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Maximum Order of Periodic Outer Automorphisms of a Free Group

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Let F_n be a free group with rank n, and denote by Out F_n its outer automorphism group. For arbitrary n, consider the orders of periodic elements in Out F_n or, equivalently, the orders of finite cyclic subgroups of Out F_n. By considering group actions on finite connected graphs, we obtained the number-theoretical characterization of these orders. Comparing the results with those for cyclic subgroups of finite symmetric groups asymptotic estimation for the maximum order c_n is derived. © 2000 Academic Press

Key Words: free group; outer automorphism.

1. INTRODUCTION

Denote by F_n a free group with rank n. Its inner automorphism group is defined by

$$\operatorname{Inn} F_n = \{f \in \operatorname{Aut} F_n;\ \exists x \in F_n,\ \text{s.t. } \forall y \in F_n,\ f(y) \equiv x^{-1}yx\}.$$

And its outer automorphism group is the group of automorphisms modulo the inner automorphism group. Namely,

$$\operatorname{Out} F_n = \operatorname{Aut} F_n / \operatorname{Inn} F_n. \tag{1.1}$$

The study of finite subgroups of Out F_n has a close relationship with the study of group actions on finite connected graphs.

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DEFINITION. An abstract *graph* consists of vertices and edges. For a finite connected graph Γ, as a topological space, its fundamental group $\pi_1(\Gamma)$ must be a free group with rank

$$n = 1 - \chi(\Gamma),$$

in which $\chi(\Gamma)$ is the *Euler number* of Γ. We call n the graph's *rank*. The *valence* of a vertex p is the number of edges connecting to p, in which the edges with both ends coincide with p will be counted twice. If there is an edge in Γ connecting two vertices x and y, we say that x and y are *adjacent*.

An *automorphism* of a graph Γ is a bijection of its vertices and edges that preserves the graph structure (here the reversal of edges is also taken into consideration). Denote by $\operatorname{Aut}\Gamma$ the group of all such automorphisms on Γ, and call it the *total automorphism group*. For any element $g \in \operatorname{Aut}\Gamma$, it induces an algebraic outer automorphism g_* on $\pi_1(\Gamma)$, the fundamental group of Γ. Noticing that $\pi_1(\Gamma) \cong F_n$, in which n is the rank of Γ, we obtain a correspondence

$$*\colon \operatorname{Aut}\Gamma \to \operatorname{Out} F_n, \qquad g \mapsto g_*.$$

This correspondence sends every subgroup $G < \operatorname{Aut}\Gamma$ homomorphically to a subgroup $G_* < \operatorname{Out} F_n$. In this case, we say that G *realizes* G_* *on* Γ. Furthermore, if the correspondence is an isomorphism, then it is called an *effective realization*.

Culler [2] and Zimmermann [7] observed independently that every finite subgroup of $\operatorname{Out} F_n$ can be realized by a group of automorphisms of certain rank n graph. Moreover, we have the following lemma, of which a proof can be found in [6].

LEMMA 1.1. *For any finite subgroup $G < \operatorname{Out} F_n$, there exists a finite connected graph Γ with rank n and no vertex of valence 1 or 2 and a subgroup $H < \operatorname{Aut}\Gamma$ realizing G effectively.*

Therefore, for analyzing finite subgroups in $\operatorname{Out} F_n$, we only need to study the automorphism groups of various graphs. Based on the above lemma, Wang and Zimmermann [6] proved that, for *finite subgroups* of $\operatorname{Out} F_n$, their maximum order is 12 when $n = 2$ and $2^n n!$ when $n \neq 2$. In [1], it is shown that the maximum order of *finite abelian subgroups* of $\operatorname{Out} F_n$ is 6 when $n = 2$ and 2^n when $n \neq 2$.

Now consider the orders of *finite cyclic subgroups* of $\operatorname{Out} F_n$ or, equivalently, the orders of *periodic outer automorphisms* of $\operatorname{Out} F_n$. In this article, also by using Lemma 1.1, number-theoretical properties are found which completely determine the set of these orders (Theorem 3.1).

This order set has a close relationship with the set of orders of permutations on an n-tuple. For any n-tuple X, the *symmetric group* S_X is the group of all permutations on it. Particularly, for $X = \{1, 2, \ldots, n\}$, S_X is denoted by S_n. $\forall \lambda \in S_n$, let $|\lambda|$ denote its order. The set of orders of elements in S_n has been discussed by many authors. Comparing these results (Lemma 2.3), the following asymptotic estimation is derived:

THEOREM 1.1 (Main Theorem). *Let c_n be the maximum order of finite cyclic subgroups of* $\operatorname{Out} F_n$ *or, equivalently, the maximum order of periodic outer automorphisms of F_n. Then*

$$c_n = \exp\bigl((1 + \theta_n)\sqrt{n \log n}\bigr), \tag{1.2}$$

in which $\{\theta_n\}$ is a number sequence that converges to 0 *when $n \to +\infty$.*

In the following discussion, we use $o(1)$ to denote any arbitrary number sequence that converges to 0 when $n \to +\infty$.

2. THE NUMBER SET A_n AND B_n

Notation. If $x_1, \ldots, x_k$ are natural numbers, denote by $\operatorname{lcm}(x_1, \ldots, x_k)$ their *lowest common multiplier*. For arbitrary $n \in \mathbb{N}$, define

$$A_n = \bigl\{\operatorname{lcm}(x_1, \ldots, x_k);\ x_1, \ldots, x_k \in \mathbb{N},\ x_1 + \cdots + x_k \le n\bigr\}, \tag{2.1}$$

$$a_n = \max\bigl\{x;\ x \in A_n\bigr\}. \tag{2.2}$$

Apparently, if $n \le m$, then $A_n \subseteq A_m$. If $y \in A_n$, $x \mid y$, then $x \in A_n$. Here “|” denotes divisibility.

