Adaptive Pole Positioning in MIMO Linear Systems by Periodic Multirate-Input Controllers

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Submitted by C. T. Leondes

Received July 31, 1998

In this paper, the certainty equivalence principle is used to combine the identification method with a control structure derived from the pole placement problem, which rely on periodic multirate-input controllers. The proposed adaptive pole placers, contain a sampling mechanism with different sampling period to each system input and rely on a periodically varying controller which suitably modulates the sampled outputs and reference signals of the plant under control. Such a control strategy allows us to arbitrarily assign the poles of the sampled closed-loop system in desired locations and does not make assumptions on the plant other than controllability and observability of the continuous and the sampled system, and the knowledge of a set of structural indices, namely the locally minimum controllability indices of the continuous-time plant. An indirect adaptive control scheme is derived, which estimates the unknown plant parameters (and consequently the controller parameters) on-line, from sequential data of the input and outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period $T_0$. Using the proposed algorithm, the controller determination is based on the transformation of the discrete analogous of the
system under control to a phase-variable canonical form, prior to the application of
the control design procedure. The solution of the problem can, then, be obtained
by a quite simple utilization of the concept of state similarity transformation.
Known indirect adaptive pole placement schemes usually resort to the computation
of dynamic controllers through the solution of a polynomial Diophantine equation,
thus introducing high order exogenous dynamics in the control loop. Moreover, in
many cases, the solution of the Diophantine equation for a desired set of closed-loop
eigenvalues might yield an unstable controller, and the overall adaptive pole
placement scheme is unstable with unstable compensators because their outputs
are unbounded. The proposed control strategy avoids these problems, since here
gain controllers are essentially needed to be designed. Moreover, persistency of
excitation and, therefore, parameter convergence, of the continuous-time plant is
provided without making any assumption either the richness of the reference
signals or on the existence of specific convex sets in which the estimated parame-
ters belong or, finally, on the coprimeness of the polynomials describing the
ARMA model, as in known adaptive pole placement schemes.

1. INTRODUCTION

Multirate sampling schemes have long been the focus of interest by
many control designers. There are several reasons to use such a sampling
scheme in digital control systems. First of all, in complex, multivariable
control systems, often it is unrealistic, or sometimes impossible, to sample
all physical signals uniformly at one single rate. In such situations, one is
forced to use multirate sampling. Furthermore, in general, one gets better
performance if one can sample and hold faster. But faster A/D and D/A
conversions mean higher cost in implementation. For signals with different
bandwidths, better tradeoffs between performance and implementation
cost can be obtained using A/D and D/A converters at different rates.
On the other hand, multirate controllers are in general time-varying. Thus
multirate control systems can achieve what singlerate cannot; e.g., gain
improvement, simultaneous stabilization, and decentralized control. Fi-
nally, multirate controllers are normally more complex than singlerate
ones; but often they are finite-dimensional and periodic in a certain sense
and hence can be implemented on microprocessors via difference equa-
tions with finitely many coefficients. Therefore, like singlerate controllers,
multirate controllers do not violate the finite memory constraint in micro-
processors.

The study of multirate systems has its origins in late 1950s [1]–[3].
Recent interests are focused on stability issues [4], stabilization and pole
assignment [5]–[8], LQG/LQR designs [9]–[12], $H^\infty$ control [13]–[15],
decentralized control [16], adaptive designs [17], [18], etc. [19], [20]. In
particular, in their excellent work [5], Araki and Hagiwara propose a
digital multirate-input controller (MRIC), which suitably modulates the sampled outputs and discrete reference signals by a multirate periodically varying matrix function, in order to solve the sampled pole placement problem for linear time-invariant continuous-time systems. MRICs contain a multirate sampling mechanism with different sampling period to each system input. They can essentially be viewed as the special class of \( m \)-input, \( p \)-output multirate sampled-data control systems, in which all output samplers operate with multiplicities 1 and the input samplers with multiplicities \( N_1, \ldots, N_m \). Note that MRICs are the dual of multirate-output controllers (MROC), presented in [6], and subsequently used in [20], in which input and output samplers have the reverse operation. In [19], the MRIC based approach has been extended to the solution of the model matching problem. A main feature of the results reported in [5] and [19] is that the pole placement or the model matching is obtained without the requirement of pole-zero cancellation.

The purpose of the present paper is to explore the possibility of extending the MRIC based approach presented in [5] and subsequently used in [19], to the control of linear time-invariant continuous-time plants with unknown parameters. In particular, we use the certainty equivalence principle to combine the identification method with a control structure derived from the pole placement problem. It is worth noticing at this point that, although the inputs of the continuous-time plane are sampled in a multirate fashion, our aim here is to achieve adaptive pole placement control, only at the sampling instants \( kT_0 \), associated with the fundamental period \( T_0 \), on the basis of which the output samplers operate. To the best of the authors’ knowledge, there are no results in the literature concerning the use of this type of multirate sampled-data controllers in order to achieve adaptive pole placement control.

Adaptive pole placement is of particular interest, since the middle of the 1970s, for obvious reasons. Several techniques based on either direct or indirect adaptive control schemes were presented to treat the problem and a very large number of papers were reported on the subject; see [21]–[29], and references therein. The feedback strategies proposed to solve the adaptive pole placement problem, are hitherto based on dynamic output feedback, thus introducing high order exogenous dynamics in the control loop. On the other hand, a common feature of these techniques is that they reduce the solution of the problem to the solution of a polynomial Diophantine equation. This approach, however, does not ensure that the compensators obtained from the solution of the Diophantine equation are necessarily stable. In the case of unstable solutions, the control scheme composed by feedforward and feedback compensators is not stable and thus is not useful. The control signals are calculated from two sets of unbounded signals that are the outputs of the compensators. In a short
time the system becomes unstable. It is worth noticing at this point, that unstable solutions of the Diophantine equation, can occur even though, the system under control possesses the parity interlacing property (p.i.p.) [30] (is strongly stabilizable). A plant is said that it possesses the p.i.p. if the number of its real poles between each pair of zeros in the unstable domain is even. In this case, it is possible to obtain a stable controller from these unstable solutions by using the approach presented in [31], which is based on an interpolation procedure. Unfortunately, as mentioned above, this approach can be applied only in cases where the system under control is strongly stabilizable. When the system under control contains unknown parameters (as in the case of adaptive pole placement control), this information of crucial importance is not available to the designer. Thus, up to now, the design of a stable and useful adaptive pole placement compensator cannot be guaranteed.

The motivation for studying an adaptive version of the particular controller structure presented in [5] and [19], is manifold. First, since it does not rely on pole-zero cancellation, it may be readily applicable for solving the adaptive pole placement problem for nonstably invertible plants. Furthermore, the degrees of freedom in the choice of the modulating function, provide a solution to the problem of assuring persistency of excitation of the continuous-time plant under control, without imposing any special assumption either on the existence of special convex sets in which the estimated parameters belong or on the coprimeness of the polynomials describing the ARMA model, as in known techniques, or finally on the richness of the reference signals (except boundedness), as in known adaptive pole placement techniques. The determination of the MRIC based adaptive pole placers sought is mainly based on the transformation of the discrete analogue of the continuous-time system under control to a phase variable canonical form, prior to the application of the control design procedure. As a consequence of this fact, the solution of the problem can be obtained by a quite simple utilization of the concept of state similarity transformation. No Diophantine equation is needed to be solved here as compared to known techniques. The designed MRIC based adaptive pole placers are always stable, since gain controllers are needed to be designed here, as compared to (possibly unstable) dynamic compensators obtained by known techniques. Therefore, the proposed adaptive scheme is readily applicable to plants which do not possess the p.i.p. As a consequence of this design philosophy, a useful globally stable indirect adaptive control scheme is derived, which estimates the unknown plant parameters (and consequently the controller parameters) on-line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period $T_0$. Finally, it is remarked that the a priori knowledge
needed in order to implement the proposed adaptive pole placers is controllability and observability of the continuous and the discretized plant under control, its order, and a set of structural indices, namely the locally minimum controllability indices of the continuous-time plant.

2. PRELIMINARIES AND DEFINITION OF THE PROBLEM

Consider the linear multi-input, multioutput system having the following state-space representation:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)
\]  

(2.1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, and \( y(t) \in \mathbb{R}^{\ell} \) is the output vector and where the matrices \( A, B, \) and \( C \) have appropriate dimensions.

With regard to system (2.1), we make the following two assumptions:

**Assumption 2.1.** (a) System (2.1) is controllable and observable and of known order \( n \). (b) There are known integers \( n_i, i \in \mathbb{J}_m, \mathbb{J}_m = \{1, 2, \ldots, m\} \), which comprise a set of locally minimum controllability indices of the pair \((A, B)\).

**Assumption 2.2.** Let \( N_i, i \in \mathbb{J}_m \) be positive integers. Also let \( N = \text{lcm}(N_1, \ldots, N_m) \), where \( \text{lcm}(\ldots, \cdot) \) denotes the least common multiplier of the arguments quoted in the braces. Then, there is a sampling period \( T_0 \in \mathbb{R}^+ \), such that the discretized systems, obtained by sampling (2.1) with periods \( T_0 \) and \( \tau = T_0/(6n - 1)N \) and having the following matrix triplets:

\[
(\Phi, \tilde{B}, C) = \left( \exp(A T_0), \int_0^{T_0} \exp(A \lambda) B d\lambda, C \right),
\]

\[
(\Phi_\tau, B_\tau, C) = \left( \exp(A \tau), \int_0^{\tau} \exp(A \lambda) B d\lambda, C \right),
\]

respectively, are controllable and observable.

Except for this prior information, the matrix triplet \((A, B, C)\) is arbitrary and unknown. It is mentioned that no assumption is made here on the relative degree of the plant or its stable invertibility.

For a controllable matrix pair \((A, B)\), with \( B \triangleq [b_1, b_2, \cdots, b_m] \), its locally minimum controllability indices (LMCsIs) are a collection of \( m \) integers
\{n_1, n_2, \ldots, n_m\}, for which the following relationships simultaneously hold:

\[ \sum_{i=1}^{m} n_i = n \quad \text{and} \quad \text{rank}\left[ \mathbf{b_1} \cdots A^{n-1} \mathbf{b_1} \cdots \mathbf{b_m} \cdots A^{n-1} \mathbf{b_m} \right] = n \]

Note that, LICCI defined as above are also known as the "Kronecker invariants" or "Kronecker indexes" of the pair \((A, B)\).

Consider now applying to system (2.1), the multirate control strategy depicted in Figure 1. With regard to the sampling mechanism, we assume that all samplers start simultaneously at \(t = 0\). The sampling periods \(T_i\) have rational ratio, i.e., \(T_i = T_0/N_i\), for \(i \in J_m\), where \(T_0\) is the common sampling period, \(N_i \in \mathbb{Z}^+\) are the input multiplicities of the sampling. The hold circuits \(H_i\) and \(H_0\) are the zero order holds with holding times \(T_i\) and \(T_0\), respectively. Let

\[ N^* = \sum_{i=1}^{m} N_i, \quad I_i = N/N_i, \quad T_N = T_0/N \]

The modulating matrix function \(F(t) \in \mathbb{R}^{m \times p}\) is assumed to be bounded, integrable and \(T_0\)-periodic, i.e.,

\[ F(t + T_0) = F(t) \quad \text{for} \quad t \in [kT_0, (k+1)T_0) \quad (2.2) \]

\[ \text{FIG. 1. Control strategy in the nonadaptive case.} \]
As it can be easily shown, the resulting closed-loop system is described by the following state-space equations:

\[
x[(k + 1)T_0] = (\Phi - K_f C)x(kT_0) + K_f w(kT_0),
\]

\[
y(kT_0) = Cx(kT_0), \quad k \geq 0
\]

where \(x(kT_0) \in \mathbb{R}^n\) is a discrete measurement vector obtained by sampling \(x(t)\) with sampling period \(T_0\) and where the matrix \(K_f \in \mathbb{R}^{n \times p}\) is defined as

\[
K_f = \int_0^{T_0} \exp[A(T_0 - \lambda)]BF(\lambda) \, d\lambda
\]  

(2.3)

The adaptive pole placement problem treated in the present paper is as follows: Find a periodic controller \(F(t)\), which when applied to system (2.1) drives the poles of the resulting closed-loop system (also called the closed-loop monodromy eigenvalues) to new desired values \(\lambda_1, \lambda_2, \ldots, \lambda_n\), where complex poles appear in conjugate pairs.

