Second cohomology groups for algebraic groups and their Frobenius kernels

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ABSTRACT
Let $G$ be a simple simply connected algebraic group scheme defined over an algebraically closed field of characteristic $p > 0$. Let $T$ be a maximal split torus in $G$, $B \supset T$ be a Borel subgroup of $G$ and $U$ its unipotent radical. Let $F : G \to G$ be the Frobenius morphism. For $r \geq 1$ define the Frobenius kernel, $G_r$, to be the kernel of the iteration of $F$ iterated with itself $r$ times. Define $U_r$ (respectively $B_r$) to be the kernel of the Frobenius map restricted to $U$ (respectively $B$). Let $X(T)$ be the integral weight lattice and $X(T)_+$ be the dominant integral weights. The computations of particular importance are $H^2(U_1, k)$, $H^2(B_r, \lambda)$ for $\lambda \in X(T)$, $H^2(G_r, X(T)_+)$ for $\lambda \in X(T)_+$, and $H^2(B, \lambda)$ for $\lambda \in X(T)$. The above cohomology groups for the case when the field has characteristic 2 are computed in this paper. These computations complete the picture started by Bendel, Nakano, and Pillen (2007) [5] for $p \geq 3$. Furthermore, the computations show $H^2(G_r, X(T)_+)$ has a good filtration.

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1. Introduction

1.1. Let $G$ be a simple simply connected affine algebraic group scheme defined over $\mathbb{F}_p$ and $k$ be an algebraically closed field of characteristic $p > 0$. Let $T$ be a maximal split torus of $G$ and $B \supset T$ be the Borel subgroup of $G$. Let $F : G \to G$ denote the Frobenius morphism and $G_1$, the first Frobenius kernel, denote the scheme-theoretic kernel of $F$. More generally, higher Frobenius kernels, $G_r$, are defined by taking the kernel of the iteration of $F$ with itself $r$ times.

It is a well-known fact that the representation theory of $G_1$ is equivalent to the representation theory of the restricted Lie algebra $g = \text{Lie}(G)$. Knowledge about the second cohomology groups is...
important because of the information it gives us about central extensions of the underlying algebraic structures. A central question in representation theory of algebraic groups is to understand the structure and the vanishing of the line bundle cohomology, \( H^n(\lambda) = H^n(G/B, \mathcal{L}(\lambda)) \) for \( \lambda \in X(T) \), where \( \mathcal{L}(\lambda) \) is the line bundle over the flag variety, \( G/B \). Fundamental to the understanding of the line bundle cohomology is the computation of the rational cohomology groups, in particular the calculation of \( H^*(B, \lambda) \). Listed below are the calculations of cohomology of Frobenius kernels that aid in the computation of the line bundle cohomology.

1.1. \( H^0(u, k) \), where \( u = \text{Lie}(U) \);
1.2. \( H^0(U_1, k) \);
1.3. \( H^0(B_1, \lambda) \), for \( \lambda \in X(T) \);
1.4. \( H^0(B_r, \lambda) \), for \( \lambda \in X(T) \);
1.5. \( H^0(B, \lambda) \), for \( \lambda \in X(T) \);
1.6. \( H^0(G_1, H^0(\lambda)) \), for \( \lambda \in X(T)_+ \);
1.7. \( H^0(G_2, H^0(\lambda)) \), for \( \lambda \in X(T)_+ \).

The goal of this paper focuses on the calculations (1.1)–(1.7) for \( n = 2 \) and \( p = 2 \), which complete the picture for \( n = 2 \). Bendel, Nakano, and Pillen [5] computed the above groups for \( n = 2 \) and \( p \geq 3 \). There are many theorems that only hold for \( p \geq 3 \), which makes the computations for \( p = 2 \) harder. Furthermore, when calculating \( H^2(U_1, k) \), an even index of connection for many types of Lie algebras creates interesting cases and case-by-case considerations. However, in the end we found that \( H^2(G_r, H^0(\lambda)) \) always has a good filtration, satisfying Donkin’s conjecture for \( V = H^0(\lambda) \). More details about the good filtrations is found in Appendix D.1

1.2. History

In 1983, Friedlander and Parshall [7] calculated various cohomology groups of algebraic groups, starting with the special case when \( G \) is the general linear group with coefficients in the adjoint representation; then extended the idea to general algebraic groups with coefficients in \( V^r \), where \( V \) is a \( G \)-module. They also calculated (1.7) for \( n = 1, 2 \) and for \( k = H^0(0) \) for \( p \neq 2, 3 \).

Andersen and Jantzen [3], determined (1.6) for \( p \geq h \), where \( h \) is the Coxeter number (i.e., \( h = \langle \rho, \alpha^\vee \rangle + 1 \)). Andersen and Jantzen also determined (1.3) for \( \lambda = w \cdot 0 + \nu \) for \( p > h \), where \( w \in W \) and \( \nu \in X(T) \). These results originally had restrictions on the type of root system involved, which were later removed by Kumar, Lairutizzi, and Thomsen [10].

In 1984, Andersen [1] calculated \( H^*(B, w \cdot 0) \), where \( w \in W \). Recently, Andersen and Rian [2] proved some general results on the behavior of \( H^*(B, \lambda) \) and developed new techniques to enable the calculation of all \( B \)-cohomology for degree at most 3 when \( p > h \). They also calculated \( H^2(B, \lambda) \) and \( H^3(B, \lambda) \) explicitly for \( \lambda \in X(T) \) and \( p > h \). For higher cohomology groups, they proved the following theorem [2, 3.1, 6.1]:

**Theorem 1.2.1.** Suppose \( p > h \). Let \( w \in W, \nu \in X(T) \). Then we have for all \( i \)

(a) \( H^0(B, w \cdot 0 + \nu \nu) \cong H^{i-\langle w \rangle}(B, \nu) \).
(b) \( H^0(B, p\lambda) = 0 \) for \( i > -2 \cdot \text{ht}(\lambda) \).

Bendel, Nakano, and Pillen [5] calculated \( H^2(B, \lambda) \) for \( \lambda \in X(T) \) and \( p \geq 3 \). This paper computes \( H^2(B, \lambda) \) for \( p = 2 \). In both of these papers, \( H^2(B, \lambda) \) was calculated by previous calculations of (1.1)–(1.4).

The focus in the past 15 years for the above calculations changed from large primes to small primes. In 1991, Jantzen [9] calculated (1.1)–(1.3), (1.6) for all primes. Jantzen used basic facts about the structure of the root systems and isomorphisms relating the different cohomology groups. Bendel,

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1 The appendices are on my webpage: [http://www.math.arizona.edu/~cwright/Research.html](http://www.math.arizona.edu/~cwright/Research.html) and on the arXiv.
Nakano, and Pillen used Jantzen’s results to get (1.4) and (1.7) for \( n = 1 \) and all \( p \) in [4]. In 2004, Bendel, Nakano, and Pillen [5] worked out (1.1)–(1.7) for \( n = 2 \) and \( p ≥ 3 \).