LEMMA 2.1. *A_n can be expressed by prime powers as follows*:

$$A_n = \bigl\{p_1^{i_1} \cdots p_k^{i_k};\ p_1, \ldots, p_k \text{ are different prime numbers},\ i_1, \ldots, i_k \in \mathbb{N},\ p_1^{i_1} + \cdots + p_k^{i_k} \le n\bigr\}. \tag{2.3}$$

Proof. Apparently the right-hand side is contained in A_n. Now, for any natural numbers $y_1, \ldots, y_l \in \mathbb{N} \setminus \{1\}$ that are coprime with each other, $y_1 \cdots y_l = \operatorname{lcm}(y_1, \ldots, y_l)$, while $y_1 + \cdots + y_l \le y_1 \cdots y_l$. This implies that, given any $x \in A_n$, one can substitute all the x_j in the definition of A_n by their prime power factors. Removing in different x_j redundant powers of each prime number, the expression in Eq. (2.3) follows. ∎

LEMMA 2.2. *A_n is exactly the set of orders of elements in the symmetric group S_n.*

Proof. In fact, given any permutation $\lambda \in S_n$, it can be decomposed into the product of disjoint cycles. (Here, for distinct numbers $i_1, \ldots, i_l \in \{1, \ldots, n\}$, a *cycle* $(i_1, \ldots, i_l)$ is an element $\mu \in S_n$ defined as follows. For j such that $1 \le j < l$, $\mu(i_j) = i_{j+1}$; $\mu(i_l) = i_1$; and μ keeps all the other numbers between 1 and n invariant.) Let the orders of these cycles be $x_1, \ldots, x_k$. Then $x_1 + \cdots + x_k \le n$, while

$$|\lambda| = \operatorname{lcm}(x_1, \ldots, x_k).$$

On the other hand, if $x = \operatorname{lcm}(x_1, \ldots, x_k)$, $x_1 + \cdots + x_k \le n$, put

$$y_0 = 0, \qquad y_j = x_1 + \cdots + x_j, \qquad 1 \le j \le k.$$

Let λ be the product of disjoint cycles $(y_{j-1} + 1, y_{j-1} + 2, \ldots, y_j)$, $j = 1, \ldots, k$. Then $\lambda \in S_n$ while $|\lambda| = x$. ∎

The study of the maximum number $a_n \in A_n$, or the maximum order of elements in S_n, began early in 1909, when Landau first established an asymptotic estimation for it (see [5] for a comprehensive review).

LEMMA 2.3. $a_n = \exp((1 + o(1))\sqrt{n \log n})$ $(n \to +\infty)$.

Notation. Similar to the above definition, define

$$B_n = \{p_1^{i_1} \cdots p_k^{i_k};\ p_1, \ldots, p_k \text{ are different prime numbers}, \\ i_1, \ldots, i_k \in \mathbb{N}, (p_1^{i_1} - p_1^{i_1 - 1}) + \cdots + (p_k^{i_k} - p_k^{i_k - 1}) \le n\}, \tag{2.4}$$

$$B_0 = \{1\}, \qquad b_n = \max\{x;\ x \in B_n\}. \tag{2.5}$$

LEMMA 2.4. *B_n can be expressed similarly to A_n as follows*:

$$B_n = \{\operatorname{lcm}(x_1, \ldots, x_k); x_1, \ldots, x_k \in \mathbb{N}, \exists m_j \mid x_j, 0 < m_j < x_j, \\ j = 1, \ldots, k,\ s.t.\ (x_1 - m_1) + \cdots + (x_k - m_k) \le n\}. \tag{2.6}$$

Proof. The right-hand side apparently contains B_n as a subset. On the other hand, suppose that $x = \operatorname{lcm}(x_1, \ldots, x_k)$ is an element in the right-hand set of Eq. (2.6). Suppose that one of the x_j, say x_1, is not a prime power. Then there is a prime number $p \mid x_j$ such that

$$x_1 = p^i q, \qquad m_1 = p^j r, \qquad i \ge j,\ p \mid q,\ p \mid r, \text{ and } q > r.$$

Apparently $x = \operatorname{lcm}(p^i, q, x_2, \ldots, x_k)$. If $i > j$, then

$$(p^i - p^j) + (q - r) \le m_1(p^{i-j} - 1) + m_1(q/r - 1) \\ \le m_1(p^{i-j} q/r - 1) = x_1 - m_1.$$

If $i = j$, then

$$(p^i - 1) + (q - r) \leq p^i(q - r) = x_1 - m_1.$$

Therefore, x_1 can be substituted by p^i and q. By induction, we may substitute every x_j by a group of prime powers. Namely,

$$x = \mathrm{lcm}(x'_1, \ldots, x'_l)$$

satisfying the requirements of the right-hand side of Eq. (2.6), and $x'_1, \ldots, x'_l$ are all prime powers. If $x'_1 = p^i$, then $x'_1 - p^{i-1} \leq x'_1 - m'_1$, so m'_1 can be substituted by p^{i-1}. This can similarly be done for other m_j. Finally, if $x_1 = p^i$, $x_2 = p^j$, $i > j$, then $x = \mathrm{lcm}(x_1, x_3, \ldots, x_k)$, so the redundant prime powers in the expression of x can all be removed. Hence, $x \in B_n$. ∎

LEMMA 2.5. *The set B_n satisfies the following properties*:

1. *If $m < n$, then $B_m \subseteq B_n$.*
2. *If $x \mid y$, $y \in B_n$, then $x \in B_n$.*
3. *If $x \in A_n$, then $2x \in B_n$.*
4. *If $m, n > 0$, $x \in B_m$, $y \in B_n$, then* $\mathrm{lcm}(x, y) \in B_{m+n}$.

Proof. (1) and (2) are direct corollaries of the definition.

(3) $\forall x \in A_n$, following Eq. (2.3), there are distinct primes $p_1, \ldots, p_k$ and indices $i_1, \ldots, i_k \in \mathbb{N}$, such that $x = p_1^{i_1} \cdots p_k^{i_k}$ while $p_1^{i_1} + \cdots + p_k^{i_k} \leq n$. If $p_1, \ldots, p_k \neq 2$, then $2x = 2 \cdot p_1^{i_1} \cdots p_k^{i_k}$, while

$$\begin{aligned}(2-1) &+ (p_1^{i_1} - p_1^{i_1-1}) + \cdots + (p_k^{i_k} - p_k^{i_k-1}) \\ &\leq 1 + (p_1^{i_1} - 1) + p_2^{i_2} + \cdots + p_k^{i_k} \leq n.\end{aligned}$$

If, say, $p_1 = 2$, then $2x = p_1^{i_1+1} p_2^{i_2} \cdots p_k^{i_k}$, while

$$\begin{aligned}(p_1^{i_1+1} - p_1^{i_1}) &+ (p_2^{i_2} - p_2^{i_2-1}) + \cdots + (p_k^{i_k} - p_k^{i_k-1}) \\ &\leq p_1^{i_1} + p_2^{i_2} + \cdots + p_k^{i_k} \leq n.\end{aligned}$$

In both cases, by definition, $2x \in B_n$.

(4) is a corollary of Eq. (2.6).

To sum up, these properties are all satisfied. ∎

LEMMA 2.6. *The growth rate of the maximum number in B_n satisfies*

$$b_n = \exp\big((1 + o(1))\sqrt{n \log n}\big), \qquad n \to +\infty. \tag{2.7}$$

Proof. By definition, there are distinct prime numbers $p_1 < \cdots < p_k$ and indices $i_1, \ldots, i_k \in \mathbb{N}$, such that $b_n = p_1^{i_1} \cdots p_k^{i_k}$, while $(p_1^{i_1} - p_1^{i_1-1}) + \cdots + (p_k^{i_k} - p_k^{i_k-1}) \le n$.