To solve the above problem, an indirect adaptive control scheme is exhibited in the sequel. In particular, we first solve the pole placement problem, namely, the assignment of the poles of the sampled system to the prespecified values \(\lambda_1, \lambda_2, \ldots, \lambda_n\), using period MRICs, for known systems. This is done in Section 3. Next, using these results, the pole placement problem is solved for the configuration of Figure 2, wherein the periodic controller \(F(t)\) is with prespecified periodic behavior and persistent excitation signals are introduced in the control loop for future identification purposes. This is done in Section 4. It is remarked that the motivation for modifying the control strategy as in Figure 2, is that it facilitates the derivation of the indirect adaptive control scheme sought, which is presented in Section 5. In Section 5, the global stability of the proposed scheme is also studied.

3. SOLUTION OF THE POLE PLACEMENT PROBLEM VIA MRICS FOR KNOWN SYSTEMS

The procedure for stabilization through pole placement using MRICs, consists in finding a periodic controller \(F(t)\), such that

\[
\det(zI - \Phi + K_f C) = \tilde{p}(z)
\]

(3.1a)
where

\[
\hat{p}(z) = \prod_{i=1}^{n} (z - \hat{\lambda}_i) = z^n + \hat{a}_1 z^{n-1} + \cdots + \hat{a}_{n-1} z + \hat{a}_n \quad (3.1b)
\]

Since, \( \det(zI - \Phi + K_s C) = \det(zI - \Phi^T + C^T K_f) \), relation (3.1a) is equivalent to the relation

\[
\det(zI - \Phi^T + C^T K_f) = \hat{p}(z) \quad (3.2)
\]

Consider now the following fictitious discrete time system:

\[
\tilde{x}[k+1]T_0 = \Phi^T \tilde{x}(kT_0) + C^T \tilde{u}(kT_0) \quad (3.3)
\]

Clearly, the pole placement problem via MRIC based control, defined in relation (3.2), is equivalent to the problem of choosing the matrix \( K_f \) in the state feedback control law

\[
\tilde{u}(kT_0) = -K_f \tilde{x}(kT_0) \quad (3.4)
\]

such that (3.2) is satisfied.

We start our analysis to this equivalent state feedback pole placement problem by first transforming system (3.3) to its equivalent input Luenberger canonical form. To this end, let \( \delta, i = 1, 2, \ldots, m \) be the controllability indices of the pair \( (\Phi^T, C^T) \) (which obviously are the observability
indices of the pair \((\Phi, C)\) and let \(P \in \mathbb{R}^{p \times n}\) be the following matrix:

\[
P = \begin{bmatrix}
c_1^T & \cdots & (\Phi^T)^{\delta_1-1}c_1^T & \cdots & c_p^T & \cdots & (\Phi^T)^{\delta_p-1}c_p^T
\end{bmatrix}
\]

where \(c_i^T, i = 1, 2, \ldots, p\) are the ordered columns of \(C^T\). Setting

\[
\gamma_j = \sum_{\rho=1}^{j} \delta_\rho, \quad j = 1, 2, \ldots, p
\]

and defining \(h_j^T\) as the \(\gamma_j\)th row of \(P^{-1}\), it can be shown that under the transformation \(\tilde{z}(kT_0) = Q \tilde{x}(kT_0)\), where \(Q \in \mathbb{R}^{n \times n}\) is the columnar stack of \(\delta_1 + \ldots + \delta_p \ (= n)\) rows, defined by

\[
Q = \begin{bmatrix}
h_1^T \\
\vdots \\
\vdots \\
h_p^T(\Phi^T)^{\delta_p-1}
\end{bmatrix}
\]

system (3.3) can be written as

\[
\tilde{z}[(k + 1)T_0] = \Phi^* \tilde{z}(kT_0) + C^* \tilde{u}(kT_0)
\]  \ (3.5)

where

\[
\Phi^* = Q \Phi^T Q^{-1}, \quad C^* = QC^T
\]

and where the matrices \(\Phi^*\) and \(C^*\) have the following respective forms:

\[
\Phi^* = \begin{bmatrix}
\Phi_{11}^* & \cdots & \Phi_{1p}^* \\
\vdots & \ddots & \vdots \\
\Phi_{p1}^* & \cdots & \Phi_{pp}^*
\end{bmatrix}, \quad C^* = \begin{bmatrix}
C_1^* \\
\vdots \\
C_p^*
\end{bmatrix}
\]

where

\[
\Phi_{ii}^* = \begin{bmatrix}
0_{\delta_i-1} & 1_{\delta_i-1} \\
0_{\delta_i} & -a_{ii}^*
\end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_i},
\]

\[
\Phi_{ij}^* = \begin{bmatrix}
0_{(\delta_i-1) \times \delta_j} \\
0_{\delta_i \times \delta_j}
\end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_j} \quad (i \neq j),
\]

\[
C_i^* = \begin{bmatrix}
0_{(\delta_i-1) \times p} \\
q_i^*
\end{bmatrix} \in \mathbb{R}^{\delta_i \times p}
\]
where
\[
\mathbf{a}_{ii}^T = \begin{bmatrix}
(a_{ii})_0 & (a_{ii})_1 & \cdots & (a_{ii})_{s-1}
\end{bmatrix},
\]
\[
\mathbf{a}_{ij}^T = \begin{bmatrix}
(a_{ij})_0 & (a_{ij})_1 & \cdots & (a_{ij})_{s-1}
\end{bmatrix} \quad (i \neq j)
\]
\[
\mathbf{q}_i^T = \begin{bmatrix}
\mathbf{0}_{s-1}^T & 1 & \tilde{q}_i^T
\end{bmatrix}, \quad \tilde{\mathbf{q}}_i^T = \begin{bmatrix}
(c_{ii})_{i+1} & (c_{ii})_{i+2} & \cdots & (c_{ii})_{p}
\end{bmatrix}
\]

Here, \( \mathbf{0}_r \), \( \mathbf{O}_{r \times q} \), and \( \mathbf{I}_r \) represent a zero \( r \)-dimensional vector, a zero \( r \times q \) matrix, and an \( r \)-dimensional identity matrix, respectively (empty if \( r \) or \( q \) is zero).

Now, let \( \mathbf{\hat{u}}(kT_0) \) be the set of inputs defined as follows:
\[
\mathbf{\hat{u}}(kT_0) = \mathbf{L} \mathbf{\hat{u}}(kT_0)
\]
where \( \mathbf{L} \) is the following upper triangular nonsingular matrix:
\[
\mathbf{L} = \begin{bmatrix}
1 & \tilde{q}_1^T \\
0 & 1 & \tilde{q}_2^T \\
& \vdots \\
0 & 0 & \cdots & 0 & 1 & \tilde{q}_{p-1}^T \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

It is now obvious that
\[
\mathbf{\hat{C}}^* \mathbf{L} = \mathbf{Q} \mathbf{C}_T^* \mathbf{L} \equiv \mathbf{\hat{C}}^* = \begin{bmatrix}
\tilde{\mathbf{c}}^*_1 \\
\vdots \\
\tilde{\mathbf{c}}^*_m
\end{bmatrix}
\]
where
\[
\tilde{\mathbf{C}}^*_i = \begin{bmatrix}
\mathbf{O}_{(s_i-1) \times p} \\
\mathbf{0}_{s_i-1}^T \\
1 & \mathbf{0}_{p-i}^T
\end{bmatrix}
\]
and that system (3.5) can be transformed to the following form:
\[
\tilde{z}[(k + 1)T_0] = \Phi^* \mathbf{\hat{z}}(kT_0) + \mathbf{\hat{C}}^* \mathbf{\hat{u}}(kT_0) \quad (3.6)
\]
which is the input Luenberger canonical form corresponding to system (3.3). In what follows, to system (3.6), we apply the following state feedback law:
\[
\mathbf{\hat{u}}(kT_0) = \mathbf{F}^* \mathbf{\hat{z}}(kT_0) \quad (3.7)
\]
in order to derive the eigenvalues of system (3.6) to desired positions \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Clearly, this is equivalent to the application of a state feedback law of the form (3.4), with

\[
K_f^T = -LF^*Q
\]

(3.8)

to system (3.3), in order to drive its eigenvalues to the desired positions \( \lambda_i \), \( i = 1, 2, \ldots, n \).

From the above analysis, it is clear that in order to solve the pole placement problem for system (3.3), under the control law (3.4), one can equivalently solve the pole placement problem for system (3.6), under the control law (3.7). The solution of this later problem can be obtained as follows: Observe first that the solution of this problem is equivalent to the problem of selecting \( F^* \) and a nonsingular transformation matrix \( T \) such that

\[
\Phi^* + \tilde{C}^*F^* = T\Pi T^{-1}
\]

(3.9)

where

\[
\Pi = \begin{cases} 
\text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_n) & \text{if the desired eigenvalues are distinct} \\
\text{blockdiag}(J_1, \ldots, J_n) & \text{if the desired eigenvalues are repeated}
\end{cases}
\]

(3.10)

with

\[
J_q = \begin{bmatrix}
\hat{\lambda}_q & 1 & 0 & \cdots & 0 \\
0 & \hat{\lambda}_q & 1 & \cdots & 0 \\
0 & 0 & \hat{\lambda}_q & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 1 \\
0 & 0 & 0 & \cdots & \hat{\lambda}_q
\end{bmatrix} \in \mathbb{R}^{r_q \times r_q}
\]

(3.11)

and where the order \( r_q \) of the \( q \)th Jordan block \( J_q \) is the multiplicity of the eigenvalue \( \hat{\lambda}_q \).

To solve (3.9) for \( F^* \) and \( T \), partition the matrices \( \Phi_{ii}^* \) and \( \Phi_{ij}^* \) as

\[
\Phi_{ii}^* = \begin{bmatrix} \hat{\Phi}_{ii}^* \\ -a_{ii}^* \end{bmatrix}, \quad \Phi_{ij}^* = \begin{bmatrix} \hat{\Phi}_{ij}^* \\ -a_{ij}^* \end{bmatrix} \quad (i \neq j)
\]

where

\[
\hat{\Phi}_{ii}^* = \begin{bmatrix} 0 \delta_{i-1} & 1 \delta_{i-1} \end{bmatrix}, \quad \hat{\Phi}_{ij}^* = O_{(\delta_i-1) \times \delta_j} \quad (i \neq j)
\]
define the matrices

\[
T = \begin{bmatrix}
T_{11} & \cdots & T_{1p} \\
\vdots & \ddots & \vdots \\
T_{p1} & \cdots & T_{pp}
\end{bmatrix}, \quad T_{ij} = \begin{bmatrix}
\hat{T}_{ij} \\
\hat{t}_{ij} \tilde{s}
\end{bmatrix},
\]

partition \( \tilde{c}_i^* \) as

\[
\tilde{c}_i^* = \begin{bmatrix}
\tilde{c}_i^+ \\
\tilde{c}_i^-
\end{bmatrix}
\]

where

\[
\tilde{c}_i^+ = \mathbf{0}_{(d, -1) \times p}, \quad \tilde{c}_i^- = \begin{bmatrix}
0_{r-1}^T & 1 & 0_{p-r}^T
\end{bmatrix}
\]

and define

\[
\Pi = \text{blockdiag}\{\Pi_1, \ldots, \Pi_p\}
\]

where each matrix \( T_{ij} \) has the dimensionality of \( \Phi_{ij}^* \), and each matrix \( \Pi_i \) has the dimensionality of \( \Phi_{ij}^* \), and may have one of the forms given in (3.10) (or its combination).

