JOURNAL OF FUNCTIONAL ANALYSIS 47, 381–418 (1982)

Green's and Dirichlet Spaces Associated with Fine Markov Processes*

E. B. DYNKIN

Department of Mathematics, Cornell University, Ithaca, New York 14850

Communicated by L. Gross

Received March 15, 1982

This is the second paper in a series devoted to Green's and Dirichlet spaces. In the first paper, we have investigated Green's space $\mathscr{H}$ and the Dirichlet space $\mathscr{H}$ associated with a symmetric Markov transition function $p_t(x, B)$. Now we assume that p is a transition function of a fine Markov process X and we prove that: (a) the space $\mathscr{H}$ can be built from functions which are right continuous along almost all paths; (b) the positive cone $\mathscr{H}^+$ in $\mathscr{H}$ can be identified with a cone M of measures on the state space; (c) the positive cone $\mathscr{H}^+$ in $\mathscr{H}$ can be interpreted as the cone of Green's potentials of measures $\mu \in M$. To every measurable set B in the state space E there correspond a subspace $\mathscr{H}(B)$ of $\mathscr{H}$ and a subspace $\mathscr{H}(B)$ of $\mathscr{H}$. The orthogonal projections of $\mathscr{H}$ onto $\mathscr{H}(B)$ and of $\mathscr{H}$ onto $\mathscr{H}(B)$ can be expressed in terms of the hitting probabilities of B by the Markov process X. As the main tool, we use additive functionals of X corresponding to measures $\mu \in M$.

1. Introduction

1.1. We denote by R^+ the open positive half-line $(0, \infty)$ and by $\mathscr{B}_{R+}$ the Borel σ-algebra in R^+.

A (stationary) *transition function* in a measurable space $(E, \mathscr{B})$ is a function $p_t(x, B)$, $t > 0$, $x \in E$, $B \in \mathscr{B}$ with the following properties:

1.1.A. For every $t, x, p_t(x, \cdot)$ is a measure on $\mathscr{B}$.

1.1.B. For every $B \in \mathscr{B}$, $p_t(x, B)$ is $\mathscr{B}_{R+} \times \mathscr{B}$-measurable in t, x.

1.1.C. $p_t(x, E) \leqslant 1$ for all t, x and it tends to 1 as $t \to 0$.

1.1.D. For all s, t, x, B

$$\int_E p_s(x, dy)\, p_t(y, B) = p_{s+t}(x, B).$$

* Research supported by NSF Grant MCS 77-03543.

0022-1236/82/090381–38$02.00/0

The corresponding operators T_t act on functions and measures by the formulas

$$T_t f(x) = \int_E p_t(x, dy) f(y), \tag{1.1}$$

$$(\mu T_t)(B) = \int_E \mu(dx)\, p_t(x, B). \tag{1.2}$$

Let m be a σ-finite measure on $(E, \mathscr{B})$. A transition function p is called *symmetric relative* to m if

$$\int_A m(dx)\, p_t(x, B) = \int_B m(dx)\, p_t(x, A) \qquad \text{for all } A, B \in \mathscr{B}. \tag{1.3}$$

In this paper *we assume that $(E, \mathscr{B})$ is a standard Borel space*, i.e., it is isomorphic to a Borel subset of a complete separable metric space. We also assume that:

1.1.E. For every $x \neq y$ there exist $t \in R^+$ and $B \in \mathscr{B}$ such that $p_t(x, B) \neq p_t(y, B)$.

We put

$$G_c = \int_0^c T_t\, dt, \qquad c \in R^+, \tag{1.4}$$

$$G = \int_{R^+} T_t\, dt. \tag{1.5}$$

It follows from 1.1.E. that the functions:

$$\{G_c f : c \in R^+, f \in \mathscr{B}, f \text{ is bounded}\} \tag{1.6}$$

separate points of E and generate $\mathscr{B}$ (see, e.g., [6, Lemma 2.1]).[1]

We consider only dissipative (transient) transition functions, i.e., we assume that:

1.1.F. $Gf(x) < \infty$ m-a.e. for every m-integrable function $f \in \mathscr{B}$.

1.2. In [10] we constructed two models K and $\mathscr{K}$ for Green's space and two models H and $\mathscr{H}$ for the Dirichlet space associated with $p_t(x, B)$. We outline the results of [10] in Subsections 1.2–1.4.

Operators T_t defined by the formula (1.1) preserve m-equivalence of functions and determine a self-adjoint contraction semi-group in $L^2(m)$.

[1] Let $\mathscr{B}$ be a σ-algebra in a space E and let f be a real-valued function on E. We write $f \in \mathscr{B}$ if f is positive and $\mathscr{B}$-measurable, and we denote by $\mu(f)$ the integral of f with respect to a measure μ. (If f is defined on a subset $E' \in \mathscr{B}$, then $\mu(f)$ means the integral of f over E').

We put $(f, g) = m(fg)$, $\|f\| = (f,f)^{1/2}$. For every $f \in \mathscr{B}^2$, $\|f\|_G^2 = (Gf, f) = \int_{R^+} (T_{t/2}f, T_{t/2}f)\, dt$.

The property 1.1.F is equivalent to:

1.2.A. There exists a strictly positive function $\rho \in \mathscr{B}$ such that $\|\rho\|_G < \infty$.

A real-valued function $\phi_t(x)$, $t > 0$, $x \in E$ is called a K-function if $m(\phi_t^2) < \infty$ for all t and $T_s\phi_t(x) = \phi_{s+t}(x)$ for all s, t, x. These conditions imply: $\phi_t(x)$ is $\mathscr{B}_{R^+} \times \mathscr{B}$-measurable in t, x; ϕ_t is a continuous $L^2(m)$-valued function which tends to 0 as $t \to \infty$.

A real-valued function $h_{s,u}(x)$, $0 \leqslant s < u$, $x \in E$ is called an H-function if $m(h_{s,u}^2) < \infty$ for all s, u, $h_{s,t} + h_{t,u} = h_{s,u}$ for all $s < t < u$ and all x and $T_t h_{s,u} = h_{s+t,u+t}$ for all s, u, t, x. We say that h is differentiable if the limit

$$h_t' = \lim_{s \uparrow t, u \downarrow t} \frac{h_{s,u}}{u - s}$$

exists in $L^2(m)$ for all $t \in R^+$. Obviously h' is a K-function.

Two K-functions ϕ and $\tilde{\phi}$ are m-equivalent if $\phi_t = \tilde{\phi}_t$ m-a.e. for each t. Analogously, two H-functions h and $\tilde{h}$ are m-equivalent if $h_{s,u} = \tilde{h}_{s,u}$ m-a.e. for all s, u.

Elements of the space K are classes of m-equivalent K-functions ϕ such that $\int_{R^+} \|\phi_t\|^2\, dt < \infty$. Elements of the space H are classes of m-equivalent H-functions h such that $\int_{R^+} \|h_t'\|^2\, dt < \infty$. K and H are Hilbert spaces with the inner products

$$(\phi, \tilde{\phi})_K = \int_0^\infty \int_E \phi_{t/2}(x)\, \tilde{\phi}_{t/2}(x)\, dt\, m(dx), \tag{1.7}$$

$$(h, \tilde{h})_H = (h', \tilde{h}')_K = \int_0^\infty \int_E h_{t/2}'(x)\, \tilde{h}_{t/2}'(x)\, dt\, m(dx). \tag{1.8}$$

They are in duality with respect to the form

$$\langle \phi, h \rangle = \int_0^\infty \int_E \phi_{t/2}(x)\, h_{t/2}'(c)\, dt\, m(dx). \tag{1.9}$$

The mapping $\phi = h'$ is an isometry of H onto K. The inverse is given by the formula

$$h_{s,u} = \int_s^u \phi_t\, dt, \qquad 0 \leqslant s < u. \tag{1.10}$$

1.3. We put $f \in \mathscr{H}$ if $T_t|f| < \infty$ m-a.e. for all t and if there exists an $h \in H$ such that

$$h_{s,u} = T_s f - T_u f \qquad m\text{-a.e. for every } 0 \leqslant s < u \tag{1.11}$$

(we set $T_s f = f$ if $s = 0$). Condition (1.11) can be rewritten in the form

$$\int_s^u \phi_t \, dt = T_s f - T_u f \qquad m\text{-a.e. for every } 0 \leqslant s < u, \tag{1.12}$$

where $\phi = h'$ is the element of K corresponding to h.

Let

$$\Phi_t(f) = (2t)^{-1} \int_E \int_E m(dx)\, p_t(x, dy)\, |f(x) - f(y)|^2 + t^{-1} \int_E m(dx)(1 - p_t(x, E)) f(x)^2. \tag{1.13}$$

It turns out that, for every $f \in \mathscr{H}$, the limit

$$(f, f)_{\mathscr{H}} = \lim_{t \downarrow 0} \Phi_t(f) \tag{1.14}$$

exists and

$$(f, f)_{\mathscr{H}} = (h, h)_H \tag{1.15}$$

if f and h are connected by (1.11). Hence $\mathscr{H}$ with the inner product (1.14) is a Hilbert space and (1.11) establishes an isometry between H and $\mathscr{H}$.[2]

It is proved in [10] that:

1.3.A. If $f \in \mathscr{H}$ and $h \in \mathscr{H}$ are connected by (1.11), then $h_{s,u} \in \mathscr{H}$ for every $0 < s < u$ and

$$f = \lim_{s \to 0,\, u \to \infty} h_{s,u}$$

in $\mathscr{H}$ and in $L^1(m^\rho)$ where $m^\rho(dx) = \rho(x)\, m(dx)$, $\|\rho\|_G < \infty$.

1.3.B. For every $f \in \mathscr{H}$,

$$m^\rho(|f|) \leqslant \|\rho\|_G \, \|f\|_{\mathscr{H}}.$$

Hence if $f_n \to f$ in $\mathscr{H}$, then a subsequence f_{n_k} converges to f m-a.e.

1.3.C. For every $f \in \mathscr{H}$, $\|T_t f - f\|_{\mathscr{H}} \to 0$ as $t \to 0$ and $\|T_t f\|_{\mathscr{H}} \to 0$ as $t \to \infty$.

[2] Abusing notations we use the same letter for a set of functions and for a set of classes of m-equivalent functions.

1.3.D. If $f \in \mathscr{H}$, then $|f| \in \mathscr{H}$ and $\| |f| \|_{\mathscr{H}} \leqslant \|f\|_{\mathscr{H}}$.
(See [10], (3.6), (5.4), (5.13), (3.9) and Corollary 1 to Theorem 5.3).

We need also the following proposition:

1.3.E. If $\|\rho\|_G < \infty$, then $q = G\rho \in \mathscr{H}$ and for every $f \in \mathscr{H}$

$$(q, f)_{\mathscr{H}} = (\rho, f).$$

To prove 1.3.E, we consider $h \in \mathscr{H}$ corresponding to f and $\psi \in K$ defined by the formula $\psi_t = T_t \rho$. We note that $T_s q - T_u q = \int_s^u \psi_t \, dt$ and, by (1.15) and (1.8),

$$(q, f)_{\mathscr{H}} = (\psi, h')_K = \int_{R^+} (\psi_{t/2}, h'_{t/2}) \, dt.$$

On the other hand,

$$(\rho, h_{s,u}) = \int_s^u (\rho, h'_t) \, dt = \int_s^u (\psi_{t/2}, h'_{t/2}) \, dt$$

and, by 1.3.A, $(\rho, h_{s,u}) \to (\rho, f)$ as $s \to 0$, $u \to \infty$.

We define $\mathscr{K}$ as the space of all linear functionals on $\mathscr{H}$. Using the duality (1.9), we map K onto $\mathscr{K}$: the functional corresponding to $\phi \subset K$ is given by the formula $k(f) = \langle h, \phi \rangle$ where h is defined by (1.11).

1.4. The positive cones in K and H are defined by the conditions: $\phi \in K^+$ if $\phi \in K$ and $\phi_t \geqslant 0$ m-a.e. for all t; $h \in H^+$ if $h \in H$ and $h_{s,u} \geqslant 0$ m-a.e. for all s, u. We note that $h \in H^+$ if and only if $h' \in K^+$.

A function $f \in \mathscr{B}$ is called *almost excessive* if $f < \infty$ m-a.e. and $T_t f \leqslant f$ m-a.e. for every t. We denote by $\mathscr{H}^+$ the set of all almost excessive functions in $\mathscr{H}$ and by $\mathscr{K}^+$ the set of positive functionals, i.e., set of $k \in \mathscr{K}$ such that $k(f) \geqslant 0$ if $f \geqslant 0$ m-a.e.

We note that $\mathscr{H}^+$ and H^+ correspond to each other under the isometry (1.11), and $\mathscr{K}^+$ and K^+ correspond to each other under the natural isometry between $\mathscr{K}$ and K. If $\phi \in K^+$ and if f is the corresponding element of $\mathscr{H}$, then by (1.12)

$$f = \int_{R^+} \phi_t \, dt. \tag{1.16}$$

It is known that:

1.4.A. If $f \leqslant g$ m-a.e. and if f, g are almost excessive functions, then $\Phi_t(f) \leqslant \Phi_t(g)$ for all t.

1.4.B. A function f belongs to $\mathscr{H}^+$ if and only if it is almost excessive and $\Phi_t(f)$ is bounded.

1.4.C. Put $f \in \mathscr{H}'$ if $f \in \mathscr{H}$ and $f \geqslant 0$ m-a.e. A function $f \in \mathscr{H}'$ belongs to $\mathscr{H}^+$ if and only if $(f, \tilde{f})_{\mathscr{H}} \geqslant 0$ for all $f \in \mathscr{H}'$.