Suppose without loss of generality that $\exists l \in \mathbb{N}$ s.t.

$$\text{when } j < l, \qquad p_j < 1 + \log n;$$
$$\text{when } j \ge l, \qquad p_j \ge 1 + \log n.$$

By the well-known asymptotic law of the distribution of prime numbers, or the *prime number theorem* (see, for example, [3, Chap. 2]), the function $\pi(\nu)$ of primes less than ν satisfies

$$\pi(\nu) = (1 + o(1))\nu/\log \nu, \qquad \nu \to +\infty. \tag{2.8}$$

So $l - 1 \le \pi(1 + \log n) = (1 + o(1)) \log n/\log\log n$. Because $p_j^{i_j} \le 2(p_j^{i_j} - p_j^{i_j-1}) \le 2n$, one sees that

$$p_1^{i_1} \cdots p_{l-1}^{i_{l-1}} \le (2n)^{l-1} \le \exp\big(o(1)\sqrt{n \log n}\big), \qquad n \to +\infty. \tag{2.9}$$

When $j \ge l$, $p_j^{i_j-1} \le (p_j^{i_j} - p_j^{i_j-1})/\log n$. Thus $p_l^{i_l-1} + \cdots + p_k^{i_k-1} \le ((p_l^{i_l} - p_l^{i_l-1}) + \cdots + (p_k^{i_k} - p_k^{i_k-1}))/\log n \le n/\log n$, which implies

$$p_l^{i_l} + \cdots + p_k^{i_k} \le n + n/\log n = n(1 + o(1)).$$

Namely, $p_l^{i_l} \cdots p_k^{i_k} \in A_{n(1+\tau_n)}$, in which $\{\tau_n\}$ is a sequence of numbers converging to 0 when $n \to +\infty$. By Lemma 2.3,

$$p_l^{i_l} \cdots p_k^{i_k} \le a_{n(1+\tau_n)} = \exp\big((1 + o(1))\sqrt{n \log n}\big), \qquad n \to +\infty. \tag{2.10}$$

Hence, we obtained $b_n = p_1^{i_1} \cdots p_k^{i_k} \le \exp((1 + o(1))\sqrt{n \log n})$.

By Lemma 2.5, $b_n \ge a_n$ which equals $\exp((1 + o(1))\sqrt{n \log n})$. Comparing this with the above inequality, the equality on the growth rate of b_n follows. ∎

3. THE ORDER SET OF GRAPH AUTOMORPHISMS

Before proving the main theorem, first define some special terms that will be used later.

DEFINITION. A *loop* is an edge with both ends coinciding with each other. If, for two vertices in a graph, there is more than one edge connecting them, then the aggregate of these edges is called *a set of multiple edges*. A *simple* graph is a connected graph without loops, multiple edges, or vertices of valences 1 or 2.

By *a chain of valence 2 vertices* in Γ we mean a set of valence 2 vertices x_1, $x_2, \ldots, x_k \in \Gamma$ such that, for each $i < k$, there is an edge in Γ connecting x_i and x_{i+1}. If $k > 1$ and there is an edge in Γ connecting x_1 and x_k, namely, closing this chain, then it is called a *closed chain*. Clearly, Γ is exactly the closed chain if it is connected and contains a closed chain.

Notation. If a graph Γ has valence 1 vertices, then one can remove these vertices and the edges connecting to them. This process can be repeated inductively to derive a graph Γ' with no valence 1 vertex. For notational convenience, we refer to the removed parts as "*leaves,*" and call Γ' a graph obtained from Γ by "*cutting leaves*."

If Γ is not a loop or closed chain and it has no valence 1 vertex, then, by merging every pair of edges joining at a valence 2 vertex, one can construct a new graph Γ' with the same rank as that of Γ. Moreover, $\operatorname{Aut}\Gamma$ is a subgroup of $\operatorname{Aut}\Gamma'$. If Γ' consists of only two vertices x, y and $n+1$ multiple edges connecting them, then call Γ "*a beam of edges*."

Given any $x, m \in \mathbb{N}$ such that $m \mid x$, the following "*lotus graph*" $G_{x,m}$ is a finite connected graph with no valence 1 vertex. It has a center point a and m other points $b_1, \ldots, b_m$, and there are x/m multiple edges $e_1^j, \ldots, e_{x/m}^j$ connecting a with each b_j (see Fig. 1). Clearly, $\operatorname{rank} G_{x,m} = x - m$. There is also a canonical automorphism $g_{x,m} \in G_{x,m}$ with order x defined as follows: $g_{x,m}(a) = a$. $g_{x,m}(b_j) = b_{j+1}$ if $j \neq m$, while $g_{x,m}(b_m) = b_1$. If $j \neq m$, $g_{x,m}(e_k^j) = e_k^{j+1}$. For $k = 1, \ldots, x/m - 1$, $g_{x,m}(e_k^m) = e_{k+1}^1$ while $g_{x,m}(e_{x/m}^m) = e_1^1$.

By Lemma 1.1, Theorem 1.1 can be induced by the following theorem that characterizes the order set of periodic outer automorphisms of F_n.

THEOREM 3.1. *Fix an* $n > 1$. *Assume arbitrarily a finite connected rank* n *graph* Γ *with no valence* 1 *vertex or closed chain and an automorphism* $g \in \operatorname{Aut}\Gamma$. *If* Γ *is a beam of edges*, *then* $\operatorname{order}(g) = \operatorname{lcm}(a, y)$ *in which*

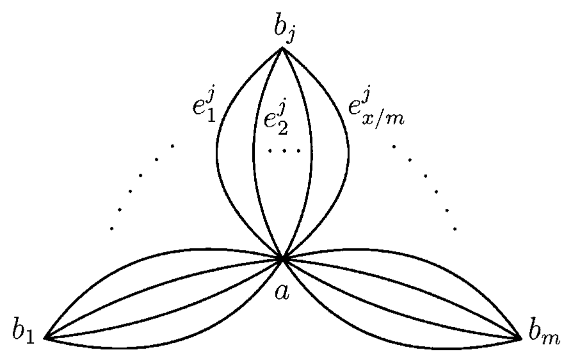

FIG. 1. The "lotus" graph $G_{x,m}$.

$a \in \{1,2\}$ and $y \in A_{n+1}$. Otherwise $\operatorname{order}(g) \in B_n$. *On the other hand, if $x \in B_n$ or $x = \operatorname{lcm}(a, y)$ in which $a \in \{1,2\}$ and $y \in A_{n+1}$, then there is a finite connected rank n graph Γ with no valence 1 vertex or closed chain, s.t. $\exists g \in \operatorname{Aut}\Gamma$ with* $\operatorname{order}(g) = x$.