Next, define \( \hat{\Phi}^* \), \( \hat{\tilde{c}}^+ \), \( \hat{\Phi}^* \), \( \hat{\tilde{c}} \), and \( \tilde{T} \) as

\[
\hat{\Phi}^* = \left[ (\hat{\Phi}_{ij}^*)_{i=1, \ldots, p} \right]_{j=1, \ldots, p}, \quad \hat{\tilde{c}}^+ = (\hat{\tilde{c}}^+)^*_{i=1, \ldots, p}, \quad (3.12a)
\]

\[
\hat{T} = \left[ \hat{T}_{ij} \right]_{i=1, \ldots, p},
\]

\[
\tilde{\Phi}^* = \left[ (\tilde{\Phi}_{ij}^*)_{i=1, \ldots, p} \right]_{j=1, \ldots, p}, \quad \tilde{\tilde{c}} = (\tilde{\tilde{c}}^+)^*_{i=1, \ldots, p}, \quad (3.12b)
\]

where parentheses define a column of blocks and brackets \([ \cdot ]\) define a row of block columns, and apply a linear transformation upon (3.9) to obtain

\[
\begin{bmatrix}
\Phi^* \\
\hat{\Phi}^*
\end{bmatrix} + \begin{bmatrix}
\tilde{c}^+ \\
\tilde{c}
\end{bmatrix} \mathbf{F}^* = \begin{bmatrix}
\hat{T} \\
\hat{T}
\end{bmatrix} \Pi \mathbf{T}^{-1} \quad (3.13)
\]
where it is noted that \( \hat{C}^+ = O_{(n-p)\times p} \) and \( \hat{C} = I_p \). From (3.13) we obtain

\[
\hat{\Phi}^* \mathbf{T} = \hat{\Pi} \mathbf{I}
\]  

(3.14)

\[
\mathbf{F}^* = -\hat{\Phi}^* + \hat{\Pi} \mathbf{I} \mathbf{I}^{-1}
\]  

(3.15)

Equations (3.14) and (3.15) show that the problem of determining \( \mathbf{F}^* \) and \( \mathbf{T} \) has been decoupled, i.e., one first finds \( \mathbf{T} \) from (3.14) and then \( \mathbf{F}^* \) from (3.15).

To find \( \mathbf{T} \), observe that

\[
\hat{\Phi}^* \mathbf{T} = \begin{bmatrix}
\hat{\Phi}^*_{11} \mathbf{T}_{11} & \hat{\Phi}^*_{11} \mathbf{T}_{12} & \cdots & \hat{\Phi}^*_{11} \mathbf{T}_{1p} \\
\hat{\Phi}^*_{22} \mathbf{T}_{21} & \hat{\Phi}^*_{22} \mathbf{T}_{22} & \cdots & \hat{\Phi}^*_{22} \mathbf{T}_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\Phi}^*_{pp} \mathbf{T}_{p1} & \hat{\Phi}^*_{pp} \mathbf{T}_{p2} & \cdots & \hat{\Phi}^*_{pp} \mathbf{T}_{pp}
\end{bmatrix}
\]  

(3.16)

\[
\hat{\Pi} = \begin{bmatrix}
\hat{\pi}_{11} \mathbf{I}_1 & \hat{\pi}_{12} \mathbf{I}_2 & \cdots & \hat{\pi}_{1p} \mathbf{I}_p \\
\hat{\pi}_{21} \mathbf{I}_1 & \hat{\pi}_{22} \mathbf{I}_2 & \cdots & \hat{\pi}_{2p} \mathbf{I}_p \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\pi}_{pp} \mathbf{I}_1 & \hat{\pi}_{pp} \mathbf{I}_2 & \cdots & \hat{\pi}_{pp} \mathbf{I}_p
\end{bmatrix}
\]  

Hence, (3.14) reduces to

\[
\hat{\Phi}^*_{ij} \mathbf{T}_{ij} = \hat{\pi}_{ij} \mathbf{I}_j \quad (i, j = 1, \ldots, p)
\]  

(3.17)

As it can be shown, the solution of (3.17) with regard to \( \mathbf{T}_{ij} \) has the form

\[
\mathbf{T}_{ij} = \begin{bmatrix}
\rho_{ij}^T \\
\rho_{ij}^T \mathbf{I}_j \\
\vdots \\
\rho_{ij}^T \mathbf{I}_{j}^{\delta_j-1}
\end{bmatrix}
\]  

(3.18)

where \( \rho_{ij}^T \) is a \( \delta_i \)-dimensional row vector with arbitrary elements for all \( i, j = 1, 2, \ldots, p \). The general form of \( \mathbf{T} \) will be

\[
\mathbf{T} = \begin{bmatrix}
(\rho_{11}^T \mathbf{I}_1)^{k=0, \ldots, \delta_1-1} & (\rho_{12}^T \mathbf{I}_2)^{k=0, \ldots, \delta_1-1} & \cdots & (\rho_{1p}^T \mathbf{I}_p)^{k=0, \ldots, \delta_1-1} \\
(\rho_{21}^T \mathbf{I}_1)^{k=0, \ldots, \delta_2-1} & (\rho_{22}^T \mathbf{I}_2)^{k=0, \ldots, \delta_2-1} & \cdots & (\rho_{2p}^T \mathbf{I}_p)^{k=0, \ldots, \delta_2-1} \\
\vdots & \vdots & \ddots & \vdots \\
(\rho_{pp}^T \mathbf{I}_1)^{k=0, \ldots, \delta_p-1} & (\rho_{pp}^T \mathbf{I}_2)^{k=0, \ldots, \delta_p-1} & \cdots & (\rho_{pp}^T \mathbf{I}_p)^{k=0, \ldots, \delta_p-1}
\end{bmatrix}
\]  

(3.19)
In (3.19), all elements of the first row of each block of $T$ are arbitrary and hence we have a total number of arbitrary elements in $T$ equal to $n \times p$. Note also that this arbitrariness is constrained by the requirement that $T$ must be invertible, i.e., $\det T \neq 0$.

To find $F^*$, observe that relations (3.12), (3.18), and (3.19) yield

$$
\tilde{T} = \begin{bmatrix}
\rho_{11}^T \Pi_{11}^\delta_i^{-1} & \rho_{12}^T \Pi_{12}^\delta_i^{-1} & \cdots & \rho_{1p}^T \Pi_{1p}^\delta_i^{-1} \\
\rho_{21}^T \Pi_{21}^\delta_i^{-1} & \rho_{22}^T \Pi_{22}^\delta_i^{-1} & \cdots & \rho_{2p}^T \Pi_{2p}^\delta_i^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{p1}^T \Pi_{p1}^\delta_i^{-1} & \rho_{p2}^T \Pi_{p2}^\delta_i^{-1} & \cdots & \rho_{pp}^T \Pi_{pp}^\delta_i^{-1}
\end{bmatrix}
$$

(3.20)

On the basis of (3.20), relation (3.15) yields

$$
F^* = -\tilde{\Phi}^* + R \Pi^* T^{-1}
$$

(3.21)

$$
R = \text{blockdiag}\{\rho_1^T, \ldots, \rho_p^T\}, \quad \Pi^* = \begin{bmatrix}
\Pi_{1i}^\delta_i \\
\vdots \\
\Pi_{pi}^\delta_i
\end{bmatrix}
$$

(3.22)

with $\rho_i^T = [\rho_{i1}^T \cdots \rho_{ip}^T]$. Note that, when $\Pi_i$ is in Jordan form, $T_{ij}$ can take the form

$$
T_{ij} = \begin{bmatrix}
(\rho_{ij}^T)_{11} & (\rho_{ij}^T)_{12} & \cdots & (\rho_{ij}^T)_{1s_i} \\
(\rho_{ij}^T)_{21} & (\rho_{ij}^T)_{22} & \cdots & (\rho_{ij}^T)_{2s_i} \\
\vdots & \vdots & \ddots & \vdots \\
(\rho_{ij}^T)_{s_i1} & (\rho_{ij}^T)_{s_i2} & \cdots & (\rho_{ij}^T)_{s_is_i}
\end{bmatrix}
$$

(3.23)

$$
\Pi_i = \text{blockdiag}\{\Pi_{1i}, \ldots, \Pi_{si}\}, \quad \rho_{ij}^T = \begin{bmatrix}
(\rho_{ij}^T)_{1} & \cdots & (\rho_{ij}^T)_{s_i}
\end{bmatrix}
$$

in which $(\rho_{ij}^T)_q$ is a row vector of dimensionality equal to that of $(\Pi_{ij})_q$. In particular if $(\Pi_{ij})_q = (\hat{\lambda})_q$, then $s_i = \delta_i$, $(\rho_{ij}^T)_q = \rho_{ijq}$,

$$
T_{ij} = \begin{bmatrix}
\rho_{i11} & \rho_{i12} & \cdots & \rho_{i1s_i} \\
\rho_{i21}(\hat{\lambda})_1 & \rho_{i22}(\hat{\lambda})_2 & \cdots & \rho_{i2s_i}(\hat{\lambda})_s_i \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{i11}(\hat{\lambda})_{\delta_i1} & \rho_{i21}(\hat{\lambda})_{\delta_i2} & \cdots & \rho_{i1s_i}(\hat{\lambda})_{\delta_is_i}
\end{bmatrix}
$$

(3.24)
It is remarked that, if we choose all the arbitrary elements $\rho_{ij}$, for $i \neq j$ equal to 0, and all elements of each $\rho_{ij}$ equal to 1, relation (3.21) can be written as

$$F^* = -\Phi^* + R_{sp} \Pi T_{sp}^{-1}$$

(3.25)

In this case, the open-loop poles contained in the subsystem determined by $\Phi_{ii}$, when closing the feedback, are shifted to the desired poles involved in the corresponding block $\Pi_i$.

In order to determine the matrix $K_f$, substitute relation (3.21) in (3.8) to yield

$$K_f = Q^T \left( \Phi^* - R_{sp} \Pi T_{sp}^{-1} \right)^T L^T$$

(3.26)

Using the matrix $K_f$ as specified by (3.26), we can readily determine the controller matrix $F(t)$, by solving (2.3). Under Assumption 2.1, on the controllability of the pair $(A, B)$, a solution of (2.3) is the following:

$$F(t) = B^T \exp[A^T (T_0 - t)] W^{-1}(A, B, T_0) K_f$$

(3.27)

where $W(A, B, T_0)$ is the controllability Grammian on $[0, T_0]$ of the pair $(A, B)$, which has the form

$$W(A, B, T_0) = \int_0^{T_0} \exp[A(T_0 - \lambda)] BB^T \exp[A^T(T_0 - \lambda)] \, d\lambda$$

Note that the controllability Grammian $W(A, B, T_0)$ is nonsingular and hence a solution of (2.3) of the form (3.27) exists if the pair $(A, B)$ is controllable.