1.4.D. If $k \in \mathscr{K}^+$ corresponds to $\phi \in K^+$, then

$$k(f) = \lim_{\varepsilon \to 0} (\phi_\varepsilon, f) \qquad \text{for all } f \in \mathscr{H}.$$

(See [10], Lemma 5.3 and 5.4, and Theorems 5.4 and 5.5).

1.5. Let μ be a σ-finite measure on $(E, \mathscr{B})$ and let μT_t be absolutely continuous with respect to m for every $t > 0$. Then the Radon–Nikodym derivative

$$\phi_t = \frac{d(\mu T_t)}{dm} \tag{1.17}$$

can be chosen to be a K-function. We call

$$(\mu, \mu)_M = \int_{R_+} m(\phi_t)^2 \, dt \tag{1.18}$$

the *energy integral for* μ and we call

$$f_\mu = \int_{R_+} \phi_t \, dt \tag{1.19}$$

Green's potential of μ. We define $\mathbf{M}$ as the set of all measures μ such that $(\mu, \mu)_M < \infty$ and we put $\mu \in \mathbf{M}(B)$ if $\mu \in \mathbf{M}$ and $\mu(E \backslash B) = 0$.

If $\mu \in \mathbf{M}$, then the restriction μ_B of μ to B belongs to $\mathbf{M}(B)$ and $(\mu_B, \mu_B)_M \leqslant (\mu, \mu)_M$. Formula (1.17) defines a natural mapping from M onto K which preserves the inner product. We have also analogous mappings from M onto $\mathscr{K}$, H and $\mathscr{H}$. By (1.16) and (1.19) the natural image of μ in $\mathscr{H}$ is Green's potential f_μ. Hence

$$(f_\mu, f_\mu)_{\mathscr{H}} = (\mu, \mu)_M. \tag{1.20}$$

We denote by $K(B)$ the minimal subspace of K which contains the natural image of $\mathbf{M}(B)$. The notations $\mathscr{K}(B)$, $H(B)$ and $\mathscr{H}(B)$ have an analogous meaning. In particular, $\mathscr{H}(B)$ is the minimal subspace of $\mathscr{H}$ which contains all Green's potentials f_μ, $\mu \in \mathbf{M}(B)$.

1.6. In this paper we investigate the space $\mathscr{H}$ and the cones $\mathscr{H}^+$ and $\mathscr{K}^+$ under the assumption that p is the transition function of a fine Markov process. The class of fine processes contains all symmetric standard processes, in particular, all symmetric diffusions. The definition which is rather close to that of a right process (cf. [6, 12, 15, 17]) is given in Section 3.

If X_t is a Markov process with transition probabilities P_x, then the transition function is given by the formula

$$p_t(x, B) = P_x\{X_t \in B\}. \tag{1.21}$$

To every measure μ on the state space $(E, \mathscr{B})$ there corresponds a measure

$$P_\mu = \int_E P_x \mu(dx). \tag{1.22}$$

We say that a set $B \subset E$ is *inaccessible* if

$$P_\mu\{X_t \in B \text{ for some } t > 0\} = 0 \qquad \text{for all } \mu \in \mathbf{M}.$$

A property holds *quasi-everywhere* (q.e.) if it holds outside an inaccessible set. A $\mathscr{B}$-measurable function f is called *right* if, for every $\mu \in \mathbf{M}$, $f(X_t)$ is right continuous in t on $[0, +\infty)$ a.s. P_μ. With every stopping time τ we associate operators

$$(\mu T_\tau)(B) = P_\mu\{X_\tau \in B\}, \tag{1.23}$$

$$(T_\tau f)(x) = P_x f(X_\tau) \tag{1.24}$$

(if $\tau = t$, then formulas (1.23), (1.24) are equivalent to (1.1)(1.2)).

Assuming that the process X_t is fine, we prove that every element f of $\mathscr{H}$ can be represented by a right function. We also prove that there exists a cone $M \subset \mathbf{M}$ with the following properties:

1.6.A. For every $\nu \in \mathbf{M}$ there exists one and only one $\mu \in M$ such that $(\nu - \mu, \nu - \mu)_M = 0$.

1.6.B. Every $f \in \mathscr{H}^+$ is Green's potential of a measure $\mu \in \mathbf{M}$.

1.6.C. The right version of Green's potential has a representation

$$f_\mu(x) = P_x a^\mu(R^+) \tag{1.25}$$

where a^μ is a continuous homogeneous additive functional of X_t (see the definition in Subsection 4.3).

1.6.D. For every $\mu, \nu \in \mathbf{M}$,

$$(\mu, \nu)_M = \mu(f_\nu) = \nu(f_\mu) = P_\mu a^\nu(R^+) = P_\nu a^\mu(R^+). \tag{1.26}$$

1.6.E. A set $B \in \mathscr{B}$ is inaccessible if and only if $\mu(B) = 0$ for all $\mu \in M$. Two functions, $f, g \in \mathscr{H}$ coincide q.e. if and only if $\mu(f) = \mu(g)$ for all $\mu \in M$.

1.6.F. A function $f \in \mathscr{H}$ is right if and only if $(f_\mu, f)_{\mathscr{H}} = \mu(f)$ for all $\mu \in M$.

1.6.G. All operators T_τ defined by (1.23) are contractions of **M**.

1.6.H. Let

$$\sigma = \inf\{t: t > 0, X_t \in B\} \tag{1.27}$$

be the first hitting time of $B \in \mathscr{B}$. If $\mu \in M$, then μT_σ is the orthogonal projection of μ on $K(B)$. If $f \in \mathscr{H}$ is right, then $T_\sigma f$ is a right version of the orthogonal projection of f onto $\mathscr{H}(B)$.

1.6.I. If $f \in \mathscr{H}$ is right, then $T_\sigma f$ is the only right function which belongs to $\mathscr{H}(B)$ and which coincides with f q.e. on B.

Under additional conditions on X_t and $(f,f)_{\mathscr{H}}$, a part of these results (in particular, 1.6.H) has been proved by a different method in [11, 16].

The relation of this paper to the literature on the subject is discussed more in the concluding Section 8.

1.7. We denote by $\mathscr{F}^P$ the completion of a σ-algebra $\mathscr{F}$ with respect to a measure P. If N is a class of measures, then $\mathscr{F}^N$ means the intersection of $\mathscr{F}^P$ over all $P \in N$.

We say that a set C is *P-negligible* in $P(C) = 0$, and we call C *P-certain* if the complement of C is P-negligible. Writing "P-a.e. on $\tilde{\Omega}$" or "P-a.s. on $\tilde{\Omega}$" means "for all $\omega \in \tilde{\Omega} \cap C$ where C is P-certain set." The expression N-a.e. is an abbreviation for P-a.e. for all $P \in N$. The expressions N-negligible and N-certain have an analogous meaning.

We deal with measures which are not necessarily finite and therefore we need the following generalization of conditioning.

Suppose that $\mathscr{A}$ is a sub-σ-algebra of $\mathscr{F}$ and let $Z \in \mathscr{F}$, $C \in \mathscr{A}$. Then

$$P\{Z \mid \mathscr{A}\} = \bar{Z} \qquad \text{a.s. } P \text{ on } C \tag{1.28}$$

means that $\bar{Z}$ is measurable with respect to the P-completion $\mathscr{A}^P$ of $\mathscr{A}$ and

$$P\{Y 1_C Z\} = P\{Y 1_C Z\} \qquad \text{for all } Y \in \mathscr{A}. \tag{1.29}$$

If $\Phi \in \mathscr{F}$ and if $0 < P\Phi < \infty$, then the formula $P^\Phi(C) = P(1_C \Phi)/P\Phi$ defines a probability measure. We note that

$$P^\Phi\{Z \mid \mathscr{A}\} = P\{Z\Phi \mid \mathscr{A}\}/P\{\Phi \mid \mathscr{A}\} \qquad \text{a.s. } P \text{ on } \{0 < P\{\Phi \mid \mathscr{A}\} < \infty\} \tag{1.30}$$

2. Measures on the Space of Paths

2.1.A. A path in E is an E-valued function on an open interval $\Delta = (\alpha, \beta)$ where the birth time α and the death time β satisfy the condition $-\infty \leqslant \alpha < \beta \leqslant -\infty$. We consider the space Ω of all paths, we put $X_t(\omega) = \omega(t)$ and we denote by $\mathscr{F}(I)$ the minimal σ-algebra in Ω which contains all sets[3]

$$\{\alpha \leqslant t\},\ \{\beta \geqslant t\} \text{ and } \{X_t \in B\}, \qquad B \in \mathscr{B} \tag{2.1}$$

with $t \in I$. We use the following notation: $\mathscr{F}_{>s} = \mathscr{F}(s, +\infty)$, $\mathscr{F}_s = \mathscr{F}(-\infty, s]$, $\mathscr{F} = \mathscr{F}(R)$. The shift operators are defined by the formula

$$(\theta_s \omega)(t) = \omega(s+t) \qquad \text{for} \quad \alpha(\omega) - s < t < \beta(\omega) - s.$$

We note that $\Omega^* = \{\alpha \leqslant 0 < \beta\} \in \mathscr{F}_{>0}$ and that the restriction of every $\mathscr{F}_{>0}$-measurable function to Ω^* depends only on the values of ω on $(0, \beta)$.

We fix a transition function p on $(E, \mathscr{B})$ and we denote by $\mathscr{N}$ the class of all measures P on $(\Omega, \mathscr{F}_{>0})$ concentrated on Ω^* such that, for every $t \in R^+$, the measure

$$\gamma_t(B) = P\{X_t \in B\}, \qquad B \in \mathscr{B} \tag{2.2}$$

is σ-finite and, for every $t > s > 0$, $B \in \mathscr{B}$,

$$P\{X_t \in B \mid \mathscr{F}_s\} = p_{t-s}(X_s, B) \qquad \text{a.s. } P \text{ on } \{s < \beta\}. \tag{2.3}$$

A measure $P \in \mathscr{N}$ is uniquely determined by its one-dimensional distributions and we write $P = P_\gamma$ if P and γ are connected by formula (2.2).

It follows from (2.2) and (2.3) that

$$\gamma_t = \gamma_s T_{t-s} \qquad \text{for all } t > s > 0. \tag{2.4}$$

An arbitrary family γ_t, $t > 0$ of σ-finite measures satisfying (2.4) is called an *entrance law*. For every entrance law γ, there exists a measure $P_\gamma \in \mathscr{N}$.

By (2.2) $\gamma_t(E) = P_\gamma\{\beta > t\}$. Hence

$$P_\gamma(\Omega) = \lim_{t \downarrow 0} \gamma_t(E). \tag{2.5}$$

2.2. For every $x \in E$, $\gamma_t^x(B) = p_t(x, B)$ is an entrance law and, by 1.1.C, $\gamma_t^x(E) \to 1$ as $t \to 0$. We use notation P_x for the probability measure P_{γ^x}. For every $s \in R^+$, $x \in E$, the formula

$$P_{s,x}(C) = P_x(\theta_{-s} C), \qquad C \in \mathscr{F}_{>s}, \tag{2.6}$$

determines a probability measure on $\mathscr{F}_{>s}$. Of course, $P_{0,x} = P_x$.

[3] $\{X_t \in B\}$ means $\{t \in \Delta,\ X_t \in B\}$.

Formula (2.3) is equivalent to each of the following properties:

2.2.A. For every $Y \in \mathcal{F}_s$ and $Z \in \mathcal{F}_{>s}$,

$$P\{Y 1_{\beta > s} Z\} = P\{Y 1_{\beta > s} P_{s,X_s} Z\}.$$

2.2.B. For every $Z \in \mathcal{F}_{>s}$,

$$P\{Z \mid \mathcal{F}_s\} = P_{s,X_s} Z \qquad \text{a.s. } P \text{ on } \{\beta > s\}.$$

2.2.C. For every $Z \in \mathcal{F}$,

$$P\{\theta_s Z \mid \mathcal{F}_s\} = P_{X_s} Z \qquad \text{a.s. } P \text{ on } \{\beta > s\}.$$

(Here $\theta_s Z(\omega) = Z(\theta_s \omega)$.)

For every measure μ on $(E, \mathcal{B})$, we put

$$P_{s,\mu}(C) = \int_E P_{s,x}(C)\,\mu(dx), \qquad C \in \mathcal{F}_{>s};$$
$$P_\mu = P_{0,\mu}. \tag{2.7}$$

It follows from 2.2.A and (2.2) that

$$P_\gamma Z 1_{\beta > s} = P_{s,\gamma_s} Z \qquad \text{for all } Z \in \mathcal{F}_{>s}. \tag{2.8}$$

2.3. Suppose that the transition function p is symmetric relative to m. Let $\phi \in K^+$. Then $\gamma_t(dx) = \phi_t(x)\,m(dx)$ is an entrance law. We denote the corresponding measure by P_ϕ and we denote by $\mathcal{M}$ the class of all measures P_ϕ, $\phi \in K^+$.

By (2.8), for every $Z \in \mathcal{F}_{>s}$,

$$P_\phi Z 1_{\beta > s} = (\phi_s, F_s), \tag{2.9}$$

where $F_s(x) = P_{s,x} Z$.