COROLLARY. *For the automorphisms of finite connected rank n graphs with no valence 1 vertex or closed chain, the set of their orders equals $B_n \cup \{\operatorname{lcm}(2, n+1)\}$. Furthermore,* $\operatorname{lcm}(2, n+1) \notin B_n$ *if and only if $n+1$ is a prime number.*

Proof. By the above theorem, this order set equals

$$\widetilde{B}_n = B_n \cup \{x = \operatorname{lcm}(a, y);\ a = 1 \text{ or } 2,\ y \in A_{n+1}\}. \tag{3.1}$$

Apparently, $A_{n+1} \subseteq B_n$. Given any $y \in A_{n+1} \setminus \{n+1\}$. If y is even, $\operatorname{lcm}(2, y) = y$. So we suppose without loss of generality that y is odd. Then, by Lemma 2.2, there is a permutation $\lambda \in S_{n+1}$, such that $y = |\lambda|$. If λ is not a cycle, then there is an $m \in \mathbb{N}$, $0 < m < n+1$, such that λ can be decomposed into two disjoint permutations λ_1, λ_2 of m- and $(n+1-m)$-tuples, respectively. Thus $2y_1 \in B_m$, $y_2 \in B_{n-m}$, while $2y = \operatorname{lcm}(2y_1, y_2)$. By Lemma 2.5, $2y \in B_n$.

Now suppose that λ is a cycle and $y = |\lambda|$ is not a prime number. Of course, $\operatorname{lcm}(2, y)$ is in $A_{n+1} \subseteq B_n$ if y is even. If y is odd, suppose that $y = p^i q$, in which p is its smallest prime factor, $p \nmid q$. If $q > 1$, then $\operatorname{lcm}(2, y) = \operatorname{lcm}(2p^i, 2q)$, while $(2p^i - p^i) + (2q - q) = p^i + q \le p^i q - 1 = n$. By Lemma 2.4, $\operatorname{lcm}(2, y) \in B_n$. If $q = 1, i > 1$, then $\operatorname{lcm}(2, y) = \operatorname{lcm}(2, p^i)$, while $(2-1) + (p^i - p) \le p^i - 2 < n$, so again $\operatorname{lcm}(2, y) \in B_n$. Therefore, the previous order set is exactly

$$B_n \cup \{\operatorname{lcm}(2, y);\ y \le n+1 \text{ is a prime number}\}.$$

Clearly, if y is prime, then $\operatorname{lcm}(2, y) \notin B_n$ if and only if $y = n+1$ and $n+1$ is an odd prime number. ∎

If $n = 1$, the only rank n graph with no valence 1 vertex and closed chain is a single loop. Thus, $c_1 = 2$. For $n = 2$, $2(2+1) = 6 > b_2 = 4$, so $c_2 = 6$. If $n > 2$, then $\operatorname{lcm}(2, n+1) \le 4(n-1)$. But $4 \in B_2$, $(n-1) \in B_{n-2}$, so $4(n-1) \in B_n$ and $\operatorname{lcm}(2, n+1) \le b_n$. Therefore, we have the following corollary.

COROLLARY. *Consider finite connected rank n graphs with no vertex of valence 1 or closed chains. Let c_n be the maximum order of automorphisms of such graphs. Then $c_1 = 2$, $c_2 = 6$, and, for $n > 2$, $c_n = b_n$.*

Proof of Theorem 1.1. By Lemma 1.1, we see that every finite subgroup $G < \operatorname{Out} F_n$ can be realized effectively on some finite connected graph Γ, in which the graph has rank n and no vertex of valence 1 or 2. Particularly, this implies that periodic outer automorphisms of F_n have the same order set as that of

$$\bigcup\bigl\{\operatorname{Aut}\Gamma;\ \operatorname{rank}(\Gamma)=n, \Gamma \text{ has no valence 1 vertex or closed chain}\bigr\}.$$

Therefore, by the above corollary, their maximum order is c_n and, for $n > 2$, $c_n = b_n$. By Lemma 2.6,

$$c_n = \exp\bigl((1+o(1))\sqrt{n\log n}\bigr), \qquad n \to +\infty. \qquad \blacksquare$$

Now we will prove Theorem 3.1. First, consider the realization problem. Given $n > 1$. If $x \in B_n$, then there are natural numbers $k, x_1, \dots, x_k, m_1, \dots, m_k \in \mathbb{N}$, such that $m_j < x_j$, $m_j \mid x_j$, $j = 1, \dots, k$, $(x_1 - m_1) + \cdots + (x_k - m_k) \le n$, while

$$x = \operatorname{lcm}(x_1, \dots, x_k).$$

Let Γ be the graph obtained by pinching lotus graphs $G_{x_1, m_1}, \dots, G_{x_k, m_k}$ at a common center point, and let g be the automorphism which restricts to g_{x_j, m_j} on each G_{x_j, m_j}. Then the rank of Γ is $(x_1 - m_1) + \cdots + (x_k - m_k) \le n$, while $g \in \operatorname{Aut}\Gamma$ has order x. Since the center point is invariant, we can attach some loops to it and obtain a new graph with rank n. Moreover, g induces an automorphism on it which restricts to identities on the loops added. This automorphism also has order x.

Therefore, every element $x \in B_n$ can be realized as the order of an automorphism g on a certain finite connected rank n graph Γ which has no valence 1 vertex or closed chain.

Now suppose that Γ is a beam of edges, and let Γ' be the graph obtained from Γ by joining each pair of edges connecting to a valence 2 vertex together. Then Γ' is a single set of multiple edges, and clearly $\operatorname{Aut}\Gamma' \supseteq \operatorname{Aut}\Gamma$. There is a flip map $\sigma \in \operatorname{Aut}\Gamma'$ that switches the two vertices and reverses all edges. Furthermore, $\operatorname{Aut}\Gamma'$ is generated by σ and permutations of the $n+1$ edges. So, in this case, the order set of $\operatorname{Aut}\Gamma'$ is exactly $\{\operatorname{lcm}(a, y);\ a \in \{1, 2\},\ y \in A_{n+1}\}$. Particularly, if $n+1$ is prime, the only element in $\operatorname{Aut}\Gamma'$ with order $\operatorname{lcm}(2, n+1)$ is the cyclic permutation of all edges composite with σ. It has no fixed point in the topological underlying space of Γ or Γ'.

Therefore, Theorem 3.1 reduces to the proof of the following statement:

CLAIM 3.2. *Suppose that* Γ *is a finite connected graph with rank* n, *having no valence* 1 *vertex or closed chain*, $n > 1$, *and* $g \in \operatorname{Aut}\Gamma$. *If* Γ *is a beam of edges*, *we require that* g *has at least one fixed point in* Γ. *Then* $\operatorname{order}(g) \in B_n$.

