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On cycle systems with specified weak chromatic number

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ABSTRACT

A weak k-colouring of an m-cycle system is a colouring of the vertices of the system with k colours in such a way that no cycle of the system has all of its vertices receive the same colour. An m-cycle system is said to be weakly k-chromatic if it has a weak k-colouring but no weak $(k-1)$-colouring. In this paper we show that for all $k \geqslant 2$ and $m \geqslant 3$ with $(k,m) \neq (2,3)$ there is a weakly k-chromatic m-cycle system of order v for all sufficiently large admissible v.

1. Introduction

An *m-cycle decomposition* of a graph G is a collection $\mathcal{D}$ of m-cycles in G such that each edge of G occurs in precisely one cycle in $\mathcal{D}$. An *m-cycle system* $(V, \mathcal{D})$ is an m-cycle decomposition, $\mathcal{D}$, of the complete graph on vertex set V and a *partial m-cycle system* $(V, \mathcal{P})$ is an m-cycle decomposition, $\mathcal{P}$, of some subgraph of the complete graph on vertex set V. The size of V is said to be the *order* of the (partial) m-cycle system. Alspach, Gavlas and Šajna [1,19,20] have shown that there exists an m-cycle system of order v if and only if v is odd, $\binom{v}{2} \equiv 0 \pmod{m}$ and either $v \geqslant m$ or $v = 1$. We shall call such integers *m-admissible*. A partial m-cycle system $(U, \mathcal{P})$ is said to be *embedded* in an m-cycle system $(V, \mathcal{D})$ if $U \subseteq V$ and $\mathcal{P} \subseteq \mathcal{D}$.

For a positive integer k, a *weak k-colouring* of a (partial) m-cycle system is a colouring of the vertices of the system with k colours in such a way that no cycle of the system has all of its vertices receive the same colour. A (partial) m-cycle system is said to be *weakly k-chromatic*, or to have *weak chromatic number k*, if it has a weak k-colouring but no weak $(k-1)$-colouring. Since weak colourings are the only colourings we will consider in this paper, we will often omit the adjectives 'weak' and 'weakly' in what follows. A set of vertices which all receive the same colour under a given colouring is referred to as *monochromatic*, and a cycle whose vertex set is monochromatic is also referred to as monochromatic.

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Weak colourings were first introduced in the context of hypergraphs, and this naturally led to the study of weak colourings of Steiner triple systems and partial Steiner triple systems. In particular, de Brandes, Phelps and Rödl [5] have shown that for all integers $k \geqslant 3$ there is an integer $n'_{k,3}$ such that for all 3-admissible integers $v \geqslant n'_{k,3}$ there is a k-chromatic Steiner triple system of order v (a simple counting argument [18] shows that there are no non-trivial 2-chromatic Steiner triple systems). Less is known, however, about weak colourings of general cycle systems. In [14] Milici and Tuza found, for all $m \geqslant 3$, an m-cycle system that could not be 2-coloured, and in [15] they found a 2-chromatic m-cycle system of order v for all $m \geqslant 4$ and $v > 1$ such that $v \equiv 1 \pmod{2m}$ if m is even and $v \equiv 1$ or $m \pmod{2m}$ if m is odd. In [3,4] Burgess and Pike showed that for all $k \geqslant 2$ and even $m \geqslant 4$ there exists a k-chromatic m-cycle system. Here we show that for all $k \geqslant 2$ and $m \geqslant 3$ with $(k,m) \neq (2,3)$ there exist k-chromatic m-cycle systems of all m-admissible orders greater than or equal to some integer $n'_{k,m}$.

Theorem 1.1. *Let k and m be integers such that $k \geqslant 2$, $m \geqslant 3$ and $(k,m) \neq (2,3)$. Then there is an integer $n'_{k,m}$ such that there exists a k-chromatic m-cycle system of order v for all m-admissible integers $v \geqslant n'_{k,m}$. Furthermore, if $n_{k,m}$ is the smallest m-admissible such value of $n'_{k,m}$ and if $u_{k,m}$ is the minimum order of a k-chromatic partial m-cycle system then $n_{k,m} \leqslant 2m(u_{k,m}+1)+1$.*

We prove this by finding embeddings of k-chromatic partial m-cycle systems in k-chromatic m-cycle systems. Since known results on weak colourings of hypergraphs imply that there exists a k-chromatic partial m-cycle system for all $k \geqslant 2$ and $m \geqslant 3$, this gives us our result. The smallest embeddings we construct (which preserve chromatic number) are almost as small as the smallest known embeddings in the case m is odd and are approximately four times as large as the smallest known embeddings in the case m is even. Furthermore, we can make use of some known bounds for weak colourings of hypergraphs to find some bounds on $n_{k,m}$.

Corollary 1.2. *Let k and m be integers such that $k \geqslant 2$, $m \geqslant 3$ and $(k,m) \neq (2,3)$. Let $u_{k,m}$ be the minimum order of a k-chromatic partial m-cycle system, and let $n_{k,m}$ be the smallest m-admissible integer such that there exists a k-chromatic m-cycle system of order v for all m-admissible integers $v \geqslant n_{k,m}$. Then*

$$\frac{1}{2m}(k-1)^{m-1} < u_{k,m} \leqslant n_{k,m} < 5m^7(k-1)^{m-1}\big(\ln(k-1)+1\big).$$

It is straightforward to verify that this implies that there are functions f_1 and f_2 of m such that

$$f_1(m)k^{m-1} \leqslant u_{k,m} \leqslant n_{k,m} \leqslant f_2(m)k^{m-1}\ln(k)$$

for all $k \geqslant 2$ and $m \geqslant 3$ with $(k,m) \neq (2,3)$. This generalises a result in [5] which proved that there are constants b_1 and b_2 such that,

$$b_1k^2 \leqslant u_{k,3} \leqslant n_{k,3} \leqslant b_2k^2\ln(k)$$

for all $k \geqslant 3$. In the later paper [16], however, this result was improved to

$$c_1k^2\ln(k) \leqslant u_{k,3} \leqslant n_{k,3} \leqslant c_2k^2\ln(k)$$

for some constants c_1 and c_2.

2. Preliminary results

We begin by introducing some notation. For a positive integer v the complete graph and empty graph of order v will be denoted by K_v and K_v^c respectively, and for positive integers a and b the complete bipartite graph with partite sets of size a and b will be denoted by $K_{a,b}$. For a given set V the complete graph and empty graph on vertex set V will be denoted by K_V and K_V^c respectively, and for sets A and B the complete bipartite graph with partition $\{A,B\}$ will be denoted by $K_{A,B}$.