We consider a function ρ subject to condition 1.2.A and we put $q = G\rho$. Formula $\psi_t = T_t \rho$ determines an element ψ of K^+ such that $\|\psi\|_K = \|\rho\|_G$. By (2.2), (1.5) and (1.7), for every $\psi \in K^+$,

$$P_\phi q(X_t) = \gamma_t(q) = (\phi_t, q) = \int_{R_+} (\phi_t, \psi_s)\,ds = \int_{R_+} (\phi_{(t+s)/2}, \psi_{(t+s)/2})\,ds$$
$$= \int_t^\infty (\phi_{s/2}, \psi_{s/2})\,ds \leqslant (\phi, \psi)_K \leqslant \|\phi\|_K \|\psi\|_K = \|\phi\|_K \|\rho\|_G. \tag{2.10}$$

2.4. In addition to the measures investigated in Subsections 2.1

through 2.3, we introduce a measure $\mathbf{P}$ on $(\Omega, \mathscr{F})$ with the following properties:

$$\mathbf{P}\{X_t \in B\} = m(B) \qquad \text{for all } t \in R, B \in \mathscr{B}, \tag{2.11}$$

$$\mathbf{P}\{X_t \in B \mid \mathscr{F}_s\} = p_{t-s}(X_s, B) \qquad \text{a.s. } \mathbf{P} \text{ on } \{s \in \Delta\}$$
$$\text{for all } s < t \in R, B \in \mathscr{B}. \tag{2.12}$$

The existence and uniqueness of $\mathbf{P}$ follows from a general theorem due to Kuznecov [13].

We use the following implications of formulas (2.11), (2.12):

2.4.A. $\mathbf{P}\theta_s Z = \mathbf{P}Z$ for all $Z \in \mathscr{F}$.

2.4.B. $\mathbf{P}f(X_s)\, g(X_t) = (f, T_{t-s} g) = (T_{t-s} f, g)$ for all $t > s > 0$, $f, g \in \mathscr{B}$ (and also for all $f, g \in L^2(m)$).

2.4.C. For every $Z \in \mathscr{F}_{>s}$,

$$\mathbf{P}\{Z \mid \mathscr{F}_s\} = P_{s, X_s} Z \qquad \text{a.s. } P \text{ on } \{s \in \Delta\}.$$

2.4.D. For every $Z \in \mathscr{F}_{>s}$,

$$\mathbf{P}Z 1_\Delta(s) = P_{s,m} Z.$$

2.4.E. All measures $P \in \mathscr{M}$ are absolutely continuous with respect to $\mathbf{P}$ on every σ-algebra $\mathscr{F}_{>s}$, $s > 0$.

Properties 2.4.A through 2.4.D are quite obvious. Let us prove 2.4.E. If $\mathbf{P}(C) = 0$ for set $C \in \mathscr{F}_{>s}$, then by 2.4.D, for every $u \in (0, s)$, $P_{u,x}(C) = 0$ m-a.e. and $P_\phi\{C, \beta > u\} = 0$ by (2.8). Hence $P_\phi(C) = 0$.

2.5. We call $P_{t,x}$ *transition probabilities*. Formulas (2.3), 2.2.A, 2.2.B, 2.2.C, (2.12) and 2.4.C express various forms of the *Markov property* which is true also in a more general setting. For instance (2.12) holds if we replace $\mathscr{F}$ by $\mathscr{F}^{\mathbf{P}}$ and adjoin to $\mathscr{F}_s$ all sets $C \in \mathscr{F}^{\mathbf{P}}$ for which $\mathbf{P}(C) = 0$. In 2.4.C we can replace $\mathscr{F}_{>s}$ by $\mathscr{F}'_{>s} = \theta_s \mathscr{F}'_{>0}$. Analogous augmentation is possible in (2.3), 2.2.A, 2.2.B, 2.2.C, and 2.4.C.

3. Fine Markov Processes

3.1. Starting from a symmetric transition function p in a standard Borel space $(E, \mathscr{B})$, we have constructed in Section 2:

(a) a measure space $(\Omega, \mathscr{F}, \mathbf{P})$;

(b) an E-valued function $X_t(\omega)$ defined for $t \in \Delta(\omega) = (\alpha(\omega), \beta(\omega)) \subset R$;

(c) transformations θ_s of Ω preserving $\mathscr{F}$ and $\mathbf{P}$ such that $X_t(\theta_s\omega) = X_{t+s}(\omega)$;

(d) two families of sub-σ-algebras of $\mathscr{F}$: $\mathscr{F}_s$ generated by sets (2.11) with $t \leqslant s$ and $\mathscr{F}_{>s}$ generated by (2.11) with $t > s$;

(e) a class $\mathscr{N}$ of measures on $(\Omega, \mathscr{F}_{>0})$ concentrated on $\Omega^* = \{\alpha \leqslant 0 < \beta\}$.

Formulas (2.12) and (2.3) establish the relation of $\mathbf{P}$ and $\mathscr{N}$ to the transition function p; (2.2) determines a 1–1 correspondence between $\mathscr{N}$ and entrance laws for p, and (2.11) shows the connection of $\mathbf{P}$ with the reference measure m.

Subject to these conditions, elements (a) through (e) determine *a Markov process X with the transition function p*. A process is *canonical* if $(\Omega, \mathscr{F})$ is the set of paths described in Section 2. Being a good starting point, the canonical process is a poor base for analytic considerations since $X_t(\omega)$ as a function of t can be completely irregular. However, for a wide class of transition functions, it is possible to define a nice function $\tilde{X}_t(\omega)$ such that $\tilde{X}_t = X_t$ a.s. $\mathbf{P}$ on $t \in \Delta$ for every $t \in R$ and $\tilde{X}_t = X_t$ a.s. $\mathscr{N}$ on $\{\alpha \leqslant t < \beta\}$ for every $t \in R^+$. If we replace X by $\tilde{X}$, then the σ-algebras $\mathscr{F}_s$ and $\mathscr{F}_{>s}$ change but their augmentations described in Subsection 2.5 do not change and $\tilde{X}$ is a Markov process with the same transition function p.

3.2. With every function f on E a stochastic process $f(X_t)$ is associated. This is a function on $R \times \Omega$ (equal to 0 for $t \notin \Delta(\omega)$). We say that f is *fine* if f is $\mathscr{B}$-measurable and if, for every $s \in R$, there exists a $\mathbf{P}$-certain set Γ such that, if $\omega \in \Gamma$ and $\alpha(\omega) < s$, then $f(X_t(\omega))$ is right continuous on (s, ω).

If f is fine, then, for $\mathbf{P}$-almost all ω, $f(X_t)$ is right continuous on all real line R except maybe $t = \alpha(\omega)$. It follows from 2.4.E that $f(X_t)$ is right continuous on R^+ a.s. $\mathscr{M}$.

We say that a Markov process is *fine* if:

3.2.A. All functions

$$f = G_c\psi, \qquad \psi \in \mathscr{B}, c \in (0, +\infty] \tag{3.1}$$

are fine.

3.2.B. $\mathscr{B}$ is generated by functions f with the property: $f(X_t(\omega))$ is right continuous on $\Delta(\omega)$ for all $\omega \in \Omega$.

From this point on we consider only fine processes.

It follows from 3.2.B that, for every $\mathscr{B}_R \times \mathscr{B}$-measurable function f and for every interval I, the function

$$Y(t, \omega) = f(t, X_t(\omega)), \qquad t \in I, \ \omega \in \Omega, \tag{3.2}$$

is $\mathscr{B}_I \times \mathscr{F}(I)$-measurable and, if $\int_I f(t, X_t)\, dt$ exists, it represents a $\mathscr{F}(I)$-measurable function. Besides all functions (3.2) are optional with respect to the filtration $\mathscr{F}_t$ (i.e., measurable with respect to the σ-algebra in $R \times \Omega$ generated by right continuous functions adapted to $\mathscr{F}_t$).

3.3. Let $\mathscr{F}_{t+}$ be the intersection of $\mathscr{F}_u$ over all $u > t$. Suppose that, for every $\omega \in \Omega$, $\tau(\omega)$ either belongs to the interval $[0, \beta(\omega))$ or is equal to $+\infty$. We say that τ is a *P-stopping time* if $\{\tau \leqslant t\} \in \mathscr{F}^P_{t+}$ for each $t \in R^+$ and we put $C \in \mathscr{F}^P_{\tau+}$ if $\{C, \tau \leqslant t\} \in \mathscr{F}^P_{t+}$ for every $t \in R^+$.

The first hitting time of a set B is defined by (1.27).

LEMMA 3.1. *Let $P \in \mathscr{N}$, $B \in \mathscr{B}$. The first hitting time σ of B is a P-stopping time. There exists a sequence τ_n of stopping times with the following properties*:

3.3.A. $X_{\tau_n} \in B$ *a.s. P on* $\{\tau_n < \infty\}$.

3.3.B. $\tau_n \downarrow \sigma$ *a.s. P.*

Proof. The first statement of Lemma 3.1 follows from [1, Chap. 3, Theorem 23].

For every n, the set $C_n = \{(t, \omega): \sigma(\omega) \leqslant t < \sigma(\omega) + 1/n, X_t(\omega) \in B\}$ is optional and its projection on Ω is $A = \{\tau < \infty\} = \{\omega: X_t(\omega) \in B$ for some $t > 0\}$. By the cross-section theorem [2, Theorem 84, Chap. IV], there exists a stopping time σ_n such that $P(A) < P\{\sigma_n < \infty\} + 2^{-n}$ and $(\sigma_n, X_{\sigma_n}) \in C_n$ for $\{\sigma_n < \infty\}$. Put $\tau_n = \sigma_1 \wedge \cdots \wedge \sigma_n$. Obviously $X_{\sigma_n} \in B$ a.s. P on $\{\sigma_n < \infty\}$ and therefore 3.3.A holds. By the Borel–Cantelli lemma, P-a.s. on A, $\sigma_n < \infty$ for all sufficiently large n. Hence $\sigma_n \to \sigma$ and $\tau_n \downarrow \sigma$ a.s. P on A. 3.3.B holds since $\sigma_n = \sigma = +\infty$ outside A.

3.4. The following theorem presents a form of the strong Markov property. We outline the proof referring to [3, 6] for detail.

THEOREM 3.1. *Let P be a probability measure of class $\mathscr{N}$. If τ is a P-stopping time, then, for every $Z \in \mathscr{F}^P$,*

$$P\{\theta_\tau Z \mid \mathscr{F}^P_{<\tau+}\} = P_{X_\tau} Z \qquad \text{a.s. } P \text{ on } \{\tau > 0\}. \tag{3.3}$$

Proof. First, we prove (3.3) for

$$Z = \int_0^c \psi(X_t)\, dt \tag{3.4}$$

with a bounded $\psi \in \mathscr{B}$. Obviously, $\theta_u Z$ is continuous in u. By Fubini's theorem,

$$P_x Z = G_c \psi(x). \tag{3.5}$$

Hence $P_{X_u}Z$ is right continuous in u a.s. P, and formula (3.3) can be proved in a routine way using the approximation of τ from above by a sequence of stopping times τ_n which take a countable set of values.

Let Y be a bounded $\mathscr{F}^P_{\tau+}$-measurable function and let $Y=0$ on $\{\tau=0\}$. By applying (3.3) to the function (3.4), we get that

$$PY\psi(X_{\tau+t}) = PYP_{X_\tau}\psi(X_t) \tag{3.6}$$

for almost all t. If ψ is fine, then both parts of (3.6) are right continuous in t and therefore (3.6) holds for all t. Since the class of bounded fine functions is closed under multiplication and generates $\mathscr{B}$, (3.6) holds for $\psi \in \mathscr{B}$.

Now using (3.6), we prove by induction that (3.3) holds for

$$Z = \psi_1(X_{t_1}) \cdots \psi_n(X_{t_n}), \qquad 0 < t_1 < \cdots < t_n,\ n = 1, 2,..., \quad \psi_1,..., \psi_n \in \mathscr{B}. \tag{3.7}$$

Hence (3.5) is true for all $Z \in \mathscr{F}^P$.

Remark. Let $\Phi \in \mathscr{F}_{0+}$. If (3.3) holds for P, then it holds for $\tilde{P}(d\omega) = \Phi(\omega)P(d\omega)$. Hence Theorem 3.1 holds for each measure $P \in \mathscr{N}$ whose restriction for $\mathscr{F}_{0+}$ is a σ-finite measure. We show in Subsection 5.4 that all $P \in \mathscr{M}$ have this property.

3.5. Let ρ satisfy condition 1.2.A and let $q = G\rho$, $E^q = \{x: q(x) = \infty\}$. For every $x \notin E^q$,

$$P_x q(X_t) = T_t q(x) \leqslant q(x) < \infty \tag{3.8}$$

and, since $(q(X_t), \mathscr{F}_t, P_x)$ is a positive right continuous supermartingale,

$$P_x\{q(X_t) = \infty \text{ for some } t > 0\} = 0.$$

Dropping E^q from the state space, we get a new fine Markov process with an additional property:

3.5.A. There exists a strictly positive function ρ such that $\|\rho\|_G < \infty$ and $q = G\rho$ is finite everywhere.

To simplify the presentation (without any substantial loss of generality), we assume that 3.5.A holds and we call ρ a *reference function.*

3.6.

THEOREM 3.2. *It is possible to extend the state space* $(E, \mathscr{B})$ *to a measurable space* $(\hat{E}, \hat{\mathscr{B}})$ *and to define mappings* $\omega \to X_0(\omega)$ *from* Ω *to* $\hat{E}$ *and* $x \to \hat{P}_x$ *from* $\hat{E}$ *to* $\mathscr{N}$ *in such a way that:*

3.6.A. *$\{X_0 \in B\} \in \mathscr{F}_{0+}$ for every $B \in \hat{\mathscr{B}}$.*

3.6.B. *$\hat{P}_x(\Omega) = 1$ for all $x \in \hat{E}$; $\hat{P}_x = P_x$ for $x \in E$.*

3.6.C. *$f(x) = \hat{P}_x Z$ is $\hat{\mathscr{B}}$-measurable for every $Z \in \mathscr{F}$.*

3.6.D. *For every $Z \in \mathscr{F}$, $P \in \mathscr{M}$,*

$$P\{Z \mid \mathscr{F}_{0+}\} = \hat{P}_{X_0} Z \qquad a.s.\ P. \tag{3.9}$$

3.6.E. *$\hat{\mathscr{B}}$ is generated by functions F with the following properties:*

3.6.E$_1$. *The restriction of F to E is a fine function.*

3.6.E$_2$. *$F(X_s) \to F(X_0)$ as $s \to 0$ a.s. P for every $P \in \mathscr{M}$.*

Proof. Let ρ be a reference function and let $q = G\rho$,

$$\eta = \int_{R^+} \rho(X_u)\, du. \tag{3.10}$$

We denote by $\mathscr{N}^\rho$ the class of all $P \in \mathscr{N}$ such that $P\eta = 1$. By (2.9), $\mathscr{N}^\rho \supset \mathscr{M}$.