1.3. Notation

Throughout the paper, the standard conventions provided in [8] is followed. Let \( G \) be a simply connected semisimple algebraic group over an algebraically closed field, \( k \), of prime characteristic, \( p > 0 \). Let \( g = \text{Lie}(G) \) be the Lie algebra of \( G \). For \( r ≥ 1 \), let \( G_r \) be the \( r \)th Frobenius kernel of \( G \). Let \( T \) be a maximal split torus in \( G \) and \( \Phi \) be the root system associated to \( (G, T) \). The positive (respectively negative) roots are \( \Phi^+ \) (respectively \( \Phi^- \)), and \( \Delta \) is the set of simple roots. Let \( B ⊃ T \) be the Borel subgroup of \( G \) corresponding to the negative roots and let \( U \) be the unipotent radical of \( B \). Let \( u = \text{Lie}(U) \) be the Lie algebra of the unipotent radical. For a given root system of rank \( n \) denote the simple roots \( \alpha_1, \alpha_2, \ldots, \alpha_n \), adhering to the ordering used in [9] (following Bourbaki).

In particular, for type \( B_n \), \( \alpha_n \) denotes the unique short simple root; for type \( C_n \), \( \alpha_n \) denotes the unique long simple root; for type \( F_4 \), \( \alpha_1 \) and \( \alpha_2 \) are the short simple roots; for type \( G_2 \), \( \alpha_1 \) is the unique short simple root. If \( \alpha \in \Phi \), and \( \alpha = \sum_{i=1}^{n} m_i \alpha_i \) then the height of \( \alpha \) is defined by \( \text{ht}(\alpha) := \sum_{i=1}^{n} m_i \).

Let \( E \) be the Euclidean space associated by \( \Phi \), and the inner product on \( E \) will be denoted by \( \langle ., . \rangle \). For any root \( \alpha \) denote the dual root by \( \alpha^* = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \). Let \( \omega_1, \omega_2, \ldots, \omega_n \) be the fundamental weights and \( X(T) \) be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by \( X(T)_+ \) and the set of \( p^r \)-restricted weights is \( X_r(T) \).

The simple modules for \( G \) are indexed by the set \( X(T)_+ \) and denoted by \( L(\lambda), \lambda \in X(T)_+ \). With \( L(\lambda) = \text{soc}_C H^G(\lambda) \), where \( \text{soc}_C H^G(\lambda) \) is the socle of the \( G \)-module \( H^G(\lambda) \) and \( H^G(\lambda) = \text{ind}_U^G \lambda \). Here \( \lambda \) denotes the one-dimensional \( B \)-module obtained by extending the character \( \lambda \in X(T)_+ \) to \( U \) trivially.

Given a \( G \)-module, \( M \), then composing a representation of \( M \) with \( F^r \) results in a new representation where \( G_r \) acts trivially, where \( M^{F^r} \) denotes the new module. For any \( \lambda \) in \( X(T) \), the \( \lambda \) weight space of \( M \) is the \( p^r \)-weight space of \( M^{F^r} \). On the other hand if \( V \) is a \( G \)-module where \( G_r \) acts trivially, then there is a unique \( G \)-module \( M \), with \( V = M^{F^r} \). We denote \( M = V^{(−r)} \).

1.4. Outline of computations

In recent work, Bendel, Nakano, and Pillen [5] calculated \( H^2(G_r, H^0(\lambda)) \) for \( p ≥ 3 \) by reducing the calculations down to \( H^2(u, k) \). Similar strategies are used to calculate \( H^2(G_r, H^0(\lambda)) \) for \( p = 2 \). The first step uses the following isomorphism to reduce the calculation to \( H^2(B_r, \lambda) \):

\[
H^2(G_r, H^0(\lambda))^{(−r)} \cong \text{ind}_B^G(H^2(B_r, \lambda)^{c(−r)}).
\]  

(1.1)

The Lyndon–Hochschild–Serre spectral sequence reduces the problem to \( H^2(B_1, \lambda) \). The problem is further reduced to the computation of \( H^2(U_1, k) \) via the isomorphism

\[
H^2(B_1, \lambda) \cong (H^2(U_1, k) \otimes \lambda)^{T_1}.
\]  

(1.2)

This isomorphism tells us that the \( B_1 \)-cohomology can easily be determined by looking at particular weight spaces of \( H^2(U_1, k) \):

\[
H^2(B_1, \lambda) \cong H^2(U_1, k)_{−\lambda}.
\]

The \( B \)-cohomology completes the calculations for the second cohomology groups as shown in [5] and Section 4. The following theorem from Bendel, Nakano, and Pillen [5] states the results for \( p ≥ 3 \):
Theorem 1.4.1. Let $p \geq 3$ and $\lambda \in X(T)$.

(a) Suppose $p > 3$ or $\Phi$ is not of type $G_2$. Then

$$H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \text{ for } w \in W \text{ and } l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\
  0 & \text{else.}
\end{cases}$$

(b) Suppose $p = 3$ and $\Phi$ is of type $G_2$. Then

$$H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \text{ for } w \in W \text{ and } l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\
  0 & \text{else.}
\end{cases}$$

where $k \neq l + 1$ if $\beta = \alpha_1$ and $\alpha = \alpha_2$, $0$ else.

$H^2(B, \lambda)$ is at most one-dimensional, as shown in [5] and Section 4.

2. Restricted Lie algebra cohomology

2.1. Observations on $U_1$-cohomology

Recall that $H^*(u, k)$ for $p \geq 3$ may be computed by the cohomology of the following complex, using the exterior algebra:

$$k \xrightarrow{d_0} u^* \xrightarrow{d_1} \Lambda^2(u)^* \xrightarrow{d_2} \Lambda^3(u)^* \to \ldots.$$

$H^*(U_1, k)$ can be computed from $H^*(u, k)$ by using the Friedlander–Parshall spectral sequence, which only holds for $p \geq 3$.

To calculate $H^*(U_1, k)$, for $p = 2$, we must take a different approach, using the restricted Lie algebra cohomology. The restricted Lie algebra cohomology may be computed by the cohomology of the following complex [8, 9.15].

$$k \xrightarrow{d_0} u^* \xrightarrow{d_1} S^2(u)^* \xrightarrow{d_2} S^3(u)^* \to \ldots.$$

The differential $d_1$ is a derivation on $S^*(u^*)$ and is thus determined by its restriction to $u^*$. Consider the following composition of maps

$$u^* \xrightarrow{d_1} S^2(u)^* \xrightarrow{\pi} \Lambda^2(u)^*$$

where $\pi$ is a surjection with kernel $\{f^2 : f \in u^*\}$ and $\partial = \pi \circ f$ being the coboundary operator for the ordinary Lie algebra cohomology, i.e. the dual of $\Lambda^2(u) \to u$ with $a \wedge b \mapsto [a, b]$. The differentials are given as follows: $d_0 = 0$ and $d_1 : u^* \to S^2(u)^*$ with

$$(d_1 \phi)(x_1 \otimes x_2) = -\phi([x_1, x_2])$$

where $\phi \in u^*$ and $x_1, x_2 \in u$. For higher differentials, identify $S^n(u)^* \cong S^n(u^*)$. The differentials are determined by the following product rule:

$$d_{i+j}(\phi \otimes \psi) = d_i(\phi) \otimes \psi + (-1)^i \phi \otimes d_j(\psi).$$
2.2. Basic results

Recall the following theorem from Jantzen, [9].

