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On k-Leaf-Connected Graphs

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A graph G is Hamilton-connected if given any two vertices u and v of G, there is a Hamilton path in G with ends u and v. In this note we consider a generalization of this property. For $k \geqslant 2$ we say that a graph $G = (VG, EG)$ is **k-leaf-connected** if $|VG| > k$ and given any subset S of VG with $|S| = k$, G has a spanning tree T such that the set S is the set of endvertices of T. Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected. This generalization is due to U. S. R. Murty. We prove that the k-leaf-connectedness property is $(|VG| + k - 1)$-stable, give sufficient conditions for a graph to be k-leaf-connected, present some necessary conditions and other related results. We show that for all naturals n, k, $2 \leqslant k < n-2$, there is a sparse k-leaf-connected graph of order n. © 1986 Academic Press, Inc.

1. DEFINITIONS AND NOTATIONS

All graphs we consider here are simple. We denote the vertex set of a graph G by VG, and the edge set by EG. An edge with ends u and v is denoted by uv. For A, $B \subset VG$, $\delta(A, B)$ denotes the set of all edges of G with one end in A and the other in B; $N_G(A)$ denotes the set of all vertices of G adjacent to vertices in A. For $v \in VG$, instead of writing $N_G(\{v\})$ we simply write $N_G(v)$, and we denote by $d_G(v)$ the degree of v in G. When the underlying graph G is clear from the context the subscript is omitted. A path in G is denoted by a sequence of its vertices. If T is a tree in G then VT and ET denote the vertex set and edge set of T, respectively. All other terms and notation not mentioned here are standard and can be found in [3 or 5].

If S is a subset of VG, an **S-spanning tree** in G is a spanning tree in G whose set of endvertices is precisely S. Given an integer $k \geqslant 2$, we say that a graph is **k-leaf-connected** or has **property P(k)**, if $|VG| > k$ and for all subsets S of VG with $|S| = k$, G has an S-spanning tree. If G has property $P(m)$

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for all m, $2 \leqslant m \leqslant k$, then we say that G is **k*-leaf-connected** or G has **property P*(k)**.

Throughout this note G denotes a graph, n stands for the order of the graph under consideration, and k is a natural number.

2. Necessary Conditions and Other Results

Theorem 1. *If G is k-leaf-connected, $2 \leqslant k < n-1$, then G is $(k+1)$-connected.*

Proof. Follows immediately from the fact that for any subset S of VG with $|S| = k$, the graph $G - S$ is connected. ∎

Corollary 1.1. *If G is k-leaf-connected, $2 \leqslant k < n-1$, then $d(v) \geqslant k+1$ for every vertex v in G.* ∎

The examples in Section 4 show that these two results are sharp.

Theorem 2. *If G is k-leaf-connected, $k \geqslant 3$ and G has at least k vertices with degree greater than $n-k$, then G is $(k-1)$-leaf-connected.*

Proof. Assume that $k < n-2$, otherwise G is complete and the result is obvious. Let $S := \{v_1, v_2, ..., v_{k-1}\}$ be a subset of VG and suppose that G does not have an S-spanning tree.

Take a vertex $x \in VG \backslash S$ such that $d(x) > n-k$. Set $S' := S \cup \{x\}$ and let T' be an S'-spanning tree in G. For each vertex $v_i \in S$, $1 \leqslant i \leqslant k-1$, let P_i be the path in T' from x to v_i. Call y_i the last vertex in P_i such that $d_{T'}(y_i) \geqslant 3$ and call y_i' the successor of y_i in P_i. Note that y_i' is not adjacent to x. Otherwise,

$$T := T' + xy_i' + y_i y_i'$$

would be an S-spanning tree in G.

It is clear that the $k-1$ vertices y_i', $1 \leqslant i \leqslant k-1$, are all distinct. Since none of them is adjacent to x, it follows that

$$d_G(x) \leqslant n-(k-1)-1 = n-k,$$

contradicting the choice of x. ∎

Theorem 3. *If G is k-leaf-connected, $k \geqslant 3$ and $k \geqslant n/2$, then G is $(k-1)$-leaf-connected.*

Proof. Follows immediately from Theorem 2 and Corollary 1.1. ∎

Remark. We do not know whether an analogous result as the one present in Corollary 2.1 is also valid for $3 \leqslant k < n/2$.

In the sequel we define the concept of m-stable property, which was introduced by Bondy and Chvátal [2] inspired by a theorem of Ore [6] on hamiltonian graphs. We prove then that the property $P(k)$ is $(n+k-1)$-stable and use this result to derive some of the sufficient conditions we present in the next section.

DEFINITION. Let P be a property defined on all graphs of order n and let m be a nonnegative integer. The property P is said to be **m-stable** if whenever G is an arbitrary graph of order n, u and v are non-adjacent vertices of G such that $d(u)+d(u) \geqslant m$ and $G+uv$ has property P, then G itself has property P.