For graphs G and H, the *lexicographic product* of G and H, denoted $G \cdot H$, is the graph with vertex set $V(G \cdot H) = V(G) \times V(H)$ in which vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either g_1 is adjacent to g_2 in G or $g_1 = g_2$ and h_1 is adjacent to h_2 in H. For vertex-disjoint graphs G and H, the *join* of G and H, denoted $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{xy\colon\ x \in V(G),\ y \in V(H)\}$. The p-cycle with vertices $x_1, x_2, \dots, x_p$ and edges $x_1x_2, x_2x_3, \dots, x_{p-1}x_p, x_px_1$ will be denoted by $(x_1, x_2, \dots, x_p)$. In what follows all operations on elements of $\mathbb{Z}_m$ are presumed to be the relevant group operations.

We will use Sotteau's characterisation of when, for even m, there exists an m-cycle decomposition of a complete bipartite graph.

Theorem 2.1. *(See [21].) Let m, a and b be positive integers such that $m \geqslant 4$ and m is even. There is an m-cycle decomposition of $K_{a,b}$ if and only if a and b are even, $a, b \geqslant \frac{m}{2}$ and $ab \equiv 0 \pmod{m}$.*

We will also need a result on cycle decompositions of graphs of the form $G \cdot K^c_{\mathbb{Z}_m}$ where G is an m-cycle.

Lemma 2.2. *Let m be an integer such that $m \geqslant 4$, and let G be the cycle $(x_1, x_2, \dots, x_m)$. Then there exists an m-cycle decomposition, $\mathcal{D}$ say, of $G \cdot K^c_{\mathbb{Z}_m}$ such that*

- $((x_1, 0), (x_2, 0), \dots, (x_m, 0)) \in \mathcal{D}$;
- *each cycle of $\mathcal{D}$ contains the vertices (x_1, i) and (x_3, i) for some $i \in \mathbb{Z}_m$; and*
- *if m is even and $m \geqslant 6$ then each cycle of $\mathcal{D}$ contains the vertices (x_a, i) and $(x_{a+1}, i + j)$ for some $a \in \{1, 3, 5\}$, $i \in \mathbb{Z}_m$ and $j \in \{0, 2, 4, \dots, m - 2\}$.*

Proof. Let p be the permutation $(0)(123 \cdots (m-1))$ of $\mathbb{Z}_m$. Let $\mathcal{D} = \{C_{i,j}\colon\ i, j \in \mathbb{Z}_m\}$ where

$$C_{i,j} = \big((x_1, i), (x_2, i + j), (x_3, i), \big(x_4, i + p(j)\big), (x_5, i), \big(x_6, i + p^2(j)\big), \dots,\\ (x_{m-1}, i), \big(x_m, i + p^{\frac{m-2}{2}}(j)\big)\big)$$

if m is even and

$$C_{i,j} = \big((x_1, i), (x_2, i + j), (x_3, i), (x_4, i + j), \dots, (x_{m-2}, i), (x_{m-1}, i + j), (x_m, i + 2j)\big)$$

if m is odd. It is routine to check that $\mathcal{D}$ is an m-cycle decomposition of $G \cdot K^c_{\mathbb{Z}_m}$ with the required properties. $\square$

Finally in this section, we require the following result which concerns m-cycle systems of small order.

Lemma 2.3. *Let m and n be integers such that $m \geqslant 4$, $m \leqslant n \leqslant \frac{7m-1}{2}$ and n is m-admissible. Let S_1 and S_2 be disjoint sets such that $|S_1| = \frac{n+1}{2}$ and $|S_2| = \frac{n-1}{2}$. Then there exists an m-cycle system $(S_1 \cup S_2, \mathcal{C})$ such that, for each $C \in \mathcal{C}$, $V(C) \nsubseteq S_1$ and $V(C) \nsubseteq S_2$.*

Proof. Let S be a set of size n. Since n is m-admissible there exists an m-cycle system on vertex set S, $(S, \mathcal{C})$ say, by the result of Alspach, Gavlas and Šajna [1,19,20]. It suffices to show that there is a partition $\{S_1, S_2\}$ of S such that $|S_1| = \frac{n+1}{2}$, $|S_2| = \frac{n-1}{2}$ and for each $C \in \mathcal{C}$, $V(C) \nsubseteq S_1$ and $V(C) \nsubseteq S_2$. It is easy to see that such a partition exists when $m \leqslant n \leqslant 2m - 1$ (noting that $\binom{2m-1}{m} > \frac{1}{m}\binom{2m-1}{2}$ in the case $n = 2m - 1$), so we may assume that $2m + 1 \leqslant n \leqslant \frac{7m-1}{2}$. There are

$$\binom{n}{\frac{n+1}{2}}$$

partitions of S into a set of size $\frac{n+1}{2}$ and a set of size $\frac{n-1}{2}$. Furthermore, $|\mathcal{C}| = \frac{1}{m}\binom{n}{2}$ and for each $C \in \mathcal{C}$ there are exactly

$$\binom{n-m}{\frac{n+1}{2}-m} + \binom{n-m}{\frac{n-1}{2}-m}$$

such partitions of S for which $V(C)$ is a subset of one of the sets in the partition. Thus, if

$$\binom{n}{\frac{n+1}{2}} > \frac{1}{m}\binom{n}{2}\left(\binom{n-m}{\frac{n+1}{2}-m} + \binom{n-m}{\frac{n-1}{2}-m}\right) \qquad (*)$$

it is easy to see that we can find the required partition of S. So it suffices to show that $(*)$ holds for all m and n such that $m \geqslant 4$, $2m+1 \leqslant n \leqslant \frac{7m-1}{2}$ and n is m-admissible.

Note that for $m = 4$, the only value of n for which $2m+1 \leqslant n \leqslant \frac{7m-1}{2}$ and n is m-admissible is $n = 9$ and that $(*)$ holds in this case. We can thus assume that $m \geqslant 5$. It is routine to show that $(*)$ is equivalent to

$$2m\frac{(n-2)!}{(n-m)!} > \frac{(\frac{n+1}{2})!}{(\frac{n+1}{2}-m)!} + \frac{(\frac{n-1}{2})!}{(\frac{n-1}{2}-m)!}$$

and hence, simplifying further, equivalent to

$$m2^m(n-2)(n-3)(n-4)\cdots(n+2-m) > (n-1)(n-3)(n-5)\cdots(n+3-2m).$$

Since $(n-2)(n-3)(n-4)\cdots(n+2-m) > (n-3)(n-5)\cdots(n+5-2m)$, it suffices to show that

$$m2^m \geqslant (n-1)(n+3-2m).$$

We will show that $m2^m - (n-1)(n+3-2m)$ is non-negative. Now

$$m2^m - (n-1)(n+3-2m) = m2^m - n(n+2-2m) - 2m + 3.$$

Since $n \geqslant 0$ we can substitute $n \leqslant \frac{7m-1}{2}$ to obtain

$$m2^m - (n-1)(n+3-2m) \geqslant m2^m - n\left(\frac{3m+3}{2}\right) - 2m + 3.$$

Since $\frac{3m+3}{2} \geqslant 0$ we can again substitute $n \leqslant \frac{7m-1}{2}$ and simplify to obtain

$$m2^m - (n-1)(n+3-2m) \geqslant \frac{1}{4}\big(m\big(2^{m+2} - 21m - 26\big) + 15\big).$$