We transform $\mathscr{N}^\rho$ into a class of probability measures using the method introduced in [3, Chap. 10, Sect. 4]. Namely, we associate with every measure $P \in \mathscr{N}^\rho$ a measure $\tilde{P}$ in the space $\tilde{\Omega} = \Omega \times R^+$ defined on the σ-algebra $\tilde{\mathscr{F}} = \mathscr{F}_{>0} \times \mathscr{B}_{R^+}$ by the formula

$$\tilde{P}(C) = \int_{\Omega \times R^+} \rho(X_u(\omega))\, 1_C(\omega, u)\, P(d\omega)\, du. \tag{3.11}$$

For every $\tilde{\mathscr{F}}$-measurable function $Z_u(\omega) \geqslant 0$,

$$\tilde{P}Z = P \int_{R^+} Z_u \rho(X_u)\, du. \tag{3.12}$$

In particular, if Z does not depend on u (in other words, if $Z \in \mathscr{F}_{>0}$), then

$$\tilde{P}Z = PZ\eta. \tag{3.13}$$

We denote by $\tilde{\mathscr{N}}$ the class of probability measures $\tilde{P}$ on $\tilde{\mathscr{F}}$ which correspond to $P \in \mathscr{N}^\rho$ by formula (3.11).

Let $\tilde{\mathscr{F}}_t$ stand for the σ-algebra in $\tilde{\Omega}$ generated by the sets

$$\{X_s \in B\} \times (s, \infty), \qquad B \in \mathscr{B} \tag{3.14}$$

with $s \leqslant t$ and let $\tilde{\mathscr{F}}_{>t}$ stand for the σ-algebra generated by the sets (3.14) with $s > t$. We check (cf. [3, Chap. 10, Sect. 4]) that a probability measure P belongs to $\tilde{\mathcal{N}}^p$ if and only if

$$\tilde{P}\{Z \mid \tilde{\mathscr{F}}_t\} = q(X_t)^{-1} P_{t,X_t} \int_t^\infty Z_u \rho(X_u)\, du \qquad \text{a.s. } P \text{ on } \{t < \beta\} \times (t, \infty) \tag{3.15}$$

for every $Z \in \tilde{\mathscr{F}}_t$ such that $Z = 0$ on $\Omega \times (-\infty, t]$. Let $\tilde{\mathscr{F}}_{0+}$ be the intersection of $\tilde{\mathscr{F}}_t$ over all $t > 0$. According to [4, Theorem 3.1] or [7, Theorem 9.1], there exists a mapping $\omega \to \tilde{Q}^\omega$ of Ω into $\tilde{\mathcal{N}}^p$ such that

$$\tilde{P}\{Z \mid \tilde{\mathscr{F}}_{0+}\} = \tilde{Q}^\omega(Z) \qquad \text{a.s. } \tilde{P} \tag{3.16}$$

for every $\tilde{P} \in \tilde{\mathcal{N}}^p$ and every $Z \in \tilde{\mathscr{F}}$. The measure $\tilde{Q}^\omega$ corresponds to an element Q^ω of $\mathcal{N}^p$ by formula (3.11). For $Z \in \mathscr{F}_{>0}$, we have, by (3.14),

$$\tilde{Q}^\omega Z = Q^\omega(Z\eta). \tag{3.17}$$

Let $Y \in \mathscr{F}_{0+}$. Then Y belongs to $\tilde{\mathscr{F}}_{0+}$ and, by (3.16) and (3.17),

$$\tilde{P}(YZ) = \tilde{P}(Y\tilde{Q}^\omega Z) = \tilde{P}(YQ^\omega(Z\eta)). \tag{3.18}$$

By (3.13) this equation is equivalent to $P(YZ\eta) = P(\eta YQ^\omega(Z\eta))$. Replacing Z by Z/η, we get

$$P(YZ) = P(\eta YQ^\omega(Z)). \tag{3.19}$$

Taking $Z = 1$, $Y = \eta$, we see that $1 = P(\eta) = P(\eta^2 Q(1))$ and therefore all measures $P \in \mathcal{N}^p$ are concentrated on the set $\Omega' = \{\omega\colon Q^\omega(1) < \infty\}$. We fix a point $c \in E$ and we put

$$\begin{aligned} \tilde{Q} &= P_c && \text{for} \quad \omega \in \Omega \backslash \Omega', \\ \tilde{Q} &= Q^\omega / Q^\omega(1) && \text{for} \quad \omega \in \Omega'. \end{aligned} \tag{3.20}$$

By (3.19), $P(Y) = P(\eta YQ^\omega(1))$. Replacing Y with $Y\hat{Q}^\omega(Z)$, we get

$$P(Y\hat{Q}^\omega(Z)) = P(\eta YQ^\omega(Z)). \tag{3.21}$$

By (3.19) and (3.22), $P(YZ) = P(Y\hat{Q}^\omega(Z))$ for every $Y \in \mathscr{F}_{0+}$. Hence, for all $P \in \mathcal{N}^p$,

$$P\{Z \mid \mathscr{F}_{0+}\} = \hat{Q}^\omega(Z) \qquad \text{a.s. } P. \tag{3.22}$$

We denote by $\hat{E}$ the set of all $P \in \mathcal{N}$ such that $P(\Omega) = 1$ and we define $\hat{\mathscr{B}}$ as the minimal σ-algebra in $\hat{E}$ with respect to which all functions

$$F_{\psi,s}(P) = P\psi(X_s), \qquad \psi \in \mathscr{B},\ s > 0, \tag{3.23}$$

are measurable. The space $(\hat{E}, \hat{\mathscr{B}})$ is standard Borel [5, Theorem 3.3]. The mapping $x \to P_x$ is a measurable injection of E into $\hat{E}$. Hence (see, e.g., [2, Chap. 111, Theorem 21] the image of every set $B \in \mathscr{B}$ belongs to $\hat{\mathscr{B}}$ and $(\hat{E}, \hat{\mathscr{B}})$ can be considered as an extension of $(E, \mathscr{B})$.

For every $\omega \in \Omega$, $\hat{Q}^\omega \in \hat{E}$, and we define a map from Ω to $\hat{E}$ by setting $X_0(\omega) = \hat{Q}(\omega)$. Since $\hat{E} \subset \mathscr{N}$, we can define the mapping $x \to \hat{P}_x$ from $\hat{E}$ to $\mathscr{N}$ as the identity mapping.

It follows from (3.12) and 2.2.C that, for every $P \in \mathscr{N}^\rho$, $\psi \in \mathscr{B}$, $s > 0$,

$$\tilde{P}\psi(X_s)\, 1_{u>s} = P\psi(X_s) \int_s^\infty \rho(X_u)\, du = P\psi(X_s)\, q(X_s). \tag{3.24}$$

Hence $Q^\omega \psi(X_s) = \tilde{Q}^\omega[(\psi(X_s)/q(X_s))\, 1_{u>s}]$ is $\mathscr{F}_{0+}$-measurable and so is $\hat{Q}^\omega \psi(X_s) = F_{\psi,s}(X_0)$. This implies 3.6.A. Property 3.6.B is obvious. To prove 3.6.C, it is sufficient to note that, if $Z \in \mathscr{F}_{>s}$, $P \in \mathscr{N}$, and if $\psi(x) = P_{s,x} Z$, then $F_{\psi,s}(P) = P\psi(X_s) = PZ$. Property 3.6.D follows from (3.22).

It remains to prove 3.6.E. We consider the family $\mathscr{S}$ of functions on $\hat{E}$ defined by the formula

$$F(P) = (P\eta)^{-1} P \int_0^c \psi(X_u)\, q(X_u)\, du, \qquad c \in R^+,\ \psi \in \mathscr{B},\ 0 \leqslant \psi \leqslant 1. \tag{3.25}$$

Let us prove that functions $F \in \mathscr{S}$ are $\hat{\mathscr{B}}$-measurable and that they generate $\hat{\mathscr{B}}$. For every bounded fine function ψ, $F_{\psi,s}(P) = P\psi(X_s)$ is right continuous in s and $\hat{\mathscr{B}}$-measurable in P, hence it is $\mathscr{B}_{R+} \times \mathscr{B}$-measurable in s, P. Since fine functions generate $\mathscr{B}$, this property holds for every bounded $\psi \in \mathscr{B}$. The function (3.25) can be represented as

$$\int_0^c F_{\psi q,u}\, du \Big/ \int_{R^+} F_{\rho,u}\, du \tag{3.26}$$

and therefore it is $\hat{\mathscr{B}}$-measurable.

To check that $\mathscr{S}$ generates $\hat{\mathscr{B}}$, it is sufficient to show that it separates points of $\hat{E}$ (see, e.g., Lemma 2.1 in [6]). Suppose $F(P_\gamma) = F(P_{\gamma'})$ for all $F \in \mathscr{S}$. Then $\gamma_t / P_\gamma \eta = \gamma'_t / P_{\gamma'} \eta$ for almost all t. By (2.5), $\gamma_t(E) \to 1$ and $\gamma'_t(E) \to 1$ as $t \to 0$, and we get $P_\gamma \eta = P_{\gamma'} \eta$. Hence $\gamma_t = \gamma'_t$ for almost all t. This implies the equality $\gamma = \gamma'$.

The restriction of F to E is equal to $G_c(\psi\rho)/G\rho$. Hence F satisfies 3.6.E$_1$.

To prove 3.6.E$_2$, we show that

$$F(X_t) = \tilde{P}\{Z^t \mid \tilde{\mathscr{F}}_t\} \qquad \text{a.s. } \tilde{P} \text{ on } \{\beta > t\} \times (t, \infty), \tag{3.27}$$

where

$$Z_u^t(\omega) = \int_t^{(c+t)u} \psi(X_s)\, ds. \tag{3.28}$$

Indeed, by (3.15),

$$\tilde{P}\{Z^t \mid \tilde{\mathscr{F}}_t\} = q(X_t)^{-1} P_{t,X_t} Y^t \qquad \text{a.s. } \tilde{P} \text{ on } \{\beta > t\} \times (t, \infty), \tag{3.29}$$

where

$$Y^t = \int_{R^+} Z_u^t \rho(X_u)\, du = \theta_t Y^0 = \int_{R^+}\int_{R^+} 1_{s<c} 1_{s<u} f(X_s)\, \rho(X_u)\, ds\, du$$

$$= \theta_t \int_0^c ds\, f(X_s) \int_s^\infty \rho(X_u)\, du. \tag{3.30}$$

By (2.6) and 2.2.C,

$$P_{t,x} Y^t = P_x Y^0 = P_x \int_0^c (\psi q)(X_s)\, ds = F(x)\, q(x). \tag{3.31}$$

Now (3.27) follows from (3.29) and (3.31).

For $\tilde{P}$-almost all (ω, u), $\tilde{P}\{Z^0 \mid \tilde{\mathscr{F}}_r\} \to P\{Z^0 \mid \mathscr{F}_{0+}\}$ as $r \to 0$ along the set of rationals. We note that $|Z^t - Z^0| \leqslant t$ and, taking into account (3.27) and (3.16), we get

$$F(X_r) \to \tilde{Q}^\omega(Z^0) \qquad \text{as } r \to 0 \text{ along rationals a.s. } \tilde{P}. \tag{3.32}$$

By (3.25), (3.19) and (3.12)

$$F(X_0) = \hat{P}_{X_0} Y^0 / \hat{P}_{X_0} \eta = Q^\omega(Y^0) = \tilde{Q}^\omega(Z^0). \tag{3.33}$$

Since F satisfies 3.6.E_1, formulas (3.32) and (3.33) imply 3.6.E_2.

4. Additive Functionals

4.1.

Theorem 4.1. *For every $h \in H$ and every $s < u$, there exists*

$$a_h(s, u) = \lim_{\delta \to 0} \int_s^u h_\delta^t(X_t)\, dt \qquad \text{in } L^2(\mathbf{P}). \tag{4.1}$$

We have

$$\mathbf{P}a_h(s,u)^2 = 2\int_s^u dt_1 \int_0^{u-t_1} \|h'_{t/2}\|^2\,dt \leqslant 2(u-s)\int_0^{u-s} \|h'_{t/2}\|^2\,dt, \quad (4.2)$$

$$\mathbf{P}f(X_r)\,a_h(s,u) = (f, h_{s-r,u-r}) \qquad \textit{for all } r \leqslant s < u, f \in L^2(m), \quad (4.3)$$

$$\mathbf{P}a_h(s,u) f(X_v) = (f, h_{v-u,v-s}) \qquad \textit{for all } s < u \leqslant v, f \in L^2(m). \quad (4.4)$$

Proof. Put $\phi_t = h'_t$, $Y_\delta = \int_s^u \phi_\delta(X_t)\,dt$. By the Fubini theorem

$$\mathbf{P}Y_\delta Y_\varepsilon = \iint_{s<t_1<t_2<u} F(t_1,t_2)\,dt_1\,dt_2,$$

where

$$F(t_1,t_2) = \mathbf{P}(\phi_\delta(X_{t_1})\,\phi_\varepsilon(X_{t_2}) + \phi_\varepsilon(X_{t_1})\,\phi_\delta(X_{t_2})).$$

By 2.4.B, $F(t_1,t_2) = 2(\phi_\delta, T_{t_2-t_1}\phi_\varepsilon) = 2\,\|\phi_{(t_2-t_1+\delta+\varepsilon)/2}\|^2$. Hence

$$\mathbf{P}Y_\delta Y_\varepsilon = 2\int_s^u dt_1 \int_{\delta+\varepsilon}^{u-t_1+\delta+\varepsilon} \|\phi_{t/2}\|^2\,dt$$

$$\to 2\int_s^u dt_1 \int_0^{u-t_1} dt\,\|\phi_{t/2}\|^2 \qquad \text{as} \quad \delta,\varepsilon \to 0.$$

Therefore $\mathbf{P}(Y_\delta - Y_\varepsilon)^2 \to 0$, the limit (4.1) exists and satisfies (4.2).
Using the Fubini theorem and 2.4.B, we get

$$\mathbf{P}f(X_r)\,Y_\delta = \int_s^u (f, T_{t-r}\phi_\delta)\,dt = \int_{s+\delta-r}^{u+\delta-r} (f,\phi_t)\,dt \to \int_{s-r}^{u-r} (f,\phi_t)\,dt.$$

Hence (4.1) implies (4.3). Formula (4.4) can be proved similarly.