THEOREM 4. *The property $P(k)$ is $(n+k-1)$-stable.*

Proof. Let u and v be any two non-adjacent vertices of a graph G such that $d(u)+d(v) \geqslant n+k-1$. Suppose $G+uv$ has property $P(k)$ but G does not.

Take $S \subset VG$ with $|S|=k$ and assume that G does not have an S-spanning tree. Let T be an S-spanning tree in $G+uv$ and call T_u and T_v the trees obtained from T by removing the edge uv. Choose T_u and T_v such that $u \in VT_u$ and $v \in VT_v$.

For any vertex x in T_u (resp. in T_v) denote by x_p the predecessor of x in the path from u (resp. v) to x in T_u (resp. T_v). Let

$$A := \{x \in VT_u : ux \in EG \backslash ET\},$$

$$B := \{x \in VT_u : x_p v \in EG \backslash ET\},$$

$$C := \{x \in VT_v : vx \in EG \backslash ET\},$$

$$D := \{x \in VT_v : x_p u \in EG \backslash ET\}.$$

It is clear that $A \cap B = \varnothing$, for if x is a vertex in $A \cap B$ then $T+ux+x_p v-x_p x-uv$ is an S-spanning tree in G, contradicting the assumption. By symmetry, it follows that $C \cap D = \varnothing$. It is also immediate that

$$|A \cup B| \leqslant |VT_u| - |N_{T_u}(u)| - 1$$

and

$$d_G(u) \leqslant |A| + |N_{T_u}(u)| + |D| + |S \cap VT_v|.$$

By symmetry,

$$|C \cup D| \leqslant |VT_v| - |N_{T_v}(v)| - 1$$

and

$$d_G(v) \leqslant |B| + |N_{T_v}(v)| + |C| + |S \cap VT_u|.$$

Therefore,

$$\begin{aligned} d_G(u) + d_G(v) &\leqslant |A \cup B| + |N_{T_u}(u)| + |N_{T_v}(v)| + |C \cup D| + k \\ &\leqslant |VT_u| + |VT_v| + k - 2 = n + k - 2, \end{aligned}$$

a contradiction. ▮

3. Sufficient Conditions

In this section we use some results given in [2] combined with our Theorem 4 to obtain the following two main theorems.

Theorem 5. *Let k and n be such that $2 \leqslant k \leqslant n-3$. Let G be a graph with degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. Suppose there is no integer i such that*

$$k - 1 < i < \frac{n+k-1}{2},$$

$$d_{i-k+1} \leqslant i,$$

$$d_{n-i} \leqslant n - i + k - 2.$$

Then G is k-leaf-connected.

Proof. Apply Proposition 2.1 and Theorem 3.1 (specialized to $t = \lceil (n+k-1)/2 \rceil$) given in [2] combined with our Theorem 4. Note that when the conditions of Theorems 3.1 are satisfied, the $(n+k-1)$-closure of G has at least t vertices of degree $n-1$, and therefore it is complete. ▮

Theorem 6. *Let k and n be such that $2 \leqslant k \leqslant n-3$. Let G be a graph with $VG = \{v_1, v_2, ..., v_n\}$. Suppose there are no indices i and j such that*

$$j \geqslant \max\{i+1, n-k-i+1\}, \qquad v_i v_j \notin EG,$$

$$d(v_i) \leqslant i + k - 1,$$

$$d(v_j) \leqslant j + k - 2,$$

$$d(v_i) + d(v_j) \leqslant n + k - 2.$$

Then G is k-leaf-connected.

Proof. Same as above, applying Theorem 3.2 instead of Theorem 3.1. ∎

Remark. Theorem 6 is stronger than Theorem 5, according to Theorem 10.1 presented in [2].

Recently, Zhu and Feng [9] obtained improvements of the results (Theorems 3.1 and 3.2 of [2]) due to Bondy and Chvátal. Applying their results one can obtain results which yield the previous theorems as corollaries. The application is straightforward and we leave it to the reader.

COROLLARY 6.1. *Suppose $2 \leqslant k < n$. If G is such that $d(u) + d(v) \geqslant n + k - 1$ for every two non-adjacent vertices u and v, then G is k-leaf-connected.*

COROLLARY 6.2. *Suppose $2 \leqslant k < n$. If G is such that*

$$|EG| \geqslant \frac{(n-1)(n-2)}{2} + k + 1,$$

then G is k-leaf-connected.

COROLLARY 6.3. *Suppose $2 \leqslant k < n$. If G is such that $d(v) \geqslant (n+k-1)/2$ for every vertex v, then G is k-leaf-connected.*

Remarks. (a) It is not difficult to verify that Theorem 5 and Theorem 6 remain valid if instead of asserting that G is k-leaf-connected we assert that G is k^*-leaf-connected. For the corollaries the validity for k^* is obvious.

(b) The given corollaries are generalizations of known results for Hamilton-connected graphs. Corollaries 6.1 and 6.2 for $k = 2$ are results due to Ore [7]. Note also that these corollaries follow immediately from Theorem 4.