It is easy to see that $\frac{1}{4}(m(2^{m+2} - 21m - 26) + 15)$ is non-negative for all $m \geqslant 5$. $\square$

3. Embeddings of partial odd-cycle systems

Our aim in this section is to prove Lemma 3.4 which gives embeddings of partial odd-cycle systems that preserve chromatic number. To do so, we will closely follow the method used in [10], which was in turn based heavily on methods employed in [11] and [12]. The embeddings constructed in this section are nearly as small as the smallest known embeddings for general odd cycle systems (see [12]). We will require the following result which is apparent from the main construction in [12] (see in particular Section 3 of [12] and note that, in the notation of that paper, we can choose $z_i = (-1)^i\lfloor\frac{i}{2}\rfloor$ which gives $z_1 = 0$, $z_2 = 1$ and $z_3 = -1$).

Lemma 3.1. *(See [12].) Let m, n and t be positive integers such that m is odd, $m \geqslant 5$ and $t \geqslant n$ and let $(\mathbb{Z}_n, \mathcal{P})$ be a partial m-cycle system of order n. Let G be the graph with vertex set $V(G) = \mathbb{Z}_{2t+1}$ and edge set $E(G) = \{xy\colon\ x, y \in \mathbb{Z}_{2t+1},\ x \neq y$ and xy is not an edge of a cycle in $\mathcal{P}\}$. Then there is an m-cycle decomposition, $\mathcal{D}$ say, of $G \cdot K^c_{\mathbb{Z}_m}$ such that*

- *if $m \geqslant 7$ then for each $C \in \mathcal{D}$ we have $(x,i), (x,i+1), (x,i+2) \in V(C)$ for some $x \in \mathbb{Z}_{2t+1}$ and $i \in \mathbb{Z}_m$; and*
- *if $m = 5$ then for each $C \in \mathcal{D}$ we have $(x,i), (x,i+1) \in V(C)$ for some $x \in \mathbb{Z}_{2t+1}$ and $i \in \mathbb{Z}_m$.*

We will also require Lemmas 3.2 and 3.3, which are stronger forms of results in [10]. Lemma 3.2 is only used in the proof of Lemma 3.3.

Lemma 3.2. *Let m and s be integers such that m is odd, s is even, $m \geqslant 5$ and $\frac{m-1}{2} \leqslant s \leqslant m-1$, let S and S' be sets and let G and G' be m-cycles such that $|S| = |S'| = \frac{s}{2}$ and S, S', $V(G)$ and $V(G')$ are pairwise disjoint. Then there is an m-cycle decomposition, $\mathcal{D}$ say, of $K^c_{S \cup S'} \vee (G \cup G')$ such that, for each $C \in \mathcal{D}$, either $|V(C) \cap S| \geqslant 1$ and $|V(C) \cap S'| \geqslant 1$ or $|V(C) \cap (S \cup S')| = 0$.*

Proof. Let A be a set such that $|A| = s$. By Lemma 3 of [10] there is an m-cycle decomposition, $\mathcal{D}$ say, of $K^c_A \vee (G \cup G')$. Furthermore, it is easy to see from the proof of that lemma that

- if $s = \frac{m-1}{2}$ then exactly one cycle in $\mathcal{D}$ is vertex disjoint from K^c_A and every other cycle in $\mathcal{D}$ intersects A in exactly $\frac{m-1}{2}$ vertices;
- if $\frac{m+1}{2} \leqslant s \leqslant m-3$ then exactly two cycles in $\mathcal{D}$ intersect A in exactly $\frac{s}{2}$ vertices and every other cycle in $\mathcal{D}$ intersects A in exactly $\frac{m-1}{2}$ vertices;
- if $s = m-1$ then every cycle in $\mathcal{D}$ intersects A in exactly $\frac{m-1}{2}$ vertices.

It suffices to show that there is a partition $\{S, S'\}$ of A such that $|S| = |S'| = \frac{s}{2}$ and for each $C \in \mathcal{D}$, either $|V(C) \cap S| \geqslant 1$ and $|V(C) \cap S'| \geqslant 1$ or $|V(C) \cap (S \cup S')| = 0$. That such a partition exists is obvious in the case $s = \frac{m-1}{2}$.

For $s \geqslant \frac{m+1}{2}$, notice that there are

$$\frac{1}{2}\binom{s}{\frac{s}{2}}$$

(unordered) partitions of A into two sets of size $\frac{s}{2}$. Furthermore, for a cycle $C \in \mathcal{D}$ such that $|V(C) \cap A| = \frac{s}{2}$ there is exactly one such partition of A for which one of the sets of the partition is disjoint from $V(C)$. Thus, since $\frac{1}{2}\binom{s}{\frac{s}{2}} > 2$ for all $s \geqslant 4$ it can be seen that the required partition exists in the case $\frac{m+1}{2} \leqslant s \leqslant m-3$.

When $s = m-1$ there are exactly $2m$ cycles in $\mathcal{D}$ so, since $\frac{1}{2}\binom{m-1}{\frac{1}{2}(m-1)} > 2m$ holds for $m \geqslant 9$, it can be seen that the required partition exists when $m \geqslant 9$ and $s = m-1$. This leaves only the cases $(m,s) = (5,4)$ and $(m,s) = (7,6)$, and in each of these cases it is easy to check that if the construction used in Lemma 2 of [10] is used to obtain the m-cycle decomposition of $K^c_A \vee (G \cup G')$ then the required partition of A exists. □

Lemma 3.3. *Let w and m be integers such that m is odd, w is even, $m \geqslant 5$ and $\frac{m-1}{2} \leqslant w \leqslant \frac{1}{2}(m-1)^2$, and let W, W', T and T' be pairwise-disjoint sets such that $|W| = |W'| = \frac{w}{2}$ and $|T| = |T'| = m$. Then there exists an m-cycle decomposition, $\mathcal{D}$ say, of $K^c_{W \cup W'} \vee (K_T \cup K_{T'})$ such that, for each $C \in \mathcal{D}$, either $|V(C) \cap W| \geqslant 1$ and $|V(C) \cap W'| \geqslant 1$ or $|V(C) \cap (W \cup W')| = 0$.*

