An almost sure central limit theorem for the overlap parameters in the Hopfield model

Barbara Gentz *
Institut für Angewandte Mathematik, Universität Zürich, Winterthurer Str. 190, CH-8057 Zürich, Switzerland

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Abstract

We consider the Hopfield model with a finite number of randomly chosen patterns above and below the critical temperature and prove an almost sure conditional central limit theorem for the vector of overlap parameters. For this purpose we analyse the almost sure asymptotic behaviour of the partition function.

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1. Introduction and results

To describe the Hopfield model, we denote by \( p \in \mathbb{N} \) the fixed number of so-called patterns and by

\[
H_n(\sigma, \xi) = -\frac{1}{2n} \sum_{\mu=1}^{p} \sum_{i,j=1}^{n} \xi_i^\mu \sigma_j \sigma_j, \quad n \in \mathbb{N},
\]

the Hamiltonian of the Hopfield model, where \((\sigma_i)_{i \in \{1, \ldots, n\}}\) with values in \{-1, 1\} \(^n \) is the spin configuration which models the neural activities, and \((\xi_i^\mu)_{i \in \{1, \ldots, n\}}\) with values in \{-1, 1\} \(^n \) is the codification of the \(\mu\)th pattern. All these random variables are assumed to be independent and identically distributed with \(P(\sigma_i = -1) = \frac{1}{2} = P(\xi_i^\mu = 1)\). Let \(\xi_i = (\xi_i^1, \ldots, \xi_i^p)\) be the vector consisting of the \(i\)th components of the \(p\) patterns, and let \(\xi = (\xi_i)_{i \in \mathbb{N}}\). We denote by \(P_{\xi} = ((\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{1})^\otimes \mathbb{N})^\otimes \mathbb{N}\) the marginal distribution of \(\xi\) and, similarly, by \(P_{\sigma} = ((\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{1})^\otimes \mathbb{N})\) the marginal distribution of \(\sigma = (\sigma_i)_{i \in \mathbb{N}}\). With the help of the vector of overlap parameters

\[
\frac{1}{n} S_n(\sigma, \xi) = \frac{1}{n} \sum_{i=1}^{n} \xi_i^\mu \sigma_i,
\]
the Hamiltonian (1.1) can be rewritten in the following convenient form:

\[ H_n(\sigma, \xi) = -\frac{n}{2} \left\| \frac{1}{n} S_n(\sigma, \xi) \right\|^2. \]

For simplicity, we will drop the explicit dependence on \( \sigma \) and \( \xi \) whenever no confusion may arise. So we will write \( S_n \) instead of \( S_n(\sigma, \xi) \), for instance. Let \( \beta > 0 \) be the inverse temperature of the model and denote by \( P_{n,\beta,\xi} \) the finite-volume Gibbs measure given by

\[ dP_{n,\beta,\xi} = e^{-\beta H_n(\sigma, \xi)} dP_\sigma / Z_{n,\beta,\xi}, \tag{1.2} \]

where the partition function \( Z_{n,\beta,\xi} \) is the appropriate normalization. Note that the case \( p = 1 \) corresponds to the Curie–Weiss model.

The Hopfield model is a model of an associative memory based on artificial binary neurons as introduced by McCulloch and Pitts (1943). Hopfield (1982) introduced the notion of an energy function to the theory of artificial neural networks, thereby relating their study to the one of spin systems in mathematical physics. In this context, the dynamics of the Hopfield model as a neural network corresponds to a Glauber single-spin dynamics on the set of spin configurations \( \{-1, 1\}^n \). In Amit et al. (1985) a generalized Glauber single-spin dynamics at finite temperature \( 1/\beta \) was introduced, governed by the Hamiltonian (1.1). This dynamics describes a reversible, irreducible Markov process, which converges for arbitrary initial spin configuration to its equilibrium distribution (1.2).

We are interested in the behaviour of the Gibbs measures \( P_{n,\beta,\xi} \) as \( n \) tends to infinity. The partition function \( Z_{n,\beta,\xi} \) has been studied via the free energy

\[ \lim_{n \to \infty} \frac{1}{n} \log Z_{n,\beta,\xi} \quad \text{resp.} \quad \lim_{n \to \infty} E_\xi \left( \frac{1}{n} \log Z_{n,\beta,\xi} \right), \]

cf. Albeverio et al. (1992) and the references given there.

Since the Hopfield model is related to the Curie–Weiss model it is helpful to introduce some notations concerning the latter. We denote by \( z^\pm(\beta) \) the largest (resp. smallest) solution \( z \in (-1, 1) \) of the Curie–Weiss equation \( \beta z = \text{artanh} z \). Recall that the free energy in the Curie–Weiss model equals

\[ f_{CW}(\beta) = \frac{\beta}{2} z^\pm(\beta)^2 - I(z^\pm(\beta)), \]

where

\[ I(z) = \begin{cases} \frac{1}{2}(1 + z) \log(1 + z) + \frac{1}{2}(1 - z) \log(1 - z) & \text{for } |z| \leq 1, \\ \infty & \text{otherwise}, \end{cases} \]

denotes the rate function which governs the large deviations of the spin per site in the Curie–Weiss model, cf. Ellis (1985, Section IV.4). To extend the definition of \( z^\pm(\beta) \) to the case of a so-called external magnetic field of strength \( h \neq 0 \), let \( z(\beta, h) \) denote the unique solution \( z \in (-1, 1) \) of \( \beta z + h = \text{artanh} z \), which satisfies \( \text{sign} z = \text{sign} h \). In addition, we extend the notation \( f_{CW}(\beta) \) for the free energy in the Curie–Weiss
model by defining

\[ f_{\mathcal{W}}(\beta, h) = \begin{cases} \frac{\beta}{2} z(\beta, h)^2 + h z(\beta, h) - l(z(\beta, h)) & \text{if } h \neq 0, \\ f_{\mathcal{W}}(\beta) & \text{otherwise.} \end{cases} \]

Consider now the distribution of the vector of overlap parameters under the Gibbs measure determined by the inverse temperature \( \beta \) and the Hamiltonian

\[ H_{n, h, l}(\sigma, \xi) = -\frac{1}{2n} \sum_{\mu=1}^{n} \sum_{i,j=1}^{n} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j - h \sum_{i=1}^{n} \xi_i^l \sigma_i, \]

where \( h \neq 0 \) and \( \xi^l \) is the preferred pattern. In Bovier et al. (1994), the weak convergence of these distributions as \( n \to \infty \) and \( h \to 0^+ \) afterwards to the Dirac measure concentrated in \( z^+(\beta)e_l \) is shown for \( P_x \)-almost all \( \xi \). Since the authors allow the number of patterns \( p \) to depend on \( n \) in a non-decreasing way satisfying \( p/n \to 0 \), \( e_l \) denotes the \( l \)th unit vector in \( \mathbb{R}^p \) resp. \( \mathbb{R}^N \). Note that \( z^\pm(\beta) = 0 \) for \( \beta \leq \beta_c = 1 \), which implies uniqueness of the limiting measure.

Considering this result, it is natural to ask for the fluctuations of the overlap parameters around \( z^+(\beta)e_l \) as \( n \to \infty \). There are two possibilities to select a preferred pattern. In Theorem 1.1, we consider the unbiased Hamiltonian (1.1) and treat the fluctuations under the condition that the vector of overlap parameters is already in a neighborhood of \( z^+(\beta)e_l \). Alternatively, the preferred pattern can be chosen by introducing an external magnetic field, which is in the spirit of Bovier et al. (1994). The latter case is treated in Corollary 1.7.

To state the central limit theorem in the above-mentioned settings, we need some additional notations. Let \( v_1, \ldots, v_d \) be an enumeration of the set \( \{-1, 1\}^p \), where \( d = 2^p \). For every \( k \in \{1, \ldots, d\} \), let

\[ N_k^p(\xi) = \sum_{i=1}^{n} 1_{\{v_k\}}(\xi_i) \]

count the number of those \( \xi_1, \ldots, \xi_n \) which attain the value \( v_k \). In the sequel, the deviation

\[ \Delta_k^p(\xi) = N_k^p(\xi) - \frac{1}{d} \]

of the relative frequency \( N_k^p/n \) from its mean \( 1/d \) will play an important rôle. Define the \( (p \times p) \)-matrix

\[ \Sigma^p(\xi) = \sum_{k=1}^{d} \Delta_k^p(\xi) v_k v_k^T. \]

We denote by \( e_1, \ldots, e_p \) the unit vectors in \( \mathbb{R}^p \), and set \( e_{-l} = -e_l \) for \( l \in \{1, \ldots, p\} \).