**Theorem 2.2.1.** $H^1(U_1, k) \cong H^1(u, k)$.

The following results, similar to those found in [5, 2.4], help identify some limitations on which linear combinations of tensor products, $\phi_\alpha \otimes \phi_\beta$, can represent cohomology classes when $\text{char } k = 2$. Using the additive property of differentials and the fact that differentials preserve the $T$ action, then we are interested in linear combinations of tensor products that have the same weight. Recall the following definition from [5].

**Definition 2.2.2.** An expression $\sum c_{\alpha, \beta} \phi_\alpha \otimes \phi_\beta \in S^2(u^*)$ is in reduced form if $c_{\alpha, \beta} \neq 0$ and for each pair $(\alpha, \beta)$ $c_{\alpha, \beta}$ appears at most once.

**Proposition 2.2.3.** Let $p = 2$ and $x = \sum c_{\alpha, \beta} \phi_\alpha \otimes \phi_\beta (\alpha, \beta \in \Phi^+)$ be an element in $S^2(u^*)$ in reduced form of weight $\gamma$ (for some $\gamma \in X(T)$ and $\gamma \notin pX(T)$). If $d_2(x) = 0$, then $d_1(\phi_\alpha) = 0$ for at least one $\alpha$ appearing in the sum.

**Proof.** For any $\alpha \in \Phi^+$, if $d_1(\phi_\alpha) = \sum c_{\beta, \eta} \phi_\delta \otimes \phi_\eta$, then $\text{ht}(\delta) < \text{ht}(\alpha)$ and $\text{ht}(\eta) < \text{ht}(\alpha)$ for all $\delta, \eta$. For all $\alpha$ and $\beta$ appearing in the sum for $x$, choose a root $\sigma$ with $\text{ht}(\sigma)$ being maximal. Without loss of generality, we may assume $\phi_\alpha$ appears in the second factor of the tensor product. Consider the corresponding term $c_{\alpha, \sigma} \phi_\alpha \otimes \phi_\sigma$. Computing $d_2(x)$, one of the components will be $c_{\alpha, \sigma} d_1(\phi_\alpha) \otimes \phi_\sigma$. By height considerations, $\phi_\sigma$ appears in no other terms, thus it is not a linear combination of the other terms. Therefore, $d_1(\phi_\alpha) = 0$. □

**Corollary 2.2.4.** Let $p = 2$.

(a) Let $x \in H^2(U_1, k)$ be a representative cohomology class in reduced form having weight $\gamma$ for some $\gamma \in X(T)$, $\gamma \notin pX(T)$. Then one of the components of $x$ is of the form $\phi_\alpha \otimes \phi_\beta$ for some simple root $\alpha \in \Delta$ and positive root $\beta \in \Phi^+$ (with $\alpha + \beta = \gamma$).

(b) Suppose $\phi_\alpha \otimes \phi_\beta$ represents a cohomology class in $H^2(U_1, k)$. Then one of three things must happen either

(i) $\alpha, \beta \in \Delta$,  
(ii) $\alpha \in \Delta$, then $d_1(\phi_\beta) = \sum c_{\sigma_1 + \sigma_2 = \beta} \phi_{\sigma_1} \otimes \phi_{\sigma_2}$, then $c_{\sigma_1, \sigma_2} = \pm 2$ for all decompositions of $\beta$ (that is the structure constant is even), or

(iii) $\alpha = \beta$ and $\alpha \in \Phi^+$.

**Proof.** Part (a) follows immediately from the previous proposition and Theorem 2.2.1, since $H^1(u, k)$ is generated by the simple roots. For part (b) let’s first assume that $\alpha = \beta$, then

$$d_2(\phi_\alpha \otimes \phi_\alpha) = d_1(\phi_\alpha) \otimes \phi_\alpha + \phi_\alpha \otimes d_1(\phi_\alpha) = 2d_1(\phi_\alpha) \otimes \phi_\alpha = 0.$$  

Now, assume $\alpha \neq \beta$ and $\alpha$ is simple. Then,

$$d_2(\phi_\alpha \otimes \phi_\beta) = d_1(\phi_\alpha) \otimes \phi_\beta + \phi_\alpha \otimes d_1(\phi_\beta) = \phi_\alpha \otimes d_1(\phi_\beta) = 0.$$  

Hence, $d_1(\phi_\beta) = 0$. Therefore, $\beta \in \Delta$ or if $\beta = \sigma_1 + \sigma_2$ then $c_{\sigma_1, \sigma_2} = \pm 2$, for all decompositions of $\beta$. □
2.3. Root sums

As previously mentioned, the computation of $H^2(U_1, k)$ involves information about $B_1$- and $B$-cohomology. In this process, certain sums involving positive roots arise. Suppose $x \in H^2(U_1, k)$ has weight $\gamma \in X(T)$. Then by Corollary 2.2.4, $\gamma = \alpha + \beta$ for $\alpha \in \Delta$ and $\beta \in \Phi^+$ and $\alpha \neq \beta$. Given such roots $\alpha$ and $\beta$, we want to know whether there exists a weight $\sigma \in X(T)$, $\beta_1, \beta_2 \in \Delta$, and integers $0 \leq i \leq p - 1$ and $m \geq 0$ such that any of the following hold:

\[
\begin{align*}
\alpha + \beta &= 2\sigma, \\
\alpha + \beta &= \beta_1 + 2\sigma, \\
\alpha + \beta &= i\beta_1 + 2^m\beta_2 + 2\sigma.
\end{align*}
\]

Given $\gamma$ a weight of $H^2(U_1, k)$, then there is a weight $v \in X(T)$ such that $H^2(B, -\gamma + pv) \neq 0$. Using results on $B$-cohomology due to Andersen, [1, 2.9], then $\gamma$ must satisfy (2.1). Note that (2.1) and (2.2) are special cases of (2.3) (i.e. when $i = 0$ and $m = 0$). Eq. (2.1) arises from the reduction $H^2(B_1, k) = H^2(U_1, k)^{T_1}$. For more details on how these equations arise see [5].

**Remark 2.3.1.** These sums noted above are only valid when 2 does not divide the index of connection.

2.4. $U_1$-cohomology

We state the theorem for $H^2(U_1, k)$ when $p = 2$, which is a summand of $T$-weights, except for $u^\ast$. In the next chapter, we explain the proof for each type.