(c) The inequality in Corollary 6.1 cannot be weakened. For each pair (n, k), where $2 \leqslant k < n - 1$, consider the graph $G_{n,k}$ which consists of a complete graph K_{n-1} and k edges joining k of its vertices to a vertex not in it. In $G_{n,k}$ every two non-adjacent vertices have degree sum $n + k - 2$ and clearly it is not k-leaf-connected.

(d) The graph $G_{n,k}$ also shows that the bound given in Corollary 6.2 is best possible. Note that the number of edges of $G_{n,k}$ is

$$\frac{n(n-1)}{2} - (n-k-1) = \frac{(n-1)(n-2)}{2} + k.$$

(e) The inequality in Corollary 6.3 is also best possible. The graph $\tilde{G}_{n,k}$ of order n (see Fig. 1) defined below for each pair (n, k), where

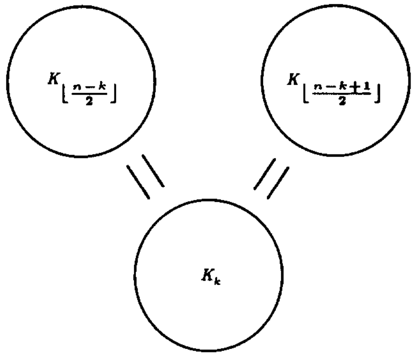

FIG. 1. The graph $\tilde{G}_{n,k}$.

$2 \leqslant k < n-1$, is such that $d(v) \geqslant \lceil (n+k-1)/2 \rceil - 1$ for every vertex v, and is not k-leaf-connected.

$\tilde{G}_{n,k}$ consists of the union of the complete graphs $K_{\lfloor (n-k)/2 \rfloor}$, $K_{\lfloor (n-k+1)/2 \rfloor}$ and K_k, and all edges with one end in K_k and other in $K_{\lfloor (n-k)/2 \rfloor} \cup K_{\lfloor (n-k+1)/2 \rfloor}$.

In the sequel we prove a result which shows that, with the additional regularity hypothesis, the degree based conditions previously presented can be slightly improved. Recently, Tomescu [8] proved the following:

THEOREM 7 (Tomescu). *Let r be a natural number, $r \geqslant 3$. If G is an r-regular graph of order $2r$, $G \neq K_{r,r}$, or an r-regular of order $2r+1$, then G is Hamilton-connected.*

This theorem can be extended in the following way:

THEOREM 8. *Let k, n be natural numbers and $r = \lfloor (n+k-2)/2 \rfloor$. If G is an r-regular graph of order n different from $K_{r,r}$ and if*

(i) $2 \leqslant k < (n+2)/3$ *in case $n+k$ is even, or*

(ii) $2 \leqslant k < (n+1)/3$ *in case $n+k$ is odd*

holds, then G is k-leaf-connected.

Proof. Case (i) $n+k$ is even. For $k=2$ the assertion is one of the statements of Theorem 7 due to Tomescu. Consider then $3 \leqslant k < (n+2)/3$.

Let S be a subset of VG with $|S| = k$ and suppose G has no S-spanning tree. Let $S' := S \backslash \{x\}$, $x \in S$. As $r = \lfloor (n+k-2)/2 \rfloor = (n+(k-1)-1)/2$, by Corollary 6.3 G has an S'-spanning tree.

Consider an S'-spanning tree T in G such that $t := d_T(x)$ is minimum, and let $x_1, x_2, \ldots, x_t$ be the vertices of T which are adjacent to x in T.

Deleting the vertex x from T we obtain t disjoint trees, say $T_1, T_2, \ldots, T_t$, such that $x_i \in VT_i$, $1 \leqslant i \leqslant t$. Set

$$S_i := S' \cap VT_i, \qquad 1 \leqslant i \leqslant t.$$

We claim that

$$N_G(x_i) \cap (VT_j \backslash S_j) = \varnothing \qquad \text{for all} \quad i \neq j,\ 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant t. \tag{1}$$

Otherwise, if there is a vertex w in the intersection then

$$\hat{T} := T + x_i w - x x_i$$

is either an S'-spanning tree with $d_{\hat{T}}(x) < t$ (contradicting the choice of T), or is an S-spanning tree (in case $d_{\hat{T}}(x) = 1$), contradicting the assumption. Thus

$$d_G(x_i) \leqslant |VT_i| - |S_i| + |S'| \qquad \text{for all } i,\ 1 \leqslant i \leqslant t, \tag{2}$$

and hence

$$\sum_{i=1}^{t} d_G(x_i) \leqslant \sum_{i=1}^{t} |VT_i| - \sum_{i=1}^{t} |S_i| + t(k-1) = n - 1 + (t-1)(k-1).$$