For convenience, let us introduce the index set \( L \) with \( L = \{-p, p\} \cap \mathbb{Z} \setminus \{0\} \) in the case \( \beta > \beta_c \), and \( L = \{1\} \), otherwise. Define \( \lambda_l = z^+(\beta)e_l \) for all \( l \in L \).

Now we are ready to state our main result, a \( P_x \)-almost sure central limit theorem for the overlap parameters. For \( \beta > \beta_c \) the centring consists not only of the deterministic
Fig. 1. The variance of the limiting Gaussian fluctuations of an overlap parameter in the Hopfield model as a function of the inverse temperature vector $\lambda_l$ as the above-mentioned result from Bovier et al. (1994) may suggest, but includes a refined $\xi$-dependent correction.

**Theorem 1.1.** Let $\beta \neq \beta_c$, $h = 0$, and fix $\varepsilon \in (0, \frac{1}{2} z^+(\beta))$ in the case $\beta > \beta_c$ and $\varepsilon > 0$ arbitrary otherwise. Let $l \in L$. Then, for $P_\xi$-almost all $\xi$, the measures

$$P_{n,\beta,\xi}(\sqrt{n} \left[ \frac{S_n}{n} - \left( \text{Id}_{\mathbb{R}^p} + \frac{1}{1 - \beta (1 - z^+(\beta)^2)} \Sigma^n \right) \lambda_l \right] \in \cdot \left\| \frac{S_n}{n} - \lambda_l \right\| < \varepsilon) \quad (1.4)$$

converge weakly as $n$ tends to infinity towards a Gaussian distribution on $\mathbb{R}^p$ with mean zero and covariance matrix $C_{\beta} = c_{\beta} \text{Id}_{\mathbb{R}^p}$, where

$$c_{\beta} = \frac{1 - z^+(\beta)^2}{1 - \beta (1 - z^+(\beta)^2)}$$

is the variance of the limiting fluctuations in the corresponding Curie–Weiss model.

**Remark 1.2.** (a) Lemma 2.1 in Section 2 proves that $1 - \beta (1 - z^+(\beta)^2) > 0$ and Fig. 1 shows a graph of the map $\beta \mapsto c_{\beta}$.

(b) For $\beta < \beta_c$ the theorem reduces to

$$P_{n,\beta,\xi}(\frac{S_n}{\sqrt{n}} \in \cdot \left\| \frac{S_n}{n} \right\| < \varepsilon) \Rightarrow \mathcal{N}(0, \frac{1}{1 - \beta} \text{Id}_{\mathbb{R}^p}),$$

which yields

$$P_{n,\beta,\xi}(\frac{S_n}{\sqrt{n}})^{-1} \Rightarrow \mathcal{N}(0, \frac{1}{1 - \beta} \text{Id}_{\mathbb{R}^p})$$

as an immediate consequence of Remark 1.4(b) below.

(c) Note that the fluctuations of the spin per site in the Curie–Weiss model, which corresponds to the overlap parameters in our model, are non-Gaussian at the critical inverse temperature $\beta_c = 1$ (cf. Ellis, 1985, Theorem V.9.5).

(d) The condition $\varepsilon < \frac{1}{4} z^+(\beta)$ in the case $\beta > \beta_c$ assures that the $2p$ balls of radius $\varepsilon$ with centres $\lambda_l$, $l \in L$, are disjoint.
(e) Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$ and $\varepsilon_n = ((1/n) \log \log n)^{1/2}$ for $n \geq 3$. In the proof of the theorem we will see that the central limit theorem holds for all $\xi \in \Xi$, where

$$\Xi = \liminf_{n \to \infty} \{ \xi \in \{-1, 1\}^P : \|\Delta^\nu(\xi)\| \leq 2\varepsilon_n/\sqrt{d} \}.$$

Lemma 2.2 below shows that $P_c(\Xi) = 1$.

The following proposition is needed for the proof of Theorem 1.1. It is stated here because it is of interest of its own.

**Proposition 1.3.** Let $\beta \neq \beta_c$, $h = 0$, and $\delta \in (0, 1/2)$.

(a) For all $\xi \in \Xi$ the partition function satisfies

$$Z_{n,\beta,\xi} = \left[ 1 - \beta(1 - z^+(\beta)^2) \right]^{-p/2} e^{ncw(\beta)} \exp \left\{ \frac{n\beta}{2} \frac{1}{1 - \beta(1 - z^+(\beta)^2)} \left\| \Sigma^\nu \lambda_\xi \right\|^2 \right\} \left[ 1 + O(n^{-\delta}) \right]$$

as $n \to \infty$.

(b) Choose $\varepsilon > 0$ as in Theorem 1.1 and let $l \in L$. Then, for all $\xi \in \Xi$,

$$P_{n,\beta,\xi} \left( \left\| \frac{S_n}{n} - \lambda_l \right\| < \varepsilon \right) = \frac{1}{Z_{n,\beta,\xi}} \left[ 1 - \beta(1 - z^+(\beta)^2) \right]^{-p/2} e^{ncw(\beta)} \exp \left\{ \frac{n\beta}{2} \frac{1}{1 - \beta(1 - z^+(\beta)^2)} \left\| \Sigma^\nu \lambda_\xi \right\|^2 \right\} \left[ 1 + O(n^{-\delta}) \right]$$

as $n \to \infty$.

**Remark 1.4.** (a) Note that the error terms in the above proposition depend on $\delta$ and $\xi$ in the following way: For all $\delta \in (0, 1/2)$ and all $\xi \in \Xi$, there exist a constant $C(\delta, \xi) > 0$ and $n_0(\delta, \xi) \in \mathbb{N}$ such that the corresponding error term is bounded by $C(\delta, \xi)n^{-\delta}$ for all $n \geq n_0(\delta, \xi)$.

(b) For $\beta < \beta_c$, the proposition states that

$$Z_{n,\beta,\xi} = \left[ 1 - \beta \right]^{-p/2} \left[ 1 + O(n^{-\delta}) \right]$$

and

$$P_{n,\beta,\xi} \left( \left\| \frac{S_n}{n} \right\| < \varepsilon \right) = 1 + O(n^{-\delta})$$

for all $\delta \in (0, 1/2)$ and all $\xi \in \Xi$.

The proofs of Theorem 1.1 and Proposition 1.3 will be given in Section 2, while in Section 3 we will study the asymptotic behaviour of integrals of the type

$$\int \exp \left\{ n \left( \frac{\beta}{2} \left\| \frac{S_n}{n} \right\|^2 + \left\langle \frac{S_n}{n}, h \lambda_l + \frac{y}{\sqrt{n}} \right\rangle \right) \right\} \, dP_\sigma,$$

where $\beta > 0$, $h \geq 0$ with $(\beta, h) \neq (\beta_c, 0)$ and $y \in \mathbb{R}^P$ are allowed to depend on $n$ in a rather restrictive way. This dependence is needed in the proof of Proposition 1.3(b).
To investigate the integral (1.5), we will transform it with the help of a Gaussian integration in such a way that the summation over \( \sigma \in \{-1, 1\}^n \) can be carried out. This provides us with a situation, where Laplace’s method can be applied, although there arise \( n \)-dependent terms which need to be handled with care. For multiple use in Section 2, we will state the result of these calculations below.

The central limit theorem and Proposition 1.3 as well as the following results for the case where an external magnetic field is present will be proved for a fixed number of patterns \( p \) only. The treatment of the case where the number \( p \) of patterns increases with the system size \( n \) is more delicate and requires refined proofs. For increasing \( p \), the influence of the random patterns cannot be controlled by the law of the iterated logarithm anymore and Laplace’s method has to be applied in an infinite-dimensional setting. We will treat this generalization in subsequent work, see Gentz (1996) for the case where \( p = p(n) \) may increase to infinity subject to the restriction \( p(n)^2/n \to 0 \) as \( n \to \infty \).