Proof. Since $\frac{m-1}{2} \leqslant w \leqslant \frac{1}{2}(m-1)^2$, it is routine to check that there are partitions $\{W_1, W_2, \ldots, W_r\}$ and $\{W'_1, W'_2, \ldots, W'_r\}$ of W and W' respectively such that $r \leqslant \frac{m-1}{2}$, $|W_i| = |W'_i|$ for $i \in \{1,2,\ldots,r\}$ and $\frac{m-1}{2} \leqslant |W_i \cup W'_i| \leqslant m-1$ for $i \in \{1,2,\ldots,r\}$. Let $\{G_1, G_2, \ldots, G_{\frac{m-1}{2}}\}$ and $\{G'_1, G'_2, \ldots, G'_{\frac{m-1}{2}}\}$ be m-cycle decompositions of K_T and $K_{T'}$ respectively (it is well known that Hamilton decompositions of complete graphs of odd order exist). By Lemma 3.2, for each $i \in \{1,2,\ldots,r\}$ there is an m-cycle decomposition, $\mathcal{D}_i$ say, of $K^c_{W_i \cup W'_i} \vee (G_i \cup G'_i)$ such that, for each $C \in \mathcal{D}_i$, either $|V(C) \cap W_i| \geqslant 1$ and $|V(C) \cap W'_i| \geqslant 1$ or $|V(C) \cap (W \cup W')| = 0$. It is easy to see that

$$\mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_r \cup \left\{ G_i, G'_i \colon i \in \left\{ r+1, r+2, \ldots, \frac{m-1}{2} \right\} \right\}$$

is an m-cycle decomposition of $K^c_{W \cup W'} \vee (K_T \cup K_{T'})$ with the required properties. □

We are now ready to prove the main result of this section.

Lemma 3.4. *Let k, m, u and v be positive integers such that m is odd, $k \geqslant 2$, $m \geqslant 5$, $(k, m) \neq (2, 5)$, $v \geqslant 2mu + \frac{3m-1}{2}$ and v is m-admissible. Then any k-chromatic partial m-cycle system of order u can be embedded in a k-chromatic m-cycle system of order v.*

Proof. We shall first deal with the case $m \geqslant 7$. The special case $m = 5$ will be dealt with afterwards.

Let $(\mathbb{Z}_u, \mathcal{P})$ be a k-chromatic partial m-cycle system and let α be a colouring of $\mathbb{Z}_u$ with the colours $c_1, c_2, \ldots, c_k$ such that no cycle of $\mathcal{P}$ is monochromatic under α. Let t and w be the integers such that $v = m(2t+1) + w$ and $\frac{m-1}{2} \leqslant w \leqslant \frac{5m-3}{2}$. Note that w is even since v and $m(2t+1)$ are odd, and that $t \geqslant u$.

Let W and W' be disjoint sets, each disjoint from $\mathbb{Z}_{2t+1} \times \mathbb{Z}_m$, with $|W| = |W'| = \frac{w}{2}$ and let $V = W \cup W' \cup (\mathbb{Z}_{2t+1} \times \mathbb{Z}_m)$ be a vertex set. Let p be the permutation $(c_1 c_2 \cdots c_k)$ of the colour set $\{c_1, c_2, \ldots, c_k\}$. Let β be a colouring of V with the colours $c_1, c_2, \ldots, c_k$ defined by $\beta(a) = c_1$ if $a \in W$, $\beta(a) = c_2$ if $a \in W'$, and

$$\beta\big((x,i)\big) = \begin{cases} \alpha(x), & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{0, 2, 4, \ldots, m-1\}; \\ p(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{1, 3, 5, \ldots, m-2\}; \\ c_1, & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{0, 2, 4, \ldots, m-1\}; \\ c_2, & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{1, 3, 5, \ldots, m-2\}. \end{cases}$$

We shall construct a collection $\mathcal{C}$ of m-cycles on the vertex set V such that $(V, \mathcal{C})$ is an m-cycle system, $\mathcal{C}$ contains a copy of $\mathcal{P}$, and β is a k-colouring of $(V, \mathcal{C})$. This will complete the proof, since the fact that $\mathcal{C}$ contains a copy of $\mathcal{P}$ will imply that the chromatic number of $(V, \mathcal{C})$ is at least k.

We construct $\mathcal{C}$ according to the following steps.

(1) For each cycle $C \in \mathcal{P}$ let $x_1, x_2, \ldots, x_m$ be vertices such that $C = (x_1, x_2, \ldots, x_m)$ and $\alpha(x_1) \neq \alpha(x_3)$ (these exist since m is odd and C is not monochromatic under α). Use Lemma 2.2, to obtain an m-cycle decomposition, $\mathcal{D}_C$ say, of $C \cdot K^c_{\mathbb{Z}_m}$ such that $((x_1, 0), (x_2, 0), \ldots, (x_m, 0)) \in \mathcal{D}_C$ and each cycle of $\mathcal{D}_C$ contains the vertices (x_1, i) and (x_3, i) for some $i \in \mathbb{Z}_m$. Since $\alpha(x_1) \neq \alpha(x_3)$, $\beta(x_1, i) \neq \beta(x_3, i)$ for all $i \in \mathbb{Z}_m$ and it is easy to see that no cycle in $\mathcal{D}_C$ is monochromatic under β. Add the cycles in each of these decompositions to $\mathcal{C}$.

(2) Let G be the graph with vertex set $V(G) = \mathbb{Z}_{2t+1}$ and edge set $E(G) = \{xy \colon x, y \in \mathbb{Z}_{2t+1},\ x \neq y$ and xy is not an edge of a cycle in $\mathcal{P}\}$. Use Lemma 3.1 to obtain an m-cycle decomposition, $\mathcal{D}$ say, of $G \cdot K^c_{\mathbb{Z}_m}$ such that for each $C \in \mathcal{D}$ we have $(x, i), (x, i+1), (x, i+2) \in V(C)$ for some $x \in \mathbb{Z}_{2t+1}$ and $i \in \mathbb{Z}_m$. Clearly, no cycle in $\mathcal{D}$ is monochromatic under β. Add the cycles in $\mathcal{D}$ to $\mathcal{C}$.

(3) For each $x \in \{0, 1, \ldots, t-1\}$ use Lemma 3.3 to obtain an m-cycle decomposition, $\mathcal{D}_x$ say, of $K^c_{W \cup W'} \vee (K_{\{2x\} \times \mathbb{Z}_m} \cup K_{\{2x+1\} \times \mathbb{Z}_m})$ such that, for each $C \in \mathcal{D}_x$, either $|V(C) \cap W| \geqslant 1$ and $|V(C) \cap W'| \geqslant 1$ or $|V(C) \cap (W \cup W')| = 0$ (we can apply Lemma 3.3 since $w \leqslant \frac{5m-3}{2}$ implies $w \leqslant \frac{1}{2}(m-1)^2$ for all $m \geqslant 7$). Since β assigns colour c_1 to the vertices in W, assigns colour c_2 to the vertices in W', and does not assign the same colour to all the vertices of $\{y\} \times \mathbb{Z}_m$ for any $y \in \mathbb{Z}_{2t+1}$, no cycle in $\mathcal{D}_x$ is monochromatic under β. Add the cycles in each of these decompositions to $\mathcal{C}$.