As $d_G(x_i) = (n+k-2)/2$, $1 \leqslant i \leqslant t$, we obtain $(t-2)n \leqslant (t-2)k$. Since $n > k$ we must have $t = 2$. Since $d_G(x_1) + d_G(x_2) = n + k - 2$ and $|VT_1| - |S_1| + |S'| + |VT_2| - |S_2| + |S'| = n + k - 2$, it follows that equality holds in (2) and

$$|VT_1| - |S_1| = |VT_2| - |S_2| = \frac{n-k}{2} =: m. \tag{3}$$

Since $t = 2$ and (1) holds, then

$$N_G(x_1) \subseteq (VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2.$$

As $d_G(x_1) = |N_G(x_1)| \leqslant |(VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2| = |VT_1| + |S_2| = d_G(x_1)$, the inclusion above must be an equality, i.e.,

$$N_G(x_1) = (VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2. \tag{4}$$

Suppose $N_G(VT_1 \backslash S_1) \cap (VT_2 \backslash S_2) = \varnothing$. We shall prove that this implies that every vertex $w \in (VT_1 \backslash S_1) \cup (VT_2 \backslash S_2)$ is adjacent to x. Assume the contrary. Then using (3) we conclude that

$$d(w) \leqslant |S_1| + |S_2| + m - 1 = \frac{n+k-4}{2},$$

a contradiction. Therefore, $N_G(x) \supseteq (VT_1 \backslash S_1) \cup (VT_2 \backslash S_2)$ and thus

$$\frac{n+k-2}{2} = d_G(x) \geqslant |(VT_1 \backslash S_1) \cup (VT_2 \backslash S_2)| = n-k,$$

which implies that $k \geqslant (n+2)/3$, contradicting the hypothesis.

Hence, $N_G(VT_1 \backslash S_1) \cap (VT_2 \backslash S_2) \neq \emptyset$ and since (1) holds there exists a vertex $w \in (VT_1 \backslash S_1)$ and a vertex $z \in (VT_2 \backslash S_2)$ such that $wz \in EG$, $w \neq x_1$, $z \neq x_2$. Let

$$\hat{T} := T - x_1 x + wz.$$

If $d_{\hat{T}}(x_1) > 1$ then $\hat{T}$ is an S-spanning tree in G. Otherwise, let P be the path in $\hat{T}$ with ends w and x_1. As $d_{\hat{T}}(w) \geqslant 3$ there is a vertex $y \in (VT_1 \backslash VP)$ such that $yw \in E\hat{T}$. Since $yx_1 \notin E\hat{T}$ and by (4) $yx_1 \in EG$, then

$$\tilde{T} := \hat{T} - yw + yx_1$$

is an S-spanning tree in G, again a contradiction. Thus the proof of Case (i) is complete.

Case (ii) $n+k$ is odd. As $r = \lfloor (n+k-2)/2 \rfloor = (n+k-3)/2$, for $k=2$ the assertion is true by Theorem 7. Consider now $3 \leqslant k < (n+1)/3$.

Since $r = (n + (k-1) - 2)/2$ and $n + (k-1)$ is even, then by the previous case, it follows that G is $(k-1)$-leaf-connected.

Let S, S', T, x, t, x_i, S_i, T_i be as defined in the Case (i), and suppose G has no S-spanning tree. Let us also assume that

$$N_G(x_i) \cap (VT_j \backslash S_j) = \emptyset \qquad \text{for all } i \neq j,\ 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant t. \tag{5}$$

Otherwise, a contradiction follows easily. Thus

$$d_G(x_i) \leqslant |VT_i| - |S_i| + |S'| \qquad \text{for all } i,\ 1 \leqslant i \leqslant t, \tag{6}$$

and hence

$$t\left(\frac{n+k-3}{2}\right) = \sum_{i=1}^{t} d_G(x_i) \leqslant n - 1 + (t-1)(k-1),$$

that is,

$$t \leqslant \frac{2(n-k)}{n-k-1}.$$

Since $3 \leqslant k < (n+1)/3$ it follows that $t < 3$, and therefore $t = 2$. But then, since $d_G(x_1) + d_G(x_2) = n + k - 3$ and $|VT_1| - |S_1| + |S'| + |VT_2| - |S_2| +$

$|S'| = n+k-2$, and since (6) holds, we may assume w.l.o.g. that $d_G(x_1) = |VT_1| - |S_1| + |S'|$ and $d_G(x_2) = |VT_2| - |S_2| + |S'| - 1$. Thus,

$$|VT_1| - |S_1| = |VT_2| - |S_2| - 1 = \frac{n-k-1}{2}.$$

From (5) we also conclude that $N_G(x_1) \subseteq (VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2$. Since $|N_G(x_1)| = |(VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2|$, then

$$N_G(x_1) = (VT_1 \backslash \{x_1\}) \cup \{x\} \cup S_2. \tag{7}$$

Let us assume now that

$$N_G(VT_1 \backslash S_1) \cap (VT_2 \backslash S_2) = \varnothing. \tag{8}$$

Thus, for any vertex $w \in VT_1 \backslash S_1$, $N_G(w) \subseteq (VT_1 \backslash \{w\}) \cup \{x\} \cup S_2$. As above we conclude that

$$N_G(w) = (VT_1 \backslash \{w\}) \cup \{x\} \cup S_2 \qquad \text{for all } w \in VT_1 \backslash S_1.$$