The proof of this statement is essentially an induction on n and the number of vertices in Γ. Suppose that it is not true. Then choose a smallest natural number n that violates the rule. Direct verification shows that this cannot happen when $n = 1$ or 2. Thus, suppose without loss of generality that $n > 2$. Corresponding to this n, there is a finite connected rank n graph Γ with no valence 1 vertex or closed chain of valence 2 vertices, s.t. $\exists g \in \operatorname{Aut}\Gamma$ with $\operatorname{order}(g) \notin B_n$. Furthermore, when Γ is a beam of edges, g must have fixed point.

From now on, assume that all the triples (n, Γ, g) we discussed are chosen as in the following lemma.

LEMMA 3.1. *As discussed above, suppose that there exists counter-examples (n, Γ, g) to the above claim. Choose such triples that minimize n, and then select in all the satisfied choices a triple with a minimum number of vertices in Γ. Then Γ has no valence 2 vertex.*

Proof. By the assumptions, Γ cannot be a single loop or closed chain. For any open chain in Γ, since the action of g on it will be determined by the action on the chain as a whole segment, one can simply merge all the edges in the chain to make a single edge. The resulting graph Γ' has rank n and a fewer number of vertices than that of Γ, while the induced action of g remains of the same order. This contradicts our assumption that (n, Γ, g) is the simplest triple. ∎

LEMMA 3.2. *For (n, Γ, g) in Lemma 3.1, Γ has no loop.*

Proof. Suppose contrarily that Γ has m loops $e_1, \dots, e_m$ $(1 \le m \le n)$. Removing these loop edges gives rise to a connected graph $\Gamma \setminus (e_1 \cup \cdots \cup e_m)$ with rank $n - m$. Denote by Γ' the subgraph obtained from it by cutting leaves. $\forall h \in \operatorname{Aut}\Gamma$ such that $h|(e_1 \cup \cdots \cup e_m)$ equals identities, and $\forall$valence 1 vertices $x \in \Gamma \setminus (e_1 \cup \cdots \cup e_m)$. Then x is the end of some e_i since Γ itself has no valence 1 vertex. Thus, $h(x) = x$, and h restricts to the identity on the unique edge in $\Gamma \setminus (e_1 \cup \cdots \cup e_m)$ connecting to x. By induction, it can be shown that h keeps all removed leaves invariant. So, if, in addition, $h|\Gamma' = \mathrm{id}_{\Gamma'}$, then $h = \mathrm{id}_\Gamma$. Particularly, this implies that

$$\operatorname{order}(g) = \operatorname{lcm}\bigl(\operatorname{order}(g|\Gamma'),\ \operatorname{order}(g|(e_1 \cup \cdots \cup e_m))\bigr). \tag{3.2}$$

There are only three cases.

Case 1. Γ' contains no closed chain and $n - m > 1$. By the assumptions, either $\operatorname{order}(g|\Gamma') \in B_{n-m}$ or $\operatorname{order}(g|\Gamma') = 2(n - m + 1)$ in which $n - m + 1 > 2$ is a prime number. Now g induces a permutation λ on the loops, and $g^{|\lambda|}$ keeps every loop invariant, although it may reverse the directions of certain loops. Thus, the order of $g|(e_1 \cup \cdots \cup e_m)$ is a divisor of $2|\lambda|$.

If $\operatorname{order}(g|\Gamma') \in B_{n-m}$, then, since $|\lambda| \in A_m$, $2|\lambda| \in B_m$, one sees from Lemma 2.5 that $\operatorname{lcm}(\operatorname{order}(g|\Gamma'), 2|\lambda|) \in B_n$. If $\operatorname{order}(g|\Gamma') = 2(n - m + 1)$ in which $n - m + 1$ is an odd prime number, then $\operatorname{lcm}(\operatorname{order}(g|\Gamma'), 2|\lambda|) = \operatorname{lcm}(n - m + 1,\ 2|\lambda|)$. Since $n - m + 1 \in B_{n-m}$, we still have

$\mathrm{lcm}(\mathrm{order}(g|\Gamma'), 2|\lambda|) \in B_n$. In both subcases, as the divisor of $\mathrm{lcm}(\mathrm{order}(g|\Gamma'), 2|\lambda|)$, $\mathrm{order}(g) \in B_n$.

Case 2. $m = n - 1$. Then Γ' is a single loop or closed chain of valence 2 vertices. Again, g induces a permutation λ on the loops, and $\mathrm{order}(g|(e_1 \cup \cdots \cup e_m))$ is a divisor of $2|\lambda|$, $|\lambda| \in A_m \subseteq A_n$. Since $g^{|\lambda|}$ induces the identity permutation on the loops, it follows from the construction of Γ' that $g^{|\lambda|}$ must fix at least one vertex in Γ' (one that has a path connecting to some loop edge, which consists of removed leaves). Therefore, $(g^{|\lambda|}|\Gamma')^2 = \mathrm{id}_{\Gamma'}$. This implies that $\mathrm{order}(g)$ is a divisor of $2|\lambda|$. Similar to Case 1, $\mathrm{order}(g) \in B_n$.

Case 3. $m = n$. Then $\Gamma \setminus (e_1 \cup \cdots \cup e_m)$ is in fact a tree graph. In this case, any element in $\mathrm{Aut}\,\Gamma$ is determined by its action on the loops. Therefore, suppose that g induces a permutation λ on the n loops. Then $|\lambda| \in A_n$ and

$$\mathrm{order}(g) = |\lambda| \text{ or } 2|\lambda| \in B_n.$$

In each case, we always get a contradiction. Hence, for Γ chosen in Lemma 3.1, Γ has no loop. ∎

Notation. Assume a graph Y. For any subset $X \subseteq Y$ and subgroup $G < \mathrm{Aut}\,Y$, *the orbit of G passing X* is defined as the set

$$O_G(X) = \bigcup\{h(X);\ h \in G\}. \tag{3.3}$$

Particularly, if G is a cyclic group generated by $h \in \mathrm{Aut}\,Y$, then denote $O_G(X)$ by $O_h(X)$.

LEMMA 3.3. *For* (n, Γ, g) *in Lemma* 3.1, Γ *has no multiple edges*.

Proof. By the previous lemma, Γ has no loop. Suppose contrarily that there is more than one edge connecting vertices $x, y \in \Gamma$. Choose an edge e_0 among these multiple edges. Suppose that the orbit of g passing e_0 contains k edges, in which m of them connect x and y. Then $m \mid k$, and for each $i \in \mathbb{N}$, there are m edges in $O_g(e_0)$ connecting $g^i(x)$ and $g^i(y)$, namely,

$$g^i(e_0), g^{i+(k/m)}(e_0), \ldots, g^{i+(m-1)(k/m)}(e_0).$$

Case 1. $O_g(e_0)$ does *not* contain all the edges connecting x and y. Then this also happens for every pair $(g^i(x), g^i(y))$. Therefore, the graph $\Gamma \setminus O_g(e_0)$ is connected with rank $n - k$ (here $k \leq n$). Denote by Γ' the graph obtained from $\Gamma \setminus O_g(e_0)$ by cutting leaves.