On the basis of (3.26) and (3.27), a solution of the pole placement problem using MRIC based control is given by

$$F(t) = B^T \exp[A^T (T_0 - t)] W^{-1}(A, B, T_0) Q^T \left( \Phi^* - R_{sp} \Pi T_{sp}^{-1} \right)^T L^T$$

(3.28)

4. A SOLUTION OF THE POLE PLACEMENT PROBLEM APPROPRIATE FOR THE ADAPTIVE CASE

In order to obtain a solution of the pole placement problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify in the sequel the control strategy of Figure 1 as it is depicted in Figure 2. In particular, we focus our attention on the
special class of the time-varying $T_0$-periodic functions $F(t)$, for which every element of $F(t)$, denoted by $f_{ij}(t)$, is piecewise constant over intervals of length $T_i$, i.e.,

$$f_{ij}(t) = f_{ij, \mu}, \quad \forall t \in [\mu T_i, (\mu + 1)T_i), \quad \mu = 0, 1, \ldots, N_i - 1$$ \hspace{1cm} (4.1)

Moreover, the persistent excitation signals $v_i(t), \forall i \in J_m$, are defined as

$$v_i(t) = d_i'(t)\psi, \quad d_i'(t) = [(d_i)_0(t) \cdots (d_i)_{N_i-1}(t)]$$ \hspace{1cm} (4.2a)

Here, $d_i(t)$ is the $T_i$-periodic vector function with elements having the form

$$(d_i)_q(t) = (d_i)_{q, \mu} \quad \text{for} \quad t \in [\mu T_i, (\mu + 1)T_i),$$

$$q = 0, 1, \ldots, N_i - 1, \mu = 0, 1, \ldots, N_i - 1$$ \hspace{1cm} (4.2b)

where $(d_i)_{q, \mu}$ are constant taking the following values:

$$(d_i)_{q, \mu} = \begin{cases} 
1 & \text{for } \mu = q, \\
0 & \text{for } \mu \neq q 
\end{cases}$$ \hspace{1cm} (4.3)

and where $\psi$ is as yet unknown. It is worth noticing that the additive term $v_i(t) = d_i'(t)\psi, \forall i \in J_m$ in each one of the inputs of the continuous-time system are used only for identification purposes and as it will be shown later, they are selected so that they will not influence the pole placement problem.

We are now able to establish the following Lemma.

**Lemma 4.1.** Consider the controllable and observable system of the form (2.1), controlled by MRJC's of the form (4.1). Furthermore, consider that persistent excitation signals of the forms (4.2) and (4.3) are introduced in each input of the system. Then, the sample closed-loop system takes the form

$$x[(k + 1)T_0] = (\Phi - \hat{B}\hat{F}C)x(kT_0) + \hat{B}\hat{F}w(kT_0) + B^*v,$$

$$y(kT_0) = Cx(kT_0) \quad \text{for } k \geq 0$$ \hspace{1cm} (4.4)

where

$$\hat{B} = \left[\begin{array}{cccc}
\hat{b}_1 & \cdots & \hat{A}^{N_i-1}_{1}\hat{b}_2 & \cdots & \hat{A}^{N_i-1}_{m}\hat{b}_m
\end{array}\right]$$ \hspace{1cm} (4.5a)

$$\hat{A}_i = \exp(A T_i) = \exp(A t_i T),$$

$$\hat{b}_i = \int_0^{T_i}\exp(A \lambda) b_i \ d\lambda = \int_0^{T_i}\exp(A \lambda) b_i \ d\lambda$$ \hspace{1cm} (4.5b)
and where the \( m \times p \) block matrix \( \hat{F} \) and the column vector \( v \in \mathbb{R}^N \) have the forms

\[
\hat{F} = \begin{bmatrix}
\hat{f}_{11} & \cdots & \hat{f}_{1p} \\
\vdots & \ddots & \vdots \\
\hat{f}_{m1} & \cdots & \hat{f}_{mp}
\end{bmatrix}, \quad \hat{f}_{ij} = \begin{bmatrix} f_{ij,N-1} \\ \vdots \\ f_{ij,0} \end{bmatrix}
\]

\[
v = \begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix}
\]

while \( \alpha_j = \sum_{k=1}^{N-1} \alpha_k \), where in general, the vector \( e_j \in \mathbb{R}^N \) is the row vector whose elements are zeros except for a unity appearing in the \( i \)th position.

**Proof.** To show that the sampled closed-loop system takes the form (4.4), we start by discretizing system (2.1) with sampling period \( T_0 \). This operation yields

\[
x[(k+1)T_0] = \Phi x(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp(A[(k+1)T_0 - \lambda])Bu(t) \, d\lambda
\]

\[
(4.7)
\]

Observing that \( u_i(t) = r(t) + d_i^T(t)v \) and taking into account the structure of the control system in Figure 2, we obtain

\[
u_i(t) = f_i^T(t)e(kT_0) + d_i^T(t)v \quad \text{for } t \in [\mu T_i, (\mu + 1)T_i)
\]

\[
(4.8a)
\]

where \( f_i^T(t) \) is the \( i \)th row of the controller matrix \( F(t) \) and \( e(kT_0) \) is given by

\[
e(kT_0) = w(kT_0) - y(kT_0) = w(kT_0) - Cx(kT_0)
\]

\[
(4.8b)
\]

Combining relations (4.7) and (4.8), we obtain the following relationship:

\[
x[(k+1)T_0] = (\Phi - K_jC)x(kT_0) + K_j w(kT_0) + \Gamma v
\]

\[
(4.9)
\]
where

\[
\Gamma = \int_{kT_0}^{(k+1)T_0} \exp\left\{A\left[\left(\begin{array}{c} T_0 \\ \lambda \end{array}\right)\right]\right\} B D(\lambda) \, d\lambda,
\]

\[D(t) = \text{blockdiag}\left\{d_i^T(t)\right\}_{i \in J_m}\]

Now, partition \(\Gamma\) as follows:

\[
\Gamma = [\Gamma_1 \quad \Gamma_2 \quad \cdots \quad \Gamma_m]^	op
\]

Then, the \((q+1)\)th column of the matrix \(\Gamma_i\), for \(i \in J_m\), denoted by \((\Gamma_i)_{q+1}\) for \(q = 0, 1, \ldots, N_i - 1\), can be expressed as

\[
(\Gamma_i)_{q+1} = \int_{0}^{T_0} \exp\left\{A(T_0 - \lambda)\right\} b_i(d_i)_{q}(\lambda) \, d\lambda \quad \text{for} \quad q = 0, 1, \ldots, N_i - 1
\]

(4.10)

Introducing relations (4.2a) and (4.3) in (4.10), we obtain

\[
(\Gamma_i)_{q+1} = \sum_{\mu=0}^{N_i-1} \int_{0}^{T_i} \exp\left\{A(T_i - \mu)\right\} b_i(d_i)_{q,\mu} \, d\lambda
\]

for \(q = 0, 1, \ldots, N_i - 1 \quad (4.11)

Relation (4.11) may further be written as

\[
(\Gamma_i)_{q+1} = \sum_{\mu=0}^{N_i-1} (d_i)_{q,\mu} \exp\left\{A(N_i - \mu - 1)T_i\right\} \int_{0}^{T_i} \exp\left\{A(T_i - \mu)\right\} b_i \, d\lambda
\]

\[
= \left( \sum_{\rho=1}^{N_i} (d_i)_{q,N_i-\rho} \hat{A}^{\rho-1}\right) \hat{b}_i
\]

Making use of relation (4.3), we arrive at the following relationship:

\[
(\Gamma_i)_{q+1} = \hat{A}_{N_i-q}^{-1} \hat{b}_i
\]

Clearly \(\Gamma = B^*\). Application of the above algorithm to the first term of (4.9) yields \(K_f = \hat{B} \hat{F}\) (see [5] for details). This completes the proof of the Lemma.

Thus far, we have established that the pole placement controller matrix \(K_f\) is related to the matrix \(\hat{F}\) via the relation \(K_f = \hat{B} \hat{F}\). It remains to
To determine $\hat{T}$, we need the following result, whose proof is given in [5].

**Lemma 4.2.** Let $(A, B)$ be a controllable pair. Let also $n_i, i \in J_m$ be a set of locally minimum controllability indices of the pair $(A, B)$. Define an analytic function $\psi(T_N)$ by

$$\psi(T_N) = \det \left[ \hat{b}_1 \quad \cdots \quad \hat{A}^{n_1-1}_{1} \hat{b}_1 \quad \cdots \quad \hat{b}_m \quad \cdots \quad \hat{A}^{n_m-1}_{m} \hat{b}_m \right]$$

Then the set of zeros of $\psi(T_N)$ does not have any limiting points except infinity, and therefore, $\psi(T_N)$ is not equal to zero for almost all $T_N$ (i.e., in a finite interval $[T^1_N, T^2_N]$), there are at most a finite number of points such that $\psi(T_N) = 0$.

Applying Lemma 4.2, we can conclude that the matrix $\hat{S}$ of the form

$$\hat{S} = \begin{bmatrix} \hat{b}_1 & \cdots & \hat{A}^{n_1-1}_{1} \hat{b}_1 & \cdots & \hat{b}_m & \cdots & \hat{A}^{n_m-1}_{m} \hat{b}_m \end{bmatrix} \quad (4.12)$$

is nonsingular for almost all $T_N \in [T^1_N, T^2_N]$. Furthermore, if the input multiplicities of the sampling $N_i$ are chosen such that $N_i \geq n_i, i \in J_m$, then, the matrices $B$ and $B^*$ have full row rank $n$ for almost all $T_N \in [T^1_N, T^2_N]$.

Now, let $E \in \mathbb{R}^{N \times N}$ be the nonsingular permutation matrix with the property $E^{-1} = E^T$, having the form

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}^T$$

where

$$E_1 = [e_1 \quad e_2 \quad \cdots \quad e_{n_1} \quad e_{N_1+1} \quad e_{N_1+2} \quad \cdots \quad e_{N_1+n_2} \quad \cdots \quad e_{N^* - N_m + n_1} \quad e_{N^* - N_m + n_2} \quad \cdots \quad e_{N^* - N_m + n_m}]$$

and

$$E_2 = [e_{n_1+1} \quad \cdots \quad e_{N_1} \quad e_{N_1+n_2+1} \quad \cdots \quad e_{N_1+N_2} \quad \cdots \quad e_{N^* - N_m + n_1+1} \quad \cdots \quad e_{N^*}]$$

where, in general, $e_j \in \mathbb{R}^{N*}$ is the column vector whose elements are zeros except for a unity appearing in the $j$th position. Also, let

$$B^* = \hat{B} E^{-1} = \begin{bmatrix} \hat{S} & \hat{Q} \end{bmatrix}$$
where the matrix $\hat{S}$ is defined by (4.13) and the matrix $\hat{Q}$ is given by

$$
\hat{Q} = \begin{bmatrix}
\hat{A}_1^{-1}b_1 & \cdots & \hat{A}_1^{N_1-1}b_1 & \cdots & \hat{A}_m^{-1}b_m & \cdots & \hat{A}_m^{N_m-1}b_m
\end{bmatrix}
$$

Furthermore, let $\Delta \in \mathbb{R}^{N^* \times N^*}$ be the nonsingular permutation matrix with the property $\Delta^{-1} = \Delta^T$, having the form

$$
\Delta = \begin{bmatrix}
\Delta_1 & \Delta_2 & \Delta_3
\end{bmatrix}^T
$$

where

$$
\Delta_1 = \begin{bmatrix}
E_{N_1-n_1+1} & \cdots & E_{N_1} & E_{N_1+N_2-n_2+1} & \cdots & E_{N_1+N_2}
\end{bmatrix} \\
\times E_{N^*-n_m+1} & \cdots & E_{N^*}
$$

$$
\Delta_2 = \begin{bmatrix}
E_{N_1-n_1} & E_{N_1+N_2-n_2} & \cdots & E_{N^*-n_m}
\Delta_3 = \begin{bmatrix}
E_1 & \cdots & E_{N_1-n_1-1} & E_{N_1+1} & \cdots & E_{N_1+N_2-n_2-1} & \cdots \\
\times E_{N^*-n_m+1} & \cdots & E_{N^*-n_m-1}
\end{bmatrix}
$$