Let $s \in S_1 \cup S_2$. Since every vertex in $VT_1 \backslash S_1$ is adjacent to s and $|VT_1 \backslash S_1| = (n-k-1)/2$, then $|N_G(s) \cap ((VT_2 \backslash S_2) \cup \{x\})| \leqslant (n+k-3)/2 - (n-k-1)/2 = k-1$. Thus

$$|\delta(S_1 \cup S_2, (VT_2 \backslash S_2) \cup \{x\})| \leqslant |S_1 \cup S_2| \cdot (k-1). \tag{9}$$

Since x is adjacent to all vertices of $VT_1 \backslash S_1$, x is adjacent to at most $k-1$ vertices of $VT_2 \backslash S_2$. This fact and (9) imply that

$$|\delta(S_1 \cup S_2 \cup \{x\}, VT_2 \backslash S_2)| \leqslant |S_1 \cup S_2| \cdot (k-1) + k - 1 = k(k-1). \tag{10}$$

On the other hand, since $|VT_2 \backslash S_2| = (n-k+1)/2$, every vertex in $VT_2 \backslash S_2$ must be adjacent to at least $(n+k-3)/2 - (n-k+1)/2 + 1 = k-1$ vertices not in $VT_2 \backslash S_2$. Thus, assuming (8),

$$|\delta(S_1 \cup S_2 \cup \{x\}, VT_2 \backslash S_2)| \geqslant (k-1)\frac{(n-k+1)}{2}. \tag{11}$$

Now (10) and (11) imply that $(k-1)((n-k+1)/2) \leqslant k(k-1)$, that is, $k \geqslant (n+1)/3$. Since this contradicts the hypothesis, we conclude that (8) cannot occur. Therefore,

$$N_G(VT_1 \backslash S_1) \cap (VT_2 \backslash S_2) \neq \varnothing,$$

and the proof follows analogously to Case (i). ∎

Remark. The upper bounds on k given in Theorem 8 are best possible. To prove this let us show that for all pairs (n, k), $n \geq 7$, such that

(i) $n+k$ is even, $(n+2)/3 \leq k \leq n-4$ and k or r is even, or

(ii) $n+k$ is odd and $(n+1)/3 \leq k \leq n-5$

holds, there exists an r-regular graph G of order n, with $r = \lfloor (n+k-2)/2 \rfloor$, $G \neq K_{r,r}$, which is not k-leaf-connected.

Let S, Y, Z be disjoint sets of vertices, with $|S| = k$, $|Z| = \lfloor (n-k)/2 \rfloor$ and $|Y| = \lfloor (n-k+1)/2 \rfloor$.

(a) Suppose (i) holds. In this case, note that $|Z| = |Y| = (n-k)/2$ and $r = (n+k-2)/2$. Let $VH := S \cup Z \cup Y$ and $EH := \delta(Z, Z \cup S) \cup \delta(Y, Y \cup S)$. Thus, $d_H(v) = r$ for all $v \in Z \cup Y$ and $d_H(v) = n-k$ for all $v \in S$.

Let $p := r-(n-k)$. Since $k \geq (n+2)/3$, then $p \geq 0$. If $p = 0$ then H is the desired graph G, otherwise let G be the graph obtained from H by completing it to an r-regular graph. That is, by adding edges with both ends in S so that $d_G(s) = d_H(s) + p = r$ for all $s \in S$. Since there is no edge in EH with both ends in S, such a completion is possible if the degree sequence $d = (d_1, d_2, \ldots, d_k)$ with $d_i = p$, $1 \leq i \leq k$, is graphic. It is easy to see that since $p < k-1$, the sequence d is graphic if k or p is even. Note that if (i) holds then in case k is odd, p is even.

Clearly, G has no S-spanning tree ($G - S$ is disconnected) and $G \neq K_{r,r}$.

(b) Suppose (ii) holds. In this case, note that $|Z| = |Y| - 1 = (n-k-1)/2$ and $r = (n+k-3)/2$. Let $VH := S \cup Z \cup Y$ and $EH := \delta(Z, Z \cup S)$. Note that to show that H can be completed to an r-regular graph G, it sufficies to show that when (ii) holds the degree sequence $d = (d_1, d_2, \ldots, d_q)$, where $q = n - |Z| = (n+k+1)/2$, $d_i = k-1$ for $1 \leq i \leq k$, and $d_i = r$ for $k < i \leq q$, is graphic. (The proof is long but not difficult and therefore it will be omitted.) It is immediate that G is not bipartite and G has no S-spanning tree ($G - S$ is disconnected).