**Lemma 1.5.** Fix \( I \in \{-p, \ldots, -1, 1, \ldots, p\} \), \( \beta > 0 \), \( \gamma > 0 \), \( h > 0 \) with \((\beta, h) \neq (\beta_c, 0)\) and \( \delta \in (0, \frac{1}{2}) \). Let \( \beta_n \uparrow \beta \) with \( 0 < \beta_n < \beta \) and \( \beta_n \neq \beta_c \) for all \( n \in \mathbb{N} \), and let \( h_n = (\beta - \beta_n) z^+(\beta) \downarrow h = 0 \) for all \( n \in \mathbb{N} \). Alternatively, let \( \beta_n = \beta \) and \( h_n = h \geq 0 \) for all \( n \in \mathbb{N} \). Let furthermore \( y_n \in \mathbb{R}^p \) with \( ||y_n|| \leq y_0 \varepsilon_n \) for some \( y_0 > 0 \). Then, for all \( \xi \in \Xi \),

\[
\int \exp \left\{ n \left[ \frac{\beta_n}{2} ||S_n||^2 + \left\langle S_n, h_n e_I + y_n \right\rangle \right] \right\} dP_\sigma
\]

\[
= \left[ 1 - \beta_n (1 - z^2) \right]^{-p/2} \exp \left\{ n \left[ f_{cw}(\beta_n, h_n) - \frac{1}{2\beta_n} ||y_n||^2 \right] \right\} \left[ 1 + \mathcal{O}(n^{-\delta}) \right]
\]

\[
\times \sum_{v \in L_0} \exp \left\{ \frac{n\beta_n}{2} \frac{1}{1 - \beta_n (1 - z^2)} \left\| y_n + \sum \varepsilon_v \right\| ^2 + n(\varepsilon_v, y_n) \right\}
\]

(1.6)

as \( n \to \infty \), where

\[
L_0 = \begin{cases} L & \text{if } h_n = 0 \ \forall n \in \mathbb{N}, \\ \{1\} & \text{otherwise} \end{cases} \quad \text{and} \quad z = \begin{cases} z^+(\beta) & \text{if } h_n = 0 \ \forall n \in \mathbb{N}, \\ z(\beta_n, h_n) & \text{otherwise} \end{cases}
\]

**Remark 1.6.** (a) Note that in the case \( \beta > \beta_c \) the definition \( h_n = (\beta - \beta_n) z^+(\beta) \) is equivalent to the condition \( z(\beta_n, h_n) = z^+(\beta) \), and that \( h_n = 0 \), otherwise. Furthermore, keep in mind that \( z = z(\beta, h) \) in the case \( h_n = h > 0 \) for all \( n \in \mathbb{N} \), and \( z = z^+(\beta) \) otherwise.

(b) Lemma 2.1 shows that \( 1 - \beta_n (1 - z^2) > 0 \).

**Corollary 1.7.** For \( \beta > 0 \), \( h > 0 \) and \( I \in \{-p, \ldots, -1, 1, \ldots, p\} \) consider the Hopfield Hamiltonian with inverse temperature \( \beta \) and external magnetic field \( he_I \). Let

\[
dP_{n, \beta, he_I, \xi} = \frac{1}{Z_{n, \beta, he_I, \xi}} \exp \left\{ n \left[ \frac{\beta}{2} \left\| S_n(\sigma, \xi) \right\|^2 + \left\langle \frac{1}{n} S_n(\sigma, \xi), he_I \right\rangle \right] \right\} dP_\sigma
\]

denote the corresponding finite-volume Gibbs measure, where \( Z_{n, \beta, he_I, \xi} \) is the appropriate normalization. Then the overlap parameters satisfy for all \( \xi \in \Xi \) a central
limit theorem with respect to $P_{n,\beta,he,\xi}$, namely the measures

$$P_{n,\beta,he,\xi}\left(\sqrt{n}\left[\frac{S_n}{n} - \left(\text{Id}_{\mathbb{R}^p} + \frac{1}{1 - \beta(1 - z(\beta, h)^2)}\Sigma^n\right)z(\beta, h)e_i\right]\right)^{-1},$$

defined for all measurable $A \subset \mathbb{R}^p$ by

$$P_{n,\beta,he,\xi}\left(\frac{S_n}{n} - \left(\text{Id}_{\mathbb{R}^p} + \frac{1}{1 - \beta(1 - z(\beta, h)^2)}\Sigma^n\right)z(\beta, h)e_i\right) \in A,$$

converge weakly as $n$ tends to infinity towards a Gaussian distribution with mean zero and covariance matrix

$$\frac{1 - z(\beta, h)^2}{1 - \beta(1 - z(\beta, h)^2)} \text{Id}_{\mathbb{R}^p}.$$

**Proof.** To derive the corollary from Lemma 1.5, it suffices to consider the Laplace transforms, because \{ $\mathbb{R}^p \ni \lambda \mapsto \exp(\lambda, y) \in \mathbb{R}$ : $y \in \mathbb{R}^p$ \} is a convergence determining class of functions; see Martin-Löf (1973, Lemma C1).

2. **Proofs of Theorem 1.1 and Proposition 1.3**

The following lemma shows that the covariance matrices in Theorem 1.1 and Corollary 1.7 are well-defined.

**Lemma 2.1.** Let $\beta > 0$, $h \geq 0$ with $(\beta, h) \neq (\beta_c, 0)$, and let $z = z^+ (\beta)$, if $h = 0$, and $z = z(\beta, h)$, otherwise. Then $1 - \beta (1 - z^2) > 0$.

**Proof.** Let us assume the contrary, i.e. $\beta \geq (1 - z^2)^{-1}$. Then $v = \text{artanh} z$ satisfies

$$v = \frac{z}{1 - z^2} + h = \tanh v \cosh^2 v + h \geq v + h,$$

where we used $\cosh^2 \text{artanh} z = (1 - z^2)^{-1}$. This estimate implies that $h = 0$ and $v = 0$, because the last inequality is strict for $v > 0$. But $v = 0$ cannot be the case unless $z = 0$ and $\beta < \beta_c = 1$ which contradicts our assumption. □

Proposition 1.3(a) is an immediate consequence of Lemma 1.5 (choose $\beta_n = \beta$, $h_n = 0$, and $y_n = 0$). For the proof of part (b) of the same proposition, we will need an estimate on $\Sigma^n$. For this purpose, we will first derive an estimate on the vector $\Delta^n$ via the law of the iterated logarithm.

**Lemma 2.2.** For $P_\xi$-almost all $\xi$ there exists an $n(\xi) \in \mathbb{N}$ such that the estimate $\|\Delta^n(\xi)\| \leq 2 \varepsilon_n / \sqrt{d}$ holds for all $n \geq n(\xi)$, where the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ is defined as in Remark 1.2(e).
Remark 2.3. The statement of the above lemma is equivalent to \( P_\xi(\Xi) = 1 \), where \( \Xi \) is defined as in Remark 1.2(e).

Proof. Let \( C(\xi) \) denote the set of accumulation points of \( (\Delta^a(\xi)/\sqrt{2\varepsilon n}))_{n \in \mathbb{N}} \). Since \( n\Delta^a \) is the sum of \( n \) independent identically distributed mean zero random variables, we can apply the law of the iterated logarithm (cf. Kuelbs, 1976, Example, p. 266, Theorem 4.1) which yields \( P_\xi(C(\xi) = K) = 1 \), where \( K \) denotes the closed unit ball in the range \( \mathcal{R} \) of the linear operator

\[
(\mathbb{R}^d)^* \ni f \mapsto Sf = E_\xi(\Delta^1 f(\Delta^1)) \in \mathbb{R}^d
\]

with respect to the norm given on \( \mathcal{R} \) by \( \|Sf\|^2 = E_\xi(f^2(\Delta^1)) \). Because \( \Delta^1 \) takes values in the hyperplane \( \mathcal{H} = \{ y \in \mathbb{R}^d : \sum_{k=1}^d y_k = 0 \} \), we know that \( \mathcal{R} \subset \mathcal{H} \). On the other hand, \( S(\langle dy, \cdot \rangle) = y \) holds for all \( y \in \mathcal{H} \). Therefore, \( \mathcal{R} = \mathcal{H} \) and

\[
\|y\|^2 = E_\xi((dy, \Delta^1)^2) = \langle dy, S(\langle dy, \cdot \rangle) \rangle = d\|y\|^2
\]

for all \( y \in \mathcal{R} \). Hence, \( K = \{ y \in \mathcal{H} : \|y\|^2 \leq 1/d \} \), which implies Lemma 2.2. \( \square \)

Corollary 2.4. For all \( \xi \in \Xi \), all \( n \geq n(\xi) \) and all \( \lambda \in \mathbb{R}^p \) the following estimate holds:

\[
\|\Sigma^a(\xi)\lambda\| \leq 2\varepsilon n^{1/2}\|\lambda\|.
\]

Proof. By Schwarz's inequality,

\[
\|\Sigma^a(\xi)\lambda\| \leq \sum_{k=1}^d |\Delta^a_k(\xi)| \|\langle \lambda, v_k \rangle v_k\| \leq \|\Delta^a\| \left( \sum_{k=1}^d \|\langle \lambda, v_k \rangle v_k\|^2 \right)^{1/2}.
\]