Finally, let

$$
\tilde{B}^* = B^*\Delta^{-1} = \begin{bmatrix}
\hat{S}^* & \hat{A}_1^{-1}b_1 & \cdots & \hat{A}_m^{-1}b_m & \hat{Q}^*
\end{bmatrix}
$$

where

$$
\hat{S}^* = \begin{bmatrix}
\hat{A}_1^{-1}b_1 & \cdots & \hat{b}_1 & \cdots & \hat{A}_m^{-1}b_m & \cdots & \hat{b}_m
\end{bmatrix} (4.13)
$$

$$
\hat{Q}^* = \begin{bmatrix}
\hat{A}_1^{N_1-1}b_1 & \cdots & \hat{A}_1^{N_1+1}b_1 & \cdots & \hat{A}_m^{N_m-1}b_m & \cdots & \hat{A}_m^{N_m+1}b_m
\end{bmatrix}
$$

Using these definitions, it is plausible to determine $\hat{F}$ by mere inspection, as

$$
\hat{F} = E^T \begin{bmatrix}
\hat{S}^{-1}Q^T(\tilde{B}^* - R_{sp}^*\Pi_{sp}^{-1})^TL^T & 0
\end{bmatrix} (4.14)
$$

It only remains to determine the appropriate vector $v$ which guarantees that the pole placement problem will not be dependent on the vector $v$. In other words

$$
v \in \ker B^* \quad \text{or} \quad B^*v = 0$$
An obvious selection of such \( v \) obtained also by inspection is the following:

\[
v = \Delta^T \begin{bmatrix}
-\hat{\mathbf{S}}^{n-1}(\hat{\mathbf{A}}_{n}^{\mathbf{1}} \hat{\mathbf{b}}_{1} + \cdots \hat{\mathbf{A}}_{m}^{\mathbf{m}} \hat{\mathbf{b}}_{m}) \\
\zeta \\
\mathbf{0}_{N^{*}-n-m}
\end{bmatrix}
\]  

(4.15)

where \( \zeta \in \mathbb{R}^{m} \) is the column vector whose elements are all equal to 1.

It is noted that the \( N^{*} \)-dimensional column vector \( v \), even though does not affects the discrete pole placement problem, it provides persistent excitation useful for the consistent identification of the system, as will be shown in the following section.

Clearly, the multirate controller matrix \( \mathbf{F}(t) \) of Figure 2 can readily be determined by making use of relations (4.1), (4.6a), (4.14), and (3.28). More precisely, the \( i \)th row \( f_{i}^{T}(t) \) of the matrix \( \mathbf{F}(t) \) and the \( i \)th block row of the matrix \( \hat{\mathbf{F}} \) are interrelated as

\[
f_{i}^{T}(t) = \begin{bmatrix}
f_{i1}(t) & \cdots & f_{ip}(t)
\end{bmatrix} = \mathbf{e}_{N_{i}-\mu}^{T} \begin{bmatrix}
\hat{f}_{i1} & \cdots & \hat{f}_{ip}
\end{bmatrix},
\]

\[
\forall \frac{\mu T_{0}}{N_{i}} \leq t < \frac{(\mu + 1) T_{0}}{N_{i}}
\]

(4.16)

for \( i \in \mathbf{J}_{m} \) and for \( \mu = 0, 1, \ldots, N_{i} - 1 \), where \( \mathbf{e}_{N_{i}-\mu} \in \mathbb{R}^{N_{i}} \) is the row vector defined as \( \mathbf{e}_{N_{i}-\mu} = e_{N_{i}-\mu}^{T} \). Note that the controller matrix \( \mathbf{F}(t) \), as specified by (4.16), is largely affected by the multirate mechanism, while the controller matrix \( \mathbf{F}(t) \) as specified by relation (3.28) is not. Furthermore, the introduction of the excitation signals \( v_{i}(t) \) in the control loop greatly facilitates the consistent estimation of the plant parameters in the case of unknown systems. For these reasons, the control strategy of Figure 2 is more appropriate than the control strategy of Figure 1 for the development of the indirect adaptive control scheme presented in the following section.

5. CONTROL STRATEGY FOR THE ADAPTIVE CASE

The control scheme presented in Section 4 has a corresponding scheme in the case where the system is unknown. For this case, the control strategy is largely based on the computation of the matrix \( \hat{\mathbf{F}} \) and of the vector \( \mathbf{v} \) from estimates of the plant parameters, and results in a globally stable closed-loop system whose poles are located to the prespecified values \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{n} \).
5.1. Plant Parameters Estimation Algorithm

The algorithm proposed here for estimating the unknown plant parameters is as follows: System (2.1), discretized with sampling period $\tau = T_0/(6n - 1)N$, takes the form

$$x[(\nu + 1)\tau] = \Phi_{\tau}x(\nu\tau) + B_u(\nu\tau), \quad y(\nu\tau) = Cx(\nu\tau), \quad \nu \geq 0 \tag{5.1}$$

where

$$\Phi_{\tau} = \exp(A\tau), \quad B_u = \int_0^T \exp(A\lambda)B d\lambda$$

Clearly, $u(\nu\tau)$ takes constant values for $\nu \in [\rho T_N, (\rho + 1)T_N)$, $\rho \geq 0$. This can be easily shown by taking into account the action of the proposed controller. Hence, iterating relation (5.1) $6n - 1$ times, we obtain

$$x[(m + 1)T_N] = \Phi_{T_N}x(mT_N) + B_{T_N}u(mT_N), \quad m \geq 0$$

where

$$\Phi_{T_N} = \Phi_\tau^N, \quad B_{T_N} = \sum_{\rho=0}^{6n-2} \Phi_\tau^\rho B_u$$

Using the same argument, we can easily conclude that

$$\hat{A}_i = \Phi_{T_N}^i, \quad \hat{b}_i = \sum_{\rho=0}^{l-1} \Phi_\tau^\rho (B_{T_N})_i$$

where $(B_{T_N})_i$ is the $i$th column of the matrix $B_{T_N}$. Introducing relation (5.2) in (5.3), yields

$$\hat{A}_i = (\Phi_\tau)^{(6n-1)i}, \quad \hat{b}_i = \sum_{j=0}^{l-1} (\Phi_\tau)^{(6n-1)j} \left( \sum_{\rho=0}^{6n-2} \Phi_\tau^\rho B_u \right)_i$$

Moreover, the matrix $F$ can be written as

$$F = \hat{A}_N^X = F_{T_N}^X = (\Phi_\tau)^{6n-1/N} \tag{5.5}$$

Therefore, $F$, $\hat{A}_i$, and $\hat{b}_i$ (which are the only matrices involved in computing $F$ and $v$) can be computed on the basis of $\Phi_{\tau}$ and $B_u$. For this reason, in what follows our aim will be the estimation of the matrix triplet
(\(\Phi_r, B_r, C\)). To this end, let the matrix \(\Omega\) be defined as

\[
\Omega = \{\Omega_{ij}\}_{i=1,2,\ldots,n, j} \quad \Omega_{ij} = C \Phi_{i+j}^j B_r
\]

(5.6)

Clearly, if one establishes estimates of the matrix \(\Omega\), then one may easily compute the desired matrix triplet \((\Phi_r, B_r, C)\), using anyone of the minimal realization algorithms reported in the literature (see, e.g., those reported in [32]–[34]). To estimate the matrix \(\Omega\), one must resort to an input–output representation (also called ARMA representation) of system (5.1). This representation is summarized in the following theorem:

**Theorem 5.1.** Suppose that there is a sampling period \(T_0 \in \mathbb{R}^+\) and input multiplicities of the sampling \(N_i, i \in J_m\), such that system (5.1), obtained by sampling the controllable and observable system (2.1), is also controllable and observable. Then, an alternative representation of system (5.1) is given by

\[
\Psi(\nu \tau) = J_1 \Psi((\nu - 2n) \tau) + J_3 W(\nu \tau) + V W((\nu - n) \tau)
\]

\[
+ V^* W((\nu - 2n) \tau)
\]

(5.7)

where

\[
\Psi(\nu \tau) = \begin{bmatrix}
\Psi((\nu - n + 1) \tau) \\
\Psi((\nu - n + 2) \tau) \\
\vdots \\
\Psi((\nu \tau))
\end{bmatrix}
\]

\[
\Psi((\nu - 2n) \tau) = \begin{bmatrix}
\Psi((\nu - 3n + 1) \tau) \\
\Psi((\nu - 3n + 2) \tau) \\
\vdots \\
\Psi((\nu - 2n) \tau)
\end{bmatrix}
\]

\[
W(\nu \tau) = \begin{bmatrix}
W((\nu - n + 1) \tau) \\
W((\nu - n + 2) \tau) \\
\vdots \\
W((\nu \tau))
\end{bmatrix}
\]

(5.8a)
\[ W[(\nu-n)\tau] = \begin{bmatrix} u[\nu-(2n+1)\tau] \\ u[\nu-(2n+2)\tau] \\ \vdots \\ u[(\nu-n)\tau] \end{bmatrix}, \]
\[ W[(\nu-2n)\tau] = \begin{bmatrix} u[\nu-(3n+1)\tau] \\ u[\nu-(3n+2)\tau] \\ \vdots \\ u[(\nu-2n)\tau] \end{bmatrix} \]  
(5.8b)

\[ J_1 = \Xi^{-1}_{n} \begin{bmatrix} \hat{j} & 0 \\ 0 & 0 \end{bmatrix} \Xi^*, \quad J_2 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \text{CB}_r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \text{CF}_{\nu-2} & \cdots & \text{CB}_r & 0 \end{bmatrix}, \]

(5.8c)

\[ V = P^*\Sigma \quad \text{and} \quad V^* = \Xi^{-1}_{n} [V^+ \ 0] \]

and where

\[ \hat{j} = P_1^* \Phi_{\nu}^{2n} P_1^*^{-1}, \quad P^* = \begin{bmatrix} C \\ C \Phi_{\nu} \\ \vdots \\ C \Phi_{\nu}^{n-1} \end{bmatrix}, \]

(5.9a)

\[ \Sigma = \begin{bmatrix} \Phi_{\nu}^{n-1} B_r & \cdots & \Phi_{\nu} B_r & B_r \end{bmatrix} \]

(5.9b)

while the nonsingular permutation matrix \( \Xi^* \in \mathbb{R}^{n \times np} \) is such that

\[ \Xi^*P^* = \begin{bmatrix} P^*_1 \\ 0 \end{bmatrix} \]

(5.10)

where \( P^*_1 \in \mathbb{R}^{n \times n} \) is the nonsingular matrix whose rows are the linearly independent rows of the matrix \( P^* \). Finally, \( U_1 \in \mathbb{R}^{n \times np} \) is the matrix containing the first \( n \) rows of the matrix

\[ U = \Xi^*J_2 \]

(5.11)

Proof. In order to prove relation (5.7), we next generalize the approach presented in [35], to the multivariable case. More precisely, from relations
(5.1) we have
\[
\begin{align*}
\mathbf{y}[\mathbf{v} - n + 1)\tau] &= \mathbf{C}\mathbf{x}[\mathbf{v} - n + 1)\tau] \\
\mathbf{y}[\mathbf{v} - n + 2)\tau] &= \mathbf{C}\Phi_1\mathbf{x}[\mathbf{v} - n + 1)\tau] + \mathbf{C}\mathbf{B}_1\mathbf{u}[\mathbf{v} - n + 1)\tau] \\
& \quad \vdots \\
\mathbf{y}(\mathbf{v}\tau) &= \mathbf{C}\Phi^{n-1}_1\mathbf{x}[\mathbf{v} - n + 1)\tau] \\
& \quad + \sum_{\rho=0}^{n-2} \mathbf{C}\Phi^{\rho}_1\mathbf{B}_\rho\mathbf{u}[\mathbf{v} - \rho - 1)\tau] \\
\end{align*}
\]
or more compactly,
\[
\Psi(\mathbf{v}\tau) = \mathbf{P}^*\mathbf{x}[\mathbf{v} - n + 1)\tau] + \mathbf{J}_2\mathbf{W}(\mathbf{v}\tau) \quad (5.12)
\]
where \(\Psi(\mathbf{v}\tau)\) and \(\mathbf{W}(\mathbf{v}\tau)\) are defined by (5.8a) and \(\mathbf{P}^*\) and \(\mathbf{J}_2\) are defined by (5.9a) and (5.8c), respectively.