**Theorem 2.4.1.** As a $T$-module,

(a) If $\Phi = A_n$, then

\[
H^2(U_1, k) \cong (u^\ast)^{(1)} \oplus \bigoplus_{\alpha, \beta \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 \oplus \bigoplus_{\alpha, \beta, \gamma \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 + 2\gamma.
\]

(b) If $\Phi = B_n$, then

\[
H^2(U_1, k) \cong (u^\ast)^{(1)} \oplus \bigoplus_{\alpha, \beta \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 \oplus \bigoplus_{\alpha, \beta, \gamma \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 + 2\gamma
\]

\[
\oplus \bigoplus_{1 \leq i \leq n-3} -(s_{\alpha_i}s_{\alpha_{n-i}}) \cdot 0 + 2\alpha_n \oplus \bigoplus_{1 \leq i \leq n-1} 2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n).
\]

(c) If $\Phi = C_n$, then

\[
H^2(U_1, k) \cong (u^\ast)^{(1)} \oplus \bigoplus_{\alpha, \beta \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 \oplus \bigoplus_{\alpha, \beta, \gamma \in \Delta} -(s_{\alpha}s_{\beta}) \cdot 0 + 2\gamma
\]

\[
\oplus \bigoplus_{1 \leq i \leq n-3} -(s_{\alpha_i}s_{\alpha_{n-i}}) \cdot 0 + 2\alpha_{n-1} \oplus \bigoplus_{1 \leq i \leq n-1} 2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n).
\]
\[\oplus - (s_{\alpha_n-1}s_{\alpha_n}) \cdot 0.\]

(d) If \( \Phi = D_n, n \geq 4 \), then

\[H^2(U_1, k) \cong \left(u^*\right)^{(1)} \oplus \bigoplus_{\alpha, \beta \in \Delta, \alpha + \beta \notin \Phi^+} \,(s_{\alpha}s_{\beta}) \cdot 0 \oplus \bigoplus_{\alpha, \beta, \gamma \in \Delta, \alpha + \beta \notin \Phi^+, \alpha + \beta + \gamma \in \Phi^+} \,(s_{\alpha}s_{\beta}) \cdot 0 + 2\gamma \]

\[\oplus - (s_{\alpha_{n-1}}s_{\alpha_{n-1}}) \cdot 0 + 2(\alpha_{n-2} + \alpha_n) \]

\[\oplus - (s_{\alpha_{n-1}}s_{\alpha_n}) \cdot 0 + 2(\alpha_{n-2} + \alpha_n) \]

\[\oplus \bigoplus_{1 \leq i \leq n-3} - (s_{\alpha_{n-1}}s_{\alpha_n}) \cdot 0 + 2(\alpha_i + \cdots + \alpha_{n-2}).\]

(e) If \( \Phi = E_n, n = 6, 7, 8 \) then

\[H^2(u, k) \cong \bigoplus_{\alpha, \beta \in \Delta, \alpha + \beta \notin \Phi^+} \,(s_{\alpha}s_{\beta}) \cdot 0 \oplus \bigoplus_{\alpha, \beta, \gamma \in \Delta, \alpha + \beta \notin \Phi^+, \alpha + \beta + \gamma \in \Phi^+} \,(s_{\alpha}s_{\beta}) \cdot 0 + 2\gamma \]

\[\bigoplus_{i=5}^{n} - (s_{\alpha_2}s_{\alpha_3}) \cdot 0 + 2(\alpha_4 + \cdots + \alpha_i) \oplus - (s_{\alpha_2}s_{\alpha_5}) \cdot 0 + 2\alpha_4 \]

\[\oplus - (s_{\alpha_3}s_{\alpha_5}) \cdot 0 + 2(\alpha_2 + \alpha_4) \oplus - (s_{\alpha_2}s_{\alpha_5}) \cdot 0 + 2(\alpha_1 + \alpha_3 + \alpha_4).\]

(f) If \( \Phi = F_4 \), then

\[H^2(U_1, k) \cong \left(u^*\right)^{(1)} \oplus \bigoplus_{\alpha, \beta \in \Delta, \alpha + \beta \notin \Phi^+} \,(s_{\alpha}s_{\beta}) \cdot 0 \oplus - (s_{\alpha_1}s_{\alpha_3}) \cdot 0 + 2\alpha_2 \]

\[\oplus - (s_{\alpha_1}s_{\alpha_3}) \cdot 0 + 2(\alpha_2 + \alpha_3) \oplus 2(\alpha_2 + \alpha_3) \oplus 2(\alpha_1 + \alpha_2 + \alpha_3) \]

\[\oplus 2(\alpha_2 + \alpha_3 + \alpha_4) \oplus 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \oplus 2(\alpha_2 + 2\alpha_3 + \alpha_4) \]

\[\oplus 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).\]

(g) If \( \Phi = G_2 \), then

\[H^2(U_1, k) \cong \left(u^*\right)^{(1)} \oplus 2(\alpha_1 + \alpha_2).\]

Refer to Appendix A (see footnote 1) to see a complete list of the cohomology classes associated with each \( w \in W \).

3. Proof of Theorem 2.4.1

3.1. Type \( A_{2n} \)

Suppose \( x \in H^2(U_1, k) \) has weight \( \gamma \in X(T) \). From Corollary 2.2.4, we know that \( \gamma = \alpha + \beta \) for some roots \( \alpha \in \Delta \) and \( \beta \in \Phi^+ \), with \( \alpha \neq \beta \). Furthermore, since 2 does not divide the index of connection, then we know \( \gamma = \alpha + \beta \) must satisfy one of Eqs. (2.1)-(2.3).
Proposition 3.1.1. Let $\Phi = A_n$ and $p = 2$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. Then there is no weight $\sigma \in X(T)$ such that $\alpha + \beta = 2\sigma$.

Proof. Consider $\alpha + \beta = 2 \sum m_i \alpha_i$, where $m_i = 0, 1$, $\forall i$. Then, $\alpha \neq \beta$, which contradicts the hypothesis. \qed

Proposition 3.1.2. Let $p = 2$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. Then, there is no simple root $\beta_1 \in \Delta$ and $\sigma \in X(T)$ such that $\alpha + \beta = \beta_1 + 2\sigma$.

Proof. Consider $\alpha + \beta = \beta_1 + 2 \sum m_i \alpha_i$. Since $\alpha = \alpha_i$ for some $i$, then $\sigma \in \Delta$. Thus the only possibility for $\beta$ is $\alpha_{i-1} + \alpha_i$ or $\alpha_i + \alpha_{i+1}$. Then $x \in H^2(U_1, k)$ has only one component and by Corollary 2.2.4(b), $\beta \in \Delta$. Hence, there does not exist $\beta_1 \in \Delta$ and $\sigma \in X(T)$ such that $\alpha + \beta = \beta_1 + 2\sigma$. \qed

Proposition 3.1.3. Let $p = 2$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. If $\alpha + \beta$ is a weight of $H^2(U_1, k)$ and there exist $\beta_1, \beta_2 \in \Delta, \sigma \in X(T), 0 < i < p$, and $m \geq 0$ such that

$$\alpha + \beta = i \beta_1 + 2m \beta_2 + 2\sigma,$$

then one of the following holds

(a) $\alpha + \beta$ is a solution to Eq. (2.1) or (2.2),

(b) if $n \geq 3$, then $\alpha + \beta = \alpha_{i-1} + \alpha_{i+1} + 2\alpha_i$ or $\alpha + \beta = \alpha_{i-1} + \alpha_{i-2} + 2\alpha_i$ or $\alpha + \beta = \alpha_{i+1} + \alpha_{i+2} + 2\alpha_i$ for $i \leq n - 2$.