Since \( \|v_k\|^2 = p \) and

\[
\frac{1}{d} \sum_{k=1}^d \langle \lambda, v_k \rangle^2 = \|\lambda\|^2,
\]

the corollary follows from Lemma 2.2. \( \square \)

Fix \( \delta \in (0, \frac{1}{2}) \) and \( \xi \in \Xi \) for the remainder of this section. Next we will give:

Proof of Proposition 1.3(b). Fix \( l \in \mathcal{L} \). First we will derive a lower estimate for the probabilities in question and then show that this estimate differs from an upper estimate by not more than a factor of \( 1 + O(n^{-\delta}) \). We have

\[
P_{n, \beta, \xi} \left( \left\| \frac{S_n}{n} - \lambda_l \right\| < \varepsilon \right) \geq \int \exp \left\{ -\frac{\sqrt{n}}{2} \left\| \frac{S_n}{n} - \lambda_l \right\|^2 \right\} dP_{n, \beta, \xi}
\]

\[
- \int \exp \left\{ -\frac{\sqrt{n}}{2} \left\| \frac{S_n}{n} - \lambda_l \right\|^2 \right\} \mathbb{1}_{\{\left\| \frac{S_n}{n} - \lambda_l \right\| \geq \varepsilon \}} dP_{n, \beta, \xi}.
\]

The second integral on the right-hand side is bounded by \( \exp\left\{ -\sqrt{n} \varepsilon^2/2 \right\} \). To calculate the first integral, we use the definition of \( P_{n, \beta, \xi} \) and apply Lemma 1.5 with \( \beta_n = \)
\( \beta - 1/\sqrt{n}, h_n = z^+ (\beta)/\sqrt{n}, y_n = 0 \) and an auxiliary \( \delta' \in (\delta, 1) \). We restrict ourselves to \( n > \beta^{-2} \), if \( \beta < \beta_c \), and \( n > (\beta - \beta_c)^{-2} \), otherwise, to assure that \( \beta_n > 0 \) and \( \beta_n \neq \beta_c \). We find that the first integral equals

\[
\frac{1}{Z_{n, \beta, \xi}} \left[ 1 - \beta_n (1 - z^+ (\beta)^2) \right]^{-p/2} \exp \left\{ n f_{cw}(\beta_n, h_n) - \frac{\sqrt{n}}{2} \| \lambda_i \| ^2 \right\} \\
\times \exp \left\{ \frac{n \beta_n}{2} \frac{1}{1 - \beta_n (1 - z^+ (\beta)^2)} \| \Sigma \lambda_i \| ^2 \right\} [1 + O(n^{-\delta'})].
\]

Applying Corollary 2.4 and Proposition 1.3(a), we get the lower estimate

\[
P_{n, \beta, \xi} \left( \left\| \frac{S_n}{n} - \lambda_i \right\| < \epsilon \right) \geq \frac{1}{Z_{n, \beta, \xi}} \left[ 1 - \beta (1 - z^+ (\beta)^2) \right]^{-p/2} e^{n f_{cw}(\beta)} \\
\times \exp \left\{ \frac{n \beta}{2} \frac{1}{1 - \beta (1 - z^+ (\beta)^2)} \| \Sigma \lambda_i \| ^2 \right\} [1 + O(n^{-\delta'})]. \tag{2.2}
\]

By the above estimate and another application of Proposition 1.3(a) with \( \delta' \) in the place of \( \delta \), we have

\[
1 \geq \sum_{l \in L} P_{n, \beta, \xi} \left( \left\| \frac{S_n}{n} - \lambda_i \right\| < \epsilon \right) \geq 1 + O(n^{-\delta'}). \tag{2.3}
\]

According to Corollary 2.4, the right-hand sides of (2.2) for different \( l \in L \) cannot differ by more than a factor of

\[
\exp \left\{ \frac{2 \beta p z^+ (\beta)^2 \log \log n}{1 - \beta (1 - z^+ (\beta)^2)} \right\} [1 + O(n^{-\delta'})] \leq O(n^{\delta' - \delta}).
\]

Together with (2.3), this implies that the lower estimate (2.2) cannot differ from an upper estimate by more than a factor of \( 1 + O(n^{-\delta'}) \). Hence, the proof of Proposition 1.3(b) is complete. \( \square \)

To prove the weak convergence stated in Theorem 1.1, we will use the following lemma.

**Lemma 2.5.** If \( \varphi \) denotes the continuous density with respect to the Lebesgue measure of a non-degenerate Gaussian distribution on \( \mathbb{R}^p \), then \( \{ \varphi(\cdot - y) : y \in \mathbb{R}^p \} \) is a convergence determining class of functions on \( \mathbb{R}^p \).

**Proof.** Let \( Q \) and \( Q_n \) for \( n \in \mathbb{N} \) be probability measures on \( \mathbb{R}^p \) such that

\[
\varphi \ast Q_n(y) = \int \varphi(x - y) Q_n(dx) \xrightarrow{n \to \infty} \int \varphi(x - y) Q(dx) = \varphi \ast Q(y) \tag{2.4}
\]

for all \( y \in \mathbb{R}^p \), where \( * \) denotes the convolution. We want to show that \( (Q_n)_{n \in \mathbb{N}} \) converges weakly to \( Q \). By Lévy’s continuity theorem (cf. Chung, 1974, Theorem 6.3.2), it is sufficient to prove pointwise convergence of the Fourier transforms \( \widehat{Q}_n \) of
$Q_n$ to the corresponding Fourier transform $\hat{Q}$ of $Q$. For this purpose, note that the uniform continuity of $\varphi$ implies the equicontinuity of the family $\{\varphi \ast Q_n\}_{n \in \mathbb{N}}$. Hence, the convergence $\varphi \ast Q_n \to \varphi \ast Q$ is uniform on compact subsets of $\mathbb{R}^p$. Because $\varphi \ast Q$ is a density of a probability measure, there exists, for any given $\varepsilon > 0$, an open ball $B$ in $\mathbb{R}^p$ such that $\int_B \varphi \ast Q(y) \, dy \geq 1 - \varepsilon$. Choose $n$ large enough to satisfy $|\varphi \ast Q_n(y) - \varphi \ast Q(y)| \leq \varepsilon / \text{Vol}(B)$ for all $y$ in the compact set $\bar{B}$. Then, we have the following estimate:

$$\|\varphi \ast Q_n - \varphi \ast Q\|_{L_1} \leq \int_B |\varphi \ast Q_n(y) - \varphi \ast Q(y)| \, dy + \int_{B^c} \varphi \ast Q(y) \, dy.$$ 

Since

$$\int_{B^c} \varphi \ast Q(y) \, dy \leq 1 - \int_B \varphi \ast Q(y) \, dy + \int_B |\varphi \ast Q_n(y) - \varphi \ast Q(y)| \, dy,$$

we get $\|\varphi \ast Q_n - \varphi \ast Q\|_{L_1} \leq 4\varepsilon$, which proves $\varphi \ast Q_n \to \varphi \ast Q$ in $L_1$. This $L_1$-convergence implies the pointwise convergence $\varphi \ast Q_n \to \varphi \ast Q$ of the Fourier transforms. Now, $\varphi \ast Q_n = \bar{\varphi} \cdot \hat{Q}_n$, $\varphi \ast Q = \bar{\varphi} \cdot \hat{Q}$, and $\bar{\varphi}$ vanishes nowhere. Therefore, $\hat{Q}_n \to \hat{Q}$ pointwise. □

With the help of the preceding lemma, we can proceed in the proof of Theorem 1.1 the way we did to achieve the estimate (2.2).