If x, y both have valence 1 in $\Gamma \setminus O_g(e_0)$, then Γ is exactly x, y together with $n + 1$ multiple edges connecting them, while $O_g(e_0)$ contains $m = n$ of the edges. Therefore, $\mathrm{order}(g)|2n \in B_n$.

If x, y both have valence greater than 1 in $\Gamma \setminus O_g(e_0)$, then $\Gamma = \Gamma' \cup O_g(e_0)$, while $\text{order}(g) = \text{lcm}(\text{order}(g|\Gamma'), |O_g(e_0)|)$. If, say, x has valence 1 and y has valence greater than 1 in $\Gamma \setminus O_g(e_0)$, then, for any $i \in \mathbb{N}$, $g^i(x) \neq y$, so if $g^i(e_0) = e_0$, then its ends $g^i(x) = x$, $g^i(y) = y$. It can be shown inductively that g^i fixes all the removed leaves. Therefore, $\text{order}(g) = \text{lcm}(\text{order}(g|\Gamma'), |O_g(e_0)|)$. In both subcases, $|O_g(e_0)| = k \in B_{k-1}$ and Γ' has rank $n - k$. Similar to the proof of Case 1 of the previous lemma, if Γ' is not a closed chain, then $\text{order}(g|\Gamma') \in B_{n-k+1}$, so $\text{order}(g) \in B_n$ by Lemma 2.5. If Γ' is a closed chain, then $k = n - 1$. Moreover, $g^k(\{x, y\}) = \{x, y\}$. Thus $g^{2k} = \text{id}_\Gamma$ and $\text{order}(g) \mid 2k \in B_n$.

Case 2. All the edges connecting x and y are in the orbit passing e_0. Consider the graph Γ'' obtained from Γ by replacing with a single edge each group of multiple edges in $O_g(e_0)$ that connect two vertices. Cut off all the leaves in Γ'', and denote the result by Γ'. It is connected with rank $n' = n - (m-1)k/m$, and $m \geq 2$.

If both x, y have valence 1 in Γ'', then Γ is exactly x, y together with $n+1$ edges connecting them, and $O_g(e_0)$ contains all the edges. Thus Γ itself is a beam of edges. By the assumptions on g, one knows that Γ cannot be a counterexample.

Otherwise, similar to the previous case, we derive that $\text{order}(g) = \text{lcm}(\text{order}(g_{\Gamma'}), k)$, in which $g_{\Gamma'}$ is the induced action of g on Γ'. In addition, the ends x, y of e_0 are both invariant under $g^{2k/m}$. There are two subcases.

If Γ' is a single loop or closed chain, then $n' = 1$, $\text{order}(g_{\Gamma'})|(2k/m)$, and $\text{order}(g)$ is a divisor of $\text{lcm}(2k/m, k)$. If $2|m$, then $2k/m|k$, so $\text{order}(g)|k$. But $k - k/m < n$, which implies that $k \in B_n$, so $\text{order}(g) \in B_n$. If $2 \nmid m$, suppose that $k/m = 2^j q, 2 \nmid q$, then $\text{order}(g)|\text{lcm}(2^{j+1}, qm)$. However,

$$(2^{j+1} - 2^j) + (qm - q) \leq 2^j(qm - q) + 1 = k - k/m + 1 = n.$$

By Lemma 2.4, $\text{lcm}(2^{j+1}, qm) \in B_n$, and consequently $\text{order}(g) \in B_n$.

If Γ' is not a closed chain, then $n' > 1$, and either $\text{order}(g_{\Gamma'}) \in B_{n'}$ or $\text{order}(g_{\Gamma'}) = 2(n'+1)$. If $\text{order}(g_{\Gamma'}) \in B_{n'}$, since $k \in B_{k-k/m}$ and $n = n' + k - k/m$, it follows from Lemma 2.5 that $\text{order}(g) \in B_n$. Otherwise Γ' is a beam of edges, $\text{order}(g_{\Gamma'}) = 2(n'+1)$, and $n'+1$ is a prime number. $(g_{\Gamma'})^{k/m}$ has at least one fixed point. It is not difficult to show from this that either $2|(k/m)$ or $(n'+1)|(k/m)$. If $2|(k/m)$, then $k/2 \in B_{k/2-k/2m} \subseteq B_{k-k/m-1}$. Thus $\text{order}(g) = \text{lcm}(n'+1, k)$ or $\text{lcm}(2(n'+1), k/2) \in B_n$. Otherwise, $n'+1 \neq 2$ will be a prime factor of k/m, so $\text{order}(g) = \text{lcm}(2, k)$. However, $k - k/m + 1 \leq n$, again by Lemma 2.4, $\text{order}(g) \in B_n$.

Hence, one always derives $\text{order}(g) \in B_n$, which contradicts the assumptions in Lemma 3.1. ∎

COROLLARY. *For* (n, Γ, g) *in Lemma* 3.1, Γ *must be a simple graph.*

LEMMA 3.4. *For* (n, Γ, g) *in Lemma* 3.1, Γ *has at least one vertex with valence* 3.

Proof. Suppose that all the vertices in Γ have valences greater than or equal to 4. Denote their total number by k_0. Then there are at least four edges connecting to each vertex, while each edge has two ends. Therefore, the total number of edges is $k_1 \geq 4k_0/2 = 2k_0$, and the Euler characteristic number $\chi(\Gamma) = k_0 - k_1 \leq -k_0$. However, $\chi(\Gamma) = 1 - n$, so $k_0 \leq n - 1$.

By the previous corollary, Γ must be simple, which implies that any $h \in \operatorname{Aut}\Gamma$ is determined by its induced permutation on the vertices of Γ. Particularly, denote by λ the induced permutation of g on the vertices of Γ. Then

$$\operatorname{order}(g) = |\lambda| \in A_{k_0} \subseteq A_n \subseteq B_n. \qquad \blacksquare$$

4. PROOF OF CLAIM 3.2

By the previous lemmas, for a triple (n, Γ, g) satisfying the requirements in Lemma 3.1, Γ must be a simple graph. Moreover, it has at least one vertex x with valence 3. Denote the three vertices in Γ adjacent to x by y_1, y_2, y_3, and denote the edge connecting x with each y_i by e_i, $i = 1, 2, 3$. Then, for any vertex $x' \in O_g(x)$ and $y' \in O_g(y_i)$, all edges in Γ connecting x' and y', if there exist such edges, must be in $O_g(\{e_1, e_2, e_3\})$.

In the following paragraphs, Claim 3.2 will be proved by reduction to absurdity from this. Consequently, Theorems 3.1 and 1.1 also hold. There are four cases.