4.2. For every finite set $\Lambda = \{t_0 < t_1 < \cdots < t\}$ we put

$$V_h(\Lambda) = \sum_{i=1}^{j} a_h(t_{i-1},t_i)^2, \qquad |\Lambda| = \max_{1\leqslant i\leqslant j}((t_i - t_{i-1}).$$

It follows from (4.2) that, if $\Lambda \subset (s,u)$, then

$$\mathbf{P}V_h(\Lambda) \leqslant 2(u-s)\int_0^{|\Lambda|} \|h'_{t/2}\|^2\,dt.$$

Hence

$$\lim \mathbf{P}V_h(\Lambda_n) = 0 \quad (4.5)$$

for every sequence $\Lambda_n \subset (s,u)$ such that $|\Lambda_n| \to 0$.

In terminology suggested by Föllmer this means that, for every s, the process $Z_t = a_h(s, t)$ has zero energy.

Suppose that $h \in H^+$, i.e., $h_{s,u} \geqslant 0$ m-a.e. for every $s < u$. Then $V_h(\Lambda)$ is a monotone function of Λ and

$$\lim V_h(\Lambda_n) = 0 \qquad \mathbf{P}\text{-a.e.} \tag{4.6}$$

if $\Lambda_1 \subset \Lambda_2 \subset \cdots \Lambda_n \subset \cdots$ and $|\Lambda_n| \to 0$.

4.3. We say that a function $a(B, \omega)$, $B \in \mathscr{B}_{R^+}$, $\omega \in \Omega$ is *an additive functional* if:

4.3.A. For every $\omega \in \Omega$, $a(\cdot, \omega)$ is a measure on $(R^+, \mathscr{B}_{R^+})$ concentrated on $\Delta(\omega) \cap R^+$.

4.3.B. For every open interval $I \subset R^+$, $a(I, \cdot)$ is measurable with respect to $\mathscr{F}_{>0}$ and with respect to $\mathscr{F}(I)^{\mathscr{M}}$ and $\mathscr{F}(I)^{\mathbf{P}}$.

An additive functional is *continuous* if the measures $a(\cdot, \omega)$ do not charge singletons. It is *homogeneous* if:

4.3.C. There exists a set $\Gamma \in \mathscr{F}_{>0}$ which is $\mathbf{P}$-certain and $\mathscr{M}$-certain and such that

$$a(I, \theta_u \omega) = a(I + u, \omega) \qquad \text{for all } I \subset R^+,\ \omega \notin \Gamma \text{ and } u > 0.$$

THEOREM 4.2. *For every $h \in H^+$ there exists a continuous homogeneous additive functional a_h such that, for every open interval $I \subset R^+$,*

$$a_h(I) = \lim_{\delta \to 0} \int_I h'_\delta(X_t)\, dt \qquad \text{in } L^2(\mathbf{P}).$$

Proof. Based on (4.6) and the following elementary considerations.

Let F be a monotone function on the set R_I of all rational points of an open interval I. We put

$$W_I(F) = \lim_{n \to \infty} \sum_{i=1}^{j_n} (F(t_i^n) - F(t_{i-1}^n))^2 = \sum_{t \in I} (F(t+) - F(t-))^2,$$

where $\Lambda_n = \{t_0^n < \cdots < t_{j_n}^n\}$ is an increasing sequence of finite sets with the union equal to R_I. F can be extended to a continuous function on I if and only if

$$W_I(F) = 0. \tag{4.7}$$

If $F(t) = \lim F_k(t)$ for every $t \in R_I$ and if $F_k(t)$ are continuous increasing functions on I, then (4.7) implies that F_k converge uniformly on every closed interval $[c_1, c_2] \subset I$.

Let $h \in H^+$ and let $t_0 \in I$. To every $\omega \in \Omega$ and every $\delta > 0$ there corresponds a continuous increasing function

$$F_\delta(t) = \int_{t_0}^{t} h'_\delta(X_s)\, ds \tag{4.8}$$

on I. By Theorem 4.1, there is a sequence $\delta_k \to 0$ such that, for **P**-almost all ω, $\lim F_{\delta_k}(t) = F(t)$ exists for all rational $t > 0$. It follows from (4.6) that (4.7) holds a.s. **P**. We put $\omega \in \Omega_I$ if (4.7) holds and if

$$F_{\delta_k}(t) \to F(t) \qquad \text{for all } t \in R_I. \tag{4.9}$$

Obviously $\Omega_I \in \mathscr{F}(I)$ and is a **P**-certain set. It follows from the last paragraph that

$$\begin{aligned} \Omega_I &= \{\omega : \mathscr{F}_\delta(t) \text{ converges uniformly on every } [c_1, c_2] \subset I\} \\ &= \Big\{\omega : \int_{t_1}^{t_2} h'_{\delta_k}(X_s)\, ds \text{ converge uniformly in } t_1, t_2 \in [c_1, c_2] \\ &\qquad \text{for every } c_1, c_2 \in I\Big\}. \end{aligned} \tag{4.10}$$

Formula

$$\begin{aligned} F^I(t) &= \lim F_{\delta_k}(t) \qquad &&\text{for} \quad \omega \in \Omega_I \\ &= 0 &&\text{for} \quad \omega \notin \Omega_I \end{aligned}$$

determines a continuous increasing function on I. There exists a unique measure a^I on $(R^+, \mathscr{B}_{R^+})$ concentrated on I such that

$$a^I(s, u] = F^I(u) - F^I(s) \qquad \text{for all } s < u \in I.$$

It is clear that it is concentrated on $I \cap \Delta$, does not charge any singleton and that

$$a^I(J) \in \mathscr{F}(I) \qquad \text{for every } J \subset I. \tag{4.11}$$

We note that

$$\Omega_I \subset \Omega_J \qquad \text{for} \quad I \subset J, \tag{4.12}$$

$$\Omega_{I_n} \downarrow \Omega_I \qquad \text{as} \quad I_n \uparrow I, \tag{4.13}$$

$$a^I(B \cap J, \omega) = a^J(B, \omega) \qquad \text{for all } I \supset J,\ B \in \mathscr{B}_{R^+},\ \omega \in \Omega_I. \tag{4.14}$$

The set Ω_I is $\mathscr{M}$-certain. This follows from 2.4.E if $I = (s, u)$ with $s > 0$. If $I = (0, u)$, we apply (4.13).

The function $a_h(B, \omega) = a^{R^+}(B, \omega)$ satisfies 4.3.A. By (4.14), $a_h(I) = a^I(I)$ on Ω_I and 4.3.B holds by virtue of (4.11). The set $\Gamma = \bigcap \Omega_{(1/n,\infty)}$ belongs to $\mathscr{F}_{>0}$ and is **P**-certain and $\mathscr{M}$-certain and it satisfies 4.3.C by virtue of (4.10).

5. Characteristic Measures of Additive Functionals. Representation of Green's Space by Measures on the State Space

5.1. To every additive functional a there corresponds a measure on $R^+ \times E$:

$$\nu(C) = \mathbf{P} \int_0^\infty 1_C(t, X_t)\, a(dt), \qquad C \in \mathscr{B}_{R^+} \times \mathscr{B}.$$

If the functional a is homogeneous, then, by 2.4.A and 4.3.C, ν is invariant with respect to the shifts $(t, x) \to (t + u, x)$ and consequently it has the form $\nu(dt, dx) = \mu(dx)\, dt$. For every positive $\mathscr{B}_{R^+} \times \mathscr{B}$-measurable function $f(t, x)$,

$$\mathbf{P} \int_0^\infty f(t, X_t)\, a(dt) = \int_0^\infty dt \int_E \mu(dx) f(t, x). \tag{5.1}$$

We call μ the *characteristic measure* of the homogeneous additive functional a.

Lemma 5.1. *If the characteristic measure μ does not charge a set D, then*

$$\int_{R^+} 1_D(X_t)\, a(dt) = 0 \qquad \text{a.s. } \mathscr{M}. \tag{5.2}$$

Proof. Put

$$Z_s = \int_s^\infty 1_D(X_t)\, a(dt).$$

By (5.1) $\mathbf{P}Z_s = 0$. Hence $Z_s = 0$ a.s. **P**. By 4.3.B, $Z_s \in \mathscr{F}^{\mathscr{M}}_{>s}$, and 2.4.E implies that $Z_s = 0$ a.s. $\mathscr{M}$ for every $s > 0$, hence $Z_0 = 0$ a.s. $\mathscr{M}$.

5.2. Our first objective is to prove the following theorem.

Theorem 5.1. *The characteristic measures of all additive functionals a_h, $h \in H^+$ are σ-finite.*

We need two lemmas.

LEMMA 5.2. *Let $f = G_c\psi$ with a bounded ψ and let $F(t, x) = T_{u-t}f(x)$. For* **P***-almost all ω, $F(t, X_t(\omega))$ is right continuous on $(-\infty, u)$ except maybe $t = \alpha$.*

Proof. For every $0 \leqslant u < v$, the function

$$G_v^u \psi = G_v \psi - G_u \psi$$

is fine. Let $\Lambda = \{u = t_0 > t_1 > \cdots\}$, $t_n \to -\infty$ and let

$$F_\Lambda(t, x) = F(t_i, x) = G_{u+c-t_i}^{u-t_i} \psi(x) \qquad \text{for} \quad t \in [t_{i+1}, t_i),\ i = 0, 1, \dots .$$

For **P**-almost all ω, $F_\Lambda(t, X_t)$ is right continuous on $(-\infty, u)$ except maybe $t = \alpha$. We have

$$F_\Lambda - F = G_{u-t}^{u-t_i} \psi - G_{u-t+c}^{u-t_i+c} \psi \qquad \text{on } [t_{i+1}, t_i).$$

Hence $|F_\Lambda - F| \leqslant 2 \sup_x |\psi(x)|\, |\Lambda|$ which implies the statement of Lemma 5.2.

LEMMA 5.3. *If μ is the characteristic measure of a_h, $h \in H^+$, then*

$$(\mu(G_u f))^2 \leqslant m(f^2)\, Pa_h(0, u)^2 \tag{5.3}$$

for every $f = G_c\psi$, $\psi \in \mathscr{B}$ and every $u > 0$.

Proof. It suffices to check (5.3) for bounded ψ.

Let $\Lambda = \{0 = t_0 < \cdots < t_j = u\}$ be a partition of $[0, u]$ and let

$$b_\Lambda(t) = t_i \qquad \text{for} \quad t \in [t_{i-1}, t_i),\ i = 1, \dots, j. \tag{5.4}$$

By (2.3) and 2.4.C

$$\mathbf{P}a(t_{i-1}, t_i) f(X_u) = \mathbf{P}a(t_{i-1}, t_i)\, F(t_i, X_{t_i}), \tag{5.5}$$

where $F(t, x) = T_{u-t}f(x)$. (For typographical convenience we drop the subscript h.) Hence

$$\mathbf{P} \int_0^u F(b_\Lambda(t), X_{b_\Lambda(t)})\, a(dt) = \mathbf{P}a(0, u) f(X_u).$$

By Lemma 5.2, $F(t, X_t)$ is right continuous on $(0, u)\backslash\{\alpha\}$ a.s. **P**. Passing to the limit along a sequence Λ_n with $|\Lambda_n| \to 0$ and using Fatou's lemma, we get

$$\mathbf{P} \int_0^u F(t, X_t)\, a(dt) \leqslant \mathbf{P}a(0, u) f(X_u). \tag{5.6}$$

Formula (5.3) follows from (5.1) and (5.6), and the Schwarz inequality.

Proof of Theorem 5.1. Let ψ be a strictly positive function in $L^2(m)$. Then $f = G_c\psi$ and $F = G_u f$ are strictly positive and belong to $L^2(m)$. By (5.3) and (4.2), $\mu(F) < \infty$.

5.3.

THEOREM 5.2. *The characteristic measure μ of an additive functional a_h, $h \in H^+$ is given by the following formula*:

$$\mu(B) = P_\phi\{X_0 \in B\}, \qquad B \in \mathscr{B}, \tag{5.7}$$

where $\phi_t = h'_t$. For all $P \in \mathscr{M}$, $P\{X_0 \notin E\} = 0$.