**Proof of Theorem 1.1.** Choose $\gamma \in (0, \beta)$ such that $\beta - \gamma \neq \beta_c$, and denote by $\varphi$ the continuous density of the Gaussian distribution on $\mathbb{R}^p$ with mean zero and covariance matrix $(1/\gamma) \text{Id}_{\mathbb{R}^p}$. Let $Q_n$ denote the measure given by (1.4). We want to prove weak convergence of $(Q_n)_{n \in \mathbb{N}}$ to a Gaussian distribution $Q$ with mean zero and covariance matrix $C_\beta$. According to Lemma 2.5, it is sufficient to show (2.4) for all $y \in \mathbb{R}^p$. Note that $\varphi \ast Q$ is the convolution of two Gaussian distributions, therefore

$$\varphi \ast Q(y) = \left(2\pi \left(\frac{1}{\gamma} + c_\beta\right)\right)^{-p/2} \exp\left\{ -\frac{1}{2} \left(\frac{1}{\gamma} + c_\beta\right) \|y\|^2 \right\}, \quad y \in \mathbb{R}^p. \quad (2.5)$$

We want to show that $\varphi \ast Q_n \to \varphi \ast Q$ pointwise. Therefore, we need to investigate $\varphi \ast Q_n$. As an abbreviation, we set $\tilde{y}_n = y/\sqrt{n} + [1 - \beta(1 - z^+(\beta)^2)]^{-1} \Sigma^* \lambda_l$. By the definition of $P_{n, \beta, \xi}$, we have

$$\varphi \ast Q_n(y)$$

$$= \left(\frac{\gamma}{2\pi}\right)^{p/2} \frac{1}{Z_{n, \beta, \xi}} \left(P_{n, \beta, \xi} \left(\frac{\|S_n - \lambda_l\|}{n} < \varepsilon\right)\right)^{-1}$$

$$\times \int \exp\left\{ \frac{\gamma}{2} \left\|\sqrt{n} \left[\frac{S_n}{n} - (\lambda_l + \tilde{y}_n)\right]\right\|^2 + \frac{n\beta}{2} \left\|\frac{S_n}{n}\right\|^2 \right\} 1_{\{\|S_n/\lambda_l\| < \varepsilon\}} \, dP_{\sigma}. \quad (2.6)$$

By Corollary 2.4, there exists a constant $\gamma_0 > 0$ such that $\|\tilde{y}_n\| \leq \gamma_0 \varepsilon_n$ for all $n \in \mathbb{N}$. In particular, $\|\tilde{y}_n\| \leq \varepsilon/2$ holds for all sufficiently large $n$. Therefore, the integral
in (2.6) can be rewritten as
\[
\exp\left\{ -\frac{ny}{2} \lambda_l + \tilde{y}_n \right\} \int \exp\left\{ n \left[ \frac{\beta - \gamma}{2} \left\| S_n \right\|^2 + \left\langle \frac{S_n}{n}, \gamma (\lambda_l + \tilde{y}_n) \right\rangle \right\} dP_\sigma \\
+ \mathcal{O}\left( \exp\left\{ -\frac{\gamma}{8 n^2} \right\} \right) Z_{n,\beta,\varepsilon}.
\]

By an application of Lemma 1.5 (with \( \beta_n = \beta - \gamma, h_n = \gamma z^+ (\beta) \), and \( y_n = \gamma \tilde{y}_n \)) and a tedious but straightforward calculation using \( z(\beta - \gamma, \gamma z^+(\beta)) = z^+(\beta) \) in the case \( \beta > \beta_c \), and \( z^+(\beta - \gamma) = z^+(\beta) \), otherwise, this proves to be equal to
\[
\left[ 1 - (\beta - \gamma)(1 - z^+(\beta))^2 \right]^{-\nu/2} e^{\nu f_{\text{cow}}(\beta)} \exp\left\{ \frac{n\beta}{2} \frac{1}{1 - \beta(1 - z^+(\beta)^2) \left\| \sum_l \lambda_l \right\|^2} \right\} \\
\times \exp\left\{ \frac{\gamma}{2} \frac{1 - \beta(1 - z^+(\beta)^2)}{1 - (\beta - \gamma)(1 - z^+(\beta)^2) \left\| y^l \right\|^2} \right\} \left[ 1 + \mathcal{O}(n^{-\delta}) \right] \\
+ \mathcal{O}\left( \exp\left\{ -\frac{\gamma}{8 n^2} \right\} \right) Z_{n,\beta,\varepsilon}.
\]

Returning to (2.6) and applying Proposition 1.3(b), we get
\[
\varphi \ast Q_n(y) = \left( \frac{\gamma}{2\pi} \right)^{\nu/2} \left[ 1 - (\beta - \gamma)(1 - z^+(\beta))^2 \right]^{-\nu/2} \\
\times \exp\left\{ \frac{\gamma}{2} \frac{1 - \beta(1 - z^+(\beta)^2)}{1 - (\beta - \gamma)(1 - z^+(\beta)^2) \left\| y^l \right\|^2} \right\} \left[ 1 + \mathcal{O}(n^{-\delta}) \right] \\
+ \mathcal{O}\left( \exp\left\{ -\frac{\gamma}{8 n^2} \right\} \right) \left( P_{n,\beta,\varepsilon} \left( \left\| \frac{S_n}{n} - \lambda_l \right\| < \varepsilon \right) \right)^{-1}.
\]

Since
\[
\frac{\gamma}{1 - (\beta - \gamma)(1 - z^+(\beta)^2)} = \left( \frac{1}{\gamma} + c_{\beta} \right)^{-1},
\]
the proof is complete (cf. (2.5)) as soon as we have shown that the error term vanishes for \( n \to \infty \). This can be seen by recalling Proposition 1.3 and Corollary 2.4. \( \square \)

3. Proof of Lemma 1.5

The aim of this section is to prove Lemma 1.5, which was the key to the proofs of Theorem 1.1 and Proposition 1.3. Recall that we want to investigate the asymptotic behaviour of
\[
\int \exp\left\{ n \left[ \frac{\beta_n}{2} \left\| S_n \right\|^2 + \left\langle \frac{S_n}{n}, h_n \lambda_l + y_n \right\rangle \right\} \right\} dP_\sigma.
\]

As a first step, we will rewrite the integrand with the help of a Gaussian integration in such a way that the exponent's dependence on \( S_n \) becomes a linear one. This provides us with a situation where the expectation with respect to \( P_\sigma \) can be calculated. Thereby, we reduce the study of (3.1) to the study of integrals of the type
\[
\int_{\mathbb{R}_p} \exp\{ n \Psi_n(\lambda) \} d\lambda.
\]
where $\Psi_n: \mathbb{R}^p \rightarrow \mathbb{R}$ depends on $\xi$ and $y_n$. We will start by investigating a deterministic function $\Phi_n$ which is close to $\Psi_n$ for almost all $\xi$. The function $\Phi_n$ is given by

$$
\Phi_n(\lambda) = -\frac{1}{2\beta_n} \| \lambda - h_n e_i \|^2 + \frac{1}{d} \sum_{k=1}^d \log \cosh(\lambda, v_k)
$$

(3.3)

and may be considered as the annealed free-energy functional in the Hopfield model, while (in the special case $y_n = 0$) $\Psi_n$ plays the role of the corresponding quenched free-energy functional.

To study the integral (3.2) we employ Laplace's method. First, we will determine the set of maximizing points of $\Phi_n$ instead of those of $\Psi_n$. Having found these points, we split the domain of integration into three parts. The so-called outer region consists of all points bounded away from the set of maximizing points by a small but fixed radius. The inner region consists of balls with the maximizing points as centres. The radius of these balls is shrinking as $n \to \infty$. The intermediate region consists of the remaining domain.

A crude estimate suffices to show that the outer region does not contribute to the asymptotic behaviour of (3.2). The treatment of the intermediate and the inner region is more delicate. It is based on a Taylor expansion of the integrand's exponent $n\Psi_n$ around the maximizing points of $\Phi_n$. To achieve that the asymptotic behaviour is determined by the inner region exclusively, the radius of the shrinking balls has to be chosen carefully. Due to the influence of the random patterns, the usual way of choosing the radius proportional to $n^{-1/2}$ does not seem to work. But choosing the shrinking radius proportional to $n^{-\delta'}$ with $\delta' < \frac{1}{2}$ will suffice.

### 3.1. Gaussian integration

Fix now $\xi \in \Xi$ for the whole section. As announced, we will start by rewriting the integrand in (1.6) with the help of a Gaussian integration:

$$
\int \exp\left\{ n \left[ \beta_n \left\| \begin{array}{c} S_n \\ n \end{array} \right\| ^2 + \left\langle \begin{array}{c} S_n \\ n \end{array}, h_n e_i + y_n \right\rangle \right] \right\} dP_o
$$

(3.4)

$$
= \left( \frac{n^{\beta_n}}{2\pi} \right)^{p/2} \int_{\mathbb{R}^p} \exp\left\{ -\frac{n\beta_n}{2} \| \lambda \|^2 + \beta_n \langle \lambda, S_n \rangle + \langle S_n, h_n e_i + y_n \rangle \right\} d\lambda dP_o. 
$$

Next, we apply Fubini's theorem and calculate the integral with respect to $P_o$ using the identity

$$
\frac{1}{2^n} \sum_{\sigma \in \{-1,1\}^n} \prod_{i=1}^n \exp\{s_i x_i\} = \prod_{i=1}^n \cosh x_i
$$

with $x_i = (\beta_n \lambda + h_n e_i + y_n, \xi_i)$. By a change of variable which replaces the expression $\beta_n \lambda + h_n e_i + y_n$ by $\lambda$ and using the definition (1.3), we find that the right-hand side of (3.4) equals

$$
\left( \frac{n}{2\pi\beta_n} \right)^{p/2} \int_{\mathbb{R}^p} \exp\{n\Psi_\lambda(\lambda)\} d\lambda,
$$

(3.5)
where
\[
\Psi_n(\lambda) = -\frac{1}{2\beta_n} \|\lambda - (h_n e_l + y_n)\|^2 + \sum_{k=1}^d \left( \frac{1}{d} + \Delta_k^n \right) \log \cosh(\lambda, v_k).
\]

We want to apply Laplace’s method to calculate the integral (3.5). The terms in \(\Psi_n\) depending on \(y_n\) and \(\Delta_k^n\), respectively, will be considered as error terms and we will start by investigating \(\Phi_n\) given by (3.3) instead of \(\Psi_n\).