(4) It can be seen that a monochromatic subset of $W \cup W' \cup (\{2t\} \times \mathbb{Z}_m)$ under β contains at most $\frac{w+m+1}{2}$ vertices. Also, at most two maximal monochromatic subsets of $W \cup W' \cup (\{2t\} \times \mathbb{Z}_m)$ under β contain at least m vertices. Thus, we can use Lemma 2.3 to find an m-cycle system on

the vertex set $W \cup W' \cup (\{2t\} \times \mathbb{Z}_m)$ which contains no cycle that is monochromatic under β (by choosing $\{S_1, S_2\}$ to be a partition of $W \cup W' \cup (\{2t\} \times \mathbb{Z}_m)$ such that $|S_1| = \frac{w+m+1}{2}$, S_1 contains one of the largest two maximal monochromatic subsets of $W \cup W' \cup (\{2t\} \times \mathbb{Z}_m)$ under β, and S_2 contains the other). Add the cycles of this system to $\mathcal{C}$.

Since, for each cycle $C = (x_1, x_2, \ldots, x_m)$ in $\mathcal{P}$, the cycle $((x_1, 0), (x_2, 0), \ldots, (x_m, 0))$ is in the decomposition of $C \cdot K^c_{\mathbb{Z}_m}$ whose cycles we add to $\mathcal{C}$ in (1), there is a copy of $\mathcal{P}$ on the vertex set $\mathbb{Z}_u \times \{0\}$ in $\mathcal{C}$. We have seen that no cycle of $\mathcal{C}$ is monochromatic under β, and it is routine to check that $(V, \mathcal{C})$ is an m-cycle system. So we are finished in the case $m \geqslant 7$.

We now consider the special case $m = 5$. Note that in this case $k \geqslant 3$. We proceed exactly as we did for the case $m \geqslant 7$, with three exceptions. Firstly, we define β by $\beta(a) = c_1$ if $a \in W$, $\beta(a) = c_2$ if $a \in W'$, and

$$\beta\big((x,i)\big) = \begin{cases} \alpha(x), & \text{if } x \in \mathbb{Z}_u,\ i \in \{0,2\}; \\ p(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{1,3\}; \\ p^2(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i = 4; \\ c_1, & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u,\ i \in \{0,2\}; \\ c_2, & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{1,3\}; \\ c_3, & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i = 4. \end{cases}$$

Secondly, in (2) we now use Lemma 3.1 to obtain a 5-cycle decomposition, $\mathcal{D}$ say, of $G \cdot K^c_{\mathbb{Z}_5}$ such that for each $C \in \mathcal{D}$ we have $(x, i), (x, i+1) \in V(C)$ for some $x \in \mathbb{Z}_{2t+1}$ and $i \in \mathbb{Z}_5$. It is easy to see that no cycle of this decomposition is monochromatic under (the new definition of) β.

Thirdly, if $v \equiv 5 \pmod{10}$, we let $w = 0$ and let t be the integer such that $v = 5(2t+1)$. Instead of performing steps (3) and (4), we simply take a 5-cycle decomposition of $K_{\{x\} \times \mathbb{Z}_5}$ for each $x \in \mathbb{Z}_{2t+1}$ (it is well known that Hamilton decompositions of complete graphs of odd order exist) and add the cycles in each of these decompositions to $\mathcal{C}$. It is easy to see that no cycle in these decompositions is monochromatic under (the new definition of) β. This means that if we are performing step (3) then, since v is 5-admissible, it must be that $v \equiv 1 \pmod{10}$ and hence $w = 6$. It follows that $w \leqslant \frac{1}{2}(m-1)^2$, so we can indeed apply Lemma 3.3 as required.

Except as noted, the arguments given in the case $m \geqslant 7$ hold without any alteration. □

4. Embeddings of partial even-cycle systems

Our aim in this section is to prove Lemma 4.3 which gives embeddings of partial even-cycle systems that preserve chromatic number. To do so, we will employ a method very similar to that used in [13] along with a technique from [2]. Note that unlike in the odd-cycle case, where we gave embeddings nearly as small as the smallest known embeddings, the smallest embeddings we construct here are approximately four times as large as the smallest known embeddings (see [9]).

Before we prove Lemma 4.3 we require a preliminary result on m-cycle decompositions of complete bipartite graphs.

Lemma 4.1. *Let m be an even integer such that $m \geqslant 4$, let S_1, S_2 and T be pairwise-disjoint sets such that $|S_1| = |S_2| = \frac{m}{2}$ and $|T| = m$. Then there exists an m-cycle decomposition, $\mathcal{D}$ say, of $K_{S_1 \cup S_2, T}$ such that, for each $C \in \mathcal{D}$, $|V(C) \cap S_1| \geqslant 1$ and $|V(C) \cap S_2| \geqslant 1$.*

Proof. It is routine to check that the result holds when $m = 4$, so we may assume $m \geqslant 6$.

Let S be a set such that $|S| = m$ and S and T are disjoint. By Theorem 2.1 there exists a decomposition, $\mathcal{D}$ say, of $K_{S,T}$. It suffices to find a partition $\{S_1, S_2\}$ of S such that $|S_1| = |S_2| = \frac{m}{2}$ and, for each $C \in \mathcal{D}$, $|V(C) \cap S_1| \geqslant 1$ and $|V(C) \cap S_2| \geqslant 1$.

Notice that there are

$$\frac{1}{2}\binom{m}{\frac{m}{2}}$$

(unordered) partitions of S into two sets of size $\frac{m}{2}$. Furthermore, $|\mathcal{D}| = m$ and for each $C \in \mathcal{D}$ there is exactly one such partition of S for which one of the sets of the partition is disjoint from $V(C)$. Thus, since for each $m \geqslant 6$ we have

$$\frac{1}{2}\binom{m}{\frac{m}{2}} > m,$$

it is easy to see that we can find the required partition. □

We also require the following result which will help us deal with the special case $m = 4$.