Proof. (1) Let us prove that

$$\mathbf{P}\int_0^c F(X_t)\,a(dt) f(X_v) = P_\phi \int_0^c F(X_0) f(X_{v-t})\,dt \tag{5.8}$$

for every $F \in \hat{\mathscr{B}}$, $f \in \mathscr{B}$, $0 < c < v$. It is sufficient to check (5.8) for $f \in L^2(m)$ and bounded F subject to conditions 3.6.E$_1$, 3.6.E$_2$. By 2.4.C, for all $0 \leqslant s < u < v$,

$$\mathbf{P}a(s,u)\,F(X_u) f(X_v) = \mathbf{P}a(s,u)\,F(X_u)\,T_{v-u} f(X_u). \tag{5.9}$$

By 2.3.B and (1.10),

$$\begin{aligned} P_\phi \int_s^u F(X_{u-t}) f(X_{v-t})\,dt &= \int_s^u P_\phi F(X_{u-t})\,T_{v-u} f(X_{u-t})\,dt \\ &= \int_s^u (\phi_{u-t}, FT_{v-u}f)\,dt \\ &= (h_{0,u-s}, FT_{v-u}f). \end{aligned} \tag{5.10}$$

It follows from (5.9), (4.4), and (5.10) that

$$\mathbf{P}a(s,u)\,F(X_u) f(X_v) = P_\phi \int_s^u F(X_{u-t}) f(X_{v-t})\,dt. \tag{5.11}$$

Now let Λ be an arbitrary partition of $[0, c]$ and let b_Λ be defined by (5.4). By (5.11),

$$\mathbf{P}\int_0^c F(X_{b_\Lambda(t)})\,a(dt) f(X_v) = P_\phi \int_0^c F(X_0) f(X_{v-t})\,dt. \tag{5.12}$$

By the dominated convergence theorem, (5.12) implies (5.8).

(2) It follows from (5.8) that, for all $0 < c < v < 1$,

$$\mathbf{P} \int_0^1 F(X_t)\, 1_{t<c, v<\beta}\, a(dt) = P_\phi \int_0^1 1_{t<c, v<t+\beta}\, dt\, F(X_0). \tag{5.13}$$

By Theorem 5.1, there exists a function $F_1 > 0$ such that $\mu(F_1) < \infty$. We consider two measures

$$\nu_1(A) = \mathbf{P} \int_0^1 1_A(t, \beta)\, F(X_t)\, a(dt),$$

$$\nu_2(A) = P_\phi \int_0^1 1_A(t, t+\beta)\, F(X_0)\, dt,$$

on the triangle $D = \{0 < t < u < 1\}$. By (5.13) they coincide on all rectangles $(0, c) \times (v, 1)$, $0 < c < v < 1$. If F/F_1 is bounded, then ν_1 is finite on all these rectangles. Thus $\nu_1(D) = \nu_2(D)$ which means that $\mu(F) = P_\phi F(X_0)$. By a monotone passage to the limit, we extend this equation to all measurable $F \geqslant 0$. Taking $F = 1_B$ we get (5.7). Since μ is concentrated on E, $X_0 \in E$ a.s. P_ϕ.

5.4. It follows from Theorems 5.1 and 5.2 that if $P \in \mathscr{M}$, then the measure

$$\mu(B) = P\{X_0 \in B\} \tag{5.14}$$

is σ-finite. Therefore the restriction of P to $\mathscr{F}_{0+}$ is σ-finite.

5.5.

THEOREM 5.3. *Formula* (5.14) *establishes a* 1–1 *mapping of* $\mathscr{M}$ *onto a subclass* M *of the class* $\mathbf{M}$ *described in Subsection* 1.5. *The inverse mapping is* $\mu \to P_\mu$. *For every* $\nu \in \mathbf{M}$ *there exists one and only one* $\mu \in M$ *such that* $\|\nu - \mu\|_M = 0$.

Proof. According to Subsection 1.5, $\mu \in \mathbf{M}$ if and only if there exists $\phi \in K$ such that

$$\frac{d(\mu T_s)}{dm} = \phi_t \qquad m\text{-a.s. for each } t > 0. \tag{5.15}$$

Formula (5.15) is equivalent to the condition $P_\mu = P_\phi$. Hence $\mu \in \mathbf{M}$ if and only if $P_\mu \in \mathscr{M}$.

Let $P \in \mathscr{M}$ and let μ be defined by (5.14). By Theorem 5.2, $X_0 \in E$ a.s. P and, using 3.6.B, we can rewrite (3.9) in the following form:

$$P\{Z \mid \mathscr{F}_{0+}\} = P_{X_0} Z \qquad \text{a.s. } P \text{ for every } Z \in \mathscr{F}_{>0}. \tag{5.16}$$

Hence $PZ = PP_{X_0}Z = P_\mu Z$. Consequently, the image M of $\mathscr{M}$ under the mapping (5.14) is a subset of $\mathbf{M}$ and (5.14) establishes a 1–1 correspondence between $\mathscr{M}$ and M.

If $\mu_1, \mu_2 \in \mathbf{M}$ and $\|\mu_1 - \mu_2\|_M = 0$, then $T_t\mu_1 = T_t\mu_2$ for all $t > 0$. Hence $P_{\mu_1} = P_{\mu_2}$. As we know, this implies $\mu_1 = \mu_2$ if μ_1 and $\mu_2 \in M$. Finally, if $\tilde{\mu} \in \mathbf{M}$ and if $\mu(B) = P_{\tilde{\mu}}\{X_0 \in B\}$, then $\mu \in M$ and $P_\mu = P_{\tilde{\mu}}$. Hence $\mu T_t = \tilde{\mu} T_t$ for all $t \to 0$ and $\|\mu - \tilde{\mu}\|_M = 0$.

5.6. Let $\mu \in M$. By Theorem 5.3, $P_\mu \in \mathscr{M}$ and

$$P_\mu\{X_0 \in B\} = \mu(B), \qquad B \in \mathscr{B}. \tag{5.17}$$

Formula (5.15) defines an isometric mapping of M onto K^+. The inverse is

$$\mu(B) = P_\Phi\{X_0 \in B\}, \qquad B \in \mathscr{B} \tag{5.18}$$

and we have $P_\Phi = P_\mu$. According to 1.4.D, the element k of $\mathscr{K}^+$ corresponding to μ is given by the formula

$$k(f) = \lim_{\varepsilon \to 0} (\phi_\varepsilon, f) = \lim_{\varepsilon \to 0} (\mu T_\varepsilon) f. \tag{5.19}$$

In particular, we have

$$k(G\rho) = \mu(G\rho) \qquad \text{if} \quad \|\rho\|_G < \infty. \tag{5.20}$$

We have also a natural isometry of M onto H^+ and $\mathscr{H}^+$ determined by the formulas

$$h_{s,u} = \frac{d(\mu G_u^s)}{dm} \qquad m\text{-a.e. for all } 0 \leqslant s < u, \tag{5.21}$$

$$f_\mu = \int_{R^+} \phi_t \, dt = \frac{d(\mu G)}{dm} \qquad m\text{-a.e.} \tag{5.22}$$

f_μ is Green's potential of μ (cf. (1.18)).

A measure $\mu(dx) = \rho(x)\, m(dx)$ belongs to M if and only if $\|\rho\|_G < \infty$ and we have

$$f_\mu = G\rho; \qquad (\mu, \mu)_M = (G\rho, \rho) = \|\rho\|_G^2. \tag{5.23}$$

It follows from (5.20) that, for all $\mu \in M$,

$$(f_\mu, G\rho)_{\mathscr{H}} = \mu(G\rho). \tag{5.24}$$

5.7. To every $\mu \in M$ there corresponds an additive functional $a^\mu = a_h$ where h is defined (5.21). By Theorem 5.2, its characteristic measure is μ.

THEOREM 5.4. *Green's potential of a measure $\mu \in M$ is given by the formula*

$$f_\mu(x) = P_x a^\mu(R^+) \qquad m\text{-a.e.} \tag{5.25}$$

For every $\mu, \tilde{\mu} \in M$

$$(\mu, \tilde{\mu})_M = P_{\tilde{\mu}} a^\mu(R^+). \tag{5.26}$$

Proof. Let $a = a^\mu = a_h$. Since $a(0, u)$ is $\mathscr{F}_{>0}$-measurable, it follows from (2.11), (2.12) that, for every $B \in \mathscr{B}$,

$$\mathbf{P} 1_B(X_0)\, a(0, u) = \mathbf{P} 1_B(X_0)\, P_{X_0} a(0, u) = \int_B P_x a(0, u)\, m(dx).$$

On the other hand, by (4.3),

$$\mathbf{P} 1_B(X_0)\, a(0, u) = (1_B, h_{0,u}) = \int_B h_{0,u}(x)\, m(dx).$$

Hence

$$h_{0,u}(x) = P_x a(0, u) \qquad m\text{-a.e.} \tag{5.27}$$

Formula (5.25) follows from (1.10), (5.24), and (5.27).

By 9.2.C, (5.25), (5.24) and 4.3.C,

$$P_{\tilde{\mu}} a(u, \infty) = \int_E m(dx)\, \tilde{\phi}_u(x)\, P_x a(R^+) = (\tilde{\phi}_u, f_\mu) = \int (\tilde{\phi}_u, \phi_t)\, dt$$

$$= \int (\tilde{\phi}_{(u+t)/2}, \phi_{(u+t)/2})\, dt = \int_{u/2}^{\infty} (\tilde{\phi}_{t/2}, \phi_{t/2})\, dt.$$

Letting $u \to 0$ and taking (1.7) and (1.18) into account, we get (5.26).

6. Dirichlet Principle. Representation of $\mathscr{H}$ by Right Functions

6.1. We say that a $\mathscr{B}$-measurable function f is *right* if $f(X_t)$ is right continuous on $[0, \infty)$ $\mathscr{M}$-a.s. A fine function is right if

$$\lim_{s \to 0} f(X_s) = f(X_0) \qquad \text{a.s. } \mathscr{M}. \tag{6.1}$$

It suffices to check condition (6.1) as s takes only rational values. Also we can restrict ourself by measures $P \in \mathscr{M}$ such that $P(\Omega) = 1$ (because every measure in $\mathscr{M}$ is equivalent to a probability measure).

LEMMA 6.1. *All functions* (1.6) *are right.*

Proof. Let $|\psi(x)| \leqslant a$ for all x and let $f = G\psi$. Put

$$Z^s = \int_s^{s+c} \psi(X_t)\, dt.$$

Suppose that $s \to 0$ along the set of rational numbers. Then, for every $P \in \mathscr{M}$,

$$P\{Z^0 \mid \mathscr{F}_s\} \to P\{Z^0 \mid \mathscr{F}_{0+}\} \qquad \text{a.s. } P.$$

Since $|Z^s - Z^0| \leqslant 2as$,

$$P\{Z^s \mid \mathscr{F}_s\} \to P\{Z^0 \mid \mathscr{F}_{0+}\} \qquad \text{a.s. } P.$$

By 9.2.C and (5.16),

$$P\{Z^s \mid \mathscr{F}_s\} = f(X_s), P\{Z^0 \mid \mathscr{F}_{0+}\} = f(X_0) \qquad \text{a.s. } P.$$

6.2. Lemma 6.1 is the first step in proving the following theorem:

THEOREM 6.1. *Every function f of $\mathscr{H}$ is m-equivalent to a right function. If $f \in \mathscr{H}$ is right, then*

$$(f_\mu, f)_{\mathscr{H}} = \mu(f) \qquad \textit{for all } \mu \in \mathscr{M}. \tag{6.2}$$

We prove simultaneously Theorems 6.1 and 6.2 which presents a general form of the Dirichlet principle.

THEOREM 6.2. *Let $B \in \mathscr{B}$ and let $\mathscr{H}_1 = \{f : f \in \mathscr{H}, f \geqslant 1_B \text{ m-a.e.}\}$. If $\mathscr{H}_1$ is nonempty, then the function.*

$$\pi_B(x) = P_x\{X_t \in B \textit{ for some } t > 0\} \tag{6.3}$$

belongs to $\mathscr{H}_1$ and $\|f\|_{\mathscr{H}} \geqslant \|\pi_B\|_{\mathscr{H}}$ for all $f \in \mathscr{H}_1$.

We note that

$$\pi_B(x) = P_x\{\sigma < \infty\}, \tag{6.4}$$

where σ is the first hitting time of B defined by (1.27). It follows from Lemma 3.1 that $\pi_B \in \mathscr{B}^\mu$ for every σ-finite measure μ. In particular π_B is m-equivalent to a $\mathscr{B}$-measurable function. We have

$$T_u \pi_B(x) = P_x\{\sigma > u\}. \tag{6.5}$$

Hence $T_u \pi_B \leqslant \pi_B$ and $T_u \pi_B \uparrow \pi_B$ as $u \downarrow 0$, i.e., π_B is an excessive function.

6.3. A set on the real line is *open from the right* if it contains, with every t, an interval $[t, u)$ for some $u > t$. We say that a set $B \in \mathscr{B}$ is *right* if the set $\{t: t \geqslant 0, X_t \in B\}$ is open from the right a.s. $\mathscr{M}$. If a $\mathscr{B}$-measurable function f is right, then all sets $\{c_1 < f < c_2\}$ are right.