3.2. Investigation of \(\Phi_n\)

First, we need the set of all \(\lambda \in \mathbb{R}^p\) maximizing \(\Phi_n\). We will describe these \(\lambda\) with the help of the corresponding one-dimensional problem. By the identity (2.1), we have
\[
\Phi_n(\lambda) = \frac{1}{d} \sum_{k=1}^d \left[ -\frac{1}{2\beta_n} (\langle \lambda, v_k \rangle - \langle h_n e_l, v_k \rangle)^2 + \log \cosh(\lambda, v_k) \right].
\]

Therefore, \(\Phi_n\) attains its maximum if and only if each \(\langle \lambda, v_k \rangle\) is chosen optimally—provided such choices are possible. To show the existence of such choices of \(\lambda\), we consider on \(\mathbb{R}\) the one-dimensional mapping
\[
u \mapsto -\frac{1}{2\beta_n}(u - \langle h_n e_l, v_k \rangle)^2 + \log \cosh u,
\]
where we distinguish three cases:

(a) In the case \(h_n > 0\), the unique maximum of the mapping (3.6) is attained in
\[u_0 = \text{artanh}(\langle h_n, v_k \rangle) = \langle e_l, v_k \rangle \text{artanh}(\langle h_n, h_n \rangle),\]
where we used \((e_l, v_k) \in \{-1, 1\}\). Therefore, choosing all the scalar products \(\langle \lambda, v_k \rangle\) optimally means solving the \(d\) linear equations
\[\langle \lambda - u_0 e_l, v_k \rangle = 0, \quad k \in \{1, \ldots, d\}.
\]
By (2.1), these equations are equivalent to \(\lambda = |u_0| e_l\).

(b) In the case \(h_n = 0\) and \(\beta > \beta_c\), we know that \(\beta_n = \beta\), and the mapping (3.6) has two maxima which are attained in \(u_0 = \pm \text{artanh} z^+(\beta)\). Instead of the linear equations (3.7), we have to solve
\[\langle \lambda, v_k \rangle = \pm \text{artanh} z^+(\beta), \quad k \in \{1, \ldots, d\},
\]
where a priori the sign can be chosen independently for each \(k\). Note that some choices lead to systems of equations which cannot be solved. Let \(\tilde{v}\) denote the element of \(\{v_1, \ldots, v_d\}\) with ones in all components. For \(v \in \{1, \ldots, p\}\), we denote by \(k_v\) the unique element of \(\{1, \ldots, d\}\) which satisfies \(e_v = \frac{1}{2}(\tilde{v} - v_k)\). By the definition of \(k_v\), we find the necessary conditions
\[\langle \lambda, e_v \rangle \in \{\langle \lambda, \tilde{v}, 0\rangle, 0\}, \quad v \in \{1, \ldots, p\},
\]
which imply that the \(2p\) vectors \(|u_0| e_v, v \in L\), are the only ones which maximize \(\Phi_n\).

(c) In the case \(h_n = 0\) and \(\beta < \beta_c\), (3.6) has a unique maximum in \(u_0 = 0\). Therefore, \(\lambda = 0\) is the unique vector maximizing \(\Phi_n\).
In the sequel, we will use $z$ as an abbreviation for $\tanh |\mu_0|$, as defined in the statement of Lemma 1.5. Note that $z = z(\beta, h)$ in the case $h_n = h > 0$ and $z = z^+(\beta)$ otherwise. Let $\mu_v = \operatorname{artanh} z e_v$ for $v \in L_0$. We can summarize the results of the cases (a), (b) and (c) by noting that $\Phi_n$ attains its maximum on the set \( \{ \mu_v : v \in L_0 \} \), which does not depend on $n$. The maximal value of $\Phi_n$ equals

$$
\Phi_n(\mu_v) = - \frac{1}{2\beta_n} [\operatorname{artanh} z - h_n]^2 + \log \cosh \operatorname{artanh} z = f_{\sum}^n(\beta_n, h_n),
$$

with arbitrary $v \in L_0$, where we used the identities

$$
\operatorname{artanh} z = \beta_n z + h_n, \quad \cosh \operatorname{artanh} z = \frac{1}{\sqrt{1 - z^2}} \quad \text{and} \quad \operatorname{artanh} z = \frac{1}{2} \log \frac{1 + z}{1 - z}.
$$

3.3. Splitting the domain of integration

Next, we will split the integral in (3.5) into three parts, namely the outer region

$$
U_r = \left( \bigcup_{v \in L_0} B_r(\mu_v) \right)^c,
$$

where $B_r(x)$ denotes the open ball of radius $r$ with centre $x$, the intermediate region

$$
V_{r,R}^n = \bigcup_{v \in L_0} B_r(\mu_v) \setminus \bigcup_{v \in L_0} B_{R_n}(\mu_v)
$$

and the inner region

$$
W_R^n = \bigcup_{v \in L_0} B_{R_n}(\mu_v).
$$

We will choose $r > 0$ sufficiently small, $R > 0$ large and a sequence $(\alpha_n)_{n \in \mathbb{N}}$ tending to zero. Usually, $\alpha_n = n^{-1/2}$ is suitable. Here we have to compensate the influence of the patterns $\xi$ and need to modify the usual approach. Therefore, we will choose $\alpha_n = n^{-\delta'}$ with an arbitrary $\delta' \in (\delta, \frac{1}{2})$. This choice yields the error term $O(n^{-\delta})$ in the statement of Lemma 1.5 (and consequently in Proposition 1.3).

3.4. Estimate for the outer region

Let $\tau = 2(3\beta + h_1 + y_0/2)$. First we will show that the integral over $B_r(0)^c$ does not contribute to the asymptotic behaviour of the integral (3.5). We only consider $n$ satisfying $n \gg n(\xi)$, with $n(\xi)$ from Lemma 2.2. Since $1/d + \Delta^a_k \geq 0$ for all $k \in \{1, \ldots, d\}$ and $\sum_{k=1}^d \Delta^a_k = 0$, we can apply the estimate $\log \cosh(\lambda, v_k) \leq |\langle \lambda, v_k \rangle|$ together with Schwarz's inequality, Lemma 2.2 and Eq. (2.1) to get

$$
\sum_{k=1}^d \left( \frac{1}{d} + \Delta^a_k \right) \log \cosh(\lambda, v_k) \leq \sqrt{1 + 4 \varepsilon_n^2 \|\lambda\|} < 2 \|\lambda\|,
$$