Since, by Assumption 2.2, the pair \((\Phi_1, \mathbf{C})\) is observable, the matrix \(\mathbf{P}^*\) has full column rank. Hence, there exists a nonsingular permutation matrix \(\Xi^* \in \mathbb{R}^{n_p \times n_p}\), such that relation (5.10) holds, where, as already mentioned, \(\mathbf{P}^*_t \in \mathbb{R}^{n \times n}\) is the nonsingular matrix whose rows are the linearly independent rows of the matrix \(\mathbf{P}^*\). It is pointed out that matrix \(\Xi^*\) can be defined as a product of two nonsingular matrices \(\Xi \in \mathbb{R}^{n_p \times n_p}\) and \(\Xi \in \mathbb{R}^{n_p \times n_p}\) via the following chain of definitions:

\[
\Xi^* = \tilde{\Xi}^*\tilde{\Xi}, \quad \tilde{\Xi}^* = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_{n_p} \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n_p - n} \end{bmatrix}, \quad \tilde{\Xi} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_{n_p} \\ \Xi_1 \end{bmatrix}
\]

where \(\Xi^*_t \in \mathbb{R}^{(n_p - n) \times n_p}\) is the matrix produced by the nonsingular matrix \(\Xi^*_t \in \mathbb{R}^{n_p \times n_p}\) of the form

\[
\Xi^*_t = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_{n_p} \end{bmatrix}
\]
by dropping the row vectors \( \mathbf{e}_i, i = j_1, j_2, \ldots, j_n \), where \( j_1, j_2, \ldots, j_n \) are the indices of the \( n \) linearly independent rows of \( \mathbf{P}^* \) defined as \( \mathbf{p}_r^{T} \), \( r = 1, 2, \ldots, n \). Note also that \( \omega_k \in \mathbb{R}^{np}, k = 1, 2, \ldots, np - n \) is the column vector of the form

\[
\omega_k = \begin{bmatrix}
(\lambda_{j_1})_k & (\lambda_{j_2})_k & \cdots & (\lambda_{j_n})_k & 0 & \cdots & 0 & -\frac{1}{(n + k)^{th}} & 0 & \cdots & 0
\end{bmatrix}
\]

where \( (\lambda_{j_0})_k, r = 1, 2, \ldots, n, k = 1, 2, \ldots, np - n \) are the coefficients of the following dependence relation holding for the rows of the matrix \( \mathbf{P}^* \):

\[
\sum_{\rho=1}^{n} (\lambda_{j_0})_k \mathbf{p}_r^{T} - \mathbf{p}_r^{T} = 0, \quad k \notin \{j_1, j_2, \ldots, j_n\}
\]

where, \( \mathbf{p}_r^{T}, k \notin \{j_1, j_2, \ldots, j_n\} \) is the \( k \)th row of the matrix \( \mathbf{P}^* \).

Now, multiplying (5.12) from the left by \( \Xi^* \), yields

\[
\mathbf{Z}^*(\nu \tau) = \begin{bmatrix}
\mathbf{P}_1^* \\
0
\end{bmatrix} \mathbf{x}[\nu - n + 1] + \mathbf{U}W(\nu \tau)
\]

where

\[
\mathbf{Z}^*(\nu \tau) = \Xi^* \Psi(\nu \tau)
\]

and where \( \mathbf{U} \) is defined by (5.11). Next, decompose \( \mathbf{Z}^*(\nu \tau) \) and \( \mathbf{U} \) as follows:

\[
\mathbf{Z}^*(\nu \tau) = \begin{bmatrix}
\mathbf{Z}_1^*(\nu \tau) \\
\mathbf{Z}_2^*(\nu \tau)
\end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix}
\mathbf{U}_1 \\
\mathbf{U}_2
\end{bmatrix}
\]

where \( \mathbf{Z}_1^*(\nu \tau) \in \mathbb{R}^n, \mathbf{Z}_2^*(\nu \tau) \in \mathbb{R}^{n(p-1)}, \mathbf{U}_1 \in \mathbb{R}^{n \times np} \) and \( \mathbf{U}_2 \in \mathbb{R}^{n(p-1) \times np} \). Clearly,

\[
\mathbf{Z}_1^*(\nu \tau) = \mathbf{P}_1^{*} \mathbf{e}[(\nu - n + 1)\tau] + \mathbf{U}_1 \mathbf{W}(\nu \tau) \quad \text{and} \quad \mathbf{Z}_2^*(\nu \tau) = \mathbf{U}_2 \mathbf{W}(\nu \tau)
\]

(5.15)

From (5.15), one may easily obtain the following relation:

\[
\mathbf{x}[(\nu - n + 1)] = \mathbf{P}_1^{* -1} [\mathbf{Z}_1^*(\nu \tau) - \mathbf{U}_1 \mathbf{W}(\nu \tau)]
\]

(5.16)

Furthermore, as it can be easily shown, the following relationship holds:

\[
\mathbf{x}[(\nu - n + 1)\tau] = \Phi_{2n}^{\nu} \mathbf{x}[(\nu - 3n + 1)\tau] + \Phi_{2n}^{\nu} \Sigma \mathbf{W}[(\nu - 2n)\tau] + \Sigma \mathbf{W}[(\nu - n)\tau]
\]

(5.17)
where $W[(\nu - n)\tau]$ and $W[(\nu - 2n)\tau]$ are given by (5.8b), and where $\Sigma$ is defined by (5.9a). Introducing appropriately relation (5.16) in relation (5.17), after some algebraic manipulations, yields

$$Z^*_1(\nu \tau) = U_1 W(\nu \tau) + \hat{j} Z^*_1[(\nu - 2n)\tau] + V^* W[(\nu - n)\tau]$$

where $\hat{j}$ and $V^*$ are defined as by (5.9a) and (5.9b), respectively. Combining relations (5.11), (5.13)–(5.15), and (5.18), we readily obtain (5.7). This completes the proof of the Theorem.

It is remarked at this point that matrix $V$ and matrix $\Omega$ are related through the following relationship:

$$\Omega = V \Lambda, \quad \Lambda = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

Relation (5.7) will be used in the sequel for the identification of the unknown matrices $J_1$, $J_2$, $V$, and $V^*$. To this end, relation (5.7) is next written in the linear regression form

$$\Psi(\nu \tau) = \Theta \phi(\nu \tau)$$

where

$$\Theta = \begin{bmatrix} J_1 & J_2 & V & V^* \end{bmatrix}$$

is the true value of the plant parameter matrix, and where

$$\phi^T(\nu \tau) = \begin{bmatrix} \Psi^T[(\nu - 2n)\tau] & W^T(\nu \tau) & W^T[(\nu - n)\tau] & W^T[(\nu - 2n)\tau] \end{bmatrix}$$

Next, define

$$Z(kT_0) = \begin{bmatrix} \phi(kT_0) & \phi(kT_0 - \tau) & \cdots & \phi[(k - 1)T_0] \end{bmatrix}$$

$$Y(kT_0) = \begin{bmatrix} \Psi(kT_0) & \Psi(kT_0 - \tau) & \cdots & \Psi[(k - 1)T_0] \end{bmatrix}$$

$$\hat{\Theta}(kT_0) = \begin{bmatrix} J_1(kT_0) & J_2(kT_0) & V(kT_0) & V^*(kT_0) \end{bmatrix}$$

where $J_1(kT_0)$, $J_2(kT_0)$, $V(kT_0)$, and $V^*(kT_0)$ are the matrices $J_1$, $J_2$, $V$, and $V^*$ evaluated at $kT_0$, through the identification procedure. Clearly, the
following relation holds:

\[ Y(kT_0) = \Theta Z(kT_0) \]

We now choose the recursive algorithm for the estimation of \( \hat{\Theta}(kT_0) \) as

\[
\hat{\Theta}(kT_0) = \hat{\Theta}[(k - 1)T_0] \\
- \left[ \hat{\Theta}[(k - 1)T_0]Z[(k - 1)T_0] - Y[(k - 1)T_0] \right] \\
\times Z^T[(k - 1)T_0] \left[ \alpha I + Z[(k - 1)T_0]Z^T[(k - 1)T_0] \right]^{-1}
\]

(5.20)

where \( \alpha \in \mathbb{R}^+ \) is arbitrary, \( \hat{\Theta}(kT_0) \) is the estimated parameter matrix \( \hat{\Theta} \) at time \( t = kT_0 \), and \( \hat{\Theta}_0 = \hat{\Theta}(kT_0)_{k=0} \) is arbitrarily specified. It is pointed out that the term \( \alpha I \) in (5.20) is added in order to avoid numerical ill conditioning, arising in the identification procedure based on the usual least-squares algorithm, when the determinant of the matrix \( Z[(k - 1)T_0]Z^T[(k - 1)T_0] \) takes small values.

Commenting on the nature of the adaptive law (5.20), we point out that, it describes an on-line estimation procedure which deals with sequential data in which the parameter estimates are recursively updated within the time limit imposed by the sampling period \( T_0 \). It is worth noticed, at this point that, in the present case, it is presumed that, a complete block of information needed for the estimation of the plant parameters, is not available prior to analysis and control, as in several off-line estimation procedures. Therefore, in our case, identification and control of the plant are performed concurrently. In order to calculate the parameters of the desired MRIC based pole placement controller, it is necessary here to update the plant parameter estimates using (5.20) and then solve the canonical equations of Sections 3 and 4 for every time step \( k \) (see the following subsection for details). This is in contrast, to the standard policy followed in cases where identification and control of the plant are performed separately, in which we solve equations for the plant and the controller parameters once, after an appropriate minimum number of observations on the basis of which, a fixed model for the controlled plant is available for further analysis (see [36, 37] for a comparative study of the two approaches).

It is worth noticing at this point that although exact solutions to the equation schemes of the paper are possible, the convergence of the identification procedure is crucial for our analysis. This is due to the fact that the adaptive law (5.20) is chosen so that \( \hat{\Theta}(kT_0) \) will satisfy the equation \( Y(kT_0) = \Theta Z(kT_0) \) \( (k \geq 0) \) asymptotically with time, i.e., for
$k \to \infty$, rather than at every time instant. In other words, in the early stages of the on-line identification procedure, the estimated parameter matrix $\hat{\Theta}(kT_0)$, obtained by (5.20), is usually far from its true value $\Theta$ and it is expected that the plant parameter estimates (and consequently the controller parameter estimates) converge to their true values, only as $k \to \infty$. Therefore, exact determination of the desired MRIC based pole placement controller through the procedures presented in Sections 3 and 4, is expected here, only after a certain step of the overall control procedure. Before this step, the calculated controllers are far from being those which guarantee the desired performance of the closed-loop system. However, it is a standard fact in all adaptive control schemes that convergence of the parameter estimates to their true values depends on the specific properties of the particular identification procedure used and crucially affects the adaptation since in cases where convergence of the estimated parameters to their true values is not guaranteed, either the calculated controllers are not the admissible ones or they cannot be computed (for instance, if $\Theta(kT_0)$, as obtained by the identification, is unbounded). So the effectiveness of our method depends on the convergence and the boundedness properties of the proposed identification procedure. These properties are summarized in the following Proposition.