Proof. First note that we only have to consider the cases $i = 0, 1$ and since $p = 2$ if $m \geq 2$, then by choosing a different $\sigma \in X(T)$, these equations reduce down to $m \leq 1$ and $i = 0, 1$. If $i = 0 = m$, then the equation reduces to (2.2), which is done. If $i = 0, m = 1$, then we have that $\alpha + \beta = 2(\beta_2 + \sigma)$ and so $\sigma = 0$, but then we are back into Eq. (2.1). If $i = 1, m = 1$, then $\sigma = 0$, which is a specific case of Eq. (2.2). The only case we have to check is $i = 1, m = 0$. Since $\alpha = \alpha_i$, then $\sigma \in \Delta$ and $\beta = \alpha_{i-2} + \alpha_{i-1} + \alpha_i, \beta = \alpha_i + \alpha_{i+1} + \alpha_{i+2}$, or $\beta = \alpha_{i-1} + \alpha_i + \alpha_{i+1}$, which are the cases above. So $\beta_1$ and $\beta_2$ are either the simple root on each side of $\sigma$ or the two simple roots to the right or left of $\alpha$. \qed

3.2. Type $A_{2n+1}$

Note that $X(T)/\mathbb{Z}\Phi = \{t\omega_1 + Z\Phi: t = 0, 1, \ldots, n\} \cong \mathbb{Z}_{n+1}$ and $n$ is odd. Since 2 divides the index of connection, we must change Eqs. (2.1)–(2.3)

$$t\omega_1 = \frac{t}{n+1} (n\alpha_1 + (n-1)\alpha_2 + \cdots + \alpha_n).$$

By revising (3.1)–(3.3), for $\alpha \in \Delta$, $\beta \in \Phi^+$ must satisfy one of the following:

$$\alpha + \beta = 2t\omega_1 + 2\sigma, \quad (3.1)$$

$$\alpha + \beta = \beta_1 + 2t\omega_1 + 2\sigma, \quad (3.2)$$

$$\alpha + \beta = i\beta_1 + 2m\beta_2 + 2t\omega_1 + 2\sigma, \quad (3.3)$$

where $\sigma \in \mathbb{Z}\Phi$. Since $2t\omega_1$ must lie in $Z\Phi$, $\frac{2t}{n+1} \in \mathbb{Z}$ and $2|n + 1$, it follows that $\frac{t}{2} \in \mathbb{Z}$, where $s := \frac{n+1}{2}$. If $2|\frac{t}{2}$ then (3.1)–(3.3) reduce to the original equations, (2.1)–(2.3) with $\sigma$ lying in the root lattice, and the arguments in Section 3.1 apply. So, we can assume that $\frac{t}{2} \neq 0 \pmod{2}$. Consider
\[ \alpha + \beta = \sum_{i=1}^{n} m_i \alpha_i, \text{ then } m_i \in \{0, 1, 2\} \text{ for } i = 1, 2, \ldots, n \text{ and } m_i = 2 \text{ for at most one } i. \] To examine the possibilities, reduce to \( \alpha_1 \) mod 2, considering sequences of 0's and 1's. The sequence looks like \((1, 0, 1, 0, \ldots, 0, 1)\), where one of the zeroes is a 2, and at most two other zeroes can be made into a one by adding a simple root (as in (3.2) or (3.3)). So, we have the following:

\[ \sum_{i=1}^{n} m_i \alpha_i = i \beta_1 + 2^m \beta_2 + (1, 0, 1, 0, \ldots, 0, 1) \pmod{2}. \]

Since the roots of \( A_n \) have consecutive 1's, then \( n \geq 9 \) has a trivial solution.

Looking at \( A_3, A_5, \) and \( A_7 \) separately it is easy to check that no additional cohomology classes occur, and the only classes that occur are weights of the form \( \alpha + \beta = s_\alpha s_\beta \cdot 0 \) and \( \alpha + 2\gamma + \beta = s_\alpha s_\beta + 2\gamma \), where \( \alpha + \beta \) is not a root and \( \alpha + \gamma + \beta \) is a root.

### 3.3. Type \( B_n \)

For type \( B_n \), \( X(T)/Z \Phi \cong \mathbb{Z}_2 \) where \( \omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n) \) is a generator, which forces us to revise (3.1)–(3.3) in the following way. We are looking for \( \alpha, \beta \in \Phi^+ \) satisfying

\[ \alpha + \beta = 2t \omega_n + 2\sigma, \quad (3.4) \]
\[ \alpha + \beta = \beta_1 + 2t \omega_n + 2\sigma, \quad (3.5) \]
\[ \alpha + \beta = i \beta_1 + 2^m \beta_2 + 2t \omega_n + 2\sigma, \quad (3.6) \]

where \( \beta_1, \beta_2, \sigma \in \mathbb{Z} \Phi \) and \( t = 0, 1 \). Furthermore, when \( \Phi \) is of type \( B_n \), we have even structure constants. The structure constants are even when the resulting root has a coefficient of 2 in the \( \alpha_n \) spot and \( \alpha_n \) is broken up between the two roots, i.e. \([\alpha_n, \alpha_i + \alpha_{i+1} + \cdots + \alpha_n] = 2(\alpha_i + \alpha_{i+1} + \cdots + 2\alpha_n)\).

**Case 1.** Suppose \( t = 0 \) and \( m_i \neq 2 \), for any \( i \): Arguments in Section 3.1 apply. Thus, we have the same classes that appear when \( \Phi \) is of type \( A_n \).

**Case 2.** Suppose \( m_i = 2 \) for some \( i \). For simplicity, the weight will be written as: \((i_1, i_2, \ldots, i_n)\), where \( i_j \) is the coefficient of the \( \alpha_j \) term.

**Case 2.1.** If \( i_j \equiv 0 \pmod{2} \) for all \( j \), then our weight is twice a root. When, this occurs, it is easy to check that this results in the two cases: (1) \( \alpha = \beta \) and (2) \( \alpha = \alpha_i, \beta = \alpha_i + 2\alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_n \).

It's easy to check that these are always in the kernel.

**Case 2.2.** If there is only one odd number, then there must be some 2's. If all of the 2's occur after the odd number, then this is a root and thus in the previous image. If all of the 2's occur before the odd number, then there can only be one 2 before it because of the pattern of the roots. In which case, there is only one term in the class, \( \phi_{\alpha_i} \otimes \phi_{\alpha_{i+1}} \).

