Q$, have the following two properties:

(a) $R$ and $Q$ are a latin pair and $R \odot Q$ has a pair of disjoint common transversals which do not intersect the main diagonal.
(b) If \( I \cup B_i \) and \( I \cup \overline{B}_i \) are the two blocks obtained from row \( i \) of \( R \), then \( I \cup B_i \) and \( I \cup B_i \) are the two blocks obtained from row \( i \) of \( Q \). Then there exists a CBHR(4n).

The extension to odd orders is straightforward.

**Theorem 2.** The existence of a symmetric skew balanced starter for \( GF(2n - 1) \) implies the existence of a CBHR(4n).

**Proof.** We need only show that there exist two CBHR(2n - 1)'s, \( R \) and \( Q \), which satisfy the conditions of Theorem 1. Let \( S \) denote the given skew balanced starter and let \( R \) and \( Q \) be obtained from \( S \) and its transpose, respectively. The skewness of \( S \) assures that \( R \odot Q \) has a pair in each cell. The fact that \( \{y_1, y_2, ..., y_{2n-1}\} = \{y_1, y_2, ..., y_{2n-1}\} \), and hence \( \{y_1, y_2, ..., y_{2n-1}\} = \{y_1, y_2, ..., y_{2n-1}\} \), assures that \( R \) and \( Q \) are a latin pair. The two common transversals exist since for each \( i \), the \( 2n - 1 \) cells \( \{j, x_i + y_i + j\} \) for \( j = 0, 1, ..., 2n - 2 \) yield a common transversal. The balanced property assures property (b) of Theorem 1.

3. The Main Results

Let \( x \) be a generator of \( GF(2n - 1) \). For every \( y \) in \( GF(2n - 1) \), define \( T(y) = z \) if \( y = x^z \). Note that \( T(uv) = T(u) + T(v) \).

**Theorem 3.** For \( 2n - 1 = 8k + 5 > 5 \) a prime power, there always exists an element \( y \) such that

(i) \( T(y) \equiv 1 \pmod{4} \),
(ii) \( T(y - 1) \equiv 1 + T(x - 1) \pmod{2} \),
(iii) \( T(y + 1) \equiv 1 + T(x + 1) \pmod{2} \).

**Proof.** Let \( A_{ij} \) denote the number of solutions of the equation

\[ 1 + x^i = x', \]

when \( x \) is a generator in \( GF(2n - 1) \) with \( s \equiv i \pmod{4} \), \( t \equiv j \pmod{4} \). Then \( A_{ij} \) is given by the array [7]
together with the relations

\[
\begin{align*}
16A &= 2n - 8 + 2s, \\
16B &= 2n + 2s - 8t, \\
16C &= 2n - 6s, \\
16D &= 2n + 2s + 8t, \\
16E &= 2n - 4 - 2s,
\end{align*}
\]

where \(2n - 1 = s^2 + 4t^2\), \(s \equiv 1 \pmod{4}\) and the sign of \(t\) depends on the choice of \(x\).

Define

\[
S_{ij} = \{ y: T(y) \equiv 1 \pmod{4}, T(y - 1) \equiv i \pmod{2}, \\
T(y + 1) \equiv j \pmod{2} \}
\]

for \(i, j \in \{0, 1\}\). Then Theorem 3 is true if \(S_{ij}\) is nonempty for every \(i, j \in \{0, 1\}\). Because then, regardless of which \(S_{ij} x\) belongs to, there always exists a \(y \in S_{i+1,j+1}\) and this \(y\) satisfies the three conditions of Theorem 3.

Note that

\[
\begin{align*}
|S_{00}| + |S_{01}| &= A_{01} + A_{11} = B + E, \\
|S_{01}| + |S_{11}| &= A_{11} + A_{12} = E + B, \\
|S_{00}| + |S_{10}| &= A_{10} + A_{12} = E + D.
\end{align*}
\]

Therefore

\[
|S_{00}| = |S_{11}|.
\]