**Proposition 5.1.** Let $\hat{\Theta}(kT_0)$ be the parameter estimation error, defined as

$$\hat{\Theta}(kT_0) = \hat{\Theta}^T(kT_0) - \Theta^T$$  \hspace{1cm} (5.21)

Then, for the parameter estimation algorithm of the form (5.20), the following properties hold:

(a) $||\hat{\Theta}(kT_0)|| \leq \mu$ for some finite $\mu \in \mathbb{R}^+$

(b) If $\lim_{k \to \infty} \sum_{\rho=0}^{k} \lambda_{\min}(Z(\rho T_0)Z^T(\rho T_0)) = \infty$ then $\lim_{k \to \infty} \hat{\Theta}(kT_0) = 0$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix.

**Proof.** (a) Taking the transpose of both sides in (5.20), introducing (5.21) in the resulting relation and taking into account the fact that $Z^T(kT_0)\Theta^T - Y^T(kT_0) = 0$, we readily obtain

$$\hat{\Theta}(kT_0) = \left(1 - \left[\alpha I + Z[(k - 1)T_0]Z^T[(k - 1)T_0]\right]^{-1}\right)Z[(k - 1)T_0]$$

$$\times Z[(k - 1)T_0]Z^T[(k - 1)T_0] \hat{\Theta}[(k - 1)T_0]$$ \hspace{1cm} (5.22)
On the basis of the matrix inversion lemma, relation (5.22) may further be written as

\[ \hat{\Theta}(kT_0) = \left(1 + \frac{1}{\alpha}Z[(k-1)T_0]Z^T[(k-1)T_0]\right)^{-1} \hat{\Theta}[(k-1)T_0] \]  

(5.23)

Therefore,

\[ \hat{\Theta}^T(kT_0)\hat{\Theta}(kT_0) = \hat{\Theta}^T[(k-1)T_0] \]

\[ \times \left(1 + \frac{1}{\alpha}Z[(k-1)T_0]Z^T[(k-1)T_0]\right)^{-2} \]

\[ \times \hat{\Theta}[(k-1)T_0] \]

\[ \leq \left(1 + \frac{\lambda_{\min}(Z[(k-1)T_0]Z^T[(k-1)T_0])}{\alpha}\right)^{-2} \]

\[ \times \hat{\Theta}^T[(k-1)T_0]\hat{\Theta}[(k-1)T_0] \]  

(5.24)

By repeatedly using the above inequality, we obtain

\[ \hat{\Theta}^T(kT_0)\tilde{\Theta}(kT_0) \leq \left[\prod_{\rho=0}^{k-1} \left(1 + \frac{\lambda_{\min}(Z(\rho T_0)Z^T(\rho T_0))}{\alpha}\right)\right]^{-2} \tilde{\Theta}^T_0\tilde{\Theta}_0 \]

\[ \leq \left[1 + \frac{1}{\alpha} \sum_{\rho=0}^{k-1} \lambda_{\min}(Z(\rho T_0)Z^T(\rho T_0))\right]^{-2} \tilde{\Theta}^T_0\tilde{\Theta}_0 \]  

(5.25)

where \( \tilde{\Theta}_0 = \hat{\Theta}_0 - \Theta^T \). Hence, \( \|\hat{\Theta}(kT_0)\| \) is uniformly bounded by \( \|\tilde{\Theta}_0\| \), and since \( \Theta \) is finite, \( \hat{\Theta}(kT_0) \) is also uniformly bounded by some finite \( \mu \in \mathbb{R}^m \).

(b) If \( \lim_{k \to \infty} \sum_{\rho=0}^{k} \lambda_{\min}(Z(\rho T_0)Z^T(\rho T_0)) = \infty \) then, from (5.24), it follows that \( \lim_{k \to \infty} \hat{\Theta}(kT_0) = 0 \), and therefore, \( \lim_{k \to \infty} \tilde{\Theta}(kT_0) = \Theta \).  

Clearly, Proposition 5.1 states that for the convergence of the plant parameters estimates \( \hat{\Theta}(kT_0) \) to their true values \( \Theta \) it is sufficient that the regression vector \( Z(kT_0) \) is persistently exciting to the amount that

\[ \lim_{k \to \infty} \sum_{\rho=0}^{k} \lambda_{\min}(Z(\rho T_0)Z^T(\rho T_0)) = \infty \]
Therefore, since adaptation and stability of the adaptive scheme depend on the convergence of the parameter estimates to their true values, it is necessary to prove excitation of $Z(kT)$. This is done in Subsection 5.3 (see Theorem 5.2, therein).

Remark 5.1. It is pointed out that although controllability and observability of the sampled system (5.1) is instrumental for our analysis, no assumption is made in the present paper on the canonical structure of the triplet $(\Phi, B, C)$. This is in contrast to the standard policy of many known adaptive schemes, in which controllability or observability canonical forms are assumed for the matrix triplet involved in the estimation procedure (see, for example, [38], [39]). The reason here for avoiding an assumption on the canonical structure for the triplet $(\Phi, B, C)$ is mainly due to the fact that canonical forms for multivariable systems are interwoven with the knowledge of a set of controllability or observability indices of the matrix triplet sought (for example, in [38], [39] a set of observability indices is needed to be known). As a consequence, when identification procedures based on canonical structures are used, much more prior knowledge relative to the structure of the controlled plant is necessary, as compared to our approach.

5.2. Algorithm for the Synthesis of the Adaptive Controller

On the basis of the estimated parameter matrix $\hat{\Theta}(kT)$ obtained by (5.20), as well as on the basis of the relations (5.4)–(5.6) and (5.19) and of anyone of the algorithms reported in the literature for the construction of a minimal realization, one can obtain the estimates which are necessary for the computation of the unknown matrices $A_i = \hat{A}_i(kT)$, $\Phi = \Phi(kT)$ and the unknown vector $b_i = \hat{b}_i(kT)$ involved in the algorithms presented in the previous sections. Moreover, since the matrices $O, \Phi, R, T, L, S, \hat{S}$ are constructed on the basis of $\hat{A}_i(kT), \Phi(kT), \Phi(kT), \hat{b}_i(kT)$, then provided that the matrix triplet $(\Phi(kT), B(kT), C(kT))$ is controllable and observable for any possible value of $\Theta(kT)$, we can obtain the following results sought:

$$\hat{F} = \hat{F}(\hat{\Theta}(kT)), \quad \nu = \nu(\hat{\Theta}(kT))$$

(5.26)

whereas no update is taken otherwise.

Overall, the procedure for the synthesis of the adaptive MRIC based adaptive pole placer, consists of the ten steps given below:

Step 1. Choose $N_i \geq n_i$ and the sampling period $\tau$ such that $\tau = T/(6n - 1)N$.

Step 2. Update the estimates of the matrix $V$ using relation (5.20).
Step 3. Find the matrix $\Omega$ using relation (5.19).

Step 4. Obtain a minimal realization for the matrix triplet $(\Phi_t, B_t, C)$ using any one of the minimal realization algorithms reported in the literature (see, e.g., the algorithms in [32]–[34]).

Step 5. Find the matrices $\hat{A}_t$ and the vectors $\hat{b}_t$, as well as the matrix $\Phi$ using relations (5.4) and (5.5), respectively.

Step 6. Use the algorithm presented in Section 3 to compute the controllability indices $\delta_i$ of the pair $(\Phi^T, C^T)$, as well as the values of the matrices $L, Q, \Phi^*, R_{xp},$ and $T_{sp}$.

Step 7. Use (3.26) to compute the controller matrix $K_f$.

Step 8. Find the matrices $\hat{S}$ and $\hat{S}^*$ using relation (4.12) and (4.13), respectively.

Step 9. Find the matrix $\hat{F}$ and the vector $v$ using relations (4.14) and (4.15), respectively.

Step 10. Find the matrix $F(t)$ of the MRIC based controller sought and the persistent excitation signals $\psi(t)$ using relations (4.16) and (4.2a), (4.2b), and (4.3), respectively.

5.3. Stability Analysis of the Adaptive Control Scheme

We now investigate the stability of the closed-loop system for arbitrary initial conditions on the plant. To this end, the following fundamental result can be established.

**Theorem 5.2.** In the closed-loop adaptive control system the regressor sequence $\phi(n\tau)$ is persistently exciting, i.e., there is $\delta > 0$, such that

$$Z(kT_0)Z^T(kT_0) = \sum_{\nu=0}^{(6n-1)N} \phi(kT_0 - \nu \tau)\phi^T(kT_0 - \nu \tau) \geq \delta I \quad (5.27)$$

**Proof.** In order to prove relation (5.27), we work as follows: Set $u_j(t) = d_j^T(t)v$. Then relation (5.7) yields

$$y_j(\nu \tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i,(n-\rho-1)p+i}y_j[(\nu - 2n - \rho)\tau]$$

$$+ \sum_{k=1}^{n-1} \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i,(n-\rho-1)p+i}y_k[(\nu - 2n - \rho)\tau]$$

$$+ \sum_{j=1}^{m} \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i,(n-\rho-2)m+i}u_j[(\nu - \rho - 1)\tau]$$

}
+ \sum_{j=1}^{m} \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{j, u}[((\nu - n - \rho)\tau)] \\
+ \sum_{j=1}^{m} \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{j, u}[((\nu - 2n - \rho)\tau)] 
(5.28)

where in general $(J_1)_{eq}$, $(J_2)_{eq}$, $(V)_{eq}$, and $(V^*)_{eq}$ are the $r - q$ elements of the matrices $J_1$, $J_2$, $V$, and $V^*$, respectively. Introducing the pseudovariables $\beta_{i, u}(\nu \tau)$, $j \in J_m$ and $\beta_{i, y_\kappa}(\nu \tau)$, $\kappa = 1, 2, \ldots, p$, $\kappa \neq i$, relation (5.28) can be decomposed as follows:

$$\beta_{i, u}(\nu \tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i, u}[((\nu - 2n - \rho)\tau)] = u_j(\nu \tau)$$

(5.29a)

$$y_{i, u}(\nu \tau) = \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i, (n-\rho-2)m+j} \beta_{i, u}[((\nu - \rho - 1)\tau)] \\
+ \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i, u}[((\nu - n - \rho)\tau)] \\
+ \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i, u}[((\nu - 2n - \rho)\tau)] 
\text{for } j \in J_m \quad (5.29b)$$

$$\beta_{i, y_\kappa}(\nu \tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i, y_\kappa}[((\nu - 2n - \rho)\tau)] = y_{i, \kappa}(\nu \tau)$$

(5.29c)

$$y_{i, y_\kappa}(\nu \tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i, y_\kappa}[((\nu - 2n - \rho)\tau)] 
\text{for } \kappa = 1, 2, \ldots, p, \kappa \neq i \quad (5.29d)$$

while

$$y_i(\nu \tau) = \sum_{j=1}^{m} y_{i, u_j}(\nu \tau) + \sum_{\kappa=1}^{p} y_{i, y_\kappa}(\nu \tau)$$