Case 1. $O_g(x) \neq O_g(y_1) \neq O_g(y_2) \neq O_g(y_3)$, or $O_g(x) \neq O_g(y_1) \neq O_g(y_2) = O_g(y_3)$, or $O_g(x) = O_g(y_3) \neq O_g(y_1) \neq O_g(y_2)$, or $O_g(x) = O_g(y_2) = O_g(y_3) \neq O_g(y_1)$. The common points of these cases are that $O_g(x) \neq O_g(y_1)$, while y_1 is the unique element in $O_g(y_1)$ that is adjacent to x. They are connected by the edge e_1. Particularly, $g^{|O_g(x)|}(y_1) = y_1$ since $g^{|O_g(x)|}(y_1)$ is also adjacent to $g^{|O_g(x)|}(x) = x$. Thus, $\exists m \in \mathbb{N}$, such that $|O_g(x)| = m|O_g(y_1)|$.

Since $g \in \operatorname{Aut}\Gamma$, each $g^j(x)$ is adjacent to exactly one vertex in $O_g(y_1)$, namely, $g^j(y_1)$. The edge connecting $g^j(x)$ and $g^j(y_1)$ is $g^j(e)$. Furthermore, each $g^j(y_1)$ is adjacent to m vertices in $O_g(x)$, i.e.,

$$g^j(x), g^{j+|O_g(y_1)|}(x), \ldots, g^{j+(m-1)|O_g(y_1)|}(x).$$

By identifying these m vertices with $g^j(y_1)$ and removing the edges

$$g^j(e_1), g^{j+|O_g(y_1)|}(e_1), \ldots, g^{j+(m-1)|O_g(y_1)|}(e_1),$$

one obtains a new graph Γ' with rank n and a fewer number of vertices than that of Γ.

Denote by $\varphi\colon \Gamma \to \Gamma'$ the canonical quotient map. Suppose that the valence of $y_1 \in \Gamma$ is j. Then, because all vertices in $O_g(x)$ have valences 3, the valence of $\varphi(y_1) \in \Gamma'$ must be $j' = j + 3m - 2m = j + m \geq 4$. Similarly, all other vertices in $\varphi(O_g(y_1))$ also have valence greater than or equal to 4. Therefore, Γ' has no vertex with valence 1 or 2. Moreover, Γ' has at least three vertices $\varphi(y_1)$, $\varphi(y_2)$, $\varphi(y_3)$, so it will not be a beam of edges.

Now g induces an action $g_{\Gamma'} \in \operatorname{Aut}\Gamma'$ defined as follows:

$$g_{\Gamma'} \circ \varphi = \varphi \circ g. \tag{4.1}$$

The above discussions show that the triple $(n, \Gamma', g_{\Gamma'})$ cannot be a counter-example to Claim 3.2. This implies that

$$\operatorname{order}(g_{\Gamma'}) \in B_n.$$

Put $h = g^{\operatorname{order}(g_{\Gamma'})}$, and arbitrarily fix a vertex $v \in \Gamma$. Then

$$\varphi(h(v)) = (g_{\Gamma'})^{\operatorname{order}(g_{\Gamma'})}(\varphi(v)) = \varphi(v).$$

If $v \notin O_g(x) \cup O_g(y_1)$, then $\varphi^{-1}(\varphi(v))$ has only one element, so $h(v) = v$. If $v \in O_g(y_1)$, then $\varphi^{-1}(\varphi(v))$ has only one element in $O_g(y_1)$, which implies that $h(v) = v$. Finally, if $v \in O_g(x)$, suppose without loss of generality that $v = x$. Then, other than e_1, there are two edges e_2, e_3 that are connected to x. Since, for $i = 2, 3$, $\varphi^{-1}(\varphi(e_i))$ has only one element, and $\varphi(h(e_i)) = (g_{\Gamma'})^{\operatorname{order}(g_{\Gamma'})}(\varphi(e_i)) = \varphi(e_i)$, it follows that $h(e_i) = e_i$. Thus, x and $h(x)$ are both common ends of e_2 and e_3. However, Γ has no multiple edges, so $h(x) = x$. To sum up, h fixes all the vertices in Γ, i.e., $h = \mathrm{id}_\Gamma$, and $\operatorname{order}(g) \mid \operatorname{order}(g_{\Gamma'})$.

Hence, $\operatorname{order}(g) \in B_n$, which contradicts our assumptions.

Case 2. $O_g(x) = O_g(y_1) \neq O_g(y_2) = O_g(y_3)$. Then $|O_g(x)|$ is even, and all the vertices in $O_g(x)$ are divided into adjacent pairs. By squeezing each pair and the edge connecting them to one point, one can construct a new graph Γ' with no vertex of valence 1 or 2. Furthermore, Γ' has rank n and a fewer number of vertices than that of Γ. Denote by $g_{\Gamma'}$ the induced action of g on Γ'. Then $(n, \Gamma', g_{\Gamma'})$ cannot be a counter-example to Claim 3.2. Since Γ' has at least three vertices, it is not a beam of edges. Therefore, $\operatorname{order}(g_{\Gamma'}) \in B_n$. However, $\operatorname{order}(g) = \operatorname{order}(g_{\Gamma'})$, which leads to a contradiction again.

By symmetry, the discussions in the above two cases imply that $O_g(y_1) = O_g(y_2) = O_g(y_3)$ whenever $\operatorname{order}(g) \notin B_n$.

Case 3. $O_g(x) = O_g(y_1) = O_g(y_2) = O_g(y_3)$. Since Γ is connected, $O_g(x)$ must contain all the vertices in Γ, which implies $\operatorname{order}(g) = |O_g(x)| = k_0$, in which k_0 is the number of vertices in Γ. Because all the vertices have valence 3, the number of edges in Γ is exactly $3k_0/2$, and $1 - n = \chi(\Gamma) = k_0 - 3k_0/2 = -k_0/2$. Hence, $\mathrm{order}(g) = k_0 = 2(n-1) \in B_n$.

Case 4. $O_g(x) \neq O_g(y_1) = O_g(y_2) = O_g(y_3)$. Let Γ'' be the connected graph that is obtained by first delete $O_g(x)$ and all the edges with at least one end in them, then insert back a single point x', and finally connecting x' with each point in $O_g(y_1)$ by one edge. Denote by Γ' the graph obtained from Γ'' by cutting leaves.

By definition, $\chi(\Gamma') = \chi(\Gamma'') = \chi(\Gamma) - (|O_g(x)| - 1) + (3|O_g(x)| - |O_g(y_1)|)$, so the rank of Γ' is

$$n' = n - \chi(\Gamma') + \chi(\Gamma) = n - 2|O_g(x)| + |O_g(y_1)| - 1. \tag{4.2}$$

Furthermore, g induces a unique action on Γ' which commutes with the quotient map from Γ to Γ'.