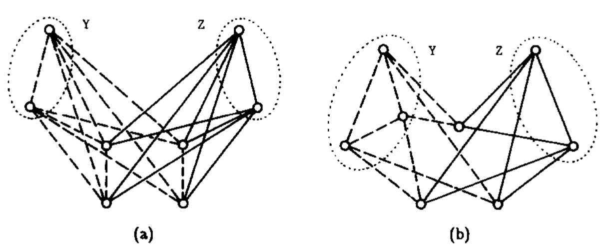

FIG. 2. (a) A 5-regular, not 4-leaf-connected graph G, $G \neq K_{5,5}$; (b) A 4-regular, not 3-leaf-connected graph G, $G \neq K_{4,4}$.

In Fig. 2 we give examples obtained by using the previous constructions. The edges added to the initial graph H to obtain G are indicated by dashed lines.

4. Families of Sparse k-Leaf-Connected Graphs

In this section we show the existence of k-leaf-connected graphs which have the least possible number of edges.

Note that by Corollary 1.1, if $k \leqslant n-2$ a k-leaf-connected graph of order n must have at least $\lceil (k+1)n/2 \rceil$ edges. So, if we denote by $\varphi(k, n)$ *the least number of edges that a k-leaf-connected graph on n vertices can have,* then we know that

$$\varphi(k, n) \geqslant \left\lceil \frac{(k+1)n}{2} \right\rceil, \qquad \text{for} \quad 2 \leqslant k \leqslant n-2. \tag{12}$$

(Clearly, $\varphi(n-2, n) = \varphi(n-1, n) = (n-1)n/2$.)

It is also known from the following theorem of Moon ([1, pp. 221]) that for $k=2$ equality holds in (12), that is,

$$\varphi(2, n) = \lceil 3n/2 \rceil, \qquad \text{for} \quad n \geqslant 4.$$

THEOREM 9 (Moon). *For every $n \geqslant 4$, there exists a Hamilton-connected graph M_n of order n with exactly $\lceil 3n/2 \rceil$ edges.*

The graphs given by Moon are shown in Fig. 3.

Now the question that remains to be answered is whether equality holds in (12) also for $3 \leqslant k < n-2$. The answer is YES, as it will be shown in the sequel. We shall consider for each pair (k, n), $3 \leqslant k < n-2$, a graph $H_{k+1,n}$ which is $(k+1)$-connected and has $\lceil (k+1)n/2 \rceil$ edges, and shall prove that $H_{k+1,n}$ is k-leaf-connected.

The construction of the graphs $H_{k+1,n}$ to be considered is due to Harary [4] and can be found in [1 and 3]. We follow the notation of [3], and consider all the additions taken modulo n.

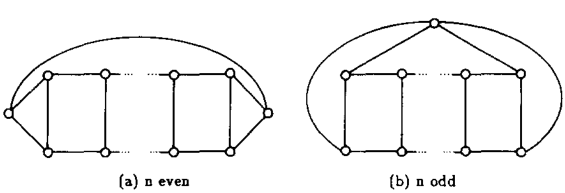

FIG. 3. Sparse Hamilton-connected graphs M_n.

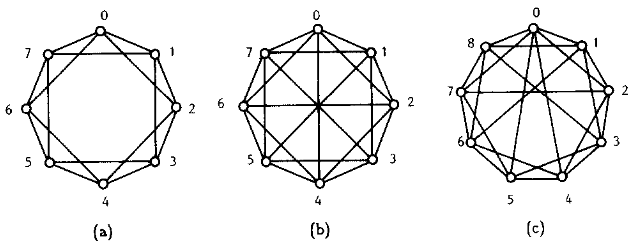

FIG. 4. (a) $H_{4,8}$; (b) $H_{5,8}$; (c) $H_{5,9}$.

Construction of $H_{m,n}$. *Case* 1. m even. Let $m = 2r$. Then $H_{2r,n}$ is constructed as follows. It has vertices 0, 1,..., $n-1$, and two vertices i and j are joined if $i - r \leqslant j \leqslant i + r$ (see Fig. 4a).

Case 2. m odd, n even. Let $m = 2r + 1$. Then $H_{2r+1,n}$ is obtained by first constructing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i + n/2$ for $1 \leqslant i \leqslant n/2$ (see Fig. 4b).

Case 3. m odd, n odd. Let $m = 2r + 1$. Then $H_{2r+1,n}$ is obtained by first constructing $H_{2r,n}$ and then adding edges joining:

(a) vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$,

(b) vertex i to vertex $i + (n+1)/2$ for $1 \leqslant i < (n-1)/2$ (see Fig. 4c).