(5.29e)
From relations (5.29b)–(5.29e), we obtain
\[
y_i(\nu \tau) = \frac{1}{p} \left\{ \sum_{j=1}^{n} \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i, (n-\rho-2)m+j} \beta_{i,u_i}[(\nu - \rho - 1)\tau] 
+ \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_i}[(\nu - \rho - 1)\tau] 
+ \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_i}[(\nu - 2n - \rho)\tau] 
+ \sum_{\kappa=1}^{p} \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+\kappa} \beta_{i,u_i}[(\nu - 2n - \rho)\tau] 
+ \sum_{\kappa=1}^{p} \sum_{\rho=0}^{n-1} \beta_{i,u_i}[(\nu - 2n - \rho)\tau] \right\} \quad (5.30)
\]
whereas relation (5.29a) yields
\[
u_j(\nu \tau) = \frac{1}{p} \sum_{i=1}^{p} \left\{ \beta_{i,u_i}(\nu \tau) - \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \times \beta_{i,u_i}[(\nu - 2n - \rho)\tau] \right\} \quad (5.31)
\]
On the basis of relations (5.7), (5.30), and (5.31), the regressor vector \( \phi(\nu \tau) \) can also be expressed as
\[
\phi(\nu \tau) = \hat{\Sigma} \hat{\beta}(\nu \tau)
\]
where
\[
\hat{\beta}^{\top}(\nu \tau) = \begin{bmatrix} \tilde{\beta}(\nu \tau) & \cdots & \tilde{\beta}[(\nu - 6n + 2)\tau] \end{bmatrix}
\]
\[
\tilde{\beta}(\rho \tau) = \begin{bmatrix} \tilde{\beta}_{1,u_1}(\rho \tau) & \cdots & \tilde{\beta}_{1,u_1}(\rho \tau) \\
\vdots & \ddots & \vdots \\
\tilde{\beta}_{p,u_p}(\rho \tau) & \cdots & \tilde{\beta}_{p,u_p}(\rho \tau) \end{bmatrix},
\]
\[
\rho = \nu - 6n + 2, \ldots, \nu
\]
\[
\tilde{\beta}_{u_j}(\rho \tau) = \begin{bmatrix} \beta_{1,u_1}(\rho \tau) & \cdots & \beta_{p,u_p}(\rho \tau) \end{bmatrix},
\]
\[
\rho = \nu - 6n + 2, \ldots, \nu, j \in J_m
\]
\[ \tilde{\beta}_\gamma (\rho \tau) = \begin{bmatrix} \beta_{2,\gamma} (\rho \tau) & \cdots & \beta_{\rho,\gamma} (\rho \tau) \end{bmatrix}, \quad \rho = \nu - 6n + 2, \ldots, \nu \]

\[ \tilde{\beta}_\kappa (\rho \tau) = \begin{bmatrix} \beta_{1,\kappa} (\rho \tau) & \cdots & \beta_{\rho-1,\kappa} (\rho \tau) \end{bmatrix}, \quad \rho = \nu - 6n + 2, \ldots, \nu, \kappa = 2, 3, \ldots, p \]

and where \( \hat{\Sigma} \in \mathbb{R}^{(3nm + np) \times (6n - 1)p(p + m - 1)} \) is a full row rank matrix. Clearly, the vector \( \phi(\nu \tau) \) is persistently exciting if \( \beta(\nu \tau) \) is also persistently exciting. So, in what follows, it suffices to investigate excitation of \( \hat{\beta}(\nu \tau) \).

To this end, observe that (5.31) can be written as

\[ u_j(\nu \tau) = \psi_j^T \hat{\beta}(\nu \tau) \quad (5.32) \]

where \( \psi_j^T \in \mathbb{R}^{(6n - 1)m(p + m - 1)} \) is a row vector whose elements are known.

In order to prove excitation of \( \hat{\beta}(\nu \tau) \), it suffices to prove that the following relationship holds:

\[ \sum_{\nu=1}^{T_{0/\tau}} \hat{\beta}(kT_0 + \nu \tau) \hat{\beta}^T(kT_0 + \nu \tau) \geq \epsilon I \quad (5.33) \]

for some \( \epsilon > 0 \). To this end, observe that from relation (5.32), we can easily obtain

\[ \sum_{\nu=1}^{T_{0/\tau}} u_j^2(kT_0 + \nu \tau) = \psi_j^T \left( \sum_{\nu=1}^{T_{0/\tau}} \hat{\beta}(kT_0 + \nu \tau) \hat{\beta}^T(kT_0 + \nu \tau) \right) \psi_j \quad (5.34) \]

Observe also that the following relation holds:

\[ u_j(kT_0 + \nu \tau) = \begin{cases} 0 & \text{if } \nu = 1, 2, \ldots, (6n - 1)(N_j - n_j - 1)l_j - 1 \\ 1 & \text{if } \nu = (6n - 1)(N_j - n_j - 1)l_j, \ldots, \\
(6n - 1)(N_j - n_j)l_j - 1 \end{cases} \]

Hence, relation (5.34) can also be written as

\[ (6n - 1)l_j + \sum_{\nu = (6n - 1)(N_j - n_j)l_j}^{T_{0/\tau}} u_j^2(kT_0 + \nu \tau) = \psi_j^T \left( \sum_{\nu=1}^{T_{0/\tau}} \hat{\beta}(kT_0 + \nu \tau) \hat{\beta}^T(kT_0 + \nu \tau) \right) \psi_j \]
We can then conclude that
\[
\psi_j^T \left( \sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0 + \nu\tau) \hat{\beta}^T(kT_0 + \nu\tau) \right) \psi_j \geq (6n - 1) l_j
\]
and that
\[
\left\{ \frac{\psi_j^T}{\|\psi_j\|} \right\} \left( \sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0 + \nu\tau) \hat{\beta}^T(kT_0 + \nu\tau) \right) \left\{ \frac{\psi_j}{\|\psi_j\|} \right\} \geq \frac{(6n - 1) l_j}{\|\psi_j\|^2}.
\]

It is now clear that the vector \(\psi_j/\|\psi_j\|\) is a vector whose norm equals unity. Hence there is a unity norm vector such that
\[
\chi^T \left( \sum_{\nu=1}^{T_0/\tau} \hat{\beta}(kT_0 + \nu\tau) \hat{\beta}^T(kT_0 + \nu\tau) \right) \chi - \frac{(6n - 1) l_j}{\|\psi_j\|^2} \geq 0.
\]

In conclusion, relation (5.33) holds. As a consequence, the vector \(\hat{\beta}(\nu\tau)\) is persistently exciting. Therefore, \(\hat{\phi}(\nu\tau)\) is also persistently exciting and hence there is a \(\delta > 0\) (which, in general, depends on the matrix \(\Sigma\)), such that relation (5.27) holds. This completes the proof of the Theorem.

We are now able to establish the stability of the adaptive control system.

**Proposition 5.2.** The closed-loop adaptive control system presented above is globally stable, i.e., for arbitrary finite initial conditions all states are uniformly bounded, and pole placement control is asymptotically attained. Furthermore, the proposed adaptive scheme provides exponential convergence of the estimated parameters.

**Proof.** Since, according to Theorem 5.2, the regressor sequence is persistently exciting, then the difference \(\hat{\Theta}(kT_0) - \Theta\) converges to zero. That is, the plant parameter estimates converge to their true values. As a consequence of this and of the fact that \(\hat{\Theta}(kT_0)\) is uniformly bounded, the controller parameter estimates (5.26) also converge to their true values. Therefore, at the sampling instants uniform boundedness of all states and discrete pole placement follow on the basis of (4.4). Uniform boundedness of \(u(t)\) and \(x(t)\) then follows from (2.1), (4.8a), (4.8b), and (4.16) and from the fact that \(w(kT_0)\) is bounded by assumption. Finally, exponential convergence of the plant parameter estimates follows from (5.23), which together with (5.27) ensures that \(\hat{\Theta}(kT_0) \rightarrow \Theta\) exponentially as \(k \rightarrow \infty\).

**Remark 5.2.** Commenting on the assumptions needed here, in order to implement the MRIC based adaptive pole placer presented above, we point out the following:

Assumption 2.1a, on the controllability and observability of the continuous-time plant as well as on the knowledge of its order, is a standard
assumption in the area of adaptive control. It is worth noticing that here, controllability of the pair \((A, B)\) is also necessary for obtaining a solution of the integral equation (3.2), with respect to the controller matrix \(F(t)\). Note also that uncontrollability (and/or unobservability) of the pair \((A, B)\) implies uncontrollability (and/or unobservability) of the plants obtained from (2.1), by discretizing with sampling periods \(T_0\), \(T_N\), and \(\tau\). From the previous analysis, however, it becomes clear that for the implementation of the adaptive control scheme, these discretized plants must be controllable and observable.

A assumption 2.1b, on the knowledge of a set of LMCI indices of the pair \((A, B)\), is instrumental for the implementation of the proposed adaptive scheme, since, on the one hand, the elements of the MRIC of the form (4.1) and the persistent excitation signals (4.2a), (4.2b), and (4.3) depend on the LMCI used, and on the other hand, the control strategy in the case of unknown systems is based on the fundamental sampling period \(\tau\), which also depends of the knowledge of a set of LMCI. Note also that, whenever Assumption 2.1b is not fulfilled, one can readily compute a set of LMCI by estimating the continuous-time system matrices \(A\) and \(B\). This can be done either using a continuous-time counterpart of the identification procedure presented in Section 5.1, or following the structural identification approach proposed in [38]. For the sake of simplicity, we assume here that the initial information about a set of LMCI of the pair \((A, B)\) is available.

A assumption 2.2 on the existence of a sampling period \(T_0\), for which controllability and observability of the matrix triplets \((\Phi, \hat{B}, C)\) and \((\Phi_x, B_x, C)\) are guaranteed, is also instrumental for our analysis. In particular, observability of the pair \((\Phi, C)\) must be guaranteed for being able to transform the pair \((\Phi^T, C^T)\) in its input Luenberger canonical form and for obtaining a solution of the pole placement control problem, in the case of unknown systems. On the other hand, controllability and observability of the matrix triplet \((\Phi_x, B_x, C)\) is necessary for resorting to the equivalent input–output representation (5.7), for the state space system of the form (5.1), as well as for being able to apply any one of the minimal realization algorithms presented in [32]–[34], which are needed here to obtain the estimates of the triplet \((\Phi_x, B_x, C)\). Note that for ensuring controllability and observability of the triplets \((\Phi, \hat{B}, C)\) and \((\Phi_x, B_x, C)\), the fundamental sampling period \(T_0\) must be selected such that simultaneously

\[
\frac{2\rho \pi j}{T_0},
\]

\(\rho = 0, 1, \ldots\) \(j = \sqrt{\mu - 1}\) is not the difference of any two eigenvalues of the matrix \(A\).
This implies that in the multirate adaptive case treated here, certain sampling frequencies must be avoided, as compared to the nonadaptive nonmultirate case. It is pointed out that conditions (5.35a) and (5.35b) are standard conditions for the selection of a regular sampling period, in order to avoid loss of controllability and observability under sampling (see [40] for a detailed analysis of this issue).

6. CONCLUSIONS

The adaptive pole placement problem of linear time-invariant continuous-time multi-input, multioutput, systems has been investigated and an indirect adaptive control scheme based on periodic multirate-input controllers has been presented for the first time. The proposed control strategy has, as compared to known related techniques, the following main advantages:

(a) It is readily applicable to nonstably invertible systems having arbitrary poles and zeros and relative degree. This is due to the fact that the approach used here to solve the adaptive pole placement problem does not rely on pole-zero cancellations.

(b) Following the proposed technique a gain controller is essentially needed to be designed, as compared to dynamic compensators or state observers needed by known indirect adaptive pole placement techniques. Consequently, the present approach avoids the problems of known adaptive pole placement techniques, interwoven with the possibly unstable solutions of the Diophantine equation. Moreover, no exogenous dynamics are introduced in the control loop by our technique, whereas in many known techniques the dynamics introduced are of high order. This fact improves the computational aspect of the problem, since the proposed technique does not require many on-line computations and its practical implementation requires computer memory only for storing the modulating matrix function $F(t)$ over one period of time.

(c) It offers a solution to the problem of ensuring persistency of excitation of the continuous-time plant under control, without imposing
any special requirement on the reference signal $w(kT_T)$ (except boundedness) and without making any assumption concerning either the existence of specific convex sets in which the estimated parameters belong or the coprimeness of the polynomials describing the ARMA model.

The present paper gives some new insights to the adaptive pole placement problem of linear systems. The present results can be extended to solve other adaptive control problems, as, for example, the problems of model reference adaptive control and adaptive decoupling using multirate-input controllers or multirate generalized sampled-data hold functions. Adaptive control schemes based on alternative parameter estimation algorithms (as, for example, the algorithm proposed in [39]) and without the need of persistent excitation signals are currently under investigation.

REFERENCES