**Case 2.3.** Suppose there are two odd numbers. Note, there can’t be more than one 2 between the two odd numbers. Otherwise, there would be no way to break up the weight into a sum of a simple and a positive root. Consider the following weight, \((0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 2)\) where the first 1 is in the \( i \)th spot, given by \( \phi_{\alpha_i} \otimes \phi_{\alpha_{i+1} + 2\alpha_n} \). This is in the kernel because the structure constant is 2 (as noted above). Besides this weight, the 1's, 2's, and 3's that appear in the weight must all be consecutive. In particular, the 2's and 3's must be consecutive, which follows from Proposition 2.2.3 and the structure constants. Furthermore, there can only be one 3, which follows from Corollary 2.2.4 and the structure of the roots. If a 3 occurs, then it cannot be in the \( \alpha_n \) spot (unless \( t = 0 \)), 2's must follow and a 1 must occur before the 3 with a 2 between the 1 and the 3. So that leaves us with a weight looking like \((0, \ldots, 0, 1, 2, 3, 2, \ldots, 2)\), with the 3 in the \( i \)th spot.
Suppose $i \leq n - 2$, then the term in the cohomology class is $\phi_{\alpha_i} \otimes \phi_{x_i}$. However, after taking the differential the following term appears $\phi_{\alpha_i} \otimes \phi_{x_1} \otimes \phi_{\alpha_i}$ for some $x_1, x_2 \in \Phi^+$, which can’t cancel out and thus not in the kernel. The only possible weight is $(0, \ldots, 0, 1, 2, 3, 2)$, which is in the kernel.

If $t = 0$, then the only weight is $(0, \ldots, 0, 1, 2, 3)$, which is also in the kernel.

**Case 2.4.** Suppose there are more than two odd numbers. Then, the weight would look like: $(0, \ldots, 0, 1, 1, 2, \ldots, 2, 3, 2, \ldots, 2)$. Suppose the 3 is in the $i$th spot and the first 1 is in the $j$th spot, then breaking this weight into 2 roots, you must have $\alpha$ contain $\alpha_j$ as part of its sum. When taking the differential, the term $\alpha_j \otimes \alpha_j \otimes \phi_{x_i}$ appears, which can’t cancel with anything else. Therefore, there can only be at most two odd numbers.

For a complete list of the cohomology classes that appear refer to Appendix B (see footnote 1).

3.4. Type $C_n$

For type $C_n$, $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_2$, where $\alpha_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n$ is a generator. This forces us to revise (3.1)–(3.3). We want $\alpha, \beta \in \Phi^+$ satisfying

\begin{align*}
\alpha + \beta &= 2t\omega_1 + 2\sigma, \quad (3.7) \\
\alpha + \beta &= \beta_1 + 2t\omega_1 + 2\sigma, \quad (3.8) \\
\alpha + \beta &= i\beta_1 + 2^n\beta_2 + 2t\omega_1 + 2\sigma, \quad (3.9)
\end{align*}

where $\sigma \in \mathbb{Z}\Phi$ and $t = 0, 1$.

**Case 1.** Suppose $t = 0$ and $\alpha = \sum m_i\alpha_i$, $m_i \neq 2$ for all $i$. Then arguments in Section 3.1 apply. Thus, we have the same classes that appear when $\Phi = A_n$ also appear when $\Phi = C_n$. Furthermore, if $\alpha + \beta = 2\sigma$, then the weight is twice a root. This results in the cases $\phi_{\alpha} \otimes \phi_{\alpha}$ and $\alpha = \alpha_1, \beta = 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-1} + \alpha_n$. It’s easy to check that both of these classes are in the kernel.

**Case 2.** Suppose there is only one odd number in the weight, then there must be a 1 and 2’s must appear. If the 2’s appear only before or after the 1, then there can only be one 2 because of the structure of the roots. Furthermore, there is only one term in the class with $\beta$ not simple and $[\alpha, \beta] \neq 0 \pmod{2}$. However, if the 2’s appear before and after the 1, then we must have $(0, \ldots, 0, 2, 1, 2)$. However, this weight doesn’t satisfy Corollary 2.2.4. Therefore, there are no weights with only one odd number.

**Case 3.** Note that outside of the weight $(1, 0, \ldots, 0, 2, 1)$ (which is in the kernel), then all 1’s, 2’s, and 3’s must be consecutive. Suppose there are at least two odd numbers. First note that $(0, \ldots, 0, 1, 2, \ldots, 2, 1, 0, \ldots, 0)$ can only be written as a sum of a simple root and a positive root when the last 1 is in the $\alpha_n$ spot, which then makes this weight a root, thus in the previous image. Suppose 3’s appear in the weight, then there can only be one 3. Suppose 3 is in the $i$th spot, where $i < n - 1$. Also, there can’t be a 1 appearing in the weight before the 3 because of the root structure. If there are 2’s appearing in the weight, $(0, \ldots, 0, 2, \ldots, 2, 3, 2, \ldots, 2, 1)$, then after taking the differential, the term $\phi_{\alpha_i} \otimes \phi_{\alpha_j} \otimes \phi_{\alpha_k}$ can’t be cancelled. Then, we see that the 3 must be the $n - 1$ spot, since it can’t be in the spot. Therefore, we have the following possibilities:

(i) $(0, \ldots, 0, 3, 1)$ which is in the kernel, since the structure constant is 2.
(ii) $(0, \ldots, 0, 1, 3, 1)$, which doesn’t satisfy Corollary 2.2.4.
(iii) \((0, \ldots, 0, 1, 2, 3, 1)\) which isn’t in the kernel.
(iv) \((0, \ldots, 0, 1, 2, \ldots, 2, 3, 1)\), which isn’t in the kernel.

For a complete list of the cohomology classes that appear refer to Appendix B (see footnote 1).

3.5. Type \(D_n\)

Suppose \(t = 0\) and \(\alpha = \sum m_i \alpha_i, m_i \neq 2\) for all \(i\). Then arguments in Section 3.1 apply. Thus, we have the same classes that appear when \(\Phi = A_n\) also appear when \(\Phi = C_n\). Furthermore, if \(\alpha + \beta = 2\sigma\), then the weight is twice a root. This results in the cases \((\alpha)\) satisfying
\[
\alpha + \beta = 2t \omega_n + 2\sigma,
\]
(3.10)
\[
\alpha + \beta = \beta_1 + 2t \omega_n + 2\sigma,
\]
(3.11)
\[
\alpha + \beta = i \beta_1 + 2^m \beta_2 + 2t \omega_n + 2\sigma,
\]
(3.12)
where \(\sigma \in \mathbb{Z}\Phi\). Since we need \(\frac{(n-2)}{2} t \in \mathbb{Z}\), then \(t \equiv 0 \pmod{2}\), then we’ll consider the cases \(t = 0, 2\).

Note that there are no weights with only one odd number.