Lemma 4.2. *Let G be the cycle (x_1, x_2, x_3, x_4). Then there exists a 4-cycle decomposition, $\mathcal{D}$ say, of $G \cdot K^c_{\mathbb{Z}_4}$ such that*

- $((x_1, 0), (x_2, 0), (x_3, 0), (x_4, 0)) \in \mathcal{D}$;
- *for each cycle $C \in \mathcal{D}$ there is an $a \in \{1, 2\}$ such that either $\{(x_a, i), (x_{a+1}, i)\} \subseteq V(C)$ for some $i \in \mathbb{Z}_4$ or one of the pairs $\{(x_a, 0), (x_{a+1}, 2)\}$, $\{(x_a, 1), (x_{a+2}, 3)\}$, or $\{(x_a, 3), (x_{a+2}, 1)\}$ is a subset of $V(C)$.*

Proof. Let $\mathcal{D} = \{A_i, A'_i\colon\ i \in \mathbb{Z}_4\} \cup \{B_j, B'_j, B''_j, B'''_j\colon\ j \in \{0, 2\}\}$ where

$$\begin{aligned}
A_i &= \big((x_1, i), (x_2, i+1), (x_3, i+2), (x_4, i+3)\big),\\
A'_i &= \big((x_1, i), (x_2, i-1), (x_3, i-2), (x_4, i-3)\big),\\
B_j &= \big((x_1, j), (x_2, j), (x_3, j), (x_4, j)\big),\\
B'_j &= \big((x_1, j), (x_2, j+2), (x_3, j), (x_4, j+2)\big),\\
B''_j &= \big((x_1, j+1), (x_2, j+3), (x_3, j+3), (x_4, j+1)\big),\\
B'''_j &= \big((x_1, j+1), (x_2, j+1), (x_3, j+3), (x_4, j+3)\big).
\end{aligned}$$

It is routine to check that $\mathcal{D}$ is a decomposition of $G \cdot K^c_{\mathbb{Z}_4}$ with the required properties. □

We are now ready to prove the main result of this section.

Lemma 4.3. *Let k, m, u and v be positive integers such that m is even, $k \geqslant 2$, $m \geqslant 4$, $(k, m) \neq (2, 4)$, $v \geqslant 2mu + m + 1$ and v is m-admissible. Then any k-chromatic partial m-cycle system of order u can be embedded in a k-chromatic m-cycle system of order v.*

Proof. We shall first deal with the case $m \geqslant 6$. The special case $m = 4$ will be dealt with afterwards.

Let $(\mathbb{Z}_u, \mathcal{P})$ be a k-chromatic partial m-cycle system and let α be a colouring of $\mathbb{Z}_u$ with the colours $c_1, c_2, \dots, c_k$ such that no cycle of $\mathcal{P}$ is monochromatic under α. Let t and w be the integers such that $v = 2mt + w + 1$ and $m \leqslant w \leqslant 3m - 1$. Note that $t \geqslant u$, that w is even since v is odd, and that $w \neq m$ since v is m-admissible, so $m + 2 \leqslant w \leqslant 3m - 2$. Let W and W' be disjoint sets, each disjoint from $\mathbb{Z}_{2t} \times \mathbb{Z}_m$, with $|W| = |W'| = \frac{w}{2}$. Let $V = \{\infty\} \cup W \cup W' \cup (\mathbb{Z}_{2t} \times \mathbb{Z}_m)$ be a vertex set. Let p be the permutation $(c_1 c_2 \cdots c_k)$ of the colour set $\{c_1, c_2, \dots, c_k\}$ and let β be a colouring of V with the colours $c_1, c_2, \dots, c_k$ defined by $\beta(a) = c_1$ if $a \in \{\infty\} \cup W$, $\beta(a) = c_2$ if $a \in W'$ and

$$\beta\big((x, i)\big) = \begin{cases} \alpha(x), & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{0, 2, 4, \dots, m-2\};\\ p(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{1, 3, 5, \dots, m-1\};\\ c_1, & \text{if } x \in \mathbb{Z}_{2t} \setminus \mathbb{Z}_u \text{ and } i \in \{0, 2, 4, \dots, m-2\};\\ c_2, & \text{if } x \in \mathbb{Z}_{2t} \setminus \mathbb{Z}_u \text{ and } i \in \{1, 3, 5, \dots, m-1\}.\end{cases}$$

We shall construct a collection $\mathcal{C}$ of m-cycles on the vertex set V such that $(V, \mathcal{C})$ is an m-cycle system, $\mathcal{C}$ contains a copy of $\mathcal{P}$, and β is a k-colouring of $(V, \mathcal{C})$. This will complete the proof, since the fact that $\mathcal{C}$ contains a copy of $\mathcal{P}$ will imply that the chromatic number of $(V, \mathcal{C})$ is at least k.

We construct $\mathcal{C}$ according to the following steps.

(1) For each cycle $C \in \mathcal{P}$ let $x_1, x_2, \ldots, x_m$ be vertices such that $C = (x_1, x_2, \ldots, x_m)$ and either $\alpha(x_1) \neq \alpha(x_3)$ or $\alpha(x_a) = \alpha(x_b)$ if and only if $a \equiv b \pmod 2$ for $a, b \in \{1, 2, \ldots, m\}$ (these exist since m is even and C is not monochromatic under α). Using Lemma 2.2, take an m-cycle decomposition, $\mathcal{D}_C$ say, of $C \cdot K^c_{\mathbb{Z}_m}$ such that
  (i) $((x_1, 0), (x_2, 0), \ldots, (x_m, 0)) \in \mathcal{D}_C$;
  (ii) each cycle of $\mathcal{D}_C$ contains the vertices (x_1, i) and (x_3, i) for some $i \in \mathbb{Z}_m$; and
  (iii) each cycle of $\mathcal{D}_C$ contains the vertices (x_a, i) and $(x_{a+1}, i + j)$ for some $a \in \{1, 3, 5\}$, $i \in \mathbb{Z}_m$ and $j \in \{0, 2, 4, \ldots, m - 2\}$.

  If $\alpha(x_1) \neq \alpha(x_3)$ then, for each $i \in \mathbb{Z}_m$, $\beta((x_1, i)) \neq \beta((x_3, i))$ and no cycle of $\mathcal{D}_C$ is monochromatic under β by (ii). Otherwise, $\alpha(x_a) = \alpha(x_b)$ if and only if $a \equiv b \pmod 2$ for $a, b \in \{1, 2, \ldots, m - 1\}$. In particular $\alpha(x_a) \neq \alpha(x_{a+1})$ for any $a \in \{1, 2, \ldots, m - 1\}$, it can be seen that $\beta((x_a, i)) \neq \beta((x_{a+1}, i + j))$ for any $a \in \{1, 2, \ldots, m - 1\}$, $i \in \mathbb{Z}_m$ and $j \in \{0, 2, 4, \ldots, m - 2\}$, and hence no cycle of $\mathcal{D}_C$ is monochromatic under β by (iii). Add the cycles in each of these decompositions to $\mathcal{C}$.

(2) For each 2-element subset $\{x, y\}$ of $\mathbb{Z}_{2t}$ such that xy is not an edge in any cycle of $\mathcal{P}$ and $\{x, y\} \neq \{z, z + t\}$ for any $z \in \{0, 1, \ldots, t - 1\}$, use Lemma 4.1 to obtain a decomposition, $\mathcal{D}_{xy}$ say, of $K_{\{x\}\times\mathbb{Z}_m,\{y\}\times\mathbb{Z}_m}$ such that for each $C \in \mathcal{D}_{xy}$, $|V(C) \cap \{(x, 0), (x, 2), (x, 4), \ldots, (x, m - 2)\}| \geqslant 1$ and $|V(C) \cap \{(x, 1), (x, 3), (x, 5), \ldots, (x, m - 1)\}| \geqslant 1$. Clearly, no cycle of $\mathcal{D}_{xy}$ is monochromatic under β. Add the cycles in each of these decompositions to $\mathcal{C}$.