LEMMA 6.2. *The statement of Theorem* 6.2 *is true for all right sets B.*

Proof. The set $\mathscr{H}_1$ is convex and closed (see 1.3.B). Hence (cf. [11, p. 62]) there exists a unique element f_0 of $\mathscr{H}_1$ with the minimal norm. If $\tilde{f} \in \mathscr{H}$ and $\tilde{f} \geqslant 0$ m-a.e., then $f_0 + \lambda \tilde{f} \in \mathscr{H}_1$ for all $\lambda \geqslant 0$. Hence $F(\lambda) = \|f_0 + \lambda \tilde{f}\|^2_{\mathscr{F}}$ attains at 0 its minimum on the positive half-line, and $2(f_0, \tilde{f})_{\mathscr{F}} = F'(0) \geqslant 0$. By 1.4.B and 1.4.C, f_0 is almost excessive and $\Phi_t(f_0)$ is bounded. Let $\mu(dx) = \rho(x)\, m(dx)$ be a probability measure. It is easy to see that $(f_0(X_t), \mathscr{F}_t, P_\mu)$ is a supermartingale. Let τ_n be the first hitting time of B by the sequence $X_{k/n}$, $k = 1, 2, \ldots$. We have $P_\mu f_0(X_{\tau_n}) \leqslant P_\mu f_0(X_{1/n}) = \mu(T_{1/n} f_0) \leqslant \mu(f_0)$. On the other hand, $P_\mu f_0(X_{\tau_n}) \geqslant P_\mu\{\tau_n < \infty\}$ since $f_0 \in \mathscr{H}_1$. If $\mu \in \mathscr{M}$, then $P_\mu\{\tau_n < \infty\} \uparrow \mu(\pi_B)$ because B is right. Therefore $\mu(\pi_B) \leqslant \mu(f_0)$ which implies: $\pi_B \leqslant f_0$ m-a.e. Since π_B and f_0 are almost excessive, $\Phi_t(\pi_B) \leqslant \Phi_t(f_0)$ for all t, by 1.4.A. By 1.4.B and 1.4.A, $\pi_B \in \mathscr{H}$ and $\|\pi_B\|_{\mathscr{F}} \leqslant \|f_0\|_{\mathscr{F}}$. Since $\pi_B \geqslant 1_B$ m-a.e., π_B belongs to $\mathscr{H}_1$ and, since the element with the minimal norm is unique, $\pi_B = f_0$ m-a.e.

LEMMA 6.3. *Formula* (6.2) *is true for every excessive* $f \in \mathscr{H}$.

Proof. Put $\rho_t = t^{-1}(f - T_t f)$. We have $G\rho_t = t^{-1} G_t f \uparrow f$ as $t \downarrow 0$. By 1.3.C, $G\rho_t \to f$ in $\mathscr{H}$, and by (5.24), $\mu(G\rho_t) = (f_\mu, G\rho_t)$. Passing to the limit, we get (6.2).

LEMMA 6.4. *If* $f \in \mathscr{H}$ *is a right function, then*

$$P_\mu\{\sup_{t \geqslant 0} |f(X_t)| > \varepsilon\} \leqslant \varepsilon^{-1} \|\mu\|_M \|f\|_{\mathscr{F}} \qquad \textit{for all } \mu \in \mathbf{M}. \tag{6.6}$$

Proof. Since f is right, the left side of (6.6) is $\leqslant \mu(\pi_B)$ where

$B = \{|f| > \varepsilon\}$. Since π_B is excessive, $\mu(\pi_B) = (f_\mu, \pi_B)_{\mathscr{F}} \leqslant \|f_\mu\|_{\mathscr{F}} \|\pi_B\|_{\mathscr{F}}$. By Lemma 6.3 the set B is right and $|\varepsilon^{-1} f| \geqslant 1_B$. By Lemma 6.2, $\|f\varepsilon^{-1}\|_{\mathscr{F}} \geqslant \|\pi_B\|_{\mathscr{F}}$.

6.4. According to Subsection 1.6, a set B is inaccessible if $X_t \notin B$ for all $t > 0$ P-a.s. for all $P \in \mathscr{M}$. This is equivalent to the condition $\pi_B = 0$ m-a.e.

LEMMA 6.5. *If a sequence of right functions $f_n \in \mathscr{H}$ converge in $\mathscr{H}$, then a subsequence f_{n_i} converges q.e. to a function f. Moreover, for every $P \in \mathscr{M}$,*

$$\sup_{t \geqslant 0} |f_{n_i}(X_t) - f(X_t)| \to 0 \qquad P\text{-a.s.} \tag{6.7}$$

Proof. Every $P \in \mathscr{M}$ has the form $P = P_\mu$ where $\mu \in \mathbf{M}$. By Lemma 6.4,

$$P\{\sup_{t \geqslant 0} |f_i(X_t) - f_j(X_t)| > 2^{-n}\} \leqslant 2^n \|\mu\|_M \|f_i - f_j\|_{\mathscr{F}}.$$

We choose n_i in such a way that $P\{Z_i > 2^{-i}\} \leqslant 2^{-i}$, where

$$Z_i = \sup_{t \geqslant 0} |f_{n_{i+1}}(X_t) - f_{n_i}(X_t)|.$$

By the Borel–Cantelli lemma, for P-almost all ω, $Z_i \geqslant 2^{-i}$ only for a finite number of i. Therefore the series $\sum_i (f_{n_{i+1}}(X_t) - f_{n_i}(X_t))$ converges absolutely and uniformly for all $t \geqslant 0$. Hence the set $B = \{x\colon \sum_i |f_{n_{i+1}}(x) - f_{n_i}(x)| = \infty\}$ is inaccessible. We put

$$f(x) = \lim f_{n_i}(x) \qquad \text{for} \quad x \notin B, \qquad f(x) = 0 \qquad \text{for} \quad x \in B.$$

Obviously (6.7) holds and therefore f is a right function.

6.5.

Proof of Theorem 6.1. Let ρ be a bounded reference function and let $\rho_B = \rho 1_B$. Denote by $\mathscr{S}$ the set of all functions $G\rho_B$, $B \in \mathscr{B}$. By (5.23), $\mathscr{S} \subset \mathscr{H}$. It follows from 1.3.E that an element of $\mathscr{H}$ which is orthogonal to $\mathscr{S}$ vanishes m-a.e. Hence $\mathscr{S}$ is everywhere dense in $\mathscr{H}$.

Put $f \in \tilde{\mathscr{H}}$ if f belongs to $\mathscr{H}$ and is m-equivalent to a right function. By Lemma 6.5, $\tilde{\mathscr{H}}$ is a closed subset of $\mathscr{H}$. It follows from 1.3.C that $G_t\rho_B \to G\rho_B$ in $\mathscr{H}$ as $t \to \infty$. By Lemma 6.1, $G_t\rho_B \in \tilde{\mathscr{H}}$. Hence $\tilde{\mathscr{H}}$ contains $\mathscr{S}$ and, being closed, it coincides with $\mathscr{H}$.

It remains to prove (6.2). Let f be a right element of $\mathscr{H}$. By 1.3.D, $|f| \in \mathscr{H}$. It follows from (5.19), Fatou's lemma and (5.17) that

$$(f_\mu, |f|)_{\mathscr{F}} = \lim_{\varepsilon \to 0} P_\mu |f(X_\varepsilon)| \geqslant P_\mu |f(X_0)| = \mu(|f|). \tag{6.8}$$

There exists a sequence of right functions $f_n \in \mathscr{S}$ such that $\|f-f_n\|_{\mathscr{H}} \to 0$. Applying (6.8) to $f-f_n$ and using 1.3.B, we get

$$\mu(|f-f_n|) \leqslant (f_\mu, |f-f_n|)_{\mathscr{H}} \leqslant \|f_\mu\|_{\mathscr{H}} \|f-f_n\|_{\mathscr{H}}.$$

Hence $\mu(f_n) \to \mu(f)$. By (5.24), $(f_\mu, f_n)_{\mathscr{H}} = \mu(f_n)$. Passing to the limit, we get (6.2).

6.6.

Proof of Theorem 6.2. Let f_0 be the function constructed in the proof of Lemma 6.2. It follows from (3.3) and (5.16) that $(f_0(X_t), \mathscr{F}^P_{t+}, P)$ is a supermartingale for every $P = P_\mu \in \mathscr{M}$. We choose f_0 to be right and we apply the optional sampling theorem (see, e.g., [14, Chap. VI, Theorem 13]) to the first hitting time σ. We get

$$Pf_0(X_\sigma) \leqslant Pf_0(X_0) = \mu(f_0).$$

Obviously $P\{\sigma < \infty,\ f_0(X_0) < 1\} = 0$. Hence $Pf_0(X_\sigma) \geqslant P\{\sigma < \infty\} = \mu(\pi_B)$. The rest of the proof is the same as for Lemma 6.2.

7. Operators T_τ

7.1. In this section we investigate the operators T_τ defined by formulas (1.23) and (1.24).

It follows from Theorem 3.1 and (5.16) that

$$P_{\mu T_\tau} Y = P_\mu \theta_\tau Y \tag{7.1}$$

for all $\mu \in M$, $Y \in \mathscr{F}_{>0}$.

If a is an additive functional, then, by 4.3.C,

$$\theta_\tau a(R^+) = a(\tau, \infty) \qquad \text{a.s. } P_\mu \tag{7.2}$$

and (7.1) implies that

$$P_{\mu T_\tau} a(R^+) = P_\mu a(\tau, \infty) \leqslant P_\mu a(R^+). \tag{7.3}$$

By applying this formula to $a(dt) = \psi(X_t)\, dt$, we get

$$\mu(T_\tau G_\psi) \leqslant \mu(G\psi), \qquad \psi \in \mathscr{B}. \tag{7.4}$$

THEOREM 7.1. *Every operator T_τ is a contraction of M.*

Proof. Let $\nu = \mu T_\tau$, $\mu \in M$. By (7.4) and (5.22), for every $C \in \mathscr{B}$,

$$(\nu G)(C) \leqslant (\mu G)(C) = \int_C f_\mu(x)\, m(dx),$$

where f_μ is Green's potential of μ. By the Radon–Nikodym theorem νG has a density ρ with respect to m and $0 \leqslant \rho \leqslant f_\mu$ m-a.e. For every $\psi \in \mathscr{B}$,

$$\nu(G\psi) = (\rho, \psi). \tag{7.5}$$

Therefore for every $t > 0$

$$(T_t\rho, \psi) = (\rho, T_t\psi) = \nu(GT_t\psi) \leqslant \nu(G\psi) = (\rho, \psi)$$

and $T_t\rho \leqslant \rho$ m-a.e. By 1.4.A, $\Phi_t(\rho) \leqslant \Phi_t(f_\mu)$ and, by 1.4.B and (1.14), $\rho \in \mathscr{H}^+$ and $\|\rho\|_{\mathscr{H}} \leqslant \|f_\mu\|_{\mathscr{H}}$. By (1.16)

$$\rho = \int_0^\infty \phi_t\, dt \tag{7.6}$$

where $\phi \in K^+$. We note that $\|\phi\|_K = \|\rho\|_{\mathscr{H}} \leqslant \|f_\mu\|_{\mathscr{H}} = \|\mu\|_M$.

If $\psi \geqslant 0$ and $(f_\mu, \psi) < \infty$, then $(\rho, \psi) < \infty$ and (7.5), (7.6) imply that

$$\int_0^u \nu(T_t\psi)\, dt = \int_0^u (\phi_t, \psi)\, dt \tag{7.7}$$

for every u. Let ρ be a reference function and let $\psi = \rho 1_B$ where $B \in \mathscr{B}$. By 1.3.B, $(f_\mu, \psi) = m^\rho(1_B f_\mu) \leqslant m^\rho(f_\mu) \leqslant \|\rho\|_G \|f_\mu\|_{\mathscr{H}} < \infty$ and (7.7) holds. Hence

$$\nu T_t = m^{\phi_t} \tag{7.8}$$

for almost all t. By (7.1), (2.8) and (2.5), Eq. (7.8) implies that

$$P_\nu \theta_t Y = P_{\nu T_t} Y = P_\phi \theta_t Y$$

for all $Y \in \mathscr{F}_{>0}$. Hence $P_\nu = P_\phi$ and, by (7.1) and (1.24),

$$P_\phi\{X_0 \in B\} = P_\nu\{X_0 \in B\} = P_\mu \theta_\tau\{X_0 \in B\} = (\mu T_\tau)(B) = \nu(B).$$

By (5.18), $\nu \in M$ and $\|\nu\|_M = \|\phi\|_K \leqslant \|\mu\|_M$.

7.2.

THEOREM 7.2. *A set $B \in \mathscr{B}$ is inaccessible if and only if $\mu(B) = 0$ for all $\mu \in M$. $\mathscr{B}$-measurable functions f and g coincide q.e. if and only if $\mu(f) = \mu(g)$ for all $\mu \in M$.*

Proof. If $\mu \in M$, then μ is the characteristic measure of a^μ and, by (5.1), $\mu(B) = \mathbf{P} \int_{R^+} e^{-t} 1_B(T_t)\, a^\mu(dt)$. Hence $\mu(B) = 0$ if B is inaccessible.

Let $\mu(B) = 0$ for all $\mu \in M$. We consider the stopping times τ_n constructed in Lemma 3.1. If $\mu \in M$, then $\mu_n = \mu T_{\tau_n} \in M$ by Theorem 7.1. Hence $P_\mu\{X_{\tau_n} \in B\} = \mu_n(B) = 0$. By 3.3.A and 3.3.B, this implies $P_\mu\{\sigma < \infty\} = 0$. Thus B is inaccessible.

Suppose that $\mu(f) = \mu(g)$ for all $\mu \in M$. Since M contains together with μ the restriction μ_B of μ to each set $B \in \mathscr{B}$, we have $\mu_B(f) = \mu_B(g)$ for all B. Hence $\mu\{f \neq g\} = 0$ and the set $\{f \neq g\}$ is inaccessible.

7.3.

THEOREM 7.3. *A function $f \in \mathscr{H}$ is right if and only if*

$$(f_\mu, f)_{\mathscr{H}} = \mu(f) \qquad \text{for all } \mu \in M. \tag{7.9}$$

The formula

$$f_\mu(x) = P_x a^\mu(R^+) \tag{7.10}$$

gives a right version of Green's potential.