**Case 1.** Suppose \(t = 0\), and there are no 3’s in the weight. We claim that if there are no 3’s involved in the weight, then there can be at most two 2’s involved. There can’t be any 2’s prior to the first 1 involved in the weight because there is no way to write the weight as a sum of a positive and a simple root. When 2’s appear after the first 1 then the other 1 must occur in either the \(\alpha_{n-1}\) or the \(\alpha_n\) spot. So, there are two possibilities for weights:

(i) \((0, \ldots, 0, 1, 2, \ldots, 2, 1)\), where the first 2 is in the \(i\)th spot. Then after taking the differential, we see that the term \(\phi_{\alpha_{n-1}} \otimes \phi_{\alpha_i} \otimes \phi_{\alpha_n}\) can only be cancelled when \(i = n - 2\).

(ii) \((0, \ldots, 0, 1, 2, \ldots, 2, 1, 2)\) where the first 2 is in the \(i\)th spot. Then the same argument holds as above.

The only extra classes that occur are \((0, \ldots, 0, 1, 2, 2, 1)\) and \((0, \ldots, 0, 1, 2, 1, 2)\).

**Case 2.** Suppose \(t = 2\) and there are no 3’s in the weight. The 2’s must only come after the first 1 in the weight, and we must have 1’s in both the \(\alpha_{n-1}\) and \(\alpha_n\) spots. Then, we have the following cases:

(i) \((0, \ldots, 0, 1, 2, \ldots, 2, 1, 1)\), which is easy to check is in the kernel.

(ii) \((0, \ldots, 0, 1, 2, \ldots, 2, 1, 1)\), which is in the previous image.

(iii) \((0, \ldots, 0, 1, 2, 1, 1, \ldots, 2, 1, 1)\), which isn’t in the kernel because the term \(\phi_{\alpha_i} \otimes \phi_{\alpha_j} \otimes \phi_{\alpha_n}\) can’t cancel with anything, when \(i\) is the place of the first 1 and \(j\) is the place of the first 2, which are not connected.

3.5.2. \(n\) even

Suppose \(n\) is even, then \(X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2\), where \(\omega_1 = \alpha_1 + \cdots + \alpha_{n-2} + \frac{1}{2} \alpha_{n-1} + \frac{1}{2} \alpha_n\) and \(\omega_n = \frac{1}{2} (\alpha_1 + 2 \alpha_2 + \cdots + (n-2) \alpha_{n-2}) + \frac{n-2}{4} \alpha_{n-1} + \frac{n}{4} \alpha_n\) are the two generators. Then for \(\alpha, \beta \in \Phi^+\)
must satisfy:

\[ \alpha + \beta = 2t\omega_1 + 2s\omega_n + 2\sigma, \quad (3.13) \]
\[ \alpha + \beta = \beta_1 + 2t\omega_1 + 2s\omega_n + 2\sigma, \quad (3.14) \]
\[ \alpha + \beta = i\beta_1 + 2^m\beta_2 + 2t\omega_1 + 2s\omega_n + 2\sigma, \quad (3.15) \]

where \( i = 0, 1, m \geq 0, s, t = 0, 1, \sigma \in \mathbb{Z}\Phi, \) and \( \alpha, \beta \in \Phi^+. \)

**Case 1.** When \( s = t = 0, \) then we can use the same proof as the case when \( t = 0 \) when \( n \) is odd.

**Case 2.** When \( s = 1, t = 0, \) the equations reduce down to the case when \( n \) is odd and \( t = 2. \)

**Case 3.** If \( t = 1, s = 0, \) then \( n \leq 12 \) because otherwise there are too many odd numbers in the weight. However the root structure tells us that if there are any 3’s in the weight, then there must be 1’s in the \( \alpha_n \) and the \( \alpha_{n-1} \) spots, then we have that \( n \leq 6, \) but since \( n \) is even, then \( n = 4, 6. \)

**Case 3.1.** When \( n = 4, \) then this is reduced down to the case when \( s = t = 0. \)

**Case 3.2.** When \( n = 6, \) then we are reduced down to the case when \( s = 1, t = 0. \)

**Case 4.** Now if \( s = t = 1, \) then the only thing that changes are the \( \alpha_{n-1} \) and \( \alpha_n \) spots, which we have already considered.

For a complete list of the cohomology classes that appear refer to Appendix B (see footnote 1).

3.6. The exceptional cases

If \( \Phi \) is one of the exceptional root systems (i.e., \( E_6, E_7, E_8, F_4, \) or \( G_2 \)), then determining the \( U_1 \) cohomology reduces to looking at finitely many cases. To do this, a program in GAP was written to calculate all different weights that satisfy Eqs. (2.1)–(2.3). Once the weights were determined, we calculated differentials to check which weights were in the kernel. Note that if \( \Phi = E_7, \) this is the only case where the index of connection is even. The program must now be run twice once when \( t = 0 \) and the other time when \( t = 1 \) with \( \omega_2 = \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7) \) as a generator.

**Remark 3.6.1.** Note, that the long simple roots in type \( F_4 \) are \( \alpha_3, \alpha_4 \) and the long simple root in type \( G_2 \) is \( \alpha_2. \)

**Remark 3.6.2.** A complete list of the possible weights and classes for the exceptional cases is seen in Appendix B (see footnote 1).

4. \( B \)-cohomology

4.1. In order to calculate \( H^2(B, \lambda) \), we first had to calculate \( H^2(B, \lambda) \). Full details on the steps taken to calculate \( H^2(B, \lambda) \) are in Appendix C (see footnote 1).

Cline, Parshall, and Scott [6] give a relationship between the \( B_r \)-cohomology and the \( B \)-cohomology:

\[ H^2(B, \lambda) \cong \lim_{\leftarrow} H^2(B, \lambda). \]
Theorem 4.1.1. Let \( p = 2 \) and \( \lambda \in X(T) \).

(a) If \( \Phi \) is simply laced, then

\[
H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^l \beta - p^t \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\
  0 & \text{else.}
\end{cases}
\]

(b) If \( \Phi \) is of type \( B_n \), then

\[
H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^l \beta - p^t \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\
  k & \text{if } \lambda = -p^{l+1}(\alpha_{n-1} + \alpha_n) - p^t \alpha_{n-1} \text{ with } 0 \leq l, \\
  0 & \text{else.}
\end{cases}
\]

(c) If \( \Phi \) is of type \( C_n \), then

\[
H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^l \beta - p^t \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\
  k & \text{if } \lambda = -p^{l+1}(\alpha_{n-1} + \alpha_n) \text{ with } 0 \leq l, \\
  0 & \text{else.}
\end{cases}
\]

(d) If \( \Phi \) is of type \( F_4 \), then

\[
H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^l \beta - p^t \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\
  k & \text{if } \lambda = -p^{l+1}(\alpha_3 + \beta) - p^t \alpha_2 \text{ with } 0 \leq l \text{ and } \beta \in \{\alpha_2, \alpha_4\}, \\
  0 & \text{else.}
\end{cases}
\]

(e) If \( \Phi \) is of type \( G_2 \), then

\[
H^2(B, \lambda) \cong \begin{cases} 
  k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\
  k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\
  k & \text{if } \lambda = -p^l \beta - p^t \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\
  k & \text{if } \lambda = -p^{l+1}(\alpha_1 + \alpha_2) - p^t \alpha_2 \text{ with } 0 \leq l, \\
  0 & \text{else.}
\end{cases}
\]

Proof. Assume that \( \lambda \in X(T) \) with \( H^2(B, \lambda) \neq 0 \), then \( \lambda \neq 0 \). Choose \( s > 0 \) such that

(i) The natural map \( H^0(B, \lambda) \to H^0(B_r, \lambda) \) is nonzero for all \( r \geq s \).
(ii) By choosing a possibly larger \( s \), we can further assume that \( |\langle \lambda, \alpha^\vee \rangle| < p^{s-1} \) for all \( \alpha \in \Delta \).