(3) For each $x \in \{0, 1, \ldots, t - 1\}$ notice that a monochromatic subset of $(\{x, x + t\} \times \mathbb{Z}_m) \cup \{\infty\}$ under β contains at most $m + 1$ vertices. Also, for each $x \in \{0, 1, \ldots, t - 1\}$, at most two maximal monochromatic subsets of $(\{x, x + t\} \times \mathbb{Z}_m) \cup \{\infty\}$ under β contain at least m vertices. Thus, we can use Lemma 2.3 to obtain an m-cycle system on the vertex set $(\{x, x + t\} \times \mathbb{Z}_m) \cup \{\infty\}$ which contains no cycle that is monochromatic under β (by choosing $\{S_1, S_2\}$ to be a partition of $(\{x, x + t\} \times \mathbb{Z}_m) \cup \{\infty\}$ such that $|S_1| = m + 1$, S_1 contains one of the two largest monochromatic subsets of $(\{x, x + t\} \times \mathbb{Z}_m) \cup \{\infty\}$ under β, and S_2 contains the other). Add the cycles in each of these systems to $\mathcal{C}$.

(4) Since $m + 2 \leqslant w \leqslant 3m - 2$ and since w is even, it is routine to check that there exist partitions $\{W_1, W_2, \ldots, W_r\}$ and $\{W'_1, W'_2, \ldots, W'_r\}$, of W and W' respectively such that $r \leqslant 5$, $|W_\ell| = |W'_\ell|$ for $\ell \in \{1, 2, \ldots, r\}$, and $\frac{m}{2} \leqslant |W_\ell \cup W'_\ell| \leqslant m - 2$ for $\ell \in \{1, 2, \ldots, r\}$. For each $\ell \in \{1, 2, \ldots, r\}$, use Theorem 2.1 to obtain an m-cycle decomposition, $\mathcal{D}_\ell$ say, of $K_{W_\ell \cup W'_\ell, \mathbb{Z}_{2t}\times\mathbb{Z}_m}$. Since each cycle in $\mathcal{D}_\ell$ contains $\frac{m}{2}$ vertices in $W_\ell \cup W'_\ell$ and $|W_\ell \cup W'_\ell| \leqslant m - 2$ it is easy to see that no cycle of $\mathcal{D}_\ell$ is monochromatic under β. Add the cycles in each of these decompositions to $\mathcal{C}$.

(5) Use Lemma 2.3 to obtain an m-cycle system on the vertex set $\{\infty\} \cup W \cup W'$ such that for each cycle C of the system $V(C) \nsubseteq \{\infty\} \cup W$ and $V(C) \nsubseteq W'$. Clearly, no cycle of the system is monochromatic under β. Add the cycles of this system to $\mathcal{C}$.

Since for each cycle $C = (x_1, x_2, \ldots, x_m)$ in $\mathcal{P}$ the cycle $((x_1, 0), (x_2, 0), \ldots, (x_m, 0))$ is in the decomposition of $C \cdot K^c_{\mathbb{Z}_m}$ whose cycles we add to $\mathcal{C}$ in (1), there is a copy of $\mathcal{P}$ on the vertex set $\mathbb{Z}_u \times \{0\}$ in $\mathcal{C}$. We have seen that no cycle of $\mathcal{C}$ is monochromatic under β, and it is routine to check that $(V, \mathcal{C})$ is an m-cycle system. So we are finished in the case $m \geqslant 6$.

We now consider the special case $m = 4$. Note that in this case $k \geqslant 3$. We proceed exactly as we did for the case $m \geqslant 6$ with two exceptions. Firstly, we define β as follows.

$$\beta\big((x,i)\big) = \begin{cases} \alpha(x), & \text{if } x \in \mathbb{Z}_u,\ i \in \{0,2\};\\ p(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i = 1;\\ p^2(\alpha(x)), & \text{if } x \in \mathbb{Z}_u \text{ and } i = 3;\\ c_1, & \text{if } x \in \mathbb{Z}_{2t} \setminus \mathbb{Z}_u,\ i \in \{0,2\};\\ c_2, & \text{if } x \in \mathbb{Z}_{2t} \setminus \mathbb{Z}_u \text{ and } i = 1;\\ c_3, & \text{if } x \in \mathbb{Z}_{2t} \setminus \mathbb{Z}_u \text{ and } i = 3. \end{cases}$$

Secondly, instead of performing step (1) we perform step (1′), as outlined below.

(1′) For each cycle $C \in \mathcal{P}$ let x_1, x_2, x_3, x_4 be vertices such that $C = (x_1, x_2, x_3, x_4)$ and either $\alpha(x_1) \neq \alpha(x_3)$ or $\alpha(x_a) = \alpha(x_b)$ if and only if $a \equiv b \pmod 2$ for $a, b \in \{1, 2, 3, 4\}$ (these exist since C is not monochromatic under α). If $\alpha(x_1) \neq \alpha(x_3)$, use Lemma 2.2 to obtain an m-cycle decomposition, $\mathcal{D}_C$ say, of $C \cdot K^c_{\mathbb{Z}_4}$ such that $((x_1,0),(x_2,0),(x_3,0),(x_4,0)) \in \mathcal{D}_C$ and each cycle of $\mathcal{D}_C$ contains the vertices (x_1,i) and (x_3,i) for some $i \in \mathbb{Z}_4$. Since $\alpha(x_1) \neq \alpha(x_3)$, $\beta((x_1,i)) \neq \beta((x_3,i))$ for each $i \in \mathbb{Z}_4$ and thus no cycle of $\mathcal{D}_C$ is monochromatic under β. Otherwise $\alpha(x_a) = \alpha(x_b)$ if and only if $a \equiv b \pmod 2$ for $a,b \in \{1,2,3,4\}$, and we use Lemma 4.2 to take a 4-cycle decomposition, $\mathcal{D}_C$ say, of $C \cdot K^c_{\mathbb{Z}_4}$ such that $((x_1,0),(x_2,0),(x_3,0),(x_4,0)) \in \mathcal{D}_C$ and for each cycle $B \in \mathcal{D}_C$ there is an $a \in \{1,2\}$ such that either $\{(x_a,i),(x_{a+1},i)\} \subseteq V(B)$ for some $i \in \mathbb{Z}_4$ or one of the pairs $\{(x_a,0),(x_{a+1},2)\}$, $\{(x_a,1),(x_{a+2},3)\}$, or $\{(x_a,3),(x_{a+2},1)\}$ is a subset of $V(B)$. It is routine to check that this implies that no cycle of $\mathcal{D}_C$ is monochromatic under β. Add the cycles in these decompositions to $\mathcal{C}$.