Proof. The "only if" part is contained in Theorem 6.1. Suppose that f satisfies (7.9). By Theorem 6.1, f is m-equivalent to a right function g. For every $\mu \in M$, $\mu(f) = (f_\mu, f)_{\mathscr{H}} = \mu(g)$. By Theorem 7.2, $f = g$ q.e. Therefore, for every $\mu \in M$ and for P_μ-almost all ω, $f(X_t)$ is right continuous on $(0, \infty)$ and tends to $g(X_0)$ as $t \to 0$. But $P_\mu\{f(X_0) \neq g(X_0)\} = \mu(f \neq g) = 0$. Hence f is right.

Let f_μ be defined by (7.10). By Theorem 5.4, for every $\nu \in M$, $(f_\nu, f_\mu)_{\mathscr{H}} = (\nu, \mu)_M = P_\nu a^\mu(R^+) = \nu(f_\mu)$. Hence f_μ is right.

7.4.

THEOREM 7.4. *Let σ be the first hitting time of $B \in \mathscr{B}$. For every $\mu \in M$, μT_σ is the orthogonal projection of μ on $K(B)$. For every right $f \in \mathscr{H}$, $T_\sigma f$ is the right version of the orthogonal projection of f onto $\mathscr{H}(B)$.*

Proof. (1) We have, for every $\mu, \nu \in M$,

$$P_\mu a^\nu(0, \tau) = P_\mu \int_{R^+} 1_{t<\tau}\, a^\nu(dt) \leqslant P_\mu \int_{R^+} 1_{E \setminus B}(X_t)\, a^\nu(dt).$$

Let $\nu \in M(B) = \mathbf{M}(B) \cap M$. Then, by Lemma 5.1, the right side is equal to 0 and, by (7.3), $P_{\mu T_\sigma} a^\nu(R^+) = P_\mu a^\nu(R^+)$. By (5.26), this is equivalent to the equation $(\mu T_\sigma, \nu)_M = (\mu, \nu)_M$. Let τ_n be stopping times defined in Lemma 3.1.

By Theorem 7.1, $\mu_n = \mu T_{\tau_n} \in M$ and $\|\mu_n\|_M \leqslant \|\mu\|_M$. It follows from 3.3.A that $\mu_n \in M(B)$. By (7.3) and 3.3.B, for every $\nu \in M$,

$$(\mu_n, \nu)_M = P_\mu a^\nu(\tau_n, \infty) \to P_\mu a^\nu(\sigma, \infty) = (\mu T_\sigma, \nu)_M.$$

Hence $\mu_n \to \mu T_\sigma$ weakly in K and $\mu T_\sigma \in K(B)$.

(2) Let f^B denote the right version of the orthogonal projection of $f \in \mathscr{H}$ on $\mathscr{H}(B)$. It follows from (1) that $f_\mu^B = f_{\mu T_\sigma}$ for all $\mu \in M$. In general, by (7.9),

$$\mu(f^B) = (f_\mu, f^B)_{\mathscr{H}} = (f_\mu^B, f)_{\mathscr{H}} = (f_{\mu T_\sigma}, f)_{\mathscr{H}} = (\mu T_\sigma)(f) = \mu(T_\sigma f)$$

and, by Theorem 7.2, $f^B \doteq T_\sigma f$ q.e. Since $T_\sigma f$ satisfies (7.9), it is right.

Remark. $T_\sigma f_\mu = f_{\mu T_\sigma}$ q.e. because both functions represent $(f_\mu)^B$.

7.5.

THEOREM 7.5. *Let σ be the first hitting time of $B \in \mathscr{B}$ and let $f \in \mathscr{H}$ be a right function. Then $T_\sigma f = f$ q.e. on B, and $T_\sigma f$ is the only right function in $\mathscr{H}(B)$ which coincides with f q.e. on B.*

Proof. By Theorem 7.4, $f - T_\sigma f$ is orthogonal to f_μ, $\mu \in M(B)$. Hence $\mu(T_\sigma f) = (f_\mu, T_\sigma f)_{\mathscr{H}} = (f_\mu, f)_{\mathscr{H}} = \mu(f)$ which implies the first statement of the theorem.

If F is a right element of $\mathscr{H}(B)$, then, by Theorem 7.4, $T_\sigma F = F$ m-a.e., hence q.e. and $\mu(F) = \mu(T_\sigma F)$ for all $\mu \in M$. If $F = f$ q.e. on B, then $(\mu T_\sigma)(F) = (\mu T_\sigma)(f)$. Hence $\mu(T_\sigma f) = \mu(F)$ for all $\mu \in M$, and $T_\sigma f = F$ q.e. by Theorem 7.2.

COROLLARY. *$P_x\{\sigma > 0\} = 0$ q.e. on B.*

Indeed, let $\rho > 0$, $\|\rho\|_G < \infty$. Then $f = G\rho$ is a right element of $\mathscr{H}$ and $T_\sigma f(x) = P_x \int_\sigma^\infty \rho(X_t)\, dt$. Hence by Theorem 7.5, $P_x \int_0^\sigma \rho(X_t)\, dt = 0$ q.e. on B.

8. CONCLUDING REMARKS

8.1. *Comparison with the Previous Work*

The relation between Markov processes and Dirichlet spaces is the subject of books by Silverstein [16] and Fukushima [11]. We refer to these books for earlier history. A substantial part of [11, 16] is devoted to the construction of a Markov process starting from a Dirichlet space—an important problem which has been left aside in the present paper.

In all the literature on Dirichlet spaces, it is assumed that the state space

E is a locally compact separable Hausdorff space and that $\mathscr{H}$ contains sufficiently large amount of continuous functions. We do not need any conditions of topological nature. The second distinction is related to the first one. We get all results on Dirichlet spaces by probabilistic methods in constrast to Silverstein and Fukushima who present, first, an analytic theory of Dirichlet forms and then apply it to probabilistic problems.

8.2. *Nondissipative Transition Functions*

For every Markov transition function $p_t(x, B)$ and every constant $\lambda > 0$, the formula

$$p_t(x, B) = e^{-\lambda t} p_t(x, B)$$

defines a dissipative transition function. This makes it possible to apply the results of this paper to the general transition functions.

8.3. *Fine Processes*

In the definition of fine processes, the family (3.1) can be replaced by the family

$$f(x) = \int_{R+} e^{-\lambda t} T_t \psi(x) \, dt, \qquad \psi \in \mathscr{B}, \ \lambda > 0$$

or by the family of all λ-excessive functions. All these conditions determine the same class of processes. Another equivalent condition is: for every $u \in R$ and every $f \in \mathscr{B}$, the function

$$T_{u-t} f(X_t) = P_{t, X_t} f(X_u)$$

is right continuous in t on $\Delta \cap (-\infty, u)$ for $\mathbf{P}$-almost all ω.

Let us discuss the relation between fine processes and right processes investigated in [6, 12, 15, 17]. In the definition of both classes, right continuity of certain functions along almost all paths is required. The difference is that "almost all" means $\mathbf{P}$-a.s. for the reference measure $\mathbf{P}$ in the case of fine processes and it means $\mathscr{N}$-a.s. in the case of right processes. As we have mentioned in Subsection 3.1, right continuity $\mathbf{P}$-a.s. implies right continuity $\mathscr{M}$-a.s. However, in general, it does not imply right continuity $\mathscr{N}$-a.s. The implication takes place under the following additional condition:

8.3.A. For every t, x, the measure $p_t(x, \cdot)$ is absolutely continuous with respect to m.

In this case the distinction between fine and right processes is insignificant.

The second part of the definition of a right process is: the state space E can be imbedded into a compact metric space $\hat{E}$ in such a way that $X_t(\omega)$ is right continuous for all ω. It is assumed in [6, 17] that E is a Borel set in $\hat{E}$. In this case our condition 3.2.B is obviously satisfied. In [12] a weaker condition—E is universally measurable—is introduced.

Condition 3.2.B can be replaced by a weaker condition:

8.3.B. There exists a $\mathcal{N}$-certain set $\Gamma \in \mathcal{F}_{>0}$ such that $\Gamma \subset \Omega^*$ and $\mathcal{B}$ is generated by functions f with the following property: $f(X_t(\omega))$ is right continuous on $(0, \beta(\omega))$ for all $\omega \in \Gamma$.

Under assumption 8.3.A, condition 8.3.B follows from 3.2.B and therefore can be dropped. This is the reason that 3.2.B has not appeared in [8, 9].

In the present paper, only symmetric transition functions are considered. A natural generalization is a pair of transition functions in duality with respect to a measure m. A reference measure **P** can be defined in the space of paths $\omega(t\pm)$ with split time. It might be interesting to investigate the class of fine processes in this more general case.

8.4. *The Entrance Space*

Dealing with measures of the class $\mathcal{M}$, it is natural to restrict the space Ω to the set $\Omega^* = \{\alpha \leqslant 0 < \beta\}$ on which all these measures are concentrated. Originally, $X_t(\omega)$, $\omega \in \Omega^*$, is defined only for $t > 0$. Theorem 3.2 shows that it is possible to define $X_0(\omega)$ with values in an extension $\hat{E}$ of E and to preserve the Markov property and the right continuity. $\hat{E}$ is a version of an entrance space. Actually, working with $\mathcal{M}$, we do not need to extend E since, according to Section 5, $X_0 \in E$ $\mathcal{M}$-a.s.

8.5. *Additive Functionals*

The results of Fukushima on positive continuous additive functionals (see [11, Sect. 5.1]) follow easily from theorems of Sections 4 and 5. Methods used in these sections work also in the case of several Markov processes, and the principal results of [8] can be proved in this way. In [8] assumption 8.3.A has been used. Now we are able to remove this restriction.

8.6. *Classes* **M** *and* M

These classes coincide if the transition function separates measures, i.e., if the condition $\mu T_t = \nu T_t$ for all $t > 0$ implies that $\mu = \nu$. In general, $\mathbf{M} \neq M$. For example, let $E = \{1, 2\}$, $p_t(1, 1) = p_t(2, 1) = e^{-t}$, $p_t(1, 2) = p_t(2, 2) = 0$; $m(1) = 1$, $m(2) = 0$; $\mu(1) = 0$, $\mu(2) = 1$. The transition function p is symmetric relative to m, and μ belongs to **M** but not to M.

8.7. *Dirichlet Problem*

Let X_t be the Brownian motion in R^3. Then

$$(f, f)_{\mathscr{H}} = \int_{R^3} [(f_x)^2 + (f_y)^2 + (f_z)^2] \, dm,$$

where f_x, f_y, f_z are weak partial derivatives of f and m is the Lebesgue measure. The space $\mathscr{H}$ consists of all f for which this integral is finite.

Let D be an open set and let B be the complement of D. A right function f belongs to $\mathscr{H}(B)$ if and only if it is harmonic in D.

We say that F is a solution of the Dirichlet problem for f in D if F is harmonic in D, is a right function and if $F = f$ q.e. on B.

It follows from Theorem 7.5 that, for every right $f \in \mathscr{H}$ there exists a unique solution of the Dirichlet problem which belongs to $\mathscr{H}$. A classical Kellog–Evans theorem follows immediately from this fact.

8.8. *Measures Vanishing on All Inaccessible Sets*

Put $\mu \in \tilde{M}$ if μ is a σ-finite measure which does not charge any inaccessible set. By Theorem 7.2, $M \subset \tilde{M}$. Is every $\tilde{\mu} \in \tilde{M}$ equivalent to a measure $\mu \in M$? The answer is positive under condition 8.3.A (see, e.g., [8, p. 66]). In general, the answer is unknown.

References

1. C. Dellacherie, "Capacités et processus stochastiques," Springer-Verlag, Berlin/Heidelberg/New York, 1972.
2. C. Dellacherie and P. A. Meyer, "Probabilities and Potential," North-Holland, Amsterdam/New York/Oxford, 1978.
3. E. B. Dynkin, "Markov Processes, Vols. I and II," Springer-Verlag, Berlin/Göttingen/Heidelberg, 1965.
4. E. B. Dynkin, Initial and final behaviour of trajectories of Markov processes, *Russian Math. Surveys* **26** (1971), 165–185.
5. E. B. Dynkin, Integral representation of excessive measures and excessive functions, *Russian Math. Surveys* **27** (1972), 43–84.
6. E. B. Dynkin, Regular Markov processes, *Russian Math. Surveys* **28** (1973), 33–64.
7. E. B. Dynkin, Sufficient statistics and extreme points, *Ann. Probab.* **6** (1978), 705–730.
8. E. B. Dynkin, Additive functionals of several time-reversible Markov processes, *J. Funct. Anal.* **42** (1981), 64–101.
9. E. B. Dynkin, Harmonic functions associated with several Markov processes, *Adv. in Appl. Math.* **2** (1981), 260–283.
10. E. B. Dynkin, Green's and Dirichlet spaces for a symmetric Markov transition function, Lecture Notes of the London Math. Soc., 1982.
11. M. Fukushima, "Dirichlet Forms and Markov Processes," North-Holland, Amsterdam/Oxford/New York, 1980.
12. R. Getoor, "Markov Processes: Ray Processes and Right Processes," Lecture Notes in Mathematics No. 440, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
13. S. E. Kuznecov, Construction of Markov processes with random birth and death times, *Theory Probab. Appl.* **18** (1974), 571–575.
14. P. A. Meyer, "Probability and Potentials," Blaisdell, Waltham, Mass./Toronto/London, 1966.
15. P. A. Meyer, "Processus de Markov," Lecture Notes in Mathematics No. 26, Springer-Verlag, Berlin/Heidelberg/New York, 1967.

16. M. L. SILVERSTEIN, "Symmetric Markov Processes," Lecture Notes in Mathematics No. 426, Springer-Verlag, Berlin/Heidelberg/New York, 1974.
17. J. B. WALSH AND P. A. MEYER, Quelques applications des résolvantes de Ray, *Invent. Math.* **14** (1971), 143–166.