The \( H^2(B_r, \lambda) \) theorem and condition (ii) give us \( H^2(B_r, \lambda) \) is one-dimensional for all \( r \geq s \). Furthermore, since \( H^2(B, \lambda) \) has trivial \( B \)-action, then \( H^2(B, \lambda) \cong k \) for all \( r \geq s \), by condition (i).
On the other hand, if there exists an integer \( s \) such that \( H^2(B_r, \lambda) \cong k \) for all \( r \geq s \), then \( H^2(B, \lambda) \cong \lim \rightarrow H^2(B_r, \lambda) \cong k. \quad \square \)

5. \( G_r \)-cohomology

5.1. The \( B_r \)-cohomology can also be used to calculate \( H^2(G_r, H^0(\lambda)) \) for \( \lambda \in X(T)_+ \). Recall the following theorem from [5, 6.1].

**Theorem 5.1.1.** Let \( \lambda \in X(T)_+ \) and \( p \) be an arbitrary prime. Then

\[
H^2(G_r, H^0(\lambda))^{(-r)} \cong \text{ind}_B^G H^2(B_r, \lambda)^{(-r)}.
\]

Using the isomorphism in Theorem 5.1.1 and the \( B_r \) cohomology results (Theorem C.2.6 in Appendix C, see footnote 1) then we get the following theorem.

**Theorem 5.1.2.** Let \( p = 2 \) and \( \lambda \in X(T)_+ \). Then \( H^2(G_r, H^0(\lambda)) \) is given by the following table.

<table>
<thead>
<tr>
<th>( H^2(G_r, H^0(\lambda)), p = 2 )</th>
<th>Root system</th>
<th>( \lambda \in X(T)_+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ind}_B^G H^2(B_1, w \cdot 0 + pv)^{(r-1)} )</td>
<td>( A_6, (n \neq 3) ), ( B_8 ), ( E_6, F_4, G_2 )</td>
<td>( p^{r-1}(w \cdot 0 + pv), l(w) = 0, 2 )</td>
</tr>
<tr>
<td>( \text{ind}_B^G H^2(B_1, w \cdot 0 + pv)^{(r-1)} )</td>
<td>( A_3 )</td>
<td>( p^r )</td>
</tr>
<tr>
<td>( \text{ind}_B^G H^2(B_1, w \cdot 0 + pv)^{(r)} )</td>
<td>( C_n, D_n )</td>
<td>( p^{r-1}(w \cdot 0 + pv), l(w) = 0, 2 )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( \text{All} )</td>
<td>( p^r v + p^r w \cdot 0 ) with ( l(w) = 2 ) and ( 0 \leq l &lt; r - 1 )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( A_2, D_n, E_6 )</td>
<td>( p^r v - p^r \alpha ) with ( 0 &lt; l &lt; r ), ( \alpha \in \Delta )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( B_3, B_4, F_4, G_2 )</td>
<td>( p^r v - p^r \alpha ) with ( 0 \leq l &lt; r ), ( \alpha \in \Delta )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( B_n, n \geq 5 )</td>
<td>( p^r v - p^r \alpha ) with ( 0 \leq l &lt; r ), ( \alpha \in \Delta, l \neq r - 1 ) if ( \alpha = \alpha_{n-1} )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( C_n )</td>
<td>( p^r v - p^r \alpha ) with ( 0 \leq l &lt; r ), ( \alpha \in \Delta, l \neq r - 1 ) if ( \alpha = \alpha_n )</td>
</tr>
<tr>
<td>( H^0(\nu)^{(r)} )</td>
<td>( \text{All} )</td>
<td>( p^r v - p^r \beta - p^r \alpha ) with ( \alpha, \beta \in \Delta )</td>
</tr>
<tr>
<td>( \text{ind}_B^G H^0(\nu)^{(r)} )</td>
<td>( A_3 )</td>
<td>( p^r v + p^r \alpha_2 - p^r \alpha ) with ( 0 \leq l &lt; r - 1, \alpha \in \Delta )</td>
</tr>
<tr>
<td>( H^0(\nu + \alpha_2)^{(r)} \oplus H^0(\nu + \alpha_2)^{(r)} )</td>
<td>( A_3 )</td>
<td>( p^r v + p^r \alpha_2 - p^r \alpha ) with ( 0 \leq l &lt; r - 1, \alpha \in \Delta )</td>
</tr>
<tr>
<td>( H^0(\nu + \alpha_3)^{(r)} \oplus H^0(\nu + \alpha_3)^{(r)} )</td>
<td>( B_3 )</td>
<td>( p^r v + p^r \alpha_2 - p^r \alpha ) with ( 0 \leq l &lt; r - 1, \alpha \in \Delta )</td>
</tr>
<tr>
<td>( \text{ind}<em>B^G (M</em>{B_r} \otimes \nu)^{(r)} )</td>
<td>( B_n )</td>
<td>( p^r v - p^r (\alpha_{n+1} + \alpha_n) - p^r \alpha_2 ) with ( 0 \leq l &lt; r - 1 )</td>
</tr>
<tr>
<td>( \text{ind}<em>B^G (A</em>{11} + M_{B_2}) \otimes \nu)^{(r)} )</td>
<td>( B_4 )</td>
<td>( p^r v - p^r \alpha_1 - p^r \alpha ) with ( \alpha \in \Delta ), ( 0 \leq l &lt; r - 1, l \in {1, 3} )</td>
</tr>
</tbody>
</table>
| \( \text{ind}_B^G (M_{B_2} \otimes \nu)^{(r)} \) | \( B_n \) | \( p^r v - p^r \alpha_{n-1} - p^r \alpha_n \) with \( 0 \leq l < r - 1 \)

(continued on next page)
The module, $M$, is taken to be the appropriate module:

- $M_{\Delta_1}$ is the module with factors $\alpha_n, k$.
- $M_{\Delta_2}$ is the module with factors $\alpha_{n-1}, k$.
- $M_{F_4}$ is the module with factors $\alpha_2, k$.
- $M_{G_2}$ is the module with factors $\alpha_1, k$.

If the $B_r$-cohomology involves an indecomposable $B$-module, then it is necessary to determine the structure of the modules for the $G_r$-cohomology, separately. To see the structure of the modules, refer to Appendix C (see footnote 1), found at the arXiv.

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**References**