Furthermore, let \(Z\) denote the set of \(z\) in \(GF(2n - 1)\) such that 
\(Z = \{ z: T(z) \equiv 2 \pmod{4}, \quad T(z - 1) \equiv 0 \pmod{2} \}\). Then for each 
\(y \in S_{00} \cup S_{11}, \quad y^2 \in Z\) since \(T(y^2) \equiv 2 \pmod{4}\) and \(T(y^2 - 1) = T(y - 1) T(y + 1) \equiv 0 \pmod{2}\). On the other hand, for every \(z \in Z\), there exist \(y\) and \(\bar{y}\) such that \(y^2 = \bar{y}^2 = z\), while \(y \equiv 1 \pmod{4}\) and \(\bar{y} \equiv 3 \pmod{4}\).
Clearly \( y \in S_{00} \cup S_{11} \), for otherwise \( z = y^2 \) cannot satisfy \( T(z - 1) \equiv 0 \) (mod 2). We have proved that

\[
|S_{00}| + |S_{11}| = |S_{00} \cup S_{11}| = |Z| = A_{02} + A_{22} = C + A = 2E.
\]

Therefore \( |S_{00}| = |S_{11}| = E, |S_{01}| = B, |S_{10}| = D \). But for \( 2n - 1 > 16 \), it is straightforward to verify that

\[
B = \frac{1}{16} [(s + 1)^2 + 4(t - 1)^2 - 4] > 0,
\]
\[
D = \frac{1}{16} [(s + 1)^2 + 4(t + 1)^2 - 4] > 0,
\]
\[
E = \frac{1}{16} [(s - 1)^2 + 4t^2 - 4] > 0.
\]

Hence Theorem 3 is true for \( 2n - 1 > 16 \). Finally, for \( 2n - 1 = 13 \), it is easily seen that \( x = 2 \) is a generator of \( GF(13) \) and \( y = 2^5 \) is an element satisfying all three conditions of Theorem 3. The proof is complete.

Bose [2] proved the following result:

**Lemma.** For \( 4k + 1 \) a prime power, let \( Q \) denote the set of quadratic residues in \( GF(4k + 1) \) and \( \bar{Q} \) the set of nonresidues. Then \( Q \) and \( \bar{Q} \) are supplementary difference sets.

We now show

**Theorem 4.** For \( 2n - 1 = 8k + 5 > 5 \) a prime power, there always exists a symmetric skew balanced starter.

**Proof.** Consider the set of \( 4k + 1 \) pairs:

\[
(x^{4i+1}, x^{4i+2}), \quad i = 0, 1, ..., 2k,
\]
\[
(x^{4i+3}, x^{4i+3}y), \quad i = 0, 1, ..., 2k, \quad (y \text{ is from Theorem 3}).
\]

Though this set has been proved to be a strong balanced starter in [4], for the sake of completeness, we prove all its properties.

(i) The \( 8k + 4 \) elements in these pairs are clearly distinct.

(ii) The \( 8k + 4 \) symmetric differences generated by the \( 4k + 2 \) pairs are

\[
\pm x^{4i+1}(x - 1), \quad i = 0, 1, ..., 2k,
\]
\[
\pm x^{4i+3}(y - 1), \quad i = 0, 1, ..., 2k.
\]
Since $T(x - 1) \equiv 1 + T(y - 1) \pmod{2}$, the differences are all distinct.

(iii) The balanced property follows from the Bose lemma.

(iv) The $8k + 4$ positive and negative sums of the $4k + 2$ pairs are

$$
\pm x^{4i+1}(x + 1), \quad i = 0, 1, \ldots, 2k,
\pm x^{4i+3}(y + 1), \quad i = 0, \ldots, 2k.
$$

Since $T(x + 1) \equiv 1 + T(y + 1) \pmod{2}$, the sums are all distinct.

(v) Since $2n - 1 \equiv 1 \pmod{4}$,

$$
\{x^{4i+1} : i = 0, 1, \ldots, 2k\} \cup \{x^{4i+3} : i = 0, 1, \ldots, 2k\}
= \{-x^{4i+1} : i = 0, 1, \ldots, 2k\}
\cup \{-x^{4i+3} : i = 0, 1, \ldots, 2k\}.
$$

From Theorems 2 and 4, we immediately obtain

**Corollary.** A CBHR$(4n)$ exists when $2n - 1 \equiv 8k + 5 > 5$ is a prime power.

The first 10 CBHR$(4n)$'s obtained by the Corollary are

\[
\begin{array}{cccccccccc}
  k & 1 & 3 & 4 & 6 & 7 & 12 & 13 & 16 & 18 & 19 \\
  4n & 28 & 60 & 76^* & 108 & 124^* & 204 & 220^* & 268^* & 300 & 316^*
\end{array}
\]

where * denotes the fact that such a CBHR had not previously been known to exist.

**References**