Except as noted the arguments given in the case $m \geqslant 6$ hold without any alteration. □

5. Proof of main theorem

Before we can prove our main theorem we must show that for all $m \geqslant 3$ and $k \geqslant 2$ there exists a k-chromatic partial m-cycle system. This is a simple consequence of the following result which is a special case of a result on weak colourings of hypergraphs from [17] (note that the value of $|V|$ does not appear in the statement of the theorem in [17] but is explicitly defined in the proof).

Theorem 5.1. *(See [17].) Let ℓ and m be positive integers such that $m \geqslant 3$ and let V be a set such that $|V| = \lfloor 2m^6\ell^{m-1}(\ln(\ell)+1)\rfloor$. Then there is a collection of m-element subsets of V, $\{S_1,S_2,\dots,S_t\}$ say, such that $|S_i \cap S_j| \leqslant 1$ for all $1 \leqslant i < j \leqslant t$ and under every ℓ-colouring of V one of $S_1,S_2,\dots,S_t$ is monochromatic.*

Lemma 5.2. *Let k and m be integers such that $k \geqslant 2$ and $m \geqslant 3$. Then there exists a k-chromatic partial m-cycle system of order $\lfloor 2m^6(k-1)^{m-1}(\ln(k-1)+1)\rfloor$.*

Proof. Let V be a set such that $|V| = \lfloor 2m^6(k-1)^{m-1}(\ln(k-1)+1)\rfloor$. By Theorem 5.1 there is a collection of m-element subsets of V, $\{S_1,S_2,\dots,S_t\}$ say, such that $|S_i \cap S_j| \leqslant 1$ for all $1 \leqslant i < j \leqslant t$ and under every $(k-1)$-colouring of V one of $S_1,S_2,\dots,S_t$ is monochromatic. Let $\mathcal{P} = \{C_1,C_2,\dots,C_t\}$, where C_i is an arbitrary m-cycle on vertex set S_i for each $i \in \{1,2,\dots,t\}$ and note that $(V,\mathcal{P})$ is a partial m-cycle system. Since under every $(k-1)$-colouring of V one of $S_1,S_2,\dots,S_t$ is monochromatic, it follows that the chromatic number of $(V,\mathcal{P})$ is at least k.

Let $k^\dagger$ be an integer such that $k^\dagger \geqslant 2$. The removal of any cycle from a $(k^\dagger)$-chromatic partial m-cycle system results in a new partial m-cycle system whose chromatic number is either $k^\dagger$ or $k^\dagger - 1$. To see this, observe that otherwise we could obtain a $(k^\dagger - 1)$-colouring of the original system by taking the $(k^\dagger - 2)$-colouring of the new system (which exists by assumption) and recolouring an arbitrary vertex on the cycle which is removed to form the new system with a colour which does not appear on any other vertex.

Thus, since the chromatic number of $(V,\mathcal{P})$ is at least k, it is easy to see that we can remove cycles one at a time from $(V,\mathcal{P})$ until we obtain a k-chromatic partial m-cycle system. □

Given Lemmas 3.4, 4.3 and 5.2 it is now a simple matter to prove our main theorem.

Proof of Theorem 1.1. By Lemma 5.2 there exists a k-chromatic partial m-cycle system, so we may let $u_{k,m}$ be the minimum order of a k-chromatic partial m-cycle system. The result is shown to be true for $m=3$ in [5] (the fact that $n_{3,k} \leqslant 6u_{3,k}+7$ is not stated in the theorem in [5] but is apparent from the proof), so we may assume $m \geqslant 4$. Note that $2m(u_{k,m}+1)+1$ is m-admissible, so it suffices to show that there exists a k-chromatic m-cycle system of order v for all m-admissible v such that $v \geqslant 2m(u_{k,m}+1)+1$. The proof now splits into two cases according to the parity of m.

Case 1. Suppose m is odd. If $(k,m) \neq (2,5)$ then by Lemma 3.4 there exists a k-chromatic m-cycle system of order v for all m-admissible v such that $v \geqslant 2m(u_{k,m}) + \frac{3m-1}{2}$ and we are finished. In the special case $(k, m) = (2, 5)$ El-Zanati and Rodger [7] have shown that there exists a 2-chromatic 5-cycle system of order v for all 5-admissible v such that $v \geqslant 5$.

Case 2. Suppose m is even. If $(k, m) \neq (2, 4)$ then by Lemma 4.3 there exists a k-chromatic m-cycle system of order v for all m-admissible v such that $v \geqslant 2m(u_{k,m}) + m + 1$ and we are finished. In the special case $(k, m) = (2, 4)$ El-Zanati and Rodger [6,7] have shown that there exists a 2-chromatic 4-cycle system of order v for all 4-admissible v such that $v \geqslant 9$. □

To obtain our lower bound on $n_{k,m}$ we will make use of another result on colourings of hypergraphs.

Theorem 5.3. *(See [8].) Let ℓ and m be integers such that $\ell \geqslant 2$ and $m \geqslant 3$ and let V be a set and let $\{S_1, S_2, \ldots, S_t\}$ be a collection of m-element subsets of V, such that each element of V is in at most $\lfloor \frac{1}{4m} \ell^{m-1} \rfloor$ of the sets $S_1, S_2, \ldots, S_t$. Then there is an ℓ-colouring of V under which none of the sets $S_1, S_2, \ldots, S_t$ is monochromatic.*

Proof of Corollary 1.2. Let $(U, \mathcal{P})$ be a partial m-cycle system of order at most $2\lfloor \frac{1}{4m}(k-1)^{m-1} \rfloor + 1$. Say $\mathcal{P} = \{C_1, C_2, \ldots, C_t\}$. Each element of U is in at most $\lfloor \frac{1}{4m}(k-1)^{m-1} \rfloor$ of the sets $V(C_1), V(C_2), \ldots, V(C_t)$, so by Theorem 5.3 there is a $(k-1)$-colouring of U under which none of the sets $V(C_1), V(C_2), \ldots, V(C_t)$ is monochromatic. This is a $(k-1)$-colouring of $(U, \mathcal{P})$. Thus no partial m-cycle system of order at most $2\lfloor \frac{1}{4m}(k-1)^{m-1} \rfloor + 1$ is k-chromatic. Combining this fact with the result of Lemma 5.2 we have

$$2\left\lfloor \frac{1}{4m}(k-1)^{m-1} \right\rfloor + 2 \leqslant u_{k,m} \leqslant 2m^6(k-1)^{m-1}\big(\ln(k-1) + 1\big).$$

Thus, by applying Theorem 1.1 and observing that $u_{k,m} \leqslant n_{k,m}$ we have that

$$2\left\lfloor \frac{1}{4m}(k-1)^{m-1} \right\rfloor + 2 \leqslant u_{k,m} \leqslant n_{k,m} \leqslant 4m^7(k-1)^{m-1}\big(\ln(k-1) + 1\big) + 2m + 1.$$

The result stated clearly follows from this. □

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