THEOREM 10 (Harary). *The graph $H_{k+1,n}$ is $(k+1)$-connected and has $\lceil (k+1)n/2 \rceil$ edges.*

LEMMA 11. *The graph $H_{k+1,n}$ is Hamilton-connected if $k \geqslant 3$.*

Proof. Since for $k \geqslant 3$ the graph $H_{k+1,n}$ contains the graph $H_{4,n}$, it suffices to prove that $H_{4,n}$ is Hamilton-connected. Let i and j be any two vertices in $H_{4,n}$. We shall prove that $H_{4,n}$ has a Hamilton path P with ends i and j. Assume that $0 \leqslant i < j - 1 \leqslant n - 2$, otherwise the result is obvious. Let $m := \lfloor (j-i)/2 \rfloor$. If $j - i$ is odd then $j - 2m = i + 1$. In this case, the claimed path is

$$P := (j, j-2, j-4, ..., j-2m, i+2, i+4, ..., i+2m, j+1, j+2, ..., i).$$

If $j - i$ is even then $j - 2(m-1) = i + 2$. In this case, let

$$P := (j, j-2, j-4, ..., j-2(m-1), i+1, i+3, ..., i+(2m+1), \\ j+2, j+3, ..., i). \quad \blacksquare$$

Remark. The graph $H_{3,n}$ is bipartite for $n=2p$, $p \geqslant 3$, p odd, and therefore non-Hamilton-connected. Also for $n=2p+1$, $p \geqslant 5$, p odd, it is easy to see that $H_{3,n}$ is non-Hamilton-connected.

Before we prove the next theorem, let us introduce the notation and terminology needed in the sequel. The hamiltonian cycle $(0, 1, ..., n-1)$ contained in $H_{k+1,n}$ is denoted by C. For any two vertices x and y in C we denote by $C(x, y)$, (resp. $C^{-1}(x, y)$), the path on C defined when one traverses C clockwise, (resp. counterclockwise), from x to y. If S is a subset of vertices and x, y are in S, we say that x S-precedes y in C if $C(x, y)$ contains no other vertex from S, except x and y. If x is a vertex, we denote by $x'$and x'' the two neighbours of x in C, choosing them in a such way that x is contained in $C(x', x'')$.

Now using an idea similar to the one used in the proof of Lemma 11, the following lemma can be easily proved.

LEMMA 12. *Consider the graph $H_{4,n}$. Let x and y be any two vertices not consecutive on C, and let $\bar{x}$ be any vertex in $C(y, x)$, $\bar{x} \neq x$. Then $H_{4,n}$ has a path from $\bar{x}$ to x containing all and only the vertices in $C(y, x)$.*

THEOREM 13. *The graph $H_{k+1,n}$ is k^*-leaf-connected if $k \geqslant 3$.*

Proof. Assume that $k \leqslant n-3$, otherwise the result is obvious. Let S be a subset of vertices with $|S| = m$, $3 \leqslant m \leqslant k$, and let us prove that $H_{k+1,n}$ has an S-spanning tree T.

Denote the elements of S by $s_1, s_2, ..., s_m$ in a such a way that s_i S-precedes s_{i+1} in C, $1 \leqslant i \leqslant m$, and $C(s_1, s_2)$ has at least 3 vertices.

The tree T will be constructed stepwise. We first construct a tree T' with leaf set S', $S' \subset S$, then (if necessary) we extend it to a tree T'' with leaf set S'', $S' \subseteq S'' \subset S$, and finally we extend T'' to T.

PROCEDURE α (Construction of T').

Input: $H_{k+1,n}$ and $S = \{s_1, ..., s_m\}$ (according to the convention).
Output: An index $j(\lceil m/2 \rceil + 1 \leqslant j \leqslant m)$ and a tree T' with leaf set

$$S' = \{s_1, s_2, ..., s_{j-1}\}.$$

(I) [Initialization]

$$T' := C(s_1, s_2); \quad S' := \{s_1, s_2\}; \quad t := 1; \quad i := 2;$$

$u := s_2'$ ("predecessor" of s_2); $v := s_2''$ ("successor" of s_2);

(II) [Iteration]
While ($i < m - 1$ and $t < \lceil k/2 \rceil$) do

```
begin T' := T' + uv;          i := i + 1;
      if v ≠ s_i then begin   T' := T' ∪ C(v, s_i);
                              S' := S' ∪ {s_i};
                              u := s'_i; t := 1;
                      end
                else          t := t + 1;
      v := s''_i;
end;
```

(III) [Returning value of j]

$j := i + 1$. [STOP.]

Note that at the end of the Procedure α, T' is a tree (with leaf set $S' = \{s_1, s_2, \ldots, s_{j-1}\}$) containing all and only the vertices of $C(s_1, s_{j-1})$. Now if $j < m$, by applying the next Procedure β (symmetric to Procedure α), we extend T' to a tree T'' with leaf set $S'' = S \setminus \{s_j\}$.

PROCEDURE β (Construction of T'').

Input: $H_{k+1,n}$, $S = \{s_1, \ldots, s_m\}$,
T', S' and j (output of Procedure α).

Output: A tree $T'' \supseteq T'$ with leaf set $S'' = S \setminus \{s_j\}$,
a vertex z in $VT'' \setminus S''$ (to be used in the construction of T).

(I) [Initialization]

$T'' := T'$; $S'' := S'$; $u := s''_1$; $v := s'_1$; $i := m$;

(II) [Iteration]

```
While i > j do begin
               T'' := T'' + uv;
               if v ≠ s_i then   begin
                                 T'' := T'' ∪ C^{-1}(v, s_i);
                                 S'' := S'' ∪ {s_i};   u := s''_i;
                                 end;
               v := s'_i;
               i := i - 1;
               end;
```

(III) [Returning the attachment vertex z]

$z := u$. [STOP].