because $\varepsilon_n < \frac{1}{2}$ for all $n \in \mathbb{N}$. Therefore,

$$
\Psi_n(\lambda) \leq - \frac{1}{2\beta_n} \|\lambda\| \left[ \|\lambda\| - 2(2\beta_n + h_1 + y_0/2) \right] \leq - \|\lambda\|
$$
for all $\lambda \in B_\varepsilon(0)^c$, which yields
\[
\int_{B_\varepsilon(0)^c} \exp\{n\Psi_n(\lambda)\} \, d\lambda \leq \int_{B_\varepsilon(0)^c} e^{-n\|\lambda\|} \, d\lambda = \omega_p \int_{\tau}^{\infty} e^{-nt} t^{p-1} \, dt, \tag{3.9}
\]
where $\omega_p$ denotes the surface area of the unit ball in $\mathbb{R}^p$. For $n \geq 2(p-1)/\tau$, we can show the following estimate by integration by parts
\[
\int_{\tau}^{\infty} e^{-nt} t^{p-1} \, dt \leq \frac{2}{n} e^{-n\tau t^{p-1}} = \mathcal{O}(e^{-nt}). \tag{3.10}
\]
Because \(\text{artanh} \, z = \beta_n z + h_n < \tau\), we can choose $r > 0$ small enough to satisfy $U_r^c \subset B_\varepsilon(0)$. In the case $|L_0| > 1$ we may as well assume that $r < \frac{1}{2} \text{artanh} \, z$ which assures that $U_r^c$ is the union of disjoint balls. Let
\[
q_{\beta_n} = \frac{1}{2} \|\lambda - \lim \nrightarrow_{n \rightarrow \infty} h_n e_1\|^2 + \frac{1}{d} \sum_{k=1}^{d} \log \cosh(\lambda, v_k).
\]
Then, there exists a constant $\varrho > 0$ such that on the compact set $U_r \cap \overline{B_\varepsilon(0)}$ the function $\Phi_\infty(\lambda)$ is bounded away from its maximal value by at least $4\varrho$, i.e. the estimate $\Phi_\infty(\lambda) - \Phi_\infty(\mu_v) \leq -4\varrho$ holds for all $\lambda \in U_r \cap \overline{B_\varepsilon(0)}$ and all $v \in L_0$. Because $\Phi_n \to \Phi_\infty$ uniformly on the compact set $\overline{B_\varepsilon(0)}$, there exists an $n_1 \in \mathbb{N}$ such that $|\Phi_n(\lambda) - \Phi_\infty(\lambda)| \leq \varrho$ for all $n \geq n_1$ and all $\lambda \in \overline{B_\varepsilon(0)}$. Therefore, for all these $n$ and all $\lambda \in U_r \cap \overline{B_\varepsilon(0)}$,
\[
\Phi_n(\lambda) - \Phi_n(\mu_v) \leq (\Phi_n(\lambda) - \Phi_\infty(\lambda)) + (\Phi_\infty(\lambda) - \Phi_\infty(\mu_v)) + (\Phi_\infty(\mu_v) - \Phi_n(\mu_v)) \leq -2\varrho.
\]
Consider those $n \geq \max\{n(\varepsilon), n_1\}$ which satisfy
\[
2\varepsilon_n \tau \leq \frac{\varrho}{2} \quad \text{and} \quad \frac{1}{\beta_1} [\tau + h_1] \|y_n\| \leq \frac{\varrho}{2}.
\]
Then, by Lemma 2.2, the above estimate and the identity (3.8),
\[
\Psi_n(\lambda) \leq \Phi_n(\lambda) + \varrho \leq \Phi_n(\mu_v) - \varrho = f_{\text{CW}}(\beta_n, h_n) - \varrho
\]
for all $\lambda \in U_r \cap \overline{B_\varepsilon(0)}$, which implies
\[
\int_{U_r \cap \overline{B_\varepsilon(0)}} \exp\{n\Psi_n(\lambda)\} \, d\lambda \leq \exp\{nf_{\text{CW}}(\beta_n, h_n) - n\varrho\} \, \text{Vol}(B_\varepsilon(0)). \tag{3.11}
\]
Combining (3.9)–(3.11) we get
\[
\int_{U_r} \exp\{n\Psi_n(\lambda)\} \, d\lambda \leq \mathcal{O}(e^{-nt}) + \mathcal{O}(e^{-n\varrho}) \exp\{nf_{\text{CW}}(\beta_n, h_n)\}. \tag{3.12}
\]
This estimate completes our investigation of the outer region.

3.5. Taylor expansion of $\Psi_n$

To investigate the intermediate and the inner region, we will replace $\Psi_n$ by its Taylor expansion with remainder. For this purpose, fix $v \in L_0$ and recall that $\mu_v$ maximizes
\( \Phi_n \), which implies grad \( \Phi_n(\mu_v) = 0 \). Convince yourself of \( \mu_v - h_n e_1 = \beta_n z e_v \), whether \( h_n \) depends on \( n \) or not. Furthermore, check \( \lambda = \frac{1}{d} \sum_{k=1}^d \langle \lambda, v_k \rangle v_k \) for \( \lambda \in \mathbb{R}^p \) (in case you need this equation) and recall the identity (3.8). With these facts in mind, the Taylor expansion of \( \Psi_n \) in \( \mu_v \) gives

\[
\Psi_n(\lambda) = f_c w(\beta_n, h_n) + \langle z e_v, y_n \rangle - \frac{1}{2\beta_n} \| y_n \|^2 + \left\langle \frac{y_n}{\beta_n} + \Sigma^n z e_v, \lambda - \mu_v \right\rangle
\]

\[
- \frac{1}{2\beta_n} [1 - \beta_n(1 - z^2)] \| \lambda - \mu_v \|^2 + R_n(\lambda - \mu_v),
\]

(3.13)

where the remainder \( R_n \) satisfies

\[
|R_n(\lambda)| \leq \frac{1}{3d} \sum_{k=1}^d |(v_k, \lambda)|^3 + \frac{1}{2} \sum_{k=1}^d |\Delta^k v_k, \lambda|^2 \leq \left[ \frac{\| \lambda \|^3}{3} + \epsilon_n \right] \sqrt{p} \| \lambda \|^2
\]

(3.14)

for all \( \lambda \in \mathbb{R}^p \), provided that \( n \geq n(\zeta) \) (see Lemma 2.2 for an estimate on \( \Delta^n \)).

3.6. Estimate for the intermediate region

First we will state some conditions on the scaling of the inner region. Let \((\alpha_n)_{n \in \mathbb{N}}\) be a decreasing sequence which satisfies \( \alpha_n \geq \epsilon_n \) for all sufficiently large \( n \in \mathbb{N} \) as well as \( \lim_{n \to \infty} \alpha_n \log \log n = 0 \) and \( \lim_{n \to \infty} (n \alpha_n^2)^{p/2} \exp\left\{-\frac{1}{2} \kappa_1 \alpha_n \right\} = 0 \), where

\[
\kappa_1 = \frac{3\beta}{1 - \beta(1 - z^2)} \kappa^2 \quad \text{with} \quad \kappa = \frac{y_0}{\beta_1} + 2\sqrt{p}\zeta.
\]

Keep in mind that the choice \( \alpha_n = n^{-\delta'} \) with \( \delta' \in (0, \delta) \) will suffice for our purpose. Consider those \( n \geq n(\xi) \) which are large enough to fulfill \( \alpha_n \geq \epsilon_n \) as well as \( \kappa_1 \alpha_n^2 > (p - 2)/n \) and

\[
\epsilon_n \sqrt{p} \leq \frac{1}{6\beta} [1 - \beta(1 - z^2)].
\]

(3.15)

Let \( r \) satisfy

\[
r \sqrt{p} \leq \frac{1}{2\beta} [1 - \beta(1 - z^2)]
\]

(3.16)

in addition to the condition needed for the treatment of the outer region. Choose

\[
R \geq \frac{6\beta}{1 - \beta(1 - z^2)} \kappa,
\]

(3.17)

and let

\[
R_0 = R - \frac{3\beta}{1 - \beta(1 - z^2)} \kappa.
\]

(3.18)
Fix $v \in L_0$ for the moment. By the Taylor expansion (3.13), the estimate (3.14) of the remainder, and the conditions (3.15) on $\varepsilon_n$ and (3.16) on $r$, we get

$$\int_{\{R_n \leq \|\lambda - m_n\| < r\}} \exp\{n\Psi_n(\lambda)\} \, d\lambda \leq \exp\{n\left[f_{\text{CW}}(\beta_n, h_n) + \langle zv, y_n \rangle - \frac{1}{2\beta_n} \|y_n\|^2\right]\} \times \int_{\{R_n \leq \|\lambda\| < r\}} \exp\left\{-\frac{n}{6\beta_n}[1 - \beta_n(1 - z^2)]\|\lambda\|^2 + n\left\langle \frac{y_n}{\beta_n} + \Sigma^n zv, \lambda \right\rangle\right\} \, d\lambda.