There are two subcases.

Subcase a. y_1 has valence 1 in Γ''; that is, any edge in Γ that is connected to y_1 must have the other end in $O_g(x)$. Let m be the total number of such edges. Then m is actually the valence of y_1, so $m \geq 3$. These properties are also satisfied by other vertices in $O_g(y_1)$ (for example, y_2 and y_3). Moreover, $O_g(x) \cup O_g(y_1)$ contains all the vertices in Γ. Since Γ is simple,

$$\mathrm{order}(g) = \mathrm{lcm}(|O_g(x)|, |O_g(y_1)|),$$

and the number of edges in Γ is $3|O_g(x)| = m|O_g(y_1)|$. However, $1 - n = \chi(\Gamma) = (|O_g(x)| + |O_g(y_1)|) - 3|O_g(x)| = |O_g(y_1)| - 2|O_g(x)|$. Thus,

$$|O_g(x)| \leq 2|O_g(x)| - |O_g(y_1)| = n - 1.$$

If $3 \mid m$, then $|O_g(y_1)|$ is a divisor of $|O_g(x)|$, so $\mathrm{order}(g) = |O_g(x)| \in B_n$. If $3 \nmid m$, suppose that $|O_g(x)| = 3^j q$ in which $3 \mid q$ and $q \geq 2$. Then $\mathrm{order}(g) = \mathrm{lcm}(3^{j+1}, q)$. Now $\forall u, v \in \mathbb{N}$, if $u, v \geq 4$, then

$$u + v \leq 2\max(u, v) \leq (\min(u, v)/2)\max(u, v) = uv/2.$$

It follows that, if $u, v \geq 2$, then $u + v \leq uv/2 + 2$. As a result,

$$(3^{j+1} - 3^j) + (q - 1) \leq (3^{j+1} - 3^j)q/2 + 2 - 1 = |O_g(x)| + 1 \leq n.$$

By Lemma 2.4, $\mathrm{order}(g) \in B_n$.

Subcase b. y_1 has valence greater than 1 in Γ''. Then all vertices in $O_g(y_1)$ also have valence greater than 1 in Γ''. Thus $\Gamma'' = \Gamma'$. For any $h = g^j$, denote by $h_{\Gamma'}$ its induced action on Γ'. Clearly, if $h_{\Gamma'} = \mathrm{id}_{\Gamma'}$, $h(x) = x$, then $h = \mathrm{id}_\Gamma$. Namely, $\mathrm{order}(g)$ is a divisor of $\mathrm{lcm}(\mathrm{order}(g_{\Gamma'}), |O_g(x)|)$.

TABLE 1
The Maximum Order c_n for Some Small n

n	c_n	n	c_n
1	2	11	$90 = 2 \cdot 3^2 \cdot 5$
2	6	12	$180 = 2^2 \cdot 3^2 \cdot 5$
3	$6 = 2 \cdot 3$	13	$210 = 2 \cdot 3 \cdot 5 \cdot 7$
4	$12 = 2^2 \cdot 3$	14	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$
5	$12 = 2^2 \cdot 3$	15	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$
6	$20 = 2^2 \cdot 5$	16	$504 = 2^3 \cdot 3^2 \cdot 7$
7	$30 = 2 \cdot 3 \cdot 5$	17	$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$
8	$60 = 2^2 \cdot 3 \cdot 5$	18	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
9	$60 = 2^2 \cdot 3 \cdot 5$	19	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
10	$84 = 2^2 \cdot 3 \cdot 7$	20	$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$

Moreover, Γ' has no closed chain, since x' has valence greater than or equal to 3.

If $|O_g(x)| \geq |O_g(y_1)|$, then

$$|O_g(x)| \leq 2|O_g(x)| - |O_g(y_1)| = n - n' - 1,$$

in which n' is the rank of Γ'. Because $n' < n$, by the "simplestness" of (n, Γ, g), order $(g_{\Gamma'}) \in B_{n'+1}$. By Lemma 2.5, order$(g) \in B_n$.

Now suppose that $|O_g(x)| < |O_g(y_1)|$. $\forall j \in \mathbb{N}$, $g^j(x) = x$ implies that $g^j(\{y_1, y_2, y_3\}) = \{y_1, y_2, y_3\}$. Since y_1, y_2, y_3 are in the same orbit of g, it follows that $|O_g(y_1)|$ is a divisor of $3|O_g(x)|$. Therefore, $|O_g(x)| = |O_g(y_1)|/3$ or $2|O_g(y_1)|/3$.

If $|O_g(x)| = |O_g(y_1)|/3$, then, for each $j \in \mathbb{N}$, the three vertices in $O_g(y_1)$ adjacent to $g^j(x)$ must be

$$g^j(y_1),\ g^{j+|O_g(x)|}(y_1),\ \text{ and }\ g^{j+2|O_g(x)|}(y_1),$$

and each $g^j(y_1)$ is adjacent to a unique vertex in $O_g(x)$. By arguments similar to the proof of Case 1, one can show that order$(g) \in B_n$.

If $|O_g(x)| = 2k,\ |O_g(y_1)| = 3k$ in which $k \in \mathbb{N}$, then

$$n - n' - 1 = 2|O_g(x)| - |O_g(y_1)| = k = |O_g(x)| - k.$$

By Lemma 2.4, $|O_g(x)| \in B_{n-n'-1}$. However, order$(g_{\Gamma'}) \in B_{n'+1}$. It follows from Lemma 2.5 that lcm(order$(g_{\Gamma'})$, $|O_g(x)|) \in B_n$. Consequently, order$(g) \in B_n$, once again a contradiction.

To sum up, all four cases are impossible. Hence, Theorem 3.1 is true. ∎

Remark. We thank the referee for pointing out that part of our results, namely, the description in Theorem 3.1, is also obtained in [4]. However, we believe that our approach here is simpler and more intrinsic. It also gives a clearer understanding of the action of elements in Out F_n on graphs as well as the realization of numbers in the order sets B_n.

5. APPENDIX: SOME CALCULATIONS FOR c_n

Theorem 3.1 reduces the problem of finding maximum orders to the study of number sets B_n. According to this theorem, for some small n, the maximum order c_n of periodic outer automorphisms of F_n can be calculated easily. Table 1 shows part of the results. In addition, it is known that, for a_n defined in Eq. (2.2),

$$\log a_n = \sqrt{n \log n}\left(1 + \frac{\log\log n + \mu_n}{2\log n}\right),$$

in which $\{\mu_n\}$ is a bounded number series. Careful analysis of B_n shows that, when $n > 2$,

$$\log c_n = \log b_n = \sqrt{n \log n}\left(1 + \frac{\log\log n + \tau_n}{2\log n}\right),$$

in which $\{\tau_n\}$ is another bounded number sequence.

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