At the end of the Procedure β, T'' is a tree with leaf set $S'' = S \setminus \{s_j\}$ containing all and only the vertices of $C(s_{j+1}, s_{j-1})$. (If $j = m$ then $T'' = T'$ and $z = s_1''$.) Now in order to obtain T it remains to include the vertices of $C(s_{j-1}'', s_{j+1}')$, assuring that s_j will become a leaf of it. To do that, we define different constructions according to the adjacency relation between s_{j+1}, s_j and s_{j-1}.

Suppose s_j and s_{j-1} are adjacent in C. If s_{j+1} and s_j are also adjacent in C then let $T := T'' + zs_j$, otherwise let $T := (T'' + zs_{j+1}') \cup C^{-1}(s_{j+1}', s_j)$. Now suppose s_j and s_{j-1} are non-adjacent in C. If s_{j+1} and s_j are also non-adjacent in C, let $x := s_j$, $y := s_{j-1}''$ and $P(x', x)$ be a path from x' to x (see Lemma 12) containing all and only the vertices in $C(y, x)$. In this case, set $T := (T'' + zs_{j+1}') \cup (C^{-1}(s_{j+1}', x'') + x''x') \cup P(x', x)$. If s_{j+1} and s_j are adjacent in C, then (since $G - S$ is connected) there is a vertex w in $VT'' \setminus S''$ which is adjacent to a vertex, say x, in $C(s_{j-1}'', s_j')$. If $x = s_{j-1}''$ let $T := (T'' + wx) \cup C(x, s_j)$; otherwise let $y := s_{j-1}''$, $P(x', x)$ be a path from x' to x containing all and only the vertices in $C(y, x)$, and set $T := (T'' + wx) \cup (P(x', x) + x'x'') \cup C(x'', s_j)$.

Clearly, in all cases above T is an S-spanning tree. Since by Lemma 11, $H_{k+1,n}$ is Hamilton-connected, the proof is complete. ∎

Now, from Theorems 9, 10, and 13 and inequality (12) we conclude that

$$\varphi(k, n) = \left\lceil \frac{(k+1)n}{2} \right\rceil, \qquad \text{for} \quad 2 \leqslant k \leqslant n-2.$$

The graphs M_n and $H_{k+1,n}$, $k \geqslant 3$, are examples of sparsest possible graphs with property $P(2)$ and $P^*(k)$, respectively. As the next theorem shows, one can combine them to obtain other sparse k-leaf-connected graphs.

THEOREM 14. *Suppose H is a graph with property $P^*(k-1)$, $k \geqslant 3$, $|VH| \geqslant 4$. Then the graph $G = H \times K_2$ obtained by the product of H and the complete graph K_2, has property $P^*(k)$.*

Proof. In order to simplify notation we consider that the graph $G = H \times K_2$ is obtained as follows. We take a graph H' isomorphic to H and let ψ: $VH \to VH'$ be the bijection that defines the isomorphism between H and H'. Then $VG := VH \cup VH'$ and $EG := EH \cup EH' \cup \{v\psi(v)\colon v \in VH\}$.

For any $v \in VH$ let $\psi(v)$ be denoted by v'. Now consider a subset S of VG with $|S| = m$, $3 \leqslant m \leqslant k$, and let us prove that G has an S-spanning tree.

By symmetry it suffices to consider the cases where $|S \cap VH| = p$, $\lceil m/2 \rceil \leqslant p \leqslant m$.

Case 1. $|S \cap VH| = p$, $\lceil m/2 \rceil \leqslant p \leqslant m-1$. Let $X := S \cap VH$. Since $2 \leqslant |X| < k$, by hypothesis there is an X-spanning tree T in H. Let

$\bar{X} := S \backslash X$. Then clearly, $|\psi^{-1}(\bar{X}) \cup X| \leqslant k$. As H has property $P(k-1)$, by Corollary 1.1, $|VH| \geqslant k+1$, and therefore, there is a vertex $v \in VH \backslash (\psi^{-1}(\bar{X}) \cup X)$.

Now, let T' be an $\bar{X}$-spanning tree in H' if $|\bar{X}| \geqslant 2$. If $\bar{X} = \{x'\}$ let T' be a Hamilton path in H' with ends x' and v'. In any case, $T \cup T' + vv'$ is an S-spanning tree in G.

Case 2. $|S \cap VH| = m$. Let x be any vertex in S, T be an $(S \backslash \{x\})$-spanning tree in H, Y be the set of neighbours of x in T, $Y' := \psi(Y)$ and T' be a Y'-spanning tree in H'. Then $T \cup T' - \{xy: y \in Y\} + \{yy': y \in Y\} + xx'$ is an S-spanning tree in G.

Now it remains to be proved that G has property $P(2)$. The proof is easy and we leave it to the reader. ▮

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