(3.19)

The integral on the right-hand side of (3.19) can be rewritten as

$$\int_{\{R_n \leq \|\lambda\| < r\}} \exp\left\{-\frac{3n\beta_n}{2[1 - \beta_n(1 - z^2)]}\|\frac{y_n}{\beta_n} + \Sigma^n zv\|^2\right\} \int_{\{R_n \leq \|\lambda\| < r\}} \exp\left\{-nF_n(\lambda)\right\} \, d\lambda,

(3.20)

where

$$F_n(\lambda) = \frac{1}{6\beta_n}[1 - \beta_n(1 - z^2)]\|\lambda - \frac{3\beta_n}{1 - \beta_n(1 - z^2)}\left\langle \frac{y_n}{\beta_n} + \Sigma^n zv\right\rangle|^2.$$

The estimate

$$\left\|\frac{y_n}{\beta_n} + \Sigma^n zv\right\| \leq \kappa \varepsilon_n,$n

(3.21)

which will be used frequently in the sequel, is a direct consequence of Corollary 2.4. By a change of variable, an application of this estimate and an enlargement of the domain of integration, we can estimate the integral in (3.20) from above by

$$\int_{\{\|\lambda\| > R_0\alpha_n\}} \exp\left\{-\frac{n}{6\beta_n}[1 - \beta_n(1 - z^2)]\|\lambda\|^2\right\} \, d\lambda \leq C_p \mathcal{O}\left((R_0\alpha_n)^p\right) \exp\left\{-\frac{n}{6\beta_n}[1 - \beta_n(1 - z^2)]R_0^2\alpha_n^2\right\},

(3.22)

where we used the estimate

$$\int_{t_0}^\infty \exp\left\{-\frac{t^2}{2\sigma^2}\right\} t^{p-1} \, dt \leq \frac{\sigma^2}{t_0^2 - (p - 2)\sigma^2} t_0^p \exp\left\{-\frac{t_0^2}{2\sigma^2}\right\},

(3.23)

which can be achieved similarly to the estimate (3.10) for all $\sigma^2 > 0$ and all $t_0 > 0$ such that $t_0^2 - (p - 2)\sigma^2 > 0$. Note that

$$\frac{n}{6\beta_n}[1 - \beta_n(1 - z^2)]R_0^2\alpha_n^2 \geq \frac{k_1}{2} n\alpha_n^2

by the definition (3.18) of $R_0$. With the help of (3.19), (3.20), and (3.22) and another application of the estimate (3.21), we get the following bound for the integral over the intermediate region:

$$\int_{\nu_{n, r}} \exp\{n\Psi_n(\lambda)\} \, d\lambda \leq \exp\left\{n\left[f_{\text{CW}}(\beta_n, h_n) - \frac{1}{2\beta_n} \|y_n\|^2\right]\right\} \sum_{v \in L_0} \exp\{n\langle zv, y_n \rangle\}

\times \mathcal{O}(\alpha_n^p) \exp\left\{-\frac{k_1}{2} n[\alpha_n^2 - \varepsilon_n^2]\right\}.

(3.24)
3.7. Estimate for the inner region

Choose again $R$ according to (3.17) and $n \geq n(\xi)$ large enough to satisfy $\alpha_n \geq \varepsilon_n$ as well as $\kappa_1 x_n^2 > (p - 2)/n$ and

$$\tilde{\alpha}_n = \left( \frac{R}{3} + 1 \right) \sqrt{p} \alpha_n \leq \frac{1}{4 \beta} [1 - \beta (1 - z^2)].$$

(3.25)

Note that for $\| \lambda \| < R \alpha_n$ the remainder in the Taylor expansion (3.13) satisfies $| R_n(\lambda) | \leq \tilde{\alpha}_n \| \lambda \|^2$. Similarly to the calculations (3.19) and (3.20), for fixed $\nu \in L_0$, we can estimate the integral

$$\int_{\{ \| \lambda - \mu \| < R \alpha_n \}} \exp \{ n \Psi_n(\lambda) \} \, d\lambda$$

(3.26)

from above (resp. below) by

$$\exp \left\{ n \left[ f_{CW}(\beta_n, h_n) + \langle \nu \omega, \frac{1}{2} \beta_n \| y_n \|^2 \right] \right\}$$

$$\times \exp \left\{ \frac{n \beta_n}{2} \left[ \frac{1}{1 - \beta_n (1 - z^2)} \right] + \frac{1}{2 \beta_n} \| \frac{y_n}{\beta_n} + \Sigma^\nu \omega \|^2 \right\}$$

$$\times \int_{B_{R \alpha_n}(0)} \exp \left\{ -n G_n^\pm(\lambda) \right\} \, d\lambda,$$

(3.27)

where

$$G_n^\pm(\lambda) = \frac{1 - \beta_n (1 - z^2) \mp 2 \beta_n \tilde{\alpha}_n}{2 \beta_n} \times \left\| \lambda - \frac{\beta_n}{1 - \beta_n (1 - z^2)} \mp \frac{2 \beta_n \tilde{\alpha}_n}{2 \beta_n} \left[ \frac{y_n}{\beta_n} + \Sigma^\nu \omega \right] \right\|^2.$$

The integral in (3.27) equals

$$\left( \frac{2 \pi \beta_n}{n} \right)^{n/2} \left[ 1 - \beta_n (1 - z^2) \mp 2 \beta_n \tilde{\alpha}_n \right]^{-n/2}$$

$$- \int_{B^c} \exp \left\{ - \frac{n}{2 \beta_n} \left[ 1 - \beta_n (1 - z^2) \mp 2 \beta_n \tilde{\alpha}_n \right] \| \lambda \|^2 \right\} \, d\lambda,$$

(3.28)

where $B$ denotes the ball of radius $R \alpha_n$ with centre

$$\frac{\beta_n}{1 - \beta_n (1 - z^2)} \mp \frac{2 \beta_n \tilde{\alpha}_n}{2 \beta_n} \left[ \frac{y_n}{\beta_n} + \Sigma^\nu \omega \right].$$

An application of the estimate (3.21) together with Condition (3.17) on $R$ and Condition (3.25) on $\tilde{\alpha}_n$ shows that the domain $B^c$ of integration in the integral (3.28) is contained in $B_{2R \alpha_n/3}(0)^c$. Therefore, another application of the estimate (3.23) yields that the integral in (3.28) is bounded above by

$$\psi(\alpha_n^2) \exp \left\{ -n \alpha_n^2 \frac{R^2}{9 \beta} [1 - \beta (1 - z^2)] \right\}.$$
By Condition (3.17) on $R$, we get
\[ n\alpha_n^2 \frac{R^2}{9\beta} [1 - \beta(1 - z^2)] \geq \frac{4}{3} \kappa_1 n\alpha_n^2. \]

Returning to (3.27) as an upper (resp. lower) bound for the integral (3.26), we find that
\[ \int \exp\{n\Psi_n(\lambda)\} d\lambda \]
is bounded above (resp. below) by
\[ \exp\left\{ n\left[ f_{cw}(\beta_n, h_n) - \frac{1}{2\beta_n} \|y_n\|^2 \right] \right\} \]
\[ \times \sum_{v \in L_0} \exp\left\{ \frac{n\beta_n}{2} \frac{1}{1 - \beta_n(1 - z^2)} \frac{1}{2\beta_n} \beta_n \right\} \left[ \frac{\gamma_n}{\beta_n} + \sum^n \gamma \right] \exp\left\{ \frac{4}{3} \kappa_1 n\alpha_n^2 \right\}. \] (3.29)

This estimate finishes our investigation of the inner region.

3.8. Conclusion

Now it remains to put the pieces together, namely the formulae (3.4), (3.5), (3.12), (3.24), and (3.29). Thereby, we get
\[ \int \exp\left\{ n\left[ \frac{\beta_n}{2} \left\| \frac{S_n}{n} \right\|^2 + \left\langle \frac{S_n}{n}, h_n e_i + y_n \right\rangle \right] \right\} dP_\sigma \]
\[ = [1 - \beta_n(1 - z^2)]^{-p/2} \exp\left\{ n\left[ f_{cw}(\beta_n, h_n) - \frac{1}{2\beta_n} \|y_n\|^2 \right] \right\} \]
\[ \times \sum_{v \in L_0} \exp\left\{ \frac{n\beta_n}{2} \frac{1}{1 - \beta_n(1 - z^2)} \frac{1}{2\beta_n} \beta_n \right\} \left[ \frac{\gamma_n}{\beta_n} + \sum^n \gamma \right] \exp\left\{ \frac{4}{3} \kappa_1 n\alpha_n^2 \right\} \]
\[ \times \left[ 1 + o\left( \exp\left\{ -n \min\{\tau, \eta\} \right\} \right) \right] + o\left( (n\alpha_n^2)^{p/2} \right) \exp\left\{ -\frac{\kappa_1}{2} n[x_n^2 - \varepsilon_n^2] \right\} \]
\[ + o(x_n \log \log n) + o\left( (n\alpha_n^2)^{p/2} \right) \exp\left\{ -\kappa_1 n\alpha_n^2 \right\}, \] (3.30)
where we used the estimate (3.21), Condition (3.25) on $x_n$, and the estimate
\[ \exp\left\{ o(n\alpha_n^2) \right\} = 1 + o(x_n \log \log n). \]

Choosing $x_n = n^{-\delta'}$ with $\delta' \in (\delta_1, 1/2)$ in (3.30) proves the lemma. \